Social Choice and Welfare
https://doi.org/10.1007/s00355-024-01542-1

ORIGINAL PAPER

Decentralized pure exchange processes on networks

Daniele Cassese[1] · **Paolo Pin**[2,3]

Received: 15 December 2023 / Accepted: 16 July 2024

Abstract

We define a class of pure exchange Edgeworth trading processes that under minimal assumptions converge to a stable set in the space of allocations, and characterise the Pareto set of these processes. Choosing a specific process belonging to this class, that we define *fair trading*, we analyse the trade dynamics between agents located on a weighted network. We determine the conditions under which there always exists a one-to-one map between the set of networks and the set of limit points of the dynamics, and derive an analog of the Second Welfare Theorem for networks. This result is used to explore what is the effect of the network topology on the trade dynamics and on the final allocation.

1 Introduction

This paper contributes to the literature on the dynamics of trade, providing a model of pure exchange trade (without production nor consumption) where agents are located on an exogenously fixed network of trading opportunities. We demonstrate the existence of a bijective relationship, given initial endowments, between the network structure and the system's convergence points of the dynamics, which are market equilibria and where the corresponding final allocations of goods belong to a subset of the Pareto set.

Previous versions of this paper have circulated under the titles "A Second Welfare Theorem for Pure Exchange Processes on Networks" and "Edgeworth trading on Networks". We thank Malbor Asllani, Timoteo Carletti, Simone Cerreia Vioglio, Luca Chiantini, Satoshi Fukuda, Edoardo Gallo, Benjamin Golub, Michael König, Thomas Mariotti, Alessandro Pavan, Joel Robbins, Alex Teytelboym and Alexander Wolitzky for helpful comments. P. Pin acknowledges funding from the Italian Ministry of Education Progetti di Rilevante Interesse Nazionale (PRIN) grant 2017ELHNNJ.

✉ Daniele Cassese
dc554@cam.ac.uk

Paolo Pin
paolo.pin@unisi.it

1 Faculty of Economics and Trinity Hall College, University of Cambridge, Cambridge, UK

2 Department of Economics and Statistics, Università di Siena, Siena, Italy

3 BIDSA, Università Bocconi, Milan, Italy

Published online: 05 August 2024

Within this framework, focusing on a Fair Trade mechanism based on the egalitarian bargaining solution (Kalai 1977), we establish a version of the Second Welfare Theorem for networks, which contributes to the analysis of how the network structure impacts final allocations and the distribution of welfare. The dynamical system that describes the path of exchanges is not analytically solvable even in the simplest case of two individuals and two goods, so we analyse the impact of trading networks on welfare via simulations. What emerges is a complex relation between the network, the initial endowments and welfare in equilibrium.

There are multiple compelling reasons to take into account the network structure of trading opportunities. One fundamental reason is that real-world transactions are shaped and influenced by the underlying relationships between agents. Due to factors such as geography, social ties, and technological compatibility, not all individuals interact with each other directly. The study of trade on networks has garnered significant attention in the field of economics, and for a comprehensive review, we refer readers to the work by Manea (2016). It is important to note that the key distinction between the contributions examined in Manea's review and our own research lies in the modelling approach. Unlike the explicit incorporation of strategic interactions among agents in the reviewed literature, our focus is primarily on characterising the dynamics of trade within a fixed network, utilizing a tractable convergent dynamical systems framework.

In the Walrasian competitive equilibrium, decentralized exchange occurs at the final equilibrium prices. However, in real market transactions, agents discover equilibrium prices by engaging in mutually beneficial trades even at disequilibrium prices (Foley 2010): to overcome the challenge posed by the absence of a real price dynamics, the concept of an "auctioneer" was introduced. In *tâtonnement* models, agents constantly engage in recontracting instead of immediate trading. As a result, prices change when the market is not in equilibrium, while quantities remain fixed (Fisher 2003). This paper takes a different approach and situates itself within the literature on out-of-equilibrium dynamics that emerged in the early 1960 s (see Petri and Hahn 2003 for a review). These models are known as *non-tâtonnement* processes or trading processes. Uzawa (1962) and Hahn (1962) introduced the "Edegworth process", where both prices and quantities adjust throughout the path. In these processes, that rely on the fundamental assumption that trade occurs only if it results in increased utility, equilibrium is path-dependent, and dynamics outside of equilibrium alter the set of equilibria. This is in contrast to the Walrasian case, where equilibrium is solely determined by initial holdings and is independent of the path taken. Both Uzawa (1962) and Hahn (1962) demonstrate that, under standard assumptions about preferences and the space of goods in the economy, these processes converge in the limit to a Pareto optimum.

