Finite Fields and Their Applications 111 (2026) 102766

Contents lists available at ScienceDirect

Finite Fields and Their Applications

journal homepage: www.elsevier.com/locate/ffa

Linear codes arising from the point-hyperplane geometry-Part I: The Segre embedding

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ARTICLE INFO

Article history:
Received 26 June 2025
Received in revised form 7 November 2025
Accepted 22 November 2025
Available online 2 December 2025
Communicated by Gary L. Mullen

MSC:
51E22
94B05
14M12

Keywords:
Segre embedding
Point-hyperplane geometry
Projective code

ABSTRACT

Let V be a vector space over the finite field $\mathbb{F}_q$ with q elements and Λ be the image of the Segre geometry $\mathrm{PG}(V) \otimes \mathrm{PG}(V^*)$ in $\mathrm{PG}(V \otimes V^*)$ under the Segre map. Consider the subvariety Λ_1 of Λ represented by the pure tensors $x \otimes \xi$ with $x \in V$ and $\xi \in V^*$ such that $\xi(x) = 0$. Regarding Λ_1 as a projective system of $\mathrm{PG}(V \otimes V^*)$, we study the linear code $\mathscr{C}(\Lambda_1)$ arising from it. We show that $\mathscr{C}(\Lambda_1)$ is a minimal code and we determine its basic parameters, its full weight list and its linear automorphism group. We also give a geometrical characterization of its minimum and second lowest weight codewords as well as of some of the words of maximum weight.

1. Introduction

Following [29], an interesting and very geometric way to construct linear codes is to start from a *projective system*, that is a spanning set of points in a finite projective space. Indeed, if Ω is a set of N distinct points of an n-dimensional projective space over a finite

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https://doi.org/10.1016/j.ffa.2025.102766

field $\mathbb{F}_q$ then it is possible to construct a projective code $\mathscr{C}(\Omega)$ out of Ω by taking the coordinate representatives (with respect to some fixed reference system) of the points of Ω as columns of a generator matrix.

In general, $\mathscr{C}(\Omega)$ is not uniquely determined by Ω, but it turns out to be unique up to monomial equivalence; as such its metric properties with respect to Hamming's distance depend only on the set of points under consideration. With a slight abuse of notation, which is however customary when dealing with such projective codes, we shall speak of $\mathscr{C}(\Omega)$ as *the* code defined by $\Omega = \{[v_1], [v_2], \dots, [v_N]\}$ where the v_i's are fixed chosen vector representatives of the points of the projective system.

The parameters $[N, k, d]$ of $\mathscr{C}(\Omega)$ depend only on the point set Ω: clearly, the length N is the size of Ω and the dimension k is the (vector) dimension of the subspace spanned by Ω. The spectrum of the cardinalities of the intersections of Ω with the hyperplanes of $\mathrm{PG}(\langle\Omega\rangle)$ is used to determine the list of the weights of the code; in particular, the minimum Hamming distance d of $\mathscr{C}(\Omega)$ is

$$d = N - \max_H |\Omega \cap H|, \tag{1}$$

as H ranges among all hyperplanes of the space $\mathrm{PG}(\langle\Omega\rangle)$; we refer to [29] for further details.

This approach for obtaining codes from projective systems has been effectively applied to construct and study several families of projective codes, such as, for example, Hermitian codes [7] Grassmann codes [19,23], Schubert codes [18,20], polar Grassmann codes [8–10,13].

In this paper we shall consider one family of projective codes arising from a special subvariety of a Segre variety corresponding to the Segre embedding of the point-hyperplane geometry of $\mathrm{PG}(V)$.

Suppose that V is an $(n+1)$-dimensional vector space over a finite field $\mathbb{F}_q$ and let V^* be its dual. The tensor product $V \otimes V^*$ can be identified with the vector space $M_{n+1}(q)$ of all square matrices of order $n+1$ with coefficients in $\mathbb{F}_q$.

The Segre variety Λ of $\mathrm{PG}(V \otimes V^*) \cong \mathrm{PG}(M_{n+1}(q))$ is the variety whose point set consists of those projective points represented by pure tensors $x \otimes \xi$ with $x \in V$ and $\xi \in V^*$, i.e. by $(n+1)$-square matrices of rank 1 in $\mathrm{PG}(M_{n+1}(q))$. Another way of defining Λ is to regard it as the image under the Segre map of the Segre geometry $\Gamma = \mathrm{PG}(V) \otimes \mathrm{PG}(V^*)$; see Section 2.4 for more information.

The linear code $\mathscr{C}(\Lambda)$ arising from Λ has been introduced and extensively studied in [5]. In particular, not only the minimum distance but also the full Hamming weight distribution of $\mathscr{C}(\Lambda)$ is known (see [5, § 4]), as well as its higher Hamming weights.

Here, we investigate the subvariety Λ_1 of the Segre variety whose point set is represented by $(n+1)$-square matrices of rank 1 having null trace, or equivalently, defined by the pure tensors $x \otimes \xi$ with $x \in V$ and $\xi \in V^*$ such that $\xi(x) = 0$.

As mentioned before, the variety Λ_1 can also be regarded as the image under the Segre embedding of the point-hyperplane geometry of $\mathrm{PG}(V)$ (also called the *long root* *geometry* $\bar{\Gamma}$ for the special linear group $\mathrm{SL}(n+1,q)$). Briefly, the points of $\bar{\Gamma}$ are the point-hyperplane pairs (p,H) of $\mathrm{PG}(V)$ where $p \in H$. We refer to Section 2.5 for more information on $\bar{\Gamma}$.

We focus on the linear code $\mathscr{C}(\Lambda_1)$ constructed as explained at the beginning of the Introduction, starting from the projective system Λ_1. We determine the parameters of $\mathscr{C}(\Lambda_1)$ and its full weight list, as well as a description of its linear automorphism group; see [21] for the definitions. We prove that $\mathscr{C}(\Lambda_1)$ is a *minimal code* and we give a complete geometric characterization of the words having minimum weight and second lowest weight. We also characterize a remarkable family of words of maximum weight.

The following are the main results of the paper.

Theorem 1.1. *Suppose V is an $(n+1)$-dimensional vector space over $\mathbb{F}_q$ and let Λ_1 be the projective system of $\mathrm{PG}(V \otimes V^*)$ whose points are represented by the pure tensors $x \otimes \xi$ such that $\xi(x) = 0$. The $[N_1, k_1, d_1]$-linear code $\mathscr{C}(\Lambda_1)$ associated to Λ_1 has parameters*

$$N_1 = \frac{(q^{n+1}-1)(q^n-1)}{(q-1)^2}, \qquad k_1 = n^2 + 2n, \qquad d_1 = q^{2n-1} - q^{n-1}.$$

A direct consequence of Theorem 1.1 is that $\lim_{n\to\infty} \frac{d_1}{N_1} = 1$; so, the asymptotic minimum distance for these codes is actually good, even if the information rate $\lim_{n\to\infty} \frac{k_1}{N_1}$ goes to 0.

The following theorem gives information on the weight list of $\mathscr{C}(\Lambda_1)$ as well as its automorphism group.

Theorem 1.2. *Let $\mathscr{C}(\Lambda_1)$ be the linear code introduced in Theorem 1.1. The following hold.*

1. *The set of weights of $\mathscr{C}(\Lambda_1)$ is in bijective correspondence with the set*

$$\{(g_1,\dots,g_t)\colon \sum_{i=1}^{t} g_i \le n+1,\ 1 \le g_1 \le \cdots \le g_t \le n+1,\ 1 \le t \le q\} \cup \{0\}.$$

2. *The code admits an automorphism group isomorphic to the central product $\mathrm{PGL}(n+1,q) \cdot \mathbb{F}_q^\star$.*

Many properties of the geometry $\bar{\Gamma}$ will play a crucial role in giving information on the structure of the code $\mathscr{C}(\Lambda_1)$ arising from Λ_1.

In particular, the concept of *geometrical hyperplane of $\bar{\Gamma}$ arising from an embedding* recalled in Section 2.2 will be used to characterize codewords having minimum, second lowest or maximum weight in geometric terms. We refer to Section 2.6 for the definition and description of the hyperplanes of $\bar{\Gamma}$ mentioned in Theorem 1.3.

Theorem 1.3. *Let $\mathscr{C}(\Lambda_1)$ be the linear code introduced in Theorem 1.1. The following hold.*

1. *$\mathscr{C}(\Lambda_1)$ is a minimal code.*
2. *The minimum weight codewords of $\mathscr{C}(\Lambda_1)$ have weight $q^{2n-1} - q^{n-1}$ and correspond to the quasi-singular but non-singular hyperplanes of the point-hyperplane geometry $\bar{\Gamma}$. These hyperplanes arise from (diagonalizable) matrices of rank 1 and non-null trace.*
   The second lowest weight codewords of $\mathscr{C}(\Lambda_1)$ have weight q^{2n-1} and correspond to the singular hyperplanes of $\bar{\Gamma}$. They arise from (non-diagonalizable) matrices of rank 1 and null trace.
3. *The maximum weight codewords of $\mathscr{C}(\Lambda_1)$ have weight $q^{n-1}(q^{n+1} - 1)/(q - 1)$ and correspond to matrices admitting no eigenvalue in $\mathbb{F}_q$. Every spread-type hyperplane of $\bar{\Gamma}$ is associated to a maximum weight codeword. Conversely, those maximum weight codewords arising from matrices having a minimal polynomial of degree 2 are associated to spread-type hyperplanes of $\bar{\Gamma}$.*

1.1. *Organization of the paper*

In Section 2 we will set the notation (Subsection 2.1) and recall all the basics we need about embeddings of geometries (Subsection 2.2), minimal codes (Subsection 2.3), the Segre geometry (Subsection 2.4), the point-hyperplane geometry $\bar{\Gamma}$ of $\mathrm{PG}(V)$, its embeddings (Subsection 2.5) and the saturation form (Subsection 2.6). We will then define the quasi-singular, singular and spread-type hyperplanes of $\bar{\Gamma}$ (Subsection 2.7).

Section 3 is focused on the code arising from the variety Λ_1 of the pure tensors $x \otimes \xi$ with $x \in V$ and $\xi \in V^*$ such that $\xi(x) = 0$ and here we will prove Theorems 1.1, 1.2 and 1.3.

