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Maximum Gain of Lossy Antennas Without and With Q-bounds

Laura Passalacqua, *Student Member, IEEE*, Cristina Yepes, *Member, IEEE*, Alejandro Murillo, Bruno Biscontini, *Member, IEEE*, Enrica Martini, *Senior Member, IEEE*, Stefano Maci, *Fellow, IEEE*

***Abstract*—Authors of this paper have recently formulated a maximum bound of super-directivity of self-resonant antennas for a given minimum Q (maximum frequency bandwidth). This paper complements the above work treating the influence of the losses. The problem is faced by assuming small losses in terms of surface resistance over the metalized surface of the minimum sphere circumscribing the antenna. The final closed form formula shows that the maximum gain is obtained by a summation that resembles the well-known Harrington's sum for maximum directivity, except that the expansion coefficients are weighted by the radiation efficiency of each spherical harmonic. The formulation is next generalized to the case of self-resonant antenna, providing a tighter bound for any losses. For small antennas, we provide a simple interpretation of the field corresponding to the maximum gain in terms of dipolar and quadrupolar source contributions, weighted by the appropriate efficiency, offering a physical insight into the phenomenon. The formulation is then extended to also account for a Q-bound, deriving a final series expression as a function of the loss resistance and of the antenna electrical size. This expression seamlessly merges to the previously derived Q-bounded maximum directivity as losses tends to zero and converges to Q-unbounded maximum Gain for Q that tends to very large values.**

***Index Terms*—Super-directivity, super-gain, antenna efficiency, antenna bandwidth, quality factor.**

I. Introduction

The relationship between super-gain and super-directivity has been a highly debated issue in antenna theory, prompting extensive research and numerous publications. Many papers have been published since the Fifties [1]-[22] aiming to provide guidance to antenna designers on achieving the maximum directivity and gain compatible with a given space constraint. Assuming sources fitting inside a minimum sphere of radius $r_{\min}$, the maximum directivity $D_{\max}$ can be found as a function of $r_{\min}$ as suggested by Harrington [5]. His method is based on the expansion of the radiated field in a finite number of spherical wave (SW) modes excited on the minimum sphere, and on the maximization of the directivity with respect to the coefficients of the expansion. This procedure leads to $D_{\max} = \sum_{n=1}^{N_{max}} (2n+1) = \left(N_{\max}\right)^2 + 2N_{\max}$ where $N_{\max} \geq 1$ is the maximum polar index of the SWs that contribute to the far field for the given minimum sphere. However, there are antennas that exhibit directivities larger than the value suggested by Harrington. These antennas are occasionally referred to as super-directive antennas. However, achieving such a high directivity often comes at the expense of a narrow bandwidth and a low efficiency. The Harrington process does not explicitly impose any constraints on bandwidth and losses, and the bound of directivity is obtained just imposing a truncation of the series at the number of Degrees of Freedom (DoF) of the field. It is not possible, indeed, to establish a rigorous limit of super-directivity without introducing a limit on bandwidth or on losses. Recently, a limit of super-directivity constrained by a maximum quality factor Q (inverse of relative bandwidth) has been provided by these authors [1].

However, [1] does not address the extent to which small losses can affect the performance of super-directive antennas. Indeed, it is widely acknowledged that super-directivity does not always translate into "super-gain", especially in cases involving high Q factors. The primary objective of this paper is to examine this specific aspect by assessing the impact of losses on the maximum super-gain.

In [22], a fundamental bound on antenna gain is found by solving a problem of convex optimization of the current density. The optimal current distribution is found expressing the antenna gain and the product bandwidth-gain as a quadratic form of the corresponding matrix operators. This procedure leads to the definition of a convex optimization problem, which is finally solved via an eigenvalue problem. However, the results in [22] do not explicitly impose frequency bandwidth maximization in the limit of maximum gain.

We articulate the presentation of this paper as follows. Section II briefly summarizes the results obtained in [1]. Section III presents a derivation of the maximum antenna gain without bandwidth limitations. A closed form formula is derived and compared with the numerical approach presented in [22]. Section IV focuses on small antennas, providing an interpretation in terms of dipolar and quadrupolar contributions and simple approximations valid for maximum gain smaller than 7. Section V formulates the gain maximization problem by introducing a limit on the total Q of the antenna and derives new exact analytical expressions. Section VI draws some conclusions.

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L. Passalacqua, C. Yepes, E. Martini, and S. Maci are with the Applied Electromagnetics Laboratory, Department of Information Engineering and Mathematics, University of Siena, 53100 Siena, Italy (e-mail: laura.passalacqua2@unisi.it, cristinayepes@ieee.org, martini@dii.unisi.it, macis@dii.unisi.it).Alejandro Murillo and Bruno Biscontini are with Huawei Technologies Co., Ltd., 80992 Munich, Germany (e-mail: alejandro.murillo.barrera@huawei.com, bruno.biscontini@huawei.co).

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II. Maximum Directivity With Q-bounds

Recently, a limit of super-directivity constrained by a minimum quality factor Q (inverse of relative bandwidth) has been provided by these authors [1] as

$$D_{\max}(\xi_0) = \sum_{n=1}^{\infty} \frac{2n+1}{\xi_0 \left(Q_n - Q\right)+1} \qquad (1)$$

where ξ_0 is the value that maximizes the series $D_{\max}(\xi)$ in the interval $\xi \in \left[0,(Q-Q_1)^{-1}\right]$ and Q_n are the quality factors of each individual harmonic according to the Fante's definition for self-resonant antennas [18]. These correspond to $Q_n = \frac{1}{2}(Q_n{'}+Q_n{''})$, where $Q_n{'}$and $Q_n{''}$ are defined in Appendix C, and will be introduced later. The analytical form (1) is obtained by applying the Lagrange multipliers method to a convex problem of directivity maximization bounded to a given constant, minimum quality factor Q. The value of ξ that minimizes the series has been approximated in [1], leading to simple closed form expressions for certain values of the antenna size, as given below.

$$D_{max} \approx \begin{cases} 3\dfrac{\left(\sqrt{\mathrm{Q}_2-\mathrm{Q}}+\sqrt{\frac{5}{3}(\mathrm{Q}-\mathrm{Q}_1)}\right)^2}{\mathrm{Q}_2-\mathrm{Q}_1} & kr_1 \le kr_{min} \le kr_{\mathrm{quad}} \\ \displaystyle\sum_{\mathrm{n}=1}^{\infty} \frac{(2n+1)}{0.16\left(\dfrac{\mathrm{Q}_n}{\mathrm{Q}}-1\right)+1} & kr_{\mathrm{quad}} \le kr_{min} \le 20 \end{cases} \qquad (2)$$

The two terms $kr_1 = 0.4\left(\dfrac{1}{\mathrm{Q}}+\dfrac{2}{\mathrm{Q}^{1/3}}\right)$ and $kr_{\mathrm{quad}} = \left(\dfrac{1}{\mathrm{Q}}+\dfrac{1}{\mathrm{Q}^{1/5}}\right)$ correspond to the values at which the maximum directivity is 3 (Chu Limit) and 7.5, respectively.

III. Maximum Gain Without Q-bounds

We present here a formulation for the maximum gain without Q-bound. This formulation resorts to Spherical Wave (SW) Expansion (SWE), using the notation of Hansen [9], summarized in Appendix A, for the representation of the electric field. According to this notation, the polar index n refers to the order of the Hankel functions, the index m refers to the azimuthal angular wave number, and the index s=1, and 2 denotes the TE and TM polarization with respect to the radial direction r, respectively. Whenever convenient, we compact the notation, using the single index i which renumbers the tern of indexes s,m,n by the rule $i = 2\left(n\left(n+1\right)+m-1\right)+s$.