In this paper, we employ a variant of the Edgeworth barter process to model dynamics outside of equilibrium, incorporating adjustments in both quantities and prices. We introduce a class of trading processes that, under a set of limited assumptions, converge to equilibrium. In this context, prices represent the instantaneous exchange rate between goods and have the flexibility to change at any point during the process. It is important to note that there is no inherent tendency for prices to gravitate towards equilibrium, as equilibria are path–dependent. Our research is connected to thef literature on planning procedures (Dreze and de la Valle Poussin 1983; Malinvaud 1972), specifically drawing on the work of Cornet (1983) regarding the neutrality of planning procedures. In Sect. 5, we delve into a more detailed discussion of this connection and its implications.

The main novelty with respect to the literature on out-of-equilibrium dynamics is the fact that only connected agents can trade: we introduce a static, weighted network determining who can trade with whom. Among the works on dynamical networks, our paper relates with Cowan and Jonard (2004) and König and Rogers (2023) who model knowledge diffusion as a barter process: agents meet their neighbours repeatedly and in case they have a differential in two dimensions of knowledge they trade, each receiving a constant share of the knowledge differential. Other related works include Flåm (2019) who study the emergence of price taking behaviour modelling trade as a sequence of bilateral exchanges where agents only trade if each exchange increases both agents' utility. Contrary to our case the network structure of agents matching is not explored but the author shows that equilibria can be path-dependent and are affected by the matching order.

Our model does not consider strategic interactions: we do not have any market game and agents do not trade with all others simultaneously but only engage in bilateral exchanges with their network contacts, which differentiates our approach from Ghosal and Morelli (2004). Moreover, we do not allow for trade frictions as for example Fleiner et al. (2019). Our model in principle can be applied to large numbers of players, even if in the current work we provide examples of small networks only. Axtell (2005) proves that decentralized exchange processes of the same class of our model have polynomial computational complexity, performing much better than Walrasian models which can be exponential in the worst case.

The paper is structured as follows: in Sect. 2 we define our family of bilateral trading processes, and we provide a characterisation of the Pareto set to which these processes converge. In Sect. 3 we expose the trading rule of choice, namely the egalitarian rule, proving that the trade so defined belongs to the family of trades of our interest. In Sect. 4 we extend trading to more agents, and we introduce the network structure as a weighted network. In Sect. 5 we prove an analog of the Second Welfare Theorem for networks, and relate our result to the literature on decentralized planning procedures.

2 The model

2.1 Pure exchange

There are $n \geq 2$ agents, we will generally refer to an agent $i \in \{1, \ldots, n\} \equiv N$, and $m \geq 2$ goods, and to a good $k \in \{1, \ldots, m\} \equiv M$. Agents can only have non-negative quantities of each good, and we are considering a pure exchange economy with no production, so that total resources in the economy are fixed and given by the sum of the agents' endowments. The endowment of agent i is a point in the positive orthant of $\mathbb{R}^m$, call this space $\mathbb{R}^m_+$, where the $k-$th coordinate represents the quantity of good k. Assume time t is continuous, with $t \in (0, \infty)$ and goods are infinitely divisible, and let $x_{ik,t}$ be the endowment of agent i at time t for good k. In this way $x_{i,t} \in \mathbb{R}^m$ is the m–dimensional vector of agent i's endowment at time t, while $x_{k,t} \in \mathbb{R}^n$ is the n–

dimensional vector of all agents' endowments of good k at time t. As we assumed there is no production, nor can the goods be disposed of, the sum of the elements of each such vector $x_{k,t}$ is constant in time. The initial allocation of the economy is then represented by the n vectors of agents' endowment at time zero, call it $x_0 = \{x_{1,0}, \dots, x_{n,0}\}$. All agents' allocations at a given point in time can then be represented by an $(m \times n)$ matrix with all non-negative entries, call it $\mathbf{X}_t$. In the following we may not express the time variable t, when it does not create ambiguity. Following Smale (1975), an unrestricted state of the economy at any time t is a point in the positive orthant of an $\mathbb{R}^{m\times n}$ space, given by the Cartesian product $(\mathbb{R}^m)^n$. As we assumed that resources are fixed in the economy at a point $w \in \mathbb{R}^m$ (where the k-th coordinate is the total quantity of good k in the economy), the state space of our interest is a subset of $\mathbb{R}^{m\times n}_+$, call it $E = \{x \in \mathbb{R}^{m\times n}_+ : \sum_{i\in N} x_{ik} = w_k \quad \forall k \in M\}$, which is an open subset of an affine subspace with compact closure in $\mathbb{R}^{m\times n}$ (Smale 1975).