2. Notation and basics

2.1. *Notation*

Let $V = V(n + 1, \mathbb{F}_q)$ be an $(n + 1)$-dimensional vector space over the finite field $\mathbb{F}_q$ and V^* its dual. Henceforth we always assume that $E = (e_i)_{i=1}^{n+1}$ is a given basis of V and $E^* = (\eta_i)_{i=1}^{n+1}$ is the dual of E; hence a basis of V^*. We shall regard a vector $x = \sum_{i=1}^{n+1} e_i x_i$ of V, represented by the $(n + 1)$-tuple $(x_i)_{i=1}^{n+1}$ of its coordinates with respect to E, as a column, namely an $(n + 1) \times 1$-matrix.

Similarly, every vector $\xi = \sum_{i=1}^{n+1} \xi_i \eta_i \in V^*$ is regarded with respect to E^* as a $1 \times (n+1)$ matrix $(\xi_1, \xi_2, \ldots, \xi_{n+1})$. Clearly, $\xi x = 0$ in terms of the usual row-by-column product if and only if $\xi(x) = 0$, that is $\sum_{i=1}^{n+1} \xi_i x_i = 0$. Henceforth we shall always use Greek lower case letters to denote vectors of V^* (regarded as *row* vectors) and Roman lower case letters to denote vectors of V (regarded as column vectors).

The tensor product $V \otimes V^*$ is isomorphic to the vector space $M_{n+1}(q)$ of the square matrices of order $n+1$ with entries in $\mathbb{F}_q$. Hence, we will freely switch from the matrix notation to the tensor notation and conversely, whenever these changes of notation will be convenient. The elements $E \otimes E^* = \{e_i \otimes \eta_j\}_{1 \leq i,j \leq n+1}$ form a basis of $V \otimes V^*$. If we map $e_i \otimes \eta_j$ to the elementary matrix $\mathbf{e}_{ij}$ whose only non-zero entry is 1 in position (i,j), we see that this gives the isomorphism $\phi\colon V \otimes V^* \to M_{n+1}(q)$ described by

$$x \otimes \xi \in V \otimes V^* \mapsto \begin{pmatrix} x_1 \\ \vdots \\ x_{n+1} \end{pmatrix} (\xi_1 \quad \dots \quad \xi_{n+1}) \in M_{n+1}(q),$$

where

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_{n+1} \end{pmatrix} \qquad \text{and} \qquad \xi = (\xi_1 \quad \dots \quad \xi_{n+1}).$$

Thus, the tensor $x \otimes \xi \in V \otimes V^*$ can be regarded as the column-times-row product $x\xi$ and for a matrix $M \in M_{n+1}(q)$, the product $\xi M x$ is the scalar obtained as the product of the row ξ times M times the column x.

Turning to projective spaces, let $\mathrm{PG}(n,q) = \mathrm{PG}(V)$ be the n-dimensional projective space defined by V. When we need to distinguish between a non-zero vector x of V and the point of $\mathrm{PG}(V)$ represented by it, we denote the latter by $[x]$. We extend this convention to subsets of V. If $X \subseteq V\setminus\{0\}$ then $[X] := \{[x] | x \in X\}$. The same conventions will be adopted for vectors and subsets of V^* and $V \otimes V^*$. In particular, if $\xi \in V^* \setminus \{0\}$ then $[\xi]$ is the point of $\mathrm{PG}(V^*)$ which corresponds to the hyperplane $[\ker(\xi)]$ of $\mathrm{PG}(V)$. In the sequel we shall freely take $[\xi]$ as a name for $[\ker(\xi)]$. Accordingly, if $0^* \in V^*$ is the null functional, we write $[0^*] = \mathrm{PG}(V)$ (or, more simply, $[0] = \mathrm{PG}(V)$).

2.2. Embeddings and hyperplanes of point-line geometries

The approach we follow to construct our codes is to start from a point-line geometry and then embed it into a projective space, so that its image will be a spanning set, hence a projective system, for the ambient projective space.

More precisely, let $\Gamma = (\mathscr{P}, \mathscr{L})$ be a point-line geometry with point set $\mathscr{P}$, line-set $\mathscr{L}$ and incidence given by inclusion. The collinearity graph G_Γ of Γ is the graph whose vertices are the points $p \in \mathscr{P}$ and whose edges are the pairs $(p,q) \in \mathscr{P} \times \mathscr{P}$ where p is collinear with q.

A *subspace* of a point-line geometry Γ is a non-empty subset X of the point set $\mathscr{P}$ of Γ such that, for every line ℓ of Γ, if $|\ell \cap X| > 1$ then $\ell \subseteq X$. A proper subspace H of Γ is said to be a *geometric hyperplane* of Γ (a *hyperplane* of Γ for short) if for every line ℓ of Γ, either $\ell \subseteq H$ or $|\ell \cap H| = 1$. By definition, hyperplanes are proper subspaces of Γ but in general they are not necessarily all *maximal* with respect to inclusion among all the proper subspaces. An important characterization of *maximal hyperplanes* is the following, see [27, Lemma 4.1.1].

Proposition 2.1. *A geometric hyperplane H of a point-line geometry Γ is a maximal subspace if and only if the collinearity graph induced by G_Γ on $\mathscr{P} \setminus H$ is connected.*

For more details we refer the reader to [26] and [27].

A *projective embedding* $\varepsilon\colon \Gamma \to \Sigma$ in a projective space Σ is an injective mapping ε from the point set $\mathscr{P}$ of Γ to the point set of a projective space Σ such that $\varepsilon(\mathscr{P})$ spans Σ and it maps lines of Γ onto projective lines of Σ. The *dimension of* ε is the vector dimension of Σ.

Given a projective embedding $\varepsilon\colon \Gamma \to \Sigma$ of Γ and a projective hyperplane W of Σ, the point set $\mathscr{W} := \varepsilon^{-1}(W) \subseteq \Gamma$ is a geometric hyperplane of Γ and $\varepsilon(\varepsilon^{-1}(W)) = W \cap \varepsilon(\mathscr{P})$.

We say that a geometric hyperplane $\mathscr{W}$ of Γ *arises from* ε if $\varepsilon(\mathscr{W})$ spans a projective hyperplane of Σ and $\mathscr{W} = \varepsilon^{-1}(W) \subseteq \Gamma$ for some hyperplane W of Σ. Note that, in general, it is possible that there are many hyperplanes of Γ which do not arise from any embedding.

2.3. Minimal codewords and minimal codes

Let $\mathscr{C} = \mathscr{C}(\Omega)$ be a projective $[N, k, d]$-code, where $\Omega \subseteq \mathrm{PG}(k-1, q)$ is the defining projective system. For any $c = (c_1, \dots, c_N) \in \mathscr{C}$ the *support* of c is the set $\mathrm{supp}(c) = \{i : c_i \neq 0\}$. Write also $\mathrm{supp}^c(c) := \{i : c_i = 0\} = \{1, \dots, N\} \setminus \mathrm{supp}(c)$.

Massey in 1993 [22] introduced the notion of *minimal codewords* (or minimal vectors) in a code, in order to devise an efficient secret sharing scheme. Minimal codewords have also numerous applications besides cryptography; e.g. they are relevant for bounding the complexity of some decoding algorithms; see [4].

Minimal codewords and minimal codes are defined as follows.

Definition 2.2. A codeword $c \in \mathscr{C}$ is *minimal* if

$$\forall c' \in \mathscr{C} : \mathrm{supp}(c') \subseteq \mathrm{supp}(c) \Rightarrow \exists \lambda \in \mathbb{F}_q : c' = \lambda c.$$

A code is *minimal* if all its codewords are minimal.

Minimal codes have been extensively investigated, since they are also amenable to efficient decoding [4], see also [15,2,3,6,12].

Ashikhmin and Barg in 1998 [4] determined a well-known and widely used sufficient condition for a code $\mathscr{C}$ to be minimal:

$$\frac{w_{\max}}{w_{\min}} < \frac{q}{q-1}, \tag{2}$$

where $w_{\min}$ and $w_{\max}$ are respectively the minimum and the maximum weight of the non-null codewords of $\mathscr{C}$. The aim is to determine codes which are minimal but do not satisfy condition (2); see e.g. [6].

The notion of *strong* or *cutting set* has been introduced in [16].

Definition 2.3. Let $\Omega \subseteq \mathrm{PG}(\langle\Omega\rangle)$ be a projective system. Then Ω is a *cutting set* (with respect to the hyperplanes) if and only if for any hyperplane H of $\mathrm{PG}(\langle\Omega\rangle)$,

$$\langle H \cap \Omega\rangle = H.$$

In 2021 Alfarano et al. [1] proved that projective minimal codes and cutting sets with respect to hyperplanes are equivalent objects (see also [28]).

Relying on the notion of cutting sets and on Proposition 2.1, we have proved in [12] the following.

Proposition 2.4. *Let $\Gamma := (\mathscr{P}, \mathscr{L})$ be a point line geometry and $\varepsilon : \Gamma \to \mathrm{PG}(V)$ a projective embedding. If the graph induced on the vertices $\mathscr{P} \setminus \varepsilon^{-1}(H)$ by the collinearity graph of Γ is connected for any hyperplane H of $\mathrm{PG}(V)$, then the projective code $\mathscr{C}(\varepsilon(\Gamma))$ is minimal.*

Proposition 2.5. *Suppose that $\Gamma = (\mathscr{P}, \mathscr{L})$ is a point-line geometry where every geometric hyperplane is a maximal subspace. Then the projective code $\mathscr{C}(\varepsilon(\Gamma))$ is minimal, for any projective embedding ε of Γ.*

2.4. The Segre geometry and its natural embedding

Suppose $\mathrm{PG}(V_1)$ and $\mathrm{PG}(V_2)$ are two given projective spaces having respective point sets $\mathscr{P}_1$ and $\mathscr{P}_2$ and respective line sets $\mathscr{L}_1$ and $\mathscr{L}_2$. The *Segre geometry* (of type $(\dim(\mathrm{PG}(V_1)), \dim(\mathrm{PG}(V_2))))$ is defined as the point-line geometry Γ having as point set the Cartesian product $\mathscr{P}_1 \times \mathscr{P}_2$ and as line set the following set:

$$\{\{p_1\} \times \ell_2 \colon p_1 \in \mathscr{P}_1, \ell_2 \in \mathscr{L}_2\} \cup \{\ell_1 \times \{p_2\} \colon \ell_1 \in \mathscr{L}_1, p_2 \in \mathscr{P}_2\};$$

incidence is given by inclusion. It is well known that this geometry, which can be regarded as the product $\mathrm{PG}(V_1) \otimes \mathrm{PG}(V_2)$, admits a projective embedding (the Segre embedding) in $\mathrm{PG}(V_1 \otimes V_2)$ mapping the point $([p], [q]) \in \mathscr{P}_1 \times \mathscr{P}_2$ to the projective point $[p \otimes q] \in \mathrm{PG}(V_1 \otimes V_2)$. This embedding lifts the automorphism group $\mathrm{PGL}(V_1) \times \mathrm{PGL}(V_2)$ of Γ to act on $\mathrm{PG}(V_1 \otimes V_2)$. The geometric hyperplanes of the Segre geometries have been fully classified and described; see [30].