A. Equivalent Currents and SW Radiation Resistances

The Love formulation of the equivalence theorem is first applied to the minimum spherical surface which includes all the sources (Fig.1(a)). The equivalent electric and magnetic currents radiate zero field inside the surface. The external field is the same as that provided by magnetic currents radiating on a perfectly conducting sphere of radius $r_{\min}$, as predicted by the Schelkunoff formulation (see Fig.1(b)). In the latter case, the induced electric currents **J** over the conducting sphere are the same as the electric currents **J** in Fig.1(a).

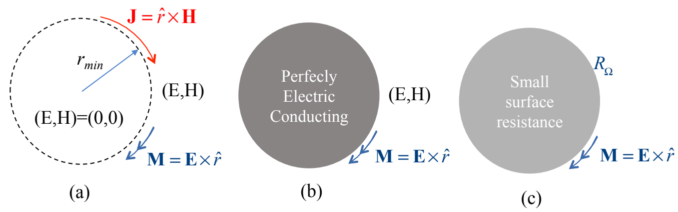

Fig. 1 Application of the equivalence principle to the minimum sphere surface. (a) Equivalent surface electric and magnetic currents radiating in free space with zero field inside (Love formulation); (b) magnetic current radiating over a perfectly electric conducting sphere; (c) Magnetic currents radiating over a metallic sphere with small losses.

In Appendix A, the conventional SWE of the radiated field is presented. It is however convenient writing the SWE of the field in a form which is related to the radiation resistance of the individual harmonics. Due to the orthonormality of the spherical waves one has $|I_i| = (\iint_{r_{\min}} |\mathbf{J}_i|^2 dA)^{1/2}$ where $\mathbf{J}_i = I_i \mathbf{f}_i$ is the electric current of the i-th SW harmonics over the minimum sphere and the orthonormal functions $\mathbf{f}_i$ are defined in Appendix A. $|I_i|$ represents the mean-square over the minimum sphere of the electric currents of the i-th SW harmonics. The radiated power can be expressed in terms of radiation resistance $R_{rad,n}^{(s)}$, associated with the i-th harmonic, as $P_r = \frac{1}{2}\sum_i R_{rad,n}^{(s)} |I_i|^2$.

This radiation resistance does not depend on the azimuthal index m, but only on the polar index n and on the polarization index s and it can be explicitly obtained through the spherical Hankel's function of the second type $h_n^{(2)}(kr_{\min})$; i.e.,

$$R_{rad,n}^{(s)} = \begin{cases} \zeta \left| \dfrac{d}{d(kr)}\left[kr_{\min} h_n^{(2)}(kr_{\min})\right]\right|^{-2} & \text{for s=1 (TE)} \\ \zeta\left(kr_{\min}\right)^{-2}\left|h_n^{(2)}(kr_{\min})\right|^{-2} & \text{for s=2 (TM)} \end{cases} \qquad (3)$$

Note that these functions have a weak dependence on $kr_{\min}$ for $n > kr_{\min}$ and tend to ζ for $n >> kr_{\min}$. We also observe that even if $P_r = \frac{1}{2}\sum_i R_{rad,n}^{(s)} |I_i|^2$ involves only the electric current coefficients I_i it represents the *total* power associated to *both* the equivalent electric and magnetic currents radiating together in free-space (Fig.1(a)) or, equivalently, the power radiated by the magnetic currents alone on top the perfectly electric surface (Fig.1(b)), on which it induces electric currents. This formulation brings to (3) as a definition of the radiation resistance and makes the radiated power congruent with a zero-field (and zero total energy) inside the minimum sphere. Considering instead only the contribution of the electric currents radiating alone in free space leads to a different expression of radiation resistance as well as a non-zero value of field inside the surface. This aspect will be investigated further in section *F*.

It is seen that for small values of $kr_{\min}$, $R_{rad,1}^{(2)}$ and $R_{rad,2}^{(2)}$ (TM) go to zero as $(kr_{\min})^2$ and $(kr_{\min})^4$, respectively, while $R_{rad,1}^{(1)}$

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and $R^{(1)}_{rad,2}$(TE) go to zero as $(kr_{\min})^4$ and $(kr_{\min})^6$, respectively.

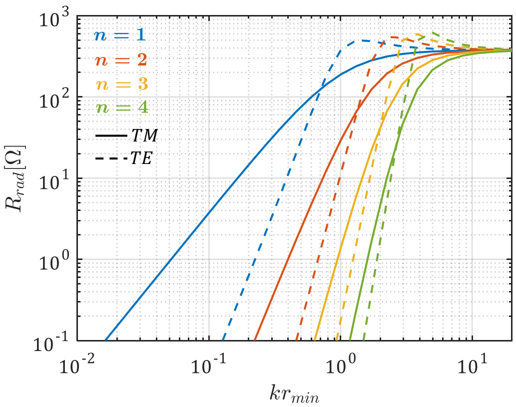

Fig. 2 Radiation resistance of individual harmonics for some values of *n* as a function of the normalized radius of the minimum sphere.

B. Gain Representation as a Function of the Current Coefficients

To estimate the gain, we assume that the magnetic currents of Fig.1(b) radiate in presence of a lossy conductor with resistivity for square-surface R_Ω. This resistance, also known as sheet resistance, has a value that depends on $\sqrt{f}$, (for copper it is approximately $R_\Omega = 2.82\times10^{-7}\sqrt{f}\ \Omega$). We assume that the electric currents induced on the conductor by the forced magnetic currents will not change significantly wrt the currents on a PEC (Fig.1(b)). The total power P_r+P_Ω (i.e., radiated plus dissipated power) is given by

$$P_r + P_\Omega = \frac{1}{2}\sum_i (R^{(s)}_{rad,n} + R_\Omega)\,|\,I_i\,|^2 \tag{4}$$

The gain is obtained in Appendix A as

$$G = 4\pi\frac{U}{P_r} = \frac{\left|\sum_i \sqrt{R^{(s)}_{rad,n}}\,I_i\mathbf{K}_i\right|^2}{\sum_i (R^{(s)}_{rad,n} + R_\Omega)\,|\,I_i\,|^2} \tag{5}$$

where $\mathbf{K}_i$ are the normalized far-field functions defined in Appendix A. In deriving (5) we have assumed that the modal currents on the surface are not perturbed by the presence of the small losses. Due to the symmetry of the problem, the gain can be maximized in an arbitrary observation direction and with an arbitrary polarization. It is therefore not restrictive to assume $(\theta,\phi)=(0,0)$ as well as a θ–polarization. Therefore, $\mathbf{K}_i$ in (5) can be substituted by its projection along θ evaluated at $(\theta,\phi)=(0,0)$; this projection depends only on the polar index *n* and will be hereinafter denoted by K_n. This value is a complex number whose magnitude is $|K_n| = \sqrt{2n+1}$. Applying the Swartz identity to the numerator of (5), yields

$$G \le \frac{\left(\sum_n \sqrt{R^{(s)}_{rad,n}}\ \,|\,I_n\,|\,|K_n|\right)^2}{\sum_n (R^{(s)}_{rad,n} + R_\Omega)\,|\,I_n\,|^2} \doteq G_0(|\,I_n\,|) \tag{6}$$

where the equality symbol is valid when the phase of $\mathbf{K}_i$ is equal to the phase of I_i^*.