Assumption 1 Any agent i is characterised by a twice continuously differentiable, strictly increasing utility function U_i from $\mathbb{R}^m_+$ to $\mathbb{R}$.

Given $x_t \in E$, a point in the space of the economy at some point in time t, call $U(x_t)$ its corresponding n–dimensional vector of utilities. Define $\mu_{ik,t} \equiv \partial U_i(x_{i,t})/\partial x_{ik,t}$ the marginal utility of agent i, with endowment $x_{i,t}$, with respect to good k, and $\mu_{i,t}$ the gradient of the utility function for agent i at time t, that is the vector of all her marginal utilities. All individual gradients are represented by an $m \times n$ matrix of all the marginal utilities at a given point in time, call it $\mathbf{M}_t$. The vector of strictly positive marginal utilities $\mu_{i,t}$, is proportional to any vector of marginal rates of substitutions with respect to any good $\ell \in \{1, \dots, m\}$. It is important to stress that for the rest of the paper we use a cardinal notion of utility, because this is the structure on which we build on our out–of–equilibrium dynamical process.
In the pure exchange economy defined above, the contract curve is given by the set of all those allocation where all marginal utilities are proportional.

Definition 1 The Pareto set $\mathcal{W}$ of the pure exchange economy is defined as:

$$\mathcal{W} = \left\{ \mathbf{X} : \forall\, i, j \in N,\ \exists k_{ij} \in \mathbb{R},\ k \neq 0,\ s.t.\ \mu_i(x_i) = k_{ij}\mu_j(x_j) \right\} \quad . \tag{1}$$

Proposition 1 (Smale 1975) *If the utility function is monotonic and indifference curves are convex, then the set of Pareto Optima is homoeomorphic to a closed* $(n-1)$ *simplex.*

For a proof of Proposition 1 see Smale (1975). The assumptions in Proposition 1 are standard in economics. Furthermore if we assume concavity of the utility function and convexity of the commodity space we have that the set of Pareto Optima is diffeomorphic to a closed $(n-1)$ simplex. It has been shown that if preferences are C^2 and convex it is possible to find utility representations that admit a convex space, for an exhaustive discussion and proofs see Mas-Colell (1990). Note that in our case the assumptions of Proposition 1 are satisfied: the state space of interest is an open subset of an affine subspace with compact closure in $\mathbb{R}^{m\times n}$ (Smale 1975). The convexity assumption makes the problem much easier to deal with, but in case this assumption is relaxed we can still characterise the Pareto set, that will be an $(n-1)$ stratified set, that is a manifold with borders and corners, see Wan (1978) and de Melo (1976). Note finally that adding an error term to equation (1) we get a diffusion process similar to the one analyzed by Anderson et al. (2004), generalized to networks by Bervoets et al. (2020) and, outside economics, by Robert and Touboul (2016).

2.2 Trading

Define *trading* between agents in N as a continuous dynamic over the endowments, which is based on marginal utilities. Formally it will be a set of differential equations of the form:

$$\frac{dx_{i,t}}{dt} = f_i\left(\mathbf{M}_t\right) \quad , \tag{2}$$

where function f_i from $\mathbb{R}_+^{n\times m}$ to $\mathbb{R}^m$, satisfies the following 3 assumptions, for any set $\mathbf{M}_t = \left(\mu_{1,t}, \mu_{2,t}, \ldots \mu_{n,t}\right)$ of feasible marginal utilities:

- **Zero sum:** the sum $\sum_{i=1}^{n} f_i$ is equal to the null vector 0.
- **Trade:** if there are at least two vectors of marginal utilities, $\mu_{i,t}$ and $\mu_{j,t}$, which are linearly independent, then at least one between f_i and f_j is different from 0.
- **Positive gradient:** for any agent i it will always be the case that $\mu_{i,t} \cdot f_i \geq 0$, with strictly positive sign if there is trade.