In this paper we shall be concerned with the case where $V_1 = V(n+1, \mathbb{F}_q)$ and $V_2 = V_1^*$. Thus, the points of Γ are all the ordered pairs $([p], [\xi])$ where $[p]$ and $[\xi]$ are respectively a point and a hyperplane of $\mathrm{PG}(n, q)$. We will denote by ε the Segre embedding of Γ in $\mathrm{PG}(V \otimes V^*)$, also called the *natural embedding of* Γ. Since $V \otimes V^*$ is isomorphic to the vector space $M_{n+1}(q)$ of the square matrices of order $n+1$ with entries in $\mathbb{F}_q$ (see Subsection 2.1), the Segre embedding is given by

$$\varepsilon\colon \Gamma \to \mathrm{PG}(M_{n+1}(q)),\ \ \varepsilon(([x],[\xi])) = [x \otimes \xi]. \tag{3}$$

The linear automorphism group $\mathrm{PGL}(V) \otimes \mathrm{PGL}(V^*)$ of Γ acts on the pure tensors of $\mathrm{PG}(M_{n+1}(q))$ as

$$([M],[N]) : [x \otimes \xi] \to [Mx \otimes \xi N],\ \ \forall [M],[N] \in \mathrm{PGL}(n+1,q)$$

and this action extends to all of $\mathrm{PG}(M_{n+1}(q))$ by linearity. The pure tensors $x \otimes \xi \in V \otimes V^*$ with $0 \neq x \in V$ and $0 \neq \xi \in V^*$ yield the matrices of $M_{n+1}(q)$ of rank 1. With x and ξ as above, let $[x]$ and $[\xi]$ be the point and the hyperplane of $\mathrm{PG}(n,q)$ represented by x and ξ. Then the point set of Γ is precisely the set $\{([x],[\xi]), [x] \in \mathrm{PG}(V), [\xi] \in \mathrm{PG}(V^*)\}$ and the image of Γ under the Segre embedding is

$$\Lambda := \varepsilon(\Gamma) = \{[x \otimes \xi]\colon [x] \in \mathrm{PG}(V), [\xi] \in \mathrm{PG}(V^*)\}, \tag{4}$$

also called the *Segre variety of* $\mathrm{PG}((n+1)^2-1,q)$. Accordingly, Λ is represented by the set of all $(n+1)$-square matrices of rank 1.

Regarding Λ as a projective system of $\mathrm{PG}(M_{n+1}(q))$, we can consider the linear code $\mathscr{C}(\Lambda)$ defined by it. This is called the *Segre code*. It is easy to determine the length N and the dimension k of it. Indeed, N is the number of point-hyperplane pairs of $\mathrm{PG}(n,q)$ and k is the dimension of the Segre embedding. The minimum distance as well as the full weight enumerator is also known.

Proposition 2.6 *(See [5]).* $\mathscr{C}(\Lambda)$ *is an* $[N,k,d]$*-code with*

$$N = \frac{(q^{n+1}-1)}{(q-1)}\frac{(q^{n+1}-1)}{(q-1)}, \qquad k = (n+1)^2, \qquad d = q^{2n}.$$

2.5. The point-hyperplane geometry of a projective space

Let $\bar{\Gamma}$ be the subgeometry of Γ having as points, all the points (p,H) of Γ with the further requirement that $p \in H$. Two points are collinear in $\bar{\Gamma}$ if and only if they are collinear in Γ; explicitly, the points (p,H) and (p',H') are collinear in $\bar{\Gamma}$ if and only if $p \in H'$ or $p' \in H$.

If $p := [x]$ and $H := [\xi]$, then the point $([x],[\xi])$ of Γ is a point of $\bar{\Gamma}$ if and only if $\xi(x) = 0$. The geometry $\bar{\Gamma}$ is called the *point-hyperplane geometry of* $\mathrm{PG}(V)$ or, also, the *long root geometry for the special linear group* $\mathrm{SL}(n+1,\mathbb{F}_q)$. The linear automorphism group of $\bar{\Gamma}$ is $\mathrm{PGL}(n+1,q)$ and it acts transitively on the points of $\bar{\Gamma}$.

The group $\mathrm{PGL}(n+1,q)$ lifts through the Segre embedding (see Definition (3)) ε to a subgroup of the automorphism group $\mathrm{PGL}(n^2+2n,q)$ of $\mathrm{PG}(M_{n+1}(q))$. In particular, it acts on the pure tensors as follows

$$g([x],[\xi]) \xrightarrow{\varepsilon} [gx \otimes \xi g^{-1}],\ \forall g \in \mathrm{PGL}(n+1,q) \tag{5}$$

and this action can be extended to all elements of $\mathrm{PG}(M_{n+1}(q))$ by linearity. Consequently, $\mathrm{PGL}(n+1,q)$ acts on the matrix representatives of the elements of $\mathrm{PG}(M_{n+1}(q))$ by conjugation and its projective orbits correspond to the conjugacy classes of matrices up to a non-zero proportionality coefficient. The following is trivial.

Proposition 2.7. *Suppose $x \in V$ and $\xi \in V^*$. The vector $x \otimes \xi \in V \otimes V^*$, regarded as an $(n+1)$-square matrix of rank 1, is null-traced if and only if $\xi(x) = 0$.*

Denote by $M^0_{n+1}(q)$ the hyperplane of $M_{n+1}(q)$ of null-traced square matrices of order $n+1$, then by Proposition 2.7 we have $\varepsilon^{-1}(M^0_{n+1}(q)) = \bar{\Gamma}$. So, according to Section 2.2, $\bar{\Gamma}$ is a geometric hyperplane of Γ arising from ε; in the terminology of [30, Lemma 3], $\bar{\Gamma}$ is a so-called *black hyperplane* of Γ; all black hyperplanes of Γ correspond to subgeometries of Γ isomorphic to $\bar{\Gamma}$, but these hyperplanes do not arise, in general, from ε. We point out that $M^0_{n+1}(q)$ is also the module which hosts the *adjoint representation* of the special linear group $\mathrm{SL}(n+1,\mathbb{F}_q)$.

Let now $\bar{\varepsilon}$ be the projective embedding of $\bar{\Gamma}$ obtained as the restriction $\varepsilon|_{\bar{\Gamma}}$ of ε to $\bar{\Gamma}$. From Proposition 2.7 we have

$$\bar{\varepsilon}\colon \bar{\Gamma} \to \mathrm{PG}(M^0_{n+1}(q)),\ \bar{\varepsilon}(([x],[\xi])) = [x \otimes \xi]. \tag{6}$$

This is a projective embedding of $\bar{\Gamma}$ with dimension $\dim(\bar{\varepsilon}) = \dim(M^0_{n+1}(q)) = (n+1)^2 - 1$. Define

$$\Lambda_1 := \bar{\varepsilon}(\bar{\Gamma}) = \{[x \otimes \xi]\colon [x] \in \mathrm{PG}(V), [\xi] \in \mathrm{PG}(V^*) \text{ and } [x] \in [\xi]\}. \tag{7}$$

Suppose now that $\mathbb{F}_q$ admits non-trivial automorphisms and take $\sigma \in \mathrm{Aut}\,(\mathbb{F}_q)$, $\sigma \neq 1$. It is possible to define a 'twisted version' $\bar{\varepsilon}_\sigma$ of $\bar{\varepsilon}$ as follows (see [17] and [25]):

$$\bar{\varepsilon}_\sigma\colon \Gamma \to \mathrm{PG}(M_{n+1}(q)),\ \bar{\varepsilon}_\sigma(([x],[\xi])) = [x^\sigma \otimes \xi],$$

where $x^\sigma := ({x_i}^\sigma)_{i=1}^{n+1}$. The map $\bar{\varepsilon}_\sigma$ is again a projective embedding of $\bar{\Gamma}$ with dimension $\dim(\bar{\varepsilon}_\sigma) = (n+1)^2$. Put now

$$\Lambda_\sigma := \bar{\varepsilon}_\sigma(\bar{\Gamma}) = \{[x^\sigma \otimes \xi]\colon [x] \in \mathrm{PG}(V), [\xi] \in \mathrm{PG}(V^*) \text{ and } [x] \in [\xi]\}. \tag{8}$$

In this paper we shall study in detail the properties of the projective system Λ_1 since we want to construct a linear code from it. In the forthcoming paper [11], which is a direct continuation of the present work, we shall focus on Λ_σ and on the code arising from it. As we will soon illustrate, many properties of the geometry $\bar{\Gamma}$ play a crucial role in providing information on the structure of the code arising from Λ_1. In particular, some families of geometric hyperplanes arising from $\bar{\varepsilon}$ will be used to characterize in geometrical terms all the codewords having minimum weight, second lowest weight and some codewords having maximum weight.

2.6. The saturation form

Let $f\colon M_{n+1}(q) \times M_{n+1}(q) \to \mathbb{F}_q$ be the non-degenerate symmetric bilinear form defined as

$$f(X,Y) = \mathrm{Tr}(XY),\ \ \forall\, X,Y \in M_{n+1}(q), \tag{9}$$

where XY is the usual row-times-column product and $\mathrm{Tr}(XY)$ is the trace of the matrix XY. So, with $X = (x_{i,j})_{i,j=1}^{n+1}$ and $Y = (y_{i,j})_{i,j=1}^{n+1}$ we have

$$f((x_{i,j})_{i,j=1}^{n+1}, (y_{i,j})_{i,j=1}^{n+1}) = \sum_{i,j} x_{i,j}y_{j,i}.$$

Note that this definition does not depend on the choice of the basis of $M_{n+1}(q)$. The form f is called *the saturation form* of $M_{n+1}(q)$ (see [25]).