C. Maximum Gain for Antennas with External Tuning.

Eq. (6) is well suited to make a maximization of the gain in absence of *Q*-bounds, but with an external tuning circuit for reaching the resonance. This is different from the case presented in [1], relevant to self-resonant antennas. In order to find the maximum value $G_0(|\,I_q\,|)$ for any possible set of amplitudes I_q, we impose $\partial G_0 / \partial |I_q| = 0$ for any *q*. This approach follows the method applied by Harrington in [3] to maximize the directivity, but without imposing a priori a truncation on the harmonic series, which was necessary in the Harrington's process to get a finite result. In Appendix B, it is demonstarated that the solution for the maximum gain $G_{max} = G_0\big|_{\frac{\partial G_0}{\partial |I_i|}=0}$ is

$$\begin{aligned} G_{max} &= \frac{1}{2}\sum_{n=1}^{\infty}(2n+1)\left(\eta_n^{TE} + \eta_n^{TM}\right) \\ \eta_n^{TE,\,TM} &= \frac{1}{1 + R_\Omega / R^{(1,2)}_{rad,n}} \end{aligned} \tag{7}$$

where $\eta_n^{(TE,TM)}$ are the radiation efficiencies of any individual TE and TM SW harmonic and $R^{(s)}_{rad,n}$ is defined in (3). It is noted that a similar formula, with a different efficiency definition, was derived in [19]. This maximum gain is obtained with the current coefficients over the minimum sphere given in Appendix B. These correspond to SWE electric field coefficients (with Hansen's normalization)

$$C_i = \delta_i \eta_n^{TE,TM}\sqrt{2n+1} \tag{8}$$

where

$$\delta_i = \begin{cases} 0 \ \text{ if } |m| \ne 1 \\ \sqrt{\zeta}\,I_0(-j)^n \ \text{ if } s{=}1, m{=}\pm1 \text{ (TE)} \\ \sqrt{\zeta}\,I_0 m(-j)^n \ \text{ if } s{=}2, m{=}\pm1 \text{ (TM)} \end{cases} \tag{9}$$

and the superscript TE, TM stands for s=1 and s=2, respectively. In deriving (8) we have used the relationship $C_i = I_i\sqrt{R^{(s)}_{rad,n}}$. The constant I_0 is arbitrary, since it disappears in calculating G_{max}. The radiation efficiency $\eta = P_r/(P_r + P_\Omega)$ in condition of maximum gain is indeed given by

$$\eta \doteq \frac{\sum_n R^{(s)}_{rad,n}\,|\,I_n\,|^2}{\sum_n \left(R^{(s)}_{rad,n} + R_\Omega\right)|\,I_n\,|^2} = \frac{\sum_{n=1}^{\infty}(2n+1)\left[\left(\eta_n^{TE}\right)^2 + \left(\eta_n^{TM}\right)^2\right]}{\sum_{n=1}^{\infty}(2n+1)\left[\eta_n^{TE} + \eta_n^{TM}\right]} \tag{10}$$

Fig. 3 shows the values of maximum gain obtained implementing (7) for various values of the surface resistance R_Ω (continuous lines). It also shows the directivity (dashed lines) obtained dividing the maximum gain in (7) by the efficiency in (10).

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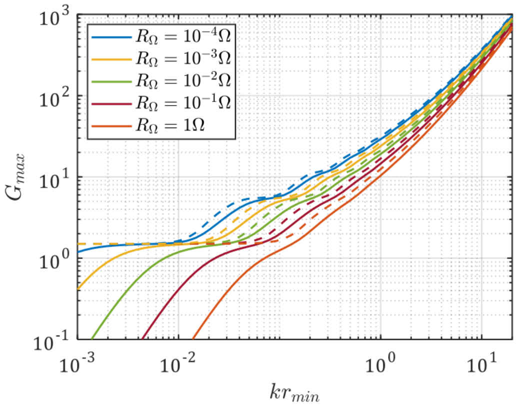

Fig. 3. Continuous lines: maximum gain of non-resonant (externally tuned) antennas calculated for different values of R_Ω as a function of kr_{min}. Dashed lines: corresponding directivity obtained by G_{max}/η.

The maximum gain and the corresponding directivity show inflection points at the maxima of the radiation efficiency. The lower frequency inflection point occurs at G_{max}=1.5. For smaller electrical sizes the directivity saturates to 1.5. It is worth noting that this value corresponds to the directivity of an electric dipole and not to the directivity of a Huygens' source. This behavior was already underlined in [22], and its meaning stems from the higher radiation resistance level exhibited by the first TM dipolar mode (electric dipole) for very small antennas. In contrast, the radiation resistance of the first TE mode is much lower, and therefore more affected by the losses. It is also worth noting that the second inflection points of the maximum gain correspond to the fact that in the summation in (7) one more mode becomes significantly excited.

D. Value of Q on the Maximum Gain Curve

The *Q*-factor can be defined in two different ways depending on weather the antenna is self-resonant or it is matched by providing an external reactive energy from a lossless circuit. In the first case, one has $W_E = W_H$, $Q = 2\omega W_E / P_r = 2\omega W_H / P_r$ where ω is the angular frequency, P_r is the radiated power and W_E and W_H are the electric and magnetic energies stored in the external region, respectively. For non-resonant antennas, one has $Q = 2\omega W_E / P_r$ for capacitive antennas and $Q = 2\omega W_H / P_r$ for inductive antennas. This definition assumes that an external energy has been added to the system to get the resonance. In both cases, the *Q*-factor can be interpreted as the reciprocal of the fractional bandwidth $BW = 1/Q$ when it is larger than 10 [16].

The calculation of the stored energies $W_{E,H}$ of a general spherical wave expansion is an old topic [27],[17],[2]-[30]. Essentially, the most used approaches are the ones provided by Chu [25], Collin and Rothschild [16] and Fante [2], the latter generalized to the case of arbitrary field internal to the minimum sphere in [28]. In [16], the quality factor of each individually tuned spherical wave is defined, for a *unit radiated power*, as

$$Q'_n = \begin{cases} 2\omega W_{E,n} & TM \text{ modes} \\ 2\omega W_{H,n} & TE \text{ modes} \end{cases} \tag{11}$$

where $W_{E,n}$ and $W_{H,n}$ are the electric and magnetic energy associated with the *n-th* mode. In [2], Fante introduced additional subdominant terms needed for the calculation of the Q-factor of a generic non-resonant antenna with unit power as

$$Q''_n = \begin{cases} 2\omega W_{E,n} & TM \text{ modes} \\ 2\omega W_{E,n} & TE \text{ modes} \end{cases} \tag{12}$$

The exact expressions of Q_n' and Q_n'' are reported in Appendix C. It is noted that they are independent from the azimuthal index *m*. A general expression of the *Q* of a non-resonant antenna is given by

$$Q = \begin{cases} \dfrac{\left(\sum_{TM} |C'_n|^2 Q'_n + \sum_{TE} |C''_n|^2 Q''_n \right)}{\sum_n |C'_n|^2 + |C''_n|^2} & W_E > W_H \\ \dfrac{\left(\sum_{TM} |C'_n|^2 Q''_n + \sum_{TE} |C''_n|^2 Q'_n \right)}{\sum_n |C'_n|^2 + |C''_n|^2} & W_H > W_E \end{cases} \tag{13}$$

where C'_n and C''_n are the coefficients of the TM and TE modes, respectively. In the specific case we are dealing with, the antenna is not resonant due to the unbalancing provided by the different efficiency of TE and TM modes. This implies that till a certain dimension of the antenna, the reactive electric energy dominates, and therefore we should always apply the first of (13); therefore, using (8) leads to

$$Q_{tot} = \frac{\sum_{n=1}^{\infty} (2n+1) \left[\left(\eta_n^{TM}\right)^2 Q'_n + \left(\eta_n^{TE}\right)^2 Q''_n \right]}{\sum_{n=1}^{\infty} (2n+1) \left[\left(\eta_n^{TM}\right)^2 + \left(\eta_n^{TE}\right)^2 \right]} \tag{14}$$

The value of *Q* on the maximum Gain curve as a function of the antenna size is presented in Fig. 4 for various values of the loss resistances. It is seen that the bandwidth (the inverse of Q_{tot}) becomes extremely small for small resistance, even if it corresponds to very high maximum gain limit. Furthermore, all the curves tend asymptotically for small electrical size to the value of Q that satisfies the Chu limit, namely

$$Q_{tot} \to Q'_1 = \frac{1}{(kr_{min})^3} + \frac{1}{(kr_{min})} \quad \text{for} \quad kr_{\min} \to 0 \tag{15}$$

It is worth noting that for resonant antennas that maximize the directivity the convergence as $kr_{\min} \to 0$ is like

$Q_{tot} \to Q_1 = \frac{1}{2}\left(Q_1' + Q_1''\right) = \frac{1}{2}(kr_{min})^{-3} + (kr_{min})^{-1}$. This behavior is due to the lower efficiency of the TE modes in the quasi-static limit.

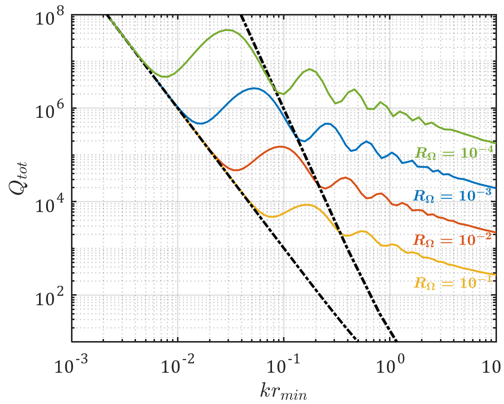

Fig. 4. Quality factor Q of spherical wave expansion with coefficients associated to the maximum gain for various values of the loss resistance (values in Ohm). The two dashed dotted lines represent Q_1' and Q_2' .

E. *Maximum Gain for Self-Resonant Antennas*

The solution in (8) is not self-resonant; namely, there should be an additional external reactive tuning circuit to reach the maximum gain. For self-resonant antennas (which is also the case treated in [1]), the maximum gain is obtained by using a method similar to the one given in the Appendix B of [1], based on the dual (Lagrange) problem. We don't repeat here the process, showing only the result, that is

$$G_{\max}(\xi_0) = \sum_n \left(\frac{\frac{1}{2}\eta_n^{TE}\left(2n+1\right)}{\left[1+\xi_0\eta_n^{TE}\left(Q_n' - Q_n''\right)\right]} + \frac{\frac{1}{2}\eta_n^{TM}\left(2n+1\right)}{\left[1-\xi_0\eta_n^{TM}\left(Q_n' - Q_n''\right)\right]} \right) \quad (16)$$

where ξ_0 is the value that minimizes the series $G_{\max}(\xi)$ in the interval

$$\xi \in \left[-\frac{1}{\max_n\left\{\eta_n^{TE}\left(Q_n' - Q_n''\right)\right\}}; \ \frac{1}{\min_n\left\{\eta_n^{TM}\left(Q_n' - Q_n''\right)\right\}} \right] \quad (17)$$

The gain in (16) is obtained with field coefficients

$$C_i = \delta_i \frac{\eta_n^{(TE,TM)}\sqrt{2n+1}}{\left[1+\xi_0\eta_n^{(TE,TM)}\left(Q_n' - Q_n''\right)\right]} \quad (18)$$

where δ_i is defined in (9). The coefficients for the current expansion are obtained dividing by the square root of the individual TE and TM *n*-indexed radiation resistance. Comparison between (16) and (7) are given in Fig. 5. The resonant antenna maximum gain is tighter than the one for externally tuned antenna, especially for small antennas and it drops rapidly to zero for gain approximately equal to 3. This aspect has been also underlined in [22].

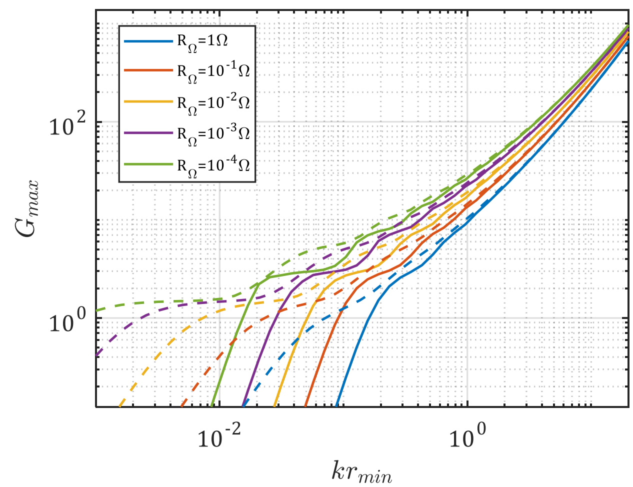

Fig. 5 Maximum antennas gain calculated for different values of R_Ω as a function of kr_{min}. Dashed lines: externally tuned antennas. Continuous lines: self-resonant antennas.

F. *Comparison with Gustafsson-Capek results*

It is important to compare our result with the one obtained by Gustafsson and Capek in [22]. The formulation presented there is based on a Methods of Moments (MoM) applied to the surface of an arbitrary metallic body and by a convex optimization procedure. They found the maximum super-gain by imposing a maximization of the power intensity with constant radiated power for any coefficient of the MoM basis functions, assuming small losses on the metallic surface.

This procedure is quite general and can be applied to arbitrary shapes. In Fig.1 of [22], the authors apply the procedure to a spherical shape, using spherical modes as basis functions. In the externally tuned case, the convex optimization is carried out without conditions on the reactance of the MoM matrix. In the self-resonant case, they impose also a vanishing reactive average power through the imaginary part of the MoM matrix. We have re-implemented the convex optimization procedure of Gustafson-Capek for both cases, and we have found results very close (but not equal) to the results provided by (7) and (15). Fig. 6 shows the discrepancy between our results and the ones in [22] for the externally tuned case (a similar situation is found for the self-resonant case). Although this has initially puzzled us, finally we have found the reason of it. We found indeed that the results from our procedure become identical to those obtained in [22] if one uses the radiation resistance produced by *electric currents only*. More specifically, we can reproduce exactly the results in [22], changing the SW radiation resistances in (7) with the following expression

$$R_{rad,J-only,n}^{(s)} = \begin{cases} \zeta\left[\dfrac{\partial}{\partial(kr)}\left(kr_{\min} j_n\left(kr_{\min}\right)\right)\right]^2 & \text{for s=1 (TE)} \\ \zeta\left[kr_{\min} j_n\left(kr_{\min}\right)\right]^2 & \text{for s=2 (TM)} \end{cases} \quad (19)$$

These radiation resistances are the ones corresponding to harmonics of electric currents only radiating in free-space. It should be noted that using the electric currents only, provides

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field (and energy) different from zero inside the minimum sphere, which leads to a slightly tighter value of the maximum gain.

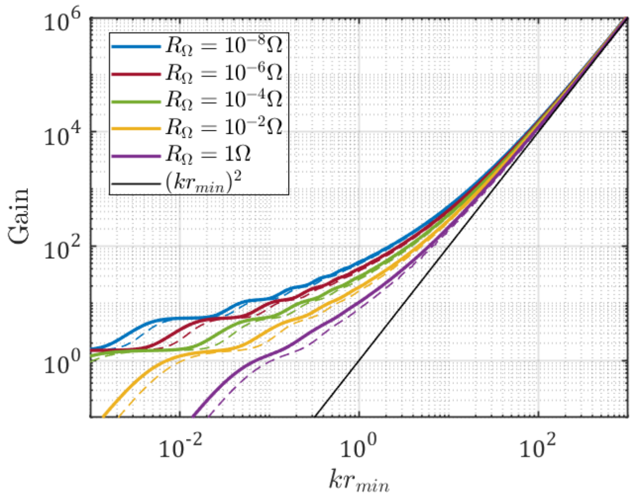

Fig. 6 Comparison between the maximum gain for the externally tuned case of our formulation (continuous lines) and the formulation in [20] (dashed lines), also obtained by using (19) in (7).