Denote by $\perp_f$ the orthogonality relation associated to f. Since f is nondegenerate, the hyperplanes of $M_{n+1}(q)$ are the orthogonal spaces $M^{\perp_f} = \{X \in M_{n+1}(q)\colon \mathrm{Tr}(XM) = 0\}$, for $M \in M_{n+1}(q) \setminus \{O\}$ and, for two matrices $M, N \in M_{n+1}(q) \setminus \{O\}$, we have $M^{\perp_f} = N^{\perp_f}$ if and only if M and N are proportional matrices.

By Definition (9), it is clear that $I^{\perp_f} = M^0_{n+1}(q)$, where I is the identity matrix. Therefore, for $M \in M_{n+1}(q)\setminus\{O\}$, we have $M^{\perp_f} = M^0_{n+1}(q)$ if and only if $M = \lambda I$ with $\lambda \in \mathbb{F}_q\setminus\{0\}$ i.e. M is a non-null scalar matrix. Hence, every hyperplane of $M^0_{n+1}(q)$ can be obtained as $M^0_{n+1}(q) \cap M^{\perp_f} = I^{\perp_f} \cap M^{\perp_f}$ for a suitable matrix $M \in M_{n+1}(q),\ M \notin \langle I\rangle$. In terms of projective spaces, for $M \in M_{n+1}(q) \setminus \langle I\rangle$, $[M^{\perp_f} \cap M^0_{n+1}(q)]$ is the projective hyperplane of $[M^0_{n+1}(q)]$ corresponding to the hyperplane $M^{\perp_f} \cap M^0_{n+1}(q)$ of $M^0_{n+1}(q)$.

The next result follows from well-known properties of polarities associated to non-degenerate reflexive bilinear forms; see [25, Proposition 2.1].

Proposition 2.8. *For $M, N \in M_{n+1}(q)\setminus\{O\}$, we have $M^{\perp_f} \cap M^0_{n+1}(q) = N^{\perp_f} \cap M^0_{n+1}(q)$ if and only if $\langle M, I\rangle = \langle N, I\rangle$.*

The orthogonal space of a pure tensor, namely the orthogonal space of a matrix of rank 1, admits an easy description. Indeed,

Proposition 2.9. *Let $x \in V \setminus \{O\}$, $\xi \in V^* \setminus \{O\}$ and $M \in M_{n+1}(q)$. Then $x \otimes \xi \in M^{\perp_f}$ if and only if $\xi M x = 0$.*

Proof. By definition, $M^{\perp_f} = \{X \in M_{n+1}(q)\colon \mathrm{Tr}(XM) = 0\}$. Hence $x \otimes \xi \in M^{\perp_f}$ if and only if $\mathrm{Tr}((x\xi)M) = 0$ if and only if $\mathrm{Tr}(x(\xi M)) = 0$. By Proposition 2.7, this is equivalent to $(\xi M)(x) = 0$. Turning to projective spaces, this means that the projective point $[x]$ belongs to the hyperplane $[\xi M]$ if $M \neq 0$ or that it (trivially) belongs to the space $\mathrm{PG}(V) = [0^*]$ if $M = 0$. $\square$

In more geometrical terms, by Proposition 2.9 we have that the point $[x \otimes \xi]$ is contained in $[M^{\perp_f}]$ if and only if the point $[x]$ is contained in the hyperplane $[\xi M]$.

2.7. Hyperplanes of $\bar{\Gamma}$

In this section we will briefly recall from [24] and [25] the most significant results related to the hyperplanes of $\bar{\Gamma}$ arising from the embedding $\bar{\varepsilon}$.

Take $M \in M_{n+1}(q) \setminus \langle I \rangle$ and let $\bar{\varepsilon}$ be the Segre embedding of $\bar{\Gamma}$, as defined in (6). Then

$$\mathscr{H}_M := \bar{\varepsilon}^{-1}([M^{\perp_f} \cap M^0_{n+1}(q)]) = \bar{\varepsilon}^{-1}(\{[X] \in \mathrm{PG}(M^0_{n+1}(q))\colon \mathrm{Tr}(XM) = 0\})$$

is a geometric hyperplane of $\bar{\Gamma}$ called a *hyperplane of plain type*, as defined in [25]. By Proposition 2.8, given any two matrices M and M' we have $\mathscr{H}_M = \mathscr{H}_{M'}$ if and only if $M = \alpha M' + \beta I$ with $(\alpha, \beta) \neq (0, 0)$.

Recall now the definition of hyperplanes of $\bar{\Gamma}$ arising from an embedding from Section 2.2.

Proposition 2.10. *[25, Corollary 1.7] The hyperplanes of $\bar{\Gamma}$ which arise from the Segre embedding $\bar{\varepsilon}$ are precisely those of plain type.*

Take $p \in \mathrm{PG}(V)$, $A \in \mathrm{PG}(V^*)$. Put $\mathscr{M}_p := \{(p, H) : p \in H\}$ and $\mathscr{M}_A := \{(x, A) : x \in A\}$. Then,

$$\mathscr{H}_{p,A} := \{(x, H)\colon (x, H) \text{ collinear (in } \bar{\Gamma}) \text{ with a point of } \mathscr{M}_p \cup \mathscr{M}_A\} \tag{10}$$

is a geometric hyperplane of $\bar{\Gamma}$, called the *quasi-singular hyperplane* defined by (p, A). If $p \in A$, then $\mathscr{H}_{p,A}$ is called *the singular hyperplane with deepest point* (p, A) and consists of all points of $\bar{\Gamma}$ not at maximal distance from (p, A) in the collinearity graph of $\bar{\Gamma}$.

Proposition 2.11. *The following hold.*

1. *The cardinality of the singular hyperplanes of $\bar{\Gamma}$ is*

$$\frac{(q^{n+1}-1)(q^{n-1}-1)}{(q-1)^2} + \frac{q^n-1}{q-1} q^{n-1}. \tag{11}$$

2. *The cardinality of the quasi-singular but not singular hyperplanes of $\bar{\Gamma}$ is*

$$\frac{(q^{n+1}-1)(q^{n-1}-1)}{(q-1)^2} + (\frac{q^n-1}{q-1} + 1) q^{n-1}. \tag{12}$$

Proof. Suppose $\mathscr{H}_{p,A}$ with $p \in \mathrm{PG}(V)$ and $A \in \mathrm{PG}(V^*)$ is a quasi-singular hyperplane of $\bar{\Gamma}$. In order to determine the cardinality $|\mathscr{H}_{p,A}|$ of $\mathscr{H}_{p,A}$, we will first count the number $|C(\mathscr{H}_{p,A})|$ of points $(r, S) \in \bar{\Gamma}$ such that $(r, S) \notin \mathscr{H}_{p,A}$. By Definition (10), $|C(\mathscr{H}_{p,A})|$ is precisely the number of points of $\bar{\Gamma}$ not collinear with any point in $\mathscr{M}_p \cup \mathscr{M}_A$. Then, $|\mathscr{H}_{p,A}|$ is the difference between the number of points of $\bar{\Gamma}$ and $|C(\mathscr{H}_{p,A})|$, that is

$$|\mathscr{H}_{p,A}| = \frac{(q^{n+1}-1)}{q-1}\frac{(q^n-1)}{q-1} - |C(\mathscr{H}_{p,A})|.$$

Suppose $p \notin A$, i.e. $\mathscr{H}_{p,A}$ is a quasi-singular, non singular hyperplane of $\bar{\Gamma}$. We have that (r, S) is not collinear with any point in $\mathscr{M}_p \cup \mathscr{M}_A$ if and only if $r \notin A$, $p \notin S$ and $r \neq p$. More in detail, the number of points $r \in \mathrm{PG}(V)$ different from p and not contained in A is $\frac{(q^{n+1}-1)-(q^n-1)}{q-1} - 1 = q^n - 1$ and the number of hyperplanes $S \in \mathrm{PG}(V^*)$ through the point r and not containing p is $\frac{(q^n-1)-(q^{n-1}-1)}{q-1} = q^{n-1}$. So, $|C(\mathscr{H}_{p,A})| = q^{2n-1} - q^{n-1}$.

Now suppose $p \in A$, i.e. $\mathscr{H}_{p,A}$ is a singular hyperplane of $\bar{\Gamma}$. We have that (r, S) is not collinear with any point in $\mathscr{M}_p \cup \mathscr{M}_A$ if and only if $r \notin A$ and $p \notin S$. So, $|C(\mathscr{H}_{p,A})| = (\frac{(q^{n+1}-1)}{q-1} - \frac{(q^n-1)}{q-1})(\frac{(q^n-1)}{q-1} - \frac{(q^{n-1}-1)}{q-1}) = q^{2n-1}$. The claim follows. □

The following theorem describes the quasi-singular hyperplanes of $\bar{\Gamma}$.

Proposition 2.12. *[25, §1.3] Take $[x] \in \mathrm{PG}(V)$ and $[\xi] \in \mathrm{PG}(V^*)$. The quasi-singular hyperplane $\mathscr{H}_{[x],[\xi]}$ is the hyperplane of plain type $\mathscr{H}_M$ where $M = x\xi$.*

By Proposition 2.12, there is a one-to-one correspondence between quasi-singular hyperplanes of $\bar{\Gamma}$ and proportionality classes of matrices of rank 1.

In particular, all quasi-singular hyperplanes are hyperplanes of plain type arising from matrices M of rank 1 and, conversely, for each matrix $M \in M_{n+1}(q)$ of rank 1 the hyperplane of plain type $\mathscr{H}_M$ is quasi-singular.

Suppose S is a *line spread* of $\mathrm{PG}(V)$, that is a family of lines of $\mathrm{PG}(V)$ such that every point of $\mathrm{PG}(V)$ belongs to exactly one member of S. We say that S *admits a dual* if there exists a line spread S^* of $\mathrm{PG}(V^*)$ such that for every line $\ell^* \in S^*$ (i.e. for every 2-codimensional subspace of $\mathrm{PG}(V)$), the members of S contained in ℓ^*, form a line spread of ℓ^*; see [25]. A line spread S admits at most one dual spread S^*, see [25, Lemma 1.9]. In [25] it is proved that if a line spread S admits a dual S^*, then it is possible to define a geometric hyperplane $\mathscr{H}_{(S,S^*)}$ of $\bar{\Gamma}$ as follows

$$\mathscr{H}_{(S,S^*)} := \{(p, H) \in \bar{\Gamma} :\ H \supset \ell_p\} = \{(p, H) \in \bar{\Gamma} :\ p \in L_H\} \tag{13}$$

where ℓ_p is the unique line of S through p and $L_H \in S^*$ is the unique 2-codimensional subspace of $\mathrm{PG}(V)$ contained in H. The hyperplane $\mathscr{H}_{(S,S^*)}$ is called a *spread-type* hyperplane of $\bar{\Gamma}$.