G. Distribution of the Coefficients of the Currents

Fig. 7 shows the histogram of the *n*-indexed current coefficients' amplitude corresponding to the maximum gain for externally tuned case in (8). The coefficients are normalized in such a way to have unitary radiated power. Fig.7(a)-(b) correspond to different values of surface resistance (R_Ω=1Ω and R_Ω=0.1Ω, respectively). Both cases of externally tuned coefficients and self-resonant coefficients are reported. The calculation is carried out for kr_{min}=2. It can be seen that the maximum coefficient amplitude is given at the index at which the ohmic losses resistance approaches the resistance of the harmonic, that is $R_\Omega \approx R_{rad,2}^{(1,2)}$, namely, when the efficiency of the harmonic is 50%, i.e. $\eta_2^{(TE,TM)} \approx \frac{1}{2}$.

It is apparent that for smaller values of the loss-resistance the optimal current coefficients are concentrated on super-reactive harmonics, it means SWs with polar index larger than $kr_{\min}$=2. This makes it challenging to achieve their excitation on the minimum sphere. Consequently, the bound described in (7) is difficult to achieve. It is also apparent from Fig. 7 that the TE optimal coefficients for the resonant case are higher, aligning with their reduced efficiency for small antennas. Furthermore, lower losses correspond to larger optimal coefficient values, as noted by the disparity in the vertical scales between Fig.7(a) and 7(b).

IV. Externally Tuned Small Antennas

For small antennas the interpretation of the previous formulas becomes interesting. We limit the analysis to the externally-tuned case, but similar considerations can be carried out for the self-resonant antennas. We first observe that the *n*-th TE and TM spherical harmonics, with azimuthal index $m = \pm 1$ have different coefficients, and this implies different impact on both bandwidth and gain.

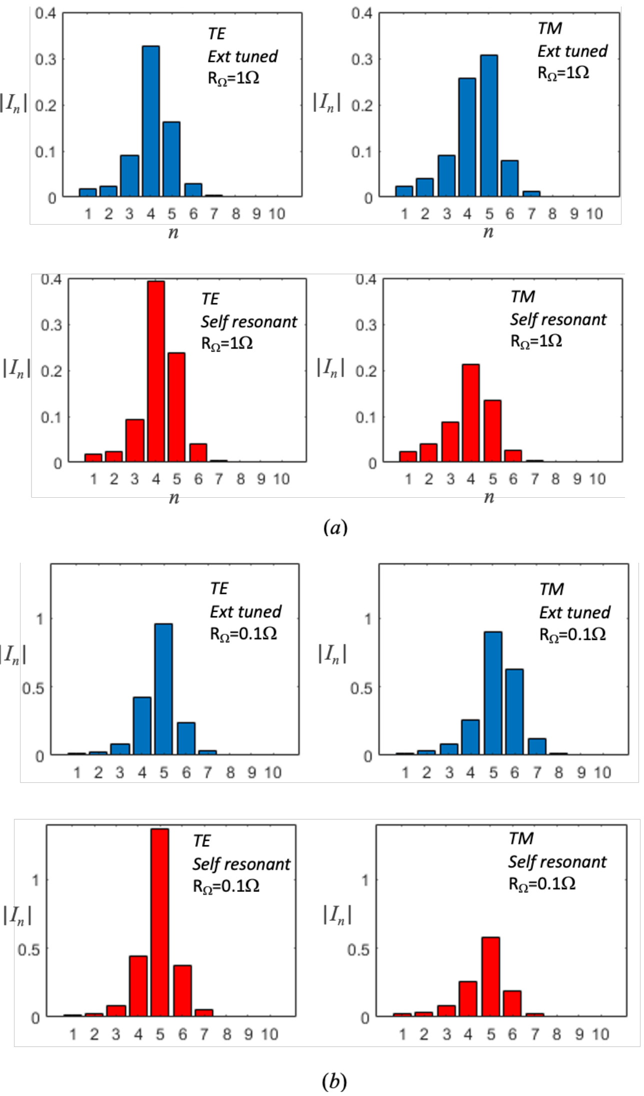

Fig. 7 Histograms of current coefficients for TE (inductive, right-hand side) and TM (capacitive, left-hand side) harmonics; both externally tuned case (blue bars) and self-resonant case (red bars) are reported. (a) $R_\Omega = 1\ \Omega$; (b) $R_\Omega = 10^{-1}\Omega$. The maximum gain is in this case approximately equal for self-resonant and tuned case. The amplitudes are normalized to have a unit radiated power (P_r=1Watt). The maximum coefficients are obtained for the *n* where the harmonics efficiency is about 50%. Note the difference of vertical scale for (a) and (b).

This is in contrast with what happens for the maximum directivity, namely in absence of losses [1]. In the latter case and for *n*=1, the maximum directivity field can be interpreted as the one produced outside the minimum sphere by an elementary Huygens' dipole (HD) located at the origin. By duality, the energy density of the HD is balanced outside the minimum sphere. In the same lossless case, the combination of the spherical wave harmonics for *n*=2 provides the field of a Huygens' Quadrupole (HQ) combined with the one of a Dual Vertical Quadrupole (DVQ) [1]. In presence of losses, the maximum gain is obtained by unbalanced coefficients, that renders the antenna non resonant. This means that outside the

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minimum sphere electric and magnetic energies are not balanced, thus, requiring the use of (13). In particular, the coefficients associated to the electric and the magnetic dipoles and quadrupoles field are weighted by their efficiency, as prescribed by the general form (8). In the far zone this leads to the electric field proportional to

$$\mathbf{h} = \frac{3}{2}\left(\eta_1^{(TM)}\mathbf{h}_1^{(TM)} + \eta_1^{(TE)}\mathbf{h}_1^{(TE)}\right) + \frac{5}{2}\left(\eta_2^{(TM)}\mathbf{h}_2^{(TM)} + \eta_2^{(TE)}\mathbf{h}_2^{(TE)}\right) \quad (20)$$

where

$$\mathbf{h}_1^{(TM)} = \left(\cos\theta\cos\phi\right)\hat{\theta} - \sin\phi\hat{\phi} \quad (21)$$

$$\mathbf{h}_1^{(TE)} = \cos\phi\hat{\theta} - \cos\theta\sin\phi\hat{\phi} \quad (22)$$

$$\mathbf{h}_2^{(TM)} = \left(\cos 2\theta\cos\phi\right)\hat{\theta} - \cos\theta\sin\phi\hat{\phi} \quad (23)$$

$$\mathbf{h}_2^{(TE)} = \left(\cos\theta\cos\phi\right)\hat{\theta} - \cos 2\theta\sin\phi\hat{\phi} \quad (24)$$

In particular, $\mathbf{h}_1^{(TM)}, \mathbf{h}_1^{(TE)}$ are the electric far-field pattern of a *x*-directed electric and *y*-directed magnetic dipoles, respectively, and $\mathbf{h}_2^{(TM)}, \mathbf{h}_2^{(TE)}$ are electric and magnetic quadrupoles, respectively. The latter are obtained by in phase combination of *x*-directed and *z*-directed electric quadrupoles, and *y*-directed and *z*-directed magnetic quadrupoles, respectively. The situation is illustrated in Fig. 8.

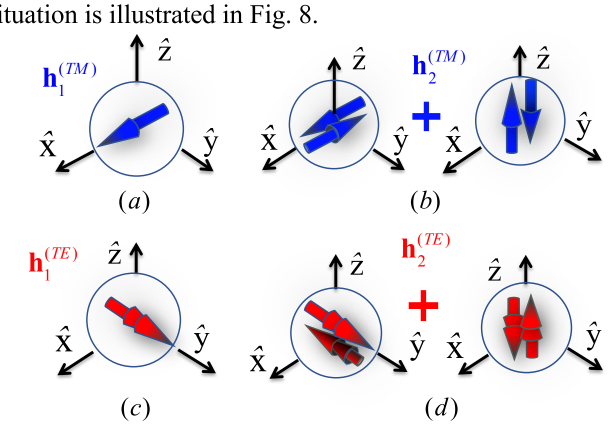

Fig. 8. Elementary sources associated with the far field pattern of dipolar $\mathbf{h}_1^{(TM)}$ (a), $\mathbf{h}_1^{(TE)}$ (c) and quadrupolar $\mathbf{h}_2^{(TM)}$ (b) $\mathbf{h}_2^{(TE)}$, (d) contributions. Electric dipoles (TM) are denoted in blue with a single arrow and magnetic dipoles (TE) in red with a double-arrow. The vertical doublet is aligned along *x* for electrical dipoles and along *y* for magnetic dipoles.

The maximum gain for this approximation is given by

$$G_{max} \approx \frac{3}{2}\left(\eta_1^{(TE)} + \eta_1^{(TM)}\right) + \frac{5}{2}\left(\eta_2^{(TE)} + \eta_2^{(TM)}\right) \quad (25)$$

which is associated to a total quality factor

$$Q_{tot} = \frac{3\left(\eta_1^{(TM)}\right)^2 Q_1' + 3\left(\eta_1^{(TE)}\right)^2 Q_1'' + 5\left(\eta_2^{(TM)}\right)^2 Q_2' + 5\left(\eta_2^{(TE)}\right)^2 Q_2''}{3\left(\eta_1^{(TM)}\right)^2 + 3\left(\eta_1^{(TE)}\right)^2 + 5\left(\eta_2^{(TM)}\right)^2 + 5\left(\eta_2^{(TE)}\right)^2} \quad (26)$$

In (17) and (18) the explicit expressions of the SW efficiency posing $x=kr_{min}$, are

$$\eta_1^{(TM)} = \frac{x^2}{x^2 + \frac{R_\Omega}{\zeta}\left(x^2+1\right)} \approx \frac{x^2}{x^2 + R_\Omega/\zeta} \quad (27)$$

$$\eta_1^{(TE)} = \frac{x^4}{x^4 + \frac{R_\Omega}{\zeta}\left[\left(x^2-1\right)^2 + x^2\right]} \approx \frac{x^4}{x^4 + R_\Omega/\zeta} \quad (28)$$

$$\eta_2^{(TM)} = \frac{x^4}{x^4 + \frac{R_\Omega}{\zeta}\left[\left(x^2-3\right)^2 + 9x^2\right]} \approx \frac{x^4}{x^4 + 9R_\Omega/\zeta} \quad (29)$$

$$\eta_2^{(TE)} = \frac{x^6}{x^6 + \frac{R_\Omega}{\zeta}\left[\left(-x^3+6x\right)^2 + \left(-3x^2+6\right)^2\right]} \approx \frac{x^6}{x^6 + 36R_\Omega/\zeta} \quad (30)$$

where the first equality is derived directly by (3); the approximation at the last members are valid for any value of x from 0 to infinity and for $R_\Omega < 10^{-1}\Omega$ with maximum relative error e_{max} limited by the following values $e_{max} < 2\times10^{-4}; e_{max} < 8\times10^{-3}; e_{max} < 8\times10^{-3}; e_{max} < 2\times10^{-3}$ for eqs. (27), (28), (29), (30), respectively. The exact values of the quality factors are given by

$$Q_1' = \frac{1}{x^3} + \frac{1}{x} \quad ; \quad Q_1'' = \frac{1}{x}$$
$$Q_2' = \frac{18}{x^5} + \frac{6}{x^3} + \frac{3}{x} \quad ; \quad Q_2'' = \frac{3}{x^3} + \frac{3}{x} \quad (31)$$

It is interesting to compare the maximum gain obtained considering two source contributions in (25) and the full series in (7). Fig. 9 shows this comparison for values $R_\Omega \in \left[10^{-4}\Omega,\ 1\,\Omega\right]$. It is seen that the two expressions coincide below G_{max} =6 and agree till G_{max} =7.

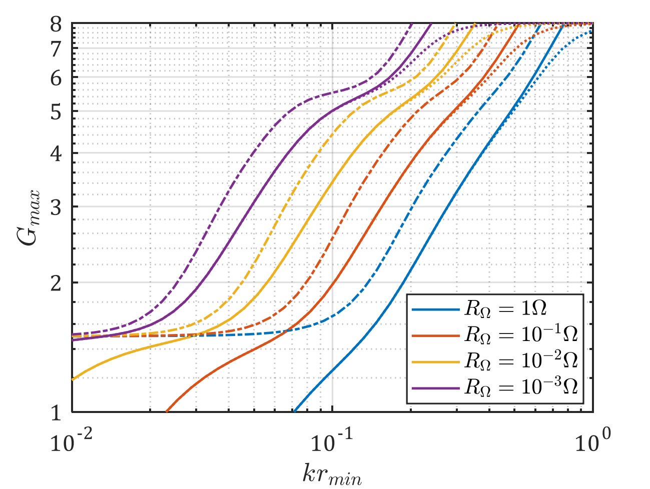

Fig. 9 Maximum gain for calculated for different values of R_Ω as a function of kr_{min} using the full series in (7) (continuous line) and the two terms approximation in (25) (dotted line). Dash-dotted lines are the corresponding directivities obtained by G_{max}/η by using the full series.

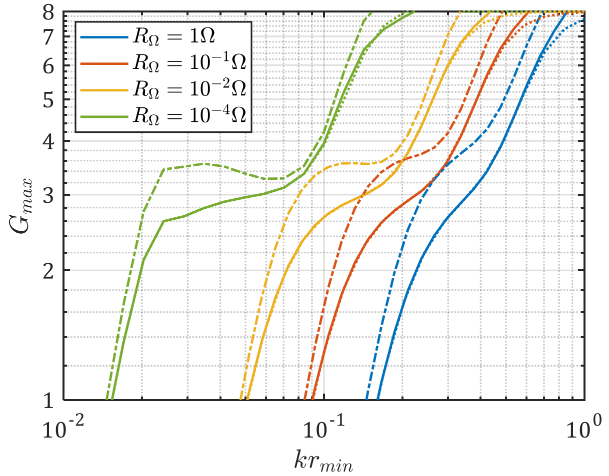

Fig. 10 Maximum gain of self-resonant antennas calculated for different values of R_Ω (in Ohm) as a function of kr_{min} using the full series in (16) (continuous lines). Dotted lines: first two terms of the series in (16). Dash-dotted lines: directivities obtained by G_{max}/η with the full series.

The directivity is also reported in the same plot. It is worth noting that the insertion in (25) of the approximation in the last expressions in (27)-(30) does not practically provide numerical differences. For completeness, Fig. 10 shows similar results as those of Fig. 9 for self-resonant antennas.