Proposition 2.13. *[25, Theorem 1.14] A hyperplane $\mathscr{H}_M$ of plain type is of spread-type if and only if M admits no eigenvalue in $\mathbb{F}_q$ and $M^2x \in \langle x, Mx\rangle$ for every non-zero vector $x \in V$.*

3. The code $\mathscr{C}(\Lambda_1)$ from the Segre embedding

In this section we consider the subcode of the Segre code $\mathscr{C}(\Lambda)$ defined by the projective system $\Lambda_1 \subset \mathrm{PG}(M^0_{n+1}(q))$; see Definition (7). We will denote by $\mathscr{C}(\Lambda_1)$ the $[N_1, k_1, d_1]$-linear code arising from Λ_1. The length of this code is the number of point-hyperplane pairs (p, H) of $\mathrm{PG}(n, q)$ with $p \in H$, that is

$$N_1 = \frac{(q^{n+1}-1)(q^n-1)}{(q-1)^2}.$$

The dimension of $\mathscr{C}(\Lambda_1)$ is the dimension of the embedding $\bar{\varepsilon}$, so

$$k_1 = n^2 + 2n.$$

To determine the weight of the codewords of $\mathscr{C}(\Lambda_1)$ we need to compute the cardinality of $\Lambda_1 \cap [W]$ where $[W]$ is a hyperplane of $[\langle\Lambda_1\rangle]$. Recall from Section 2.6 that any hyperplane of $\mathrm{PG}(V \otimes V^*)$ can be regarded as the orthogonal subspace $[M^{\perp_f}]$ of an $(n+1)\times(n+1)$-matrix M with respect to the saturation form f. The next lemma is crucial. Observe that in this paper, when we speak of eigenvectors of a matrix M we always mean *left* eigenvectors; also by $\ker(M)$ we mean the set of row vectors ξ such that $\xi M = 0$.

Definition 3.1. For any matrix $M \in M_{n+1}(q)$, denote by ν_M the number of eigenvectors of M and by $\theta_M := \frac{\nu_M}{q-1}$ the number of projective points of $\mathrm{PG}(V^*)$ whose row representatives are eigenvectors for M.

Lemma 3.2. *Let $[M^{\perp_f}]$ be a hyperplane of $\mathrm{PG}(V \otimes V^*)$, for $M \in M_{n+1}(q) \setminus \langle I\rangle$. Then*

$$|\Lambda_1 \cap [M^{\perp_f}]| = \frac{(q^{n+1}-1)(q^{n-1}-1)}{(q-1)^2} + \theta_M \cdot q^{n-1} \tag{14}$$

where θ_M is given by Definition 3.1.

Proof. First note that we can write $\Lambda_1 = \{\bar{\varepsilon}(([x],[\xi]))\colon [x] \in [\xi]\}$ as a disjoint union

$$\Lambda_1 = \bigsqcup_{[\xi]\in\mathrm{PG}(V^*)} \{[x \otimes \xi] : [x] \in [\xi]\}.$$

So,

$$[M^{\perp_f}] \cap \Lambda_1 = \bigsqcup_{[\xi]\in \mathrm{PG}(V^*)} (\{[x\otimes \xi] : [x] \in [\xi]\} \cap [M^{\perp_f}]).$$

By Proposition 2.9, $[x\otimes\xi] \in [M^{\perp_f}]$ if and only if $[x] \in [\xi M]$. Hence, $[x\otimes\xi] \in \Lambda_1\cap[M^{\perp_f}]$ if and only if $[x] \in [\xi]$ and $[x] \in [\xi M]$, i.e.

$$\Lambda_1 \cap [M^{\perp_f}] = \bigsqcup_{[\xi]\in \mathrm{PG}(V^*)} (\{[x\otimes \xi] : [x] \in ([\xi] \cap [\xi M])\}).$$

Turning to cardinalities,

$$|\Lambda_1 \cap [M^{\perp_f}]| = \sum_{[\xi]\in \mathrm{PG}(V^*)} |[\xi] \cap [\xi M]|. \tag{15}$$

Note that $|[\xi]\cap[\xi M]| = |[\xi]| = \frac{(q^n-1)}{(q-1)}$ if $[\xi] \subseteq [\xi M]$ and $|[\xi]\cap[\xi M]| = \frac{(q^{n-1}-1)}{(q-1)}$ otherwise. On the other hand, $[\xi] \cap [\xi M] = [\xi]$ if and only if ξM is a scalar multiple of ξ, that is ξ is an eigenvector of M. Since θ_M denotes the number of points of $\mathrm{PG}(V^*)$ whose vector representatives are eigenvectors of M we have

$$|\Lambda_1 \cap [M^{\perp_f}]| = \theta_M \cdot \frac{q^n-1}{q-1} + \left(\frac{q^{n+1}-1}{q-1} - \theta_M\right) \cdot \frac{q^{n-1}-1}{q-1} = \frac{(q^{n+1}-1)(q^{n-1}-1)}{(q-1)^2} + \theta_M \cdot q^{n-1}. \quad \square$$

3.1. The codewords of $\mathscr{C}(\Lambda_1)$

Suppose $\Lambda_1 := \{[X_1], [X_2], \dots, [X_N]\} \subseteq \mathrm{PG}(M^0_{n+1}(q))$ and denote by $M^*_{n+1}(q)$ the dual of the vector space $M_{n+1}(q)$. For any functional $\mathfrak{m} \in M^*_{n+1}(q)$, there exists a unique matrix $M \in M_{n+1}(q)$ such that $\mathfrak{m} = \mathfrak{m}_M$ and

$$\mathfrak{m}\colon M_{n+1}(q) \to \mathbb{F}_q, \quad \mathfrak{m}(X) = \mathrm{Tr}(XM)$$

for all $X \in M_{n+1}(q)$. Consider now the N-tuple

$$c_{\mathfrak{m}} = (\mathfrak{m}(X_1), \dots, \mathfrak{m}(X_N)) \tag{16}$$

with

$$\mathfrak{m}(X_i) = \mathrm{Tr}(X_i M),\ 1 \le i \le N, \tag{17}$$

where $M \in M_{n+1}(q)$ is associated to $\mathfrak{m}$ as before. In this setting,

$$\mathscr{C}(\Lambda_1) = \{c_{\mathfrak{m}} : \mathfrak{m} \in M^*_{n+1}(q)\}.$$

In general there are more than one functional $\mathfrak{m}$ (resp. matrix M) defining one codeword $c_{\mathfrak{m}}$.

Lemma 3.3. *Let M be a matrix such that $\mathrm{Tr}(XM) = 0$ for all $X \in M^0_{n+1}(q)$. Then $M \in \langle I \rangle$.*

Proof. The claim is straightforward because $M^0_{n+1}(q)^{\perp_f} = (I^{\perp_f})^{\perp_f} = \langle I \rangle$. $\square$

Proposition 3.4. *$\mathscr{C}(\Lambda_1)$ is vectorially isomorphic to the quotient space $M_{n+1}(q)/\langle I \rangle$.*

Proof. Define the evaluation function

$$ev : M^*_{n+1}(q) \to \mathscr{C}(\Lambda_1), \qquad \mathfrak{m} \mapsto c_{\mathfrak{m}}.$$

By [29], ev is linear and surjective and $\mathscr{C}(\Lambda_1) = \{c_{\mathfrak{m}} \colon \mathfrak{m} \in M^*_{n+1}(q)\} = ev(M_{n+1}(q))$. The kernel of ev is given by all $\mathfrak{m}$ such that $c_{\mathfrak{m}} = 0$, i.e., by (17), the kernel of ev can be identified with the space of all $M \in M_{n+1}(q)$ such that $\mathrm{Tr}(XM) = 0$ for all $X \in M^0_{n+1}(q)$, since $\langle \Lambda_1 \rangle = \mathrm{PG}(M^0_{n+1}(q))$.

By Lemma 3.3, $\ker(ev) = \{\mathfrak{m}_{\alpha I} \colon \alpha \in \mathbb{F}_q\}$, where $\mathfrak{m}_{\alpha I} \colon M_{n+1}(q) \to \mathbb{F}_q, \mathfrak{m}_{\alpha I}(X) = \mathrm{Tr}(X\alpha I) = \alpha \mathrm{Tr}(X)$. So, the function ev induces the vector space isomorphism

$$M^*_{n+1}(q)/\langle \mathfrak{m}_I \rangle \cong \mathscr{C}(\Lambda_1).$$

Since the vector space $M_{n+1}(q)$ is isomorphic to $M^*_{n+1}(q)$, we have that

$$M_{n+1}(q)/\langle I \rangle \cong M^*_{n+1}(q)/\langle \mathfrak{m}_I \rangle$$

and the claim follows. $\square$

Corollary 3.5. *Suppose $\Lambda_1 = \{[X_i] \colon i = 1, \dots, N\}$.*

1. *If $p \nmid (n+1)$, then*

$$\mathscr{C}(\Lambda_1) = \{c_{\mathfrak{m}} \colon \mathfrak{m}(X) = \mathrm{Tr}(XM) \text{ with } M \in M^0_{n+1}(q)\}$$
$$= \{(\mathrm{Tr}(X_1M), \dots, \mathrm{Tr}(X_NM)) \colon [X_i] \in \Lambda_1,\ M \in M^0_{n+1}(q)\}.$$

2. *If $p|(n+1)$, then*

$$\mathscr{C}(\Lambda_1) = \{(\mathrm{Tr}(X_1M), \dots, \mathrm{Tr}(X_NM)) \colon [X_i] \in \Lambda_1, M \in M_{n+1}(q) \text{ with } m_{1,1} = 0\}.$$

Proof. By Proposition 3.4, the function $ev/\langle I \rangle \colon M_{n+1}(q)/\langle I \rangle \to \mathscr{C}(\Lambda_1)$ is a vector space isomorphism. Suppose $p \nmid (n+1)$. Each class $[\![M]\!] = M + \langle I \rangle$ in $M_{n+1}(q)/\langle I \rangle$ contains exactly one matrix $M_0 \in M_{n+1}(q)$ of trace 0; we choose that matrix as a canonical[1] representative for the coset in $M_{n+1}(q)/\langle I\rangle$.