V. Maximum Gain With Minimum-Q bound

In [1], the exact formula for minimum-Q and maximum directivity has been derived through a convex optimization for the radiation density bounded to have a unit power and a minimum Q. Following the same approach, we can obtain the Q-bounded super-gain applying the Lagrange's multiplier method shown in Appendix C of [1]. In obtaining the bounded maximum gain, we assume that the condition $W_E>W_H$ is respected, so as to impose as a bound the first expression in (13). Since the procedure is like the one in [1], we omit the demonstration, and we present only the final expression, that is,

$$G_{\max}(\nu_0)=\sum_{n=1}^{\infty}\left(\frac{\frac{1}{2}\eta_n^{TM}(2n+1)}{\left[1+\nu_0\left(\eta_n^{TM}Q_n'-Q\right)\right]}+\frac{\frac{1}{2}\eta_n^{TE}(2n+1)}{\left[1+\nu_0\left(\eta_n^{TE}Q_n''-Q\right)\right]}\right) \quad (32)$$

where ν_0 is the values that minimize the series $G_{\max}(\nu)$ in the interval $\nu\in\left[0,(Q-\eta_1^{(TE)}Q_1')^{-1}\right]$)]. In (32), Q is the maximum desired total quality factor of the antenna. It is worth noting that the above expression changes when $W_E < W_H$; in such a case, in accordance with (13), it is sufficient to interchange the Q_n' and Q_n''. In the implementation one should therefore switch formula at the resonance (W_E=W_H). The field coefficients for the Q-bounded maximum gain are given by

$$C_i=\delta_i\frac{\eta_n^{TE,TM}\sqrt{2n+1}}{\nu_0(\eta_n^{TE,TM}Q_n''-Q)+1} \quad (33)$$

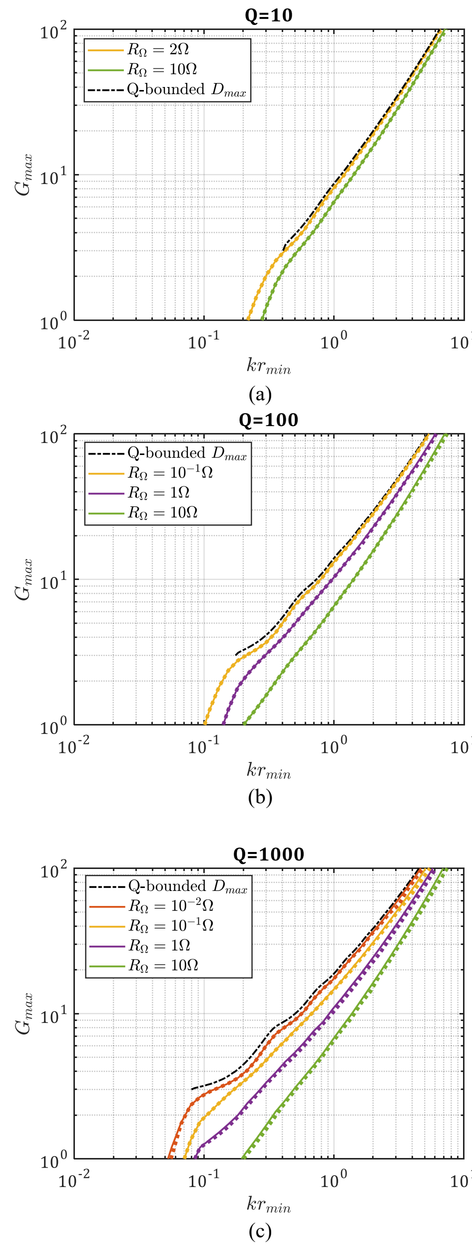

Fig. 11 Q-bounded maximum gain G_{max} as a function of the antenna size calculated for Q=10 (a), Q=100 (b), and Q=1000 (c) and different values of the ohmic losses resistance R_Ω; the curves tend smoothly to the Q-bounded maximum directivity (black line) when the losses tend to zero. R_Ω=0 corresponds to the maximum Q-bounded superdirectivity D_{max} as in (1).

The curves obtained with eq. (32) are compared with a numerical convex optimization obtained by expanding electric and magnetic currents in terms of small-domain basis functions (dotted lines); excellent agreement has been found. We should highlight that the intricate interplay between bandwidth and losses implies that, for a fixed antenna size, a larger required bandwidth may necessitate a slight increase in losses to minimize the disparity between maximum gain and maximum directivity. Eventually, the Q-bounded parametric curves of maximum gain gradually tend to the Q-bounded maximum directivity in (1) for vanishing losses (dash-dotted lines in all Fig. 11(a)-(c)), with level of admissible losses depending on the required Q. More precisely, the discrepancy between maximum Directivity and maximum Gain vanishes for smaller losses when Q is larger. This aspect is quantified in Fig. 12, where the percentage error $\varepsilon = (D_{\max} - G_{\max})/D_{\max} = 1-\eta$ between the Q-bounded gain in (32) and the Q-bounded directivity in (1) has been calculated as a function of the antenna size for different value of R_Ω. Fig. 12(a),(b),(c) correspond to Q=10, 100, 1000, respectively. A more exhaustive analysis revealed that ε less than 7% (which corresponds to an overall radiation efficiency larger than 93%) is found for $Q < 9/R_\Omega^{0.8}+3$. This means that in this range one can calculate the maximum gain with the simple formula in eq. (2) for maximum directivity. The situation is illustrated in with the help of a picture Fig. 13. Finally, we should also emphasize the fact that the direct use of (32) may provide inconsistencies in the region $Q < 9/R_\Omega^{0.8}+3$, since (32) is not valid when $W_E < W_H$, where one has to interchange Q'_n and Q''_n to get the rigth gain maximization.

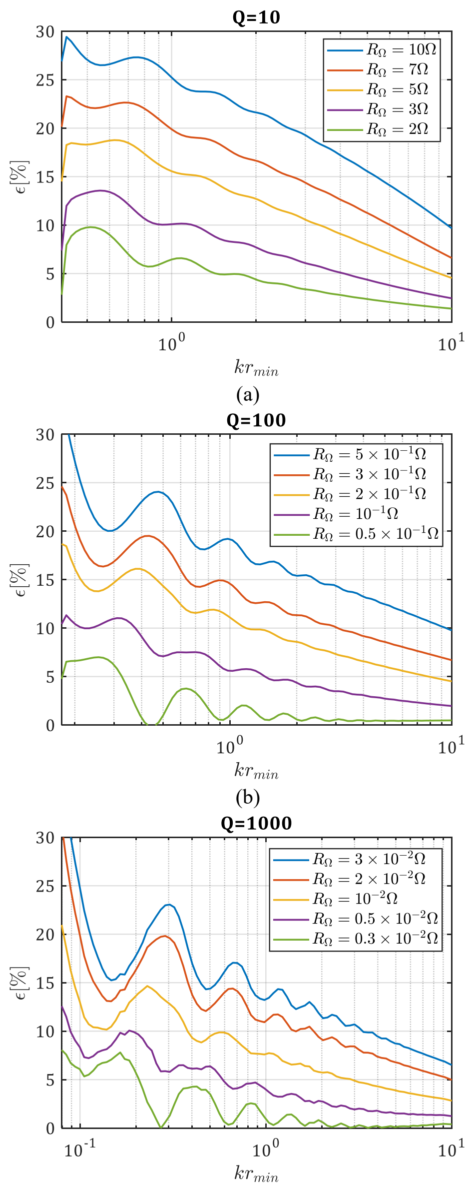

Fig. 12 Percentage difference between maximum Q-bounded directivity and maximum Q-bounded gain for different values of the loss resistence. The green curves correspond to an efficiency larger than 93% in the overall range from the Chu limit. (a) Q=10; (b) Q=100; (c) Q=1000. The horizontal scale start form the Chu limit.