If $p \mid (n+1)$, then all matrices M in the same class $[\![M]\!] = M + \langle I\rangle$ have the same trace. We can now choose as representative of $[\![M]\!]$ the only matrix $N \in [\![M]\!]$ given by $N := M - m_{1,1}I$ whose entry in position $(1,1)$ is 0. □

As a consequence of Proposition 3.4 and Theorem 1.2, it is easy to define an efficient encoding function for $\mathscr{C}(\Lambda_1)$, without the need of explicitly writing out a generator matrix; in particular if $[\![M]\!] \in M_{n+1}(q)/\langle I\rangle$ and $X_1, X_2, \dots, X_N$ are matrix representatives of the points $[X_1], \dots, [X_N]$ of the projective system of Λ_1, then the codeword corresponding to $[\![M]\!]$ is given by $(\mathrm{Tr}(X_1M), \dots, \mathrm{Tr}(X_NM))$.

The following is straightforward from Lemma 3.2, considering that for any codeword $c \in \mathscr{C}(\Lambda_1)$, the weight of c is $wt(c) = N_1 - |[M^{\perp_f}] \cap \Lambda_1|$, where M is the matrix associated to c.

Corollary 3.6. *The spectrum of weights of $\mathscr{C}(\Lambda_1)$ is*

$$\{q^{n-1}\frac{(q^{n+1}-1)}{(q-1)} - q^{n-1}\theta_M \colon M \in M_{n+1}(q)\}$$

where θ_M is given by Definition 3.1.

3.2. Proof of Theorem 1.1

Lemma 3.7. *If M is a diagonalizable matrix having $t > 2$ eigenspaces of dimensions $g_1 \geq g_2, \dots \geq g_t$ then it is always possible to construct a matrix M' with $t-1$ eigenspaces of dimension respectively $g'_1 \geq g'_2 \geq \dots \geq g'_{t-1}$, with $g'_1 = g_1 + g_2$, $g'_i = g_{i+1}$, $2 \leq i \leq t-1$ so that $\nu_{M'} > \nu_M$.*

Proof. The number of eigenvectors of M is $\nu_M = \sum_{i=1}^{t}(q^{g_i} - 1)$ with $g_i \leq n$. For $i = 1, \dots t$, let λ_i be the eigenvalue of M corresponding to the eigenspace having dimension g_i. Define as follows a diagonal matrix M' which has $\lambda_2, \dots, \lambda_t$ as eigenvalues:

$$M' := diag(\underbrace{\lambda_2, \dots, \lambda_2}_{g_1+g_2}, \underbrace{\lambda_3, \dots, \lambda_3}_{g_3}, \dots, \underbrace{\lambda_t, \dots, \lambda_t}_{g_t}).$$

Clearly, the dimensions of the eigenspaces of M' are $g'_1 \geq g'_2 \geq \dots \geq g'_{t-1}$, with $g'_1 = g_1 + g_2$, $g'_i = g_{i+1}$, $2 \leq i \leq t-1$ and $\nu_{M'} = \sum_{i=1}^{t-1}(q^{g'_i} - 1) = (q^{g_1+g_2} - 1) + \sum_{i=3}^{t-1}(q^{g_i} - 1)$. We have $\nu_{M'} - \nu_M = (q^{g_1+g_2} - 1) - (q^{g_1} - 1) - (q^{g_2} - 1) > 0$ if and only if $q^{g_1+g_2} - q^{g_2} = q^{g_2}(q^{g_1} - 1) > q^{g_1} - 1$, that is $q^{g_2} > 1$. Since $q > 1$ and $g_2 > 0$, it follows $\nu_{M'} > \nu_M$. □

[1] In more formal terms, the map $\pi : M_{n+1}/\langle I\rangle \to M^0_{n+1}(q)$ given by $\pi([\![M]\!]) := M - \mathrm{Tr}(M)I$ is a well defined vector space isomorphism, which commutes with matrix conjugation, in the sense that for all $g \in \mathrm{GL}(n+1,q)$, $\pi([\![M]\!]^g) = (\pi([\![M]\!]))^g$.

Lemma 3.8. *A non-scalar* $(n+1)$*-square matrix has a maximum number* $\nu_{max} = q^n + q - 2$ *of eigenvectors if and only if it admits exactly two eigenspaces of respective dimensions* n *and* 1*.*

Proof. Let M be a non scalar $(n+1)$-square matrix. If M cannot be diagonalized, then the sum of the dimensions of its eigenspaces is at most n and, consequently it has at most $q^n - 1$ eigenvectors. Suppose now that M can be diagonalized and let t be the number of eigenspaces of M and $g_1 \geq g_2 \geq \cdots \geq g_t$ be the respective dimensions of the eigenspaces. Since $M \notin \langle I \rangle$, we have $t \geq 2$. By recursively applying Lemma 3.7, we see that the maximum number of eigenvectors for a non-scalar diagonalizable $(n+1)$-square matrix can be attained only for $t = 2$. So, suppose we have a matrix M with just two eigenspaces of dimensions $g \leq n$ and $n+1-g$. Assume $g \geq n+1-g$. The number of eigenvectors of M is then $\nu_M = (q^g - 1) + (q^{n+1-g} - 1)$. This is maximum when $g = n$ and gives $\nu_{\max} = q^n + q - 2 \geq q^n > q^n - 1$. The converse follows immediately. $\square$

The length and dimension of $\mathscr{C}(\Lambda_1)$ are computed at the beginning of Section 3.

By Corollary 3.6 and Lemma 3.8, since $\theta_{max} = \nu_{max}/(q-1)$, we have that the minimum distance of $\mathscr{C}(\Lambda_1)$ is

$$d_1 = q^{n-1}\frac{(q^{n+1}-1)}{(q-1)} - q^{n-1}\frac{q^n+q-2}{q-1} = q^{2n-1} - q^{n-1}.$$

Theorem 1.1 is proved. $\square$

3.3. Proof of Theorem 1.2

Let M be a non-scalar matrix of $M_{n+1}(q)$ and let t denote the number of eigenspaces of M. Since the number of eigenspaces of any matrix is, clearly, the same as the number of its eigenvalues (which ranges in $\mathbb{F}_q$), we have $t \leq q$ and since M is non-scalar, $t \leq n$; so $t \leq \min(n, q)$.

Consider the following sets, where θ_M is given by Definition 3.1:

$$E = \{\theta_M \colon M \in M_{n+1}(q) \setminus \langle I \rangle\} \tag{18}$$

and

$$D = \{0\} \cup \{(g_1, \ldots, g_t) \colon \sum_{i=1}^{t} g_i \leq n+1, \\ 1 \leq g_1 \leq \cdots \leq g_i \leq g_{i+1} \leq \cdots \leq g_t \leq n+1, 1 \leq t \leq q\}. \tag{19}$$

Observe first that if $m(x)$ is a monic irreducible polynomial over $\mathbb{F}_q$ of degree $n+1$, then its companion matrix has order $n+1$ and does not have any eigenvalue in $\mathbb{F}_q$. Clearly, such a companion matrix is not a scalar matrix; so $0 \in E$.

Assume $1 \leq t \leq q$. Take t distinct elements $\lambda_1, \dots, \lambda_t \in \mathbb{F}_q$ and for any t-tuple $(g_1, \dots, g_t) \in D$, define a matrix $M_{(g_1,\dots,g_t)} \in M_{n+1}(q) \setminus \langle I \rangle$ as a block matrix of the form

$$M_{(g_1,\dots,g_t)} = diag(\underbrace{\lambda_1, \dots, \lambda_1}_{g_1}, \underbrace{\lambda_2, \dots, \lambda_2}_{g_2}, \dots \underbrace{\lambda_t, \dots, \lambda_t}_{g_t - 1}, R_{\lambda_t})$$

where R_{λ_t} is the Jordan block of order $n + 2 - \sum_{i=1}^{t} g_i$ of the form

$$R_{\lambda_t} = \begin{pmatrix} \lambda_t & 1 & 0 & \dots & 0 \\ 0 & \lambda_t & 1 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \dots & \lambda_t & 1 \\ 0 & 0 & \dots & 0 & \lambda_t \end{pmatrix}.$$

Then $M_{(g_1,\dots,g_t)}$ has exactly t eigenspaces $V_1, \dots, V_t$ of respective dimensions $g_1, \dots, g_t$ and the number of its eigenvectors is $\nu_{(g_1,\dots,g_t)} = \sum_{i=1}^{t}(q^{g_i} - 1)$. Since $\theta_{(g_1,\dots,g_t)} := \nu_{(g_1,\dots,g_t)}/(q-1)$ we see that the map

$$\begin{aligned} &\varphi\colon D \to E \\ &\varphi((g_1, \dots, g_t)) = \theta_{(g_1,\dots,g_t)} \\ &\varphi(0) = 0, \end{aligned} \tag{20}$$

is well defined.

Lemma 3.9. *Take $1 \leq t, t' \leq q$ and let $1 \leq \alpha_1 \leq \dots \leq \alpha_t$ and $1 \leq \beta_1 \leq \dots \leq \beta_{t'}$ be integers such that*

$$\sum_{i=1}^{t}(q^{\alpha_i} - 1) = \sum_{i=1}^{t'}(q^{\beta_i} - 1). \tag{21}$$

Then $t = t'$ and $\alpha_i = \beta_i$ for all $i = 1, \dots, t$.

Proof. Reducing (21) modulus q we obtain $-t \equiv -t' \pmod{q}$. As $1 \leq t, t' \leq q$ this implies $t = t'$. Since $t = t'$, we can now rewrite (21) as

$$\sum_{i=1}^{t} q^{\alpha_i} = \sum_{i=1}^{t} q^{\beta_i}. \tag{22}$$

In order to prove the claim, we now proceed by induction on the number t of terms in (22).

- If $t = 2$, suppose $q^{\alpha_1} + q^{\alpha_2} = q^{\beta_1} + q^{\beta_2}$, that is

$$q^{\alpha_1}(1 + q^{\alpha_2-\alpha_1}) = q^{\beta_1}(1 + q^{\beta_2-\beta_1}).$$

Assume by contradiction $\alpha_1 \neq \beta_1$; so we can take without loss of generality $\alpha_1 > \beta_1$. So we get

$$q^{\alpha_1-\beta_1}(1 + q^{\alpha_2-\alpha_1}) = (1 + q^{\beta_2-\beta_1}), \tag{23}$$

where all the exponents are non-negative. If we reduce (23) modulus q we get $(1 + q^{\beta_2-\beta_1}) \equiv 0 \pmod q$. If $q^{\beta_2-\beta_1} \neq 1$ this gives $1 \equiv 0 \pmod q$, a contradiction. So it must be $\beta_2 = \beta_1$ and we get $2 \equiv 0 \pmod q$, which is possible only if $q = 2$. However in this case (23) becomes

$$2^{\alpha_1-\beta_1}(1 + 2^{\alpha_2-\alpha_1}) = 2,$$

which forces $\alpha_2 = \alpha_1$ and $\alpha_1 = \beta_1$, contradicting the hypothesis. So $\alpha_1 = \beta_1$ and, consequently, $\alpha_2 = \beta_2$.
- Suppose $2 < t < q - 1$ By induction hypothesis, the condition

$$\sum_{i=1}^{t} q^{\alpha_i} = \sum_{i=1}^{t} q^{\beta_i} \Leftrightarrow (\alpha_1, \dots, \alpha_t) = (\beta_1, \dots, \beta_t)$$

holds. We claim that it also holds for $t + 1 \le q$ terms. If $\sum_{i=1}^{t+1} q^{\alpha_i} = \sum_{i=1}^{t+1} q^{\beta_i}$ with $\alpha_{t+1} = \beta_{t+1}$, then, subtracting on the left and right hand side $q^{\alpha_{t+1}}$ and then applying the inductive hypothesis we get $(\alpha_1, \dots, \alpha_t) = (\beta_1, \dots, \beta_t)$ and we are done. Suppose then $\alpha_{t+1} \neq \beta_{t+1}$ and assume without loss of generality $\beta_{t+1} \le \alpha_{t+1} - 1$ and that, by contradiction,

$$\sum_{i=1}^{t+1} q^{\alpha_i} = \sum_{i=1}^{t+1} q^{\beta_i}.$$

Observe that for all $1 \le i \le t + 1$ we have $q^{\beta_i} \le q^{\beta_{t+1}} \le q^{\alpha_{t+1}-1}$ and $q^{\alpha_i} \ge q^0 = 1$, so

$$q^{\alpha_{t+1}} = \sum_{i=1}^{t+1} q^{\beta_i} - \sum_{i=1}^{t} q^{\alpha_i} \le \sum_{i=1}^{t+1} q^{\alpha_{t+1}-1} - \sum_{i=1}^{t} 1 = (t + 1)q^{\alpha_{t+1}-1} - t.$$

Since $0 < t + 1 \le q$ this implies $q^{\alpha_{t+1}} \le q^{\alpha_{t+1}} - t$ which is a contradiction. It follows that it must be $\alpha_{t+1} = \beta_{t+1}$. This completes the proof. $\square$

Proposition 3.10. *The sets E and D defined in (18) and (19) are in bijective correspondence.*

Proof. We need to prove that the map defined in (20) is a bijective correspondence.

Injectivity follows from Lemma 3.9. For the surjectivity, $0 \in D$ is uniquely mapped to $0 \in E$. If $\theta \in E$ and $\theta \geq 1$, by definition of D there exists a matrix $M \in M_{n+q}(q) \setminus \langle I \rangle$ with $\theta_M = \theta$ where $\nu_M = (q-1)\theta_M$ is the number of its eigenvectors. By Lemma 3.9 the list $(g_1, \dots g_t)$ of the dimensions of the eigenspaces of M is uniquely determined by ν_M. In particular $g_1, \dots, g_t$ must satisfy $1 \leq t \leq q$ and $\sum_{i=1}^{t} g_i \leq n+1$. We can take without loss of generality $g_i \leq g_j$ if $i \leq j$. So there is $(g_1, \dots, g_t) \in D$ such that $\varphi(g_1, \dots, g_t) = \theta_M$. □

Part 1 of Theorem 1.2 follows from Proposition 3.10 and Corollary 3.6.

We now prove Part 2 of Theorem 1.2. Recall that an automorphism of a code $\mathscr{C}(\Lambda)$ is a linear map $\mathscr{C}(\Lambda) \to \mathscr{C}(\Lambda)$ preserving the weights of all of the codewords; see [21].

Proposition 3.11. *The code $\mathscr{C}(\Lambda_1)$ admits the group* $\mathrm{PGL}(n+1, q)$ *as an automorphism group acting transitively on the components of the codewords.*

Proof. We need to distinguish two cases.

(A) $\boxed{p \nmid (n+1)}$ In this case $M_{n+1}(q)/\langle I \rangle \cong M^0_{n+1}(q)$; see [14, Lemma 1.5] Since the group $\mathrm{GL}(n+1, q)$ acts by conjugation on $M^0_{n+1}(q)$ and the kernel of this action is given by the scalar matrices $\langle I \rangle$, the group of transformations induced by $\mathrm{GL}(n+1, q)$ on $M^0_{n+1}(q)$ is isomorphic to $\mathrm{PGL}(n+1, q)$. By Proposition 3.4 and the proof of Corollary 3.5, $\mathscr{C}(\Lambda_1)$ and $M^0_{n+1}(q)$ are isomorphic as vector spaces and for any codeword $c \in \mathscr{C}(\Lambda_1)$ there exists $M \in M^0_{n+1}(q)$ such that $c = c_{\mathfrak{m}}$, where $\mathfrak{m} \in M^*_{n+1}(q)$ and $\mathfrak{m}(X) = \mathrm{Tr}(XM)$.

For $g \in \mathrm{GL}(n+1, q)$, consider the action ϕ on $\mathscr{C}(\Lambda_1)$ given by $c \to c^g$ where c^g is $c^g = c_{\mathfrak{m}^g}$ with $\mathfrak{m}^g(X) := \mathrm{Tr}(Xg^{-1}Mg)$. Clearly, $\mathrm{Tr}(M) = \mathrm{Tr}(g^{-1}Mg) = 0$; so $c^g \in \mathscr{C}(\Lambda_1)$.

The map ϕ is linear, since for any two codewords c_1, c_2 induced respectively by matrices M_1 and M_2 with functionals $\mathfrak{m}_1$ and $\mathfrak{m}_2$ we have, for any $\alpha, \beta \in \mathbb{F}_q$

$$
\begin{aligned}
(\alpha c_1 + \beta c_2)^g &= ((\alpha\mathfrak{m}_1 + \beta\mathfrak{m}_2)^g(X_1), \dots (\alpha\mathfrak{m}_1 + \beta\mathfrak{m}_2)^g(X_N)) = \\
&(\mathrm{Tr}(X_1 g^{-1}(\alpha M_1 + \beta M_2)g), \dots, (\mathrm{Tr}(X_N g^{-1}(\alpha M_1 + \beta M_2)g)) = \\
&(\mathrm{Tr}(\alpha X_1 g^{-1} M_1 g + \beta X_1 g^{-1} M_2 g), \dots, (\mathrm{Tr}(\alpha X_N g^{-1} M_1 g + \beta X_N g^{-1} M_2 g)) = \\
&\alpha(\mathrm{Tr}(X_1 g^{-1} M_1 g), \dots, \mathrm{Tr}(X_N g^{-1} M_1 g)) + \\
&\beta(\mathrm{Tr}(X_1 g^{-1} M_2 g), \dots, \mathrm{Tr}(X_N g^{-1} M_2 g)) = \\
&\alpha(\mathfrak{m}_1^g(X_1), \dots, \mathfrak{m}_1^g(X_N)) + \beta(\mathfrak{m}_2^g(X_1), \dots, \mathfrak{m}_2^g(X_N)) = \alpha c_1^g + \beta c_2^g.
\end{aligned}
$$

Since the number of eigenvectors of M and that of $g^{-1}Mg$ is the same and, by Corollary 3.6, the weight of a codeword depends on the number of eigenvectors defining it, the weight of c^g is the same as that of c. Finally, since the group $\mathrm{PGL}(n+1,q)$ acts flag-transitively on the geometry $\mathrm{PG}(V)$, it is also transitive on the geometry $\bar{\Gamma}$ and, consequently, by the homogeneity of the embedding $\bar{\varepsilon}$, also on the projective system Λ_1. This completes the proof of this case.

(B) $\boxed{p \mid (n+1)}$ By Proposition 3.4, $ev/\langle I\rangle\colon M_{n+1}(q)/\langle I\rangle \to \mathscr{C}(\Lambda_1)$ is a vector space isomorphism but in this case we cannot identify $M_{n+1}(q)/\langle I\rangle$ with $M^0_{n+1}(q)$. The group $\mathrm{GL}(n+1,q)$ acts on the space $M_{n+1}(q)$ by conjugation and since $\langle I\rangle$ is fixed (element-wise) by this action, it also acts on $M_{n+1}(q)/\langle I\rangle$ by considering the action on the equivalence classes. Indeed for any $[\![M]\!] \in M_{n+1}(q)/\langle I\rangle$ we have $[\![M]\!] = \{M+\lambda I : \lambda \in \mathbb{F}_q\}$; as $g^{-1}(M+\lambda I)g = g^{-1}Mg+\lambda I$ for any $g \in \mathrm{GL}(n+1,q)$, we have that the map $[\![M]\!] \to [\![M]\!]^g := [\![g^{-1}Mg]\!]$ is well defined. The kernel of this action is given by $\langle I\rangle$; so $\mathrm{PGL}(n+1,q) = \mathrm{GL}(n+1)/\langle I\rangle$ acts faithfully on $M_{n+1}(q)/\langle I\rangle$. As in Case (A), this induces an action on the components X_i of the codeword $c_{[\![M]\!]} = (\mathrm{Tr}(X_1M),\dots,\mathrm{Tr}(X_NM))$ by $c_{[\![M]\!]} \to c^g_{[\![M]\!]} := c_{[\![M]\!]^g}$ and this action is linear. It remains to prove that the action is an isometry. However, the weight of the codeword $c_{[\![M]\!]}$ depends only on the number of eigenvectors of any matrix in $[\![M]\!]$ (and all such matrices have the same number of eigenvectors). As conjugate matrices have also the same number of eigenvectors, it follows that the weight of $c^g_{[\![M]\!]} = c_{[\![M]\!]^g}$ is the same as the weight of $c_{[\![M]\!]}$. This completes the proof. □

Remark 1. The automorphism group of a code over $\mathbb{F}_q$ always contains a cyclic subgroup isomorphic to $\mathbb{F}_q^*$ in its center, acting on the words by scalar multiplication. As $\mathrm{PGL}(n+1,q)$ has a trivial center, we can see that the automorphism group of $\mathscr{C}(\Lambda)$ must be the direct product $\mathrm{PGL}(n+1,q)\cdot\mathbb{F}_q^*$. Observe however that the action of this group is not the action of $\mathrm{GL}(n+1,q)$ by conjugation and that the action of $\mathrm{PGL}(n+1,q)$ on the codewords is different in the case $p \nmid (n+1)$ and $p \mid (n+1)$.