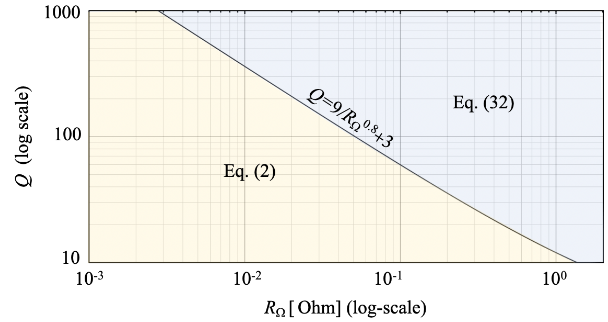

Fig. 12 Region of validity of the formulas for Q-bounded maximum gain with losses (eq. 32) and without losses (eq. 2). For $Q<9/R_\Omega^{0.8}+3$ eq. (2) can be used with a maximum error less than 7% for all antenna sizes.

VI. Conclusions

This paper has delineated various fundamental characteristics of the maximum gain exhibited by arbitrary antennas with specified electrical sizes. Analytical expressions for the maximum antenna gain, contingent upon losses and subject to Q bounds, have been presented for both resonant and non-resonant (externally tuned) cases. The exact solution is given in terms of a series that should be minimized wrt a single parameter. For smaller antennas, a concise closed form has been furnished, delivering accuracy up to a maximum gain of 7. Additionally, an intriguing interpretation in terms of magnetic and electric dipoles has been provided. The findings showcased in this study have also unveiled the range of losses and Q values wherein super-directivity corresponds to super-gain.

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Appendix A: Spherical Wave Expansion

The equivalent electric and magnetic currents of the Love formulation (Fig. 1a) are given in terms of SWE by $\mathbf{J}=jk/\sqrt{\zeta}\sum_{s,m,n} C_{s,m,n}\hat{r}\times\mathbf{F}^{(3)}_{3-s,m,n}$ and $\mathbf{M}=k\sqrt{\zeta}\sum_{s,m,n} C_{s,m,n}\mathbf{F}^{(3)}_{s,m,n}\times\hat{r}$, respectively, where ζ is the free space impedance, $C_{s,m,n}$ are the expansion coefficients of the electric field, $\mathbf{F}^{(3)}_{s,m,n}$ are the spherical wave functions evaluated over the minimum sphere, as defined in the Hansen's book [9], except for a different time dependency (which is here exp($j\omega t$)). In this notation, the superscript "3" corresponds to the spherical Hankel second-type r-dependent function. The *polar* index n refers to the order of the Hankel function and the index m refers to the azimuthal angular wave number. The subscript s=1,2 denotes TE and TM polarization with respect to the radial direction r, respectively.

The proper normalization of the spherical wave functions of Hansen and their orthogonality imply that the radiated power is given by the summation of the squared amplitude of the coefficients, $P_r = \frac{1}{2}\sum_{s,m,n}|C_{s,m,n}|^2$. The expansion of the currents can be rewritten through the (θ,ϕ)-dependent functions $\mathbf{f}_i=(1/r_{min})\mathbf{T}_{s,m,n}(\theta,\phi)\doteq(1/r_{min})\hat{r}\times\mathbf{F}^{(3)}_{s,m,n}\times\hat{r}/R^{(3)}_{s,n}$ where $R^{(3)}_{s,n}(kr)$ are defined through the spherical Hankel functions [9]. To compact the notation, we have used, the index i which renumbers the tern of indexes s,m,n in a single index $i=2\big(n\big(n+1\big)+m-1\big)+s$.

The far field radiation density for unit solid angle can be written as $U=\lim_{r\to\infty}\frac{1}{2\zeta}\left|r\mathbf{E}\right|^2=\frac{1}{8\pi}|\sum_i C_i\mathbf{K}_i|^2$, where $\mathbf{K}_i=4\pi\ \mathbf{T}_i\lim_{r\to\infty} kr\ R^{(3)}_{3-s,n}$. The same quantity can be expressed in terms of the electric current coefficients I_i through $\left|C_i\right|=\left|I_i\right|\sqrt{R^{(s)}_{rad,n}}$ where $R^{(s)}_{rad,n}$ is the radiation resistance of the individual harmonics given in (3). Using the above expression the antenna gain in direction $\left(\theta,\phi\right)$ is given by (5).

APPENDIX B: MAXIMUM GAIN COEFFICIENTS

The maximization of the function $G_0(|I_q|)$ in (6) with respect to the coefficients $|I_q|$ is obtained by imposing the vanishment of the derivative wrt $|I_q|$; namely

$$\frac{\partial G_0}{\partial\left|I_q\right|}=\frac{\partial}{\partial\left|I_q\right|}\frac{\left(\sum_i\sqrt{R^{(s)}_{rad,n}}\ \ |I_n|\left|K_n\right|\right)^2}{\sum_i(R^{(s)}_{rad,n}+R_\Omega)\,|I_n|^2}=0 \tag{34}$$

The above is equivalent to

$$\begin{aligned}&\left(2\left|K_q\right|\sqrt{R^{(s)}_{rad,q}}\right)\sum_n(R^{(s)}_{rad,n}+R_\Omega)\,|I_n|^2=\\&2\left|I_q\right|(R^{(s)}_{rad,q}+R_\Omega)\left(\sum_n\sqrt{R^{(s)}_{rad,n}}\ \ |I_n|\left|K_n\right|\right)\end{aligned} \tag{35}$$

which is respected if and only if

$$\left|K_q\right|\sqrt{R^{(s)}_{rad,q}}=\left|I_q\right|(R^{(s)}_{rad,q}+R_s) \tag{36}$$

for any q. When substituted in (6), (36) leads to (7), and together with the phase condition $\angle I_i+\angle K_i=0$, yields the current coefficients

$$I_i=\begin{cases}0 & \text{if } |m|\neq 1\\ I_0\left(-j\right)^n\sqrt{2n+1}\,\dfrac{\sqrt{\zeta R^{(1)}_{rad,n}}}{R^{(1)}_{rad,n}+R_\Omega} & \text{if } s{=}1, m{=}\pm 1\\ I_0 m(-j)^n\sqrt{2n+1}\,\dfrac{\sqrt{\zeta R^{(2)}_{rad,n}}}{R^{(2)}_{rad,n}+R_\Omega} & \text{if } s{=}2, m{=}\pm 1\end{cases} \tag{37}$$

The relation between current coefficients and electric fields coefficients $\left|C_i\right|=\left|I_i\right|\sqrt{R^{(s)}_{rad,n}}$ leads (8).

APPENDIX C: Q-FACTORS FOR SPHERICAL WAVES

The analytical expression of the Fante's dominant and subdominant Q-factors of n-indexed spherical waves is given by

$$\begin{aligned}Q'_n=x-\left|h_n(x)\right|^2\left[\tfrac{1}{2}x^3+x(n+1)\right]-\tfrac{1}{2}x^3\left|h_{n+1}(x)\right|^2\\+\tfrac{1}{2}x^2\left(2n+3\right)\left(j_n(x)j_{n+1}(x)+y_n(x)y_{n+1}(x)\right)\end{aligned} \tag{38}$$

$$Q''_n=x-\tfrac{1}{2}x^3\left[\left|h_n(x)\right|^2-j_{n-1}(x)j_{n+1}(x)-y_{n-1}(x)y_{n+1}(x)\right] \tag{39}$$

where $x=kr_{\min}$ and h_n, j_n, y_n are the spherical Hankel of second kind, Bessel, and Neumann functions of order n, respectively. For n=1 and 2 they assume the exact expressions in (31).

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