3.4. Proof of Theorem 1.3

In 2025, Pasini [25, Theorem 1.5] proved that all hyperplanes of the point-hyperplane geometry $\bar{\Gamma}$ of $\mathrm{PG}(V)$ are maximal subspaces; so point 1 of Theorem 1.3 follows immediately from Proposition 2.5.

Remark 2. We point out that the minimality of $\mathscr{C}(\Lambda_1)$ does not follow directly from the Ashikhmin and Barg Condition (2) of Section 2.3. Indeed, in order to apply Condition (2) to test a code for minimality we need to know the minimum and the maximum weight of the codewords. These are known or $\mathscr{C}(\Lambda_1)$, thanks to Theorem 1.2 and Corollary 3.6 and turn out to be $w_{\max} = q^{n-1}(q^{n+1}-1)/(q-1)$ and $w_{\min} = q^{n-1}(q^n-1)$. However,

$$\frac{w_{\max}}{w_{\min}} = \frac{(q^{n+1}-1)}{(q-1)(q^n-1)} > \frac{q}{q-1}.$$

The following is an elementary remark on matrices of rank 1.

Lemma 3.12. *A rank* 1 *matrix* $M \in M_{n+1}(q)$ *is diagonalizable if and only if* $\mathrm{Tr}(M) \neq 0$.

Proof. Suppose M is diagonalizable of rank 1. Clearly, the eigenspace V_0 of 0 must have dimension n. So there must also be another eigenspace V_λ of dimension 1 associated to a non-zero eigenvalue λ. The trace of the diagonalized matrix is thus $\lambda \neq 0$ and, since similar matrices have the same trace, also $\mathrm{Tr}(M) = \lambda \neq 0$. Suppose now M has rank 1 and it is not diagonalizable. Choose a basis $(e_1, \dots, e_{n+1})$ of $\mathbb{F}_q^{n+1}$ where the first n vectors belong to the kernel of M. So, up to conjugation, we can assume without loss of generality that only the last column of M contains non-zero entries. The characteristic polynomial $p_M(\lambda)$ of such a matrix M is $(-\lambda)^n(m_{n+1,n+1} - \lambda)$. If it were $m_{n+1,n+1} \neq 0$, then M would have one further eigenvalue with eigenspace of dimension (at least) 1; so M would be diagonalizable, a contradiction. It follows that it must be $m_{n+1,n+1} = 0$; consequently, $\mathrm{Tr}(M) = 0$. □

Proposition 3.13. *Any minimum weight codeword of* $\mathscr{C}(\Lambda_1)$ *corresponds to a hyperplane of the form* $[N^{\perp_f}]$ *where* N *is a matrix of rank* 1 *and trace different from* 0.

Proof. By Lemma 3.8, any minimum weight codeword corresponds to a hyperplane $[M^{\perp_f}]$ where M is a diagonalizable non-scalar $(n+1)$-square matrix having two eigenspaces of dimensions n and 1. Denote by λ_1 and λ_2 the two eigenvalues of M. By Proposition 2.8 we have $[M^{\perp_f}] \cap \Lambda_1 = [(M - \lambda_1 I)^{\perp_f}] \cap \Lambda_1$. Put $N := M - \lambda_1 I$. So, by Proposition 3.4, N and M determine the same codeword. Clearly, N has eigenvalues $0 = \lambda_1 - \lambda_1$ and $\lambda_2 - \lambda_1$ with multiplicities n and 1. Thus, N is diagonalizable with rank 1. By Lemma 3.12, this implies that N has trace different from 0. □

Proposition 3.14. *Any second lowest weight codeword of* $\mathscr{C}(\Lambda_1)$ *corresponds to a hyperplane of the form* $[N^{\perp_f}]$ *where* N *is a (non-diagonalizable) matrix of rank* 1 *and trace* 0.

Proof. By Lemma 3.8 and Proposition 3.13 the minimum weight codewords occur when M has rank 1 and trace different from 0, i.e. M has rank 1 and is diagonalizable; in this case $\theta = \theta_{\min} = \frac{q^n-1}{q-1} + 1$. Suppose now that N is a non-diagonalizable matrix of rank 1. By Lemma 3.12, this is the same to say that N has rank 1 and trace 0. Then, $\theta_N = \frac{q^n-1}{q-1} = \theta_{\min} - 1$. It follows from Corollary 3.6 that N determines a codeword with second lowest weight.

Conversely, suppose that N defines a codeword with the second lowest weight. By Corollary 3.6, $\theta_N = \theta_{\min} - 1 = \frac{q^n-1}{q-1}$. By Proposition 3.4, we can replace N with $N - \lambda I$ where λ is the eigenvalue of N whose eigenspace has the highest dimension. If N has rank 1, i.e. just one eigenspace, then we are done. Otherwise, suppose that N has $t > 1$ eigenspaces and let $d_1 = (n+1) - \mathrm{rank}\,(N) \geq d_2 \geq \cdots \geq d_t$ be the corresponding dimensions. Clearly, $d_1 < n$. Counting the eigenvectors of N we have that the following must hold $\nu_N = \sum_{i=1}^{t}(q^{d_i}-1) = q^n - 1$, that is $\nu_n = q^{d_1}+q^{d_2}+\cdots+q^{d_t} = q^n+t-1$. On the other hand

$$q^{d_1}+q^{d_2}+\cdots+q^{d_t} \leq q^{d_1}+q^{d_1-1}+\cdots+1 = \frac{q^{d_1+1}-1}{q-1} \leq \frac{q^n-1}{q-1} < q^n-1+t.$$

This is a contradiction. So, it must be $t = 1$ and N has rank 1. □

Propositions 3.13 and 3.14 give a geometrical interpretation for the minimum and the second lowest weight codewords. Note also that a matrix N of rank 1 determines a minimum weight codeword of $\mathscr{C}(\Lambda_1)$ if and only if $[N] \notin \Lambda_1$ while it determines a codeword with the second lowest weight if and only if $[N] \in \Lambda_1$.

By Section 2.7, the quasi-singular hyperplanes of $\bar{\Gamma}$ are in correspondence with matrices of rank 1. By Proposition 3.13, any minimum weight codeword corresponds to a matrix of rank 1 and trace different from 0. Hence any minimum weight codeword of $\mathscr{C}(\Lambda_1)$ corresponds to a quasi-singular hyperplane $\mathscr{H} := \mathscr{H}_{([x],[\xi])}$ of $\bar{\Gamma}$ defined by a matrix $x \otimes \xi$ with non-null trace. This last condition amounts to say that $[x] \notin [\xi]$, i.e. $\mathscr{H}$ is a quasi-singular but not singular hyperplane of $\bar{\Gamma}$.

Take now a codeword having second lowest weight. By Proposition 3.14, it is in correspondence with a quasi-singular hyperplane $\mathscr{H}' := \mathscr{H}_{([y],[\eta])}$ of $\bar{\Gamma}$ defined by a matrix $y \otimes \eta$ with null trace. This last condition amounts to say that $[y] \in [\eta]$, i.e. $\mathscr{H}'$ is a singular hyperplane of $\bar{\Gamma}$. Part 2 of Theorem 1.3 is proved.

We now focus on the maximum weight codewords. By Corollary 3.6 we immediately have

Proposition 3.15. *Maximum weight codewords have weight $q^{n-1}(q^{n+1}-1)/(q-1)$ and correspond to hyperplanes of the form $[M^{\perp_f}]$ where M has no eigenvalues in $\mathbb{F}_q$.*

By Proposition 2.13, a hyperplane $\mathscr{H}_M$ of $\bar{\Gamma}$ is of spread-type if and only if M is a matrix having no eigenvalue in $\mathbb{F}_q$ and $M^2x \in \langle x, Mx\rangle$ for any $x \in V \setminus \{0\}$. By Proposition 3.15, maximum weight codewords have weight $q^{n-1}(q^{n+1}-1)(q-1)$ achieved for $\theta_M = 0$ and correspond to hyperplanes $[M^{\perp_f}]$ where M has no eigenvalues in $\mathbb{F}_q$. So, any spread-type hyperplane of $\bar{\Gamma}$ arising from the Segre embedding $\bar{\varepsilon}$ of $\bar{\Gamma}$ is associated to a maximum weight codeword.

The following proposition provides conditions for the converse.

Proposition 3.16. *Any maximum weight codeword of $\mathscr{C}(\Lambda_1)$ corresponding to a matrix $M \in M_{n+1}(q) \setminus \langle I\rangle$ such that the minimal polynomial of M is irreducible of degree 2 is a spread-type hyperplane of $\bar{\Gamma}$.*

Proof. Since M is associated to a maximum weight codeword, by Proposition 3.15 M has no eigenvalue in $\mathbb{F}_q$. By hypothesis, the minimal polynomial of M is an irreducible polynomial over $\mathbb{F}_q$ of the form $p_M(x) := x^2+\alpha x+\beta$, for $\alpha, \beta \in \mathbb{F}_q$. Then $M^2+\alpha M+\beta I = 0$. So, $M^2 v = -\alpha M v - \beta v, \forall v \in V$. Hence, $M^2 v \in \langle Mv, v\rangle$ and, by Proposition 2.13, $\mathscr{H}_M$ is a spread-type hyperplane of $\bar{\Gamma}$. $\square$

Theorem 1.3 is proved. $\square$

Acknowledgments

Both authors are affiliated with GNSAGA of INdAM (Italy) whose support they kindly acknowledge. This work was partially supported by the project "CONSTR: a COllectionless-based Neuro-Symbolic Theory for learning and Reasoning", PARTENARIATO ESTESO "Future Artificial Intelligence Research - FAIR", SPOKE 1 "Human-Centered AI" Università di Pisa, "NextGenerationEU", CUP I53C22001380006.

Data availability

No data was used for the research described in the article.

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