The assumption of *zero sum trade* guarantees that we are in a pure exchange economy without consumption nor production of new goods, as the amount of all the goods remain unchanged at any step of the process. The assumption of *trade* guarantees that there is actually exchange, unless we are in a Pareto optimal allocation, where the marginal rate of substitution between any two goods would be the same for any couple of agents. Finally, the assumption of *positive gradient* guarantees that any marginal exchange represents a Pareto improvement. That is because

$$\begin{aligned} \frac{dU_i}{dt} &= \sum_{k=1}^{m} \frac{\partial U_i}{\partial x_{ik}} \frac{dx_{ik}}{dt} \\ &= \mu_{i,t} \cdot f_i\left(\mathbf{M}_t\right) \geq 0 \end{aligned} \tag{3}$$

An allocation $\mathbf{X}^*$ (the $(n \times m)$ matrix representing quantities of each of the m goods for each of the n agents) is an *equilibrium* of the system in Eq. (2) if $f_i(\mathbf{X}^*) = 0$ for all $i = 1, \ldots, n$ and a solution is a function $\hat{x}(t, \mathbf{X}_0) : \mathbb{R} \times \mathbb{R}_+^{n\times m} \rightarrow \mathbb{R}_+^{n\times m}$ where $\mathbf{X}_0$ is the initial condition at time 0.

Definition 2 The solution $\hat{x}(t, \mathbf{X}_0^*)$ is *stable* if for every $\epsilon > 0$ there exists a $\delta > 0$ such that:

$$|\mathbf{X}_0 - \mathbf{X}_0^*| \leq \delta \implies |\hat{x}(t, \mathbf{X}_0) - \hat{x}(t, \mathbf{X}_0^*)| \leq \epsilon \,, \ \forall t \geq 0 \ . \tag{4}$$

Generalizing (Hahn 1982), it is easy to show that all and only the fixed points of the dynamical system defined in Eq. (2), are Pareto optimal allocations. That is because the function

$$\bar{U}(\mathbf{X}_t) \equiv \sum_{i=1}^{n} U_i(x_{i,t})$$

can be seen as a potential. It is bounded in its dominion of all possible allocations, it strictly increases as long as there is trade (i.e. out of equilibrium), and it is stable when there are no two agents who could both profitably exchange goods between them. At the limit $\bar{U}$ will converge for sure to a value, say $\bar{U}^*$, corresponding to an allocation $\mathbf{X}^*$. As preferences are strictly convex, there will be no trade in $\mathbf{X}^*$.
The fixed points of the above dynamical system, which are market equilibria, are reached by a sequence of utility increasing, infinitesimally small trades from an initial state, hence the set of the solutions of any such trade mechanism is an open subset of the Pareto set $\mathcal{W}$ defined in Eq. (1) (Smale 1975).
Note that at this stage there are no assumptions restricting endowments not to become negative, that is to say we are not requiring a condition like $\frac{dx_{ik}}{dt} > 0$ as $x_{ik} \to 0$. This will depend on the initial endowment $\mathbf{X}_0$ of the agents and on their utility functions.

Assumption 2 As any marginal exchange represents a Pareto improvement, we assume that any Pareto improvement starting from the initial conditions will lie in the convex set of the initial endowments, that lies in the non-negative orthant.

Examples that satisfy these properties are the classical Walrasian *tâtonnement* process, as well as non-*tâtonnement* processes, as can be found in Hahn (1982) and Hurwicz et al. (1975a, b).

3 *Fair trading* between two agents

Let us start by considering $n = 2$. There is an entire family of trading mechanisms satisfying the very general assumptions of zero sum, trade and positive gradient. As we choose a trading mechanism we are implicitly making assumptions on some bargaining rule that has been fixed by the agents participating in the trade. This is a restriction to some extent, still we can choose different trading mechanisms corresponding to different bargaining solutions that satisfy the assumptions. We define a mechanism that we call *fair trading*, that is based on the egalitarian solution by Kalai (1977): whenever there is room for a Pareto improvement, agents trade if and only if they equally split the gains in utility from the trade.

As in Kalai (1977) we consider utility as cardinal, in other words it does not only represent an ordering among alternatives, it also attaches a precise value to alternatives on the same indifference curve. Moreover, we are allowing for interpersonal comparisons and we are assuming that agents have full knowledge of each others' preferences. This imply first of all that our results are not invariant under monotone transformation of the utility function: unless the utilities of all agents are rescaled by the same factor, the trade path changes and consequently the equilibria change. In this respect we also do not explicitly consider strategic misrepresentation of preferences, in other words we are assuming that agents are always revealing their true utility function. It is worth remembering that Shapley (1969) showed that there is no strongly individually ratio-

nal ordinal solution to bilateral bargaining problems, and here we are considering a trade mechanism based on a bilateral bargaining solution, as in Kalai (1977).
Trading is bilateral, $N = \{1, 2\}$, and $m \geq 2$ goods. By the zero sum property we have that $f_1 = -f_2$. We are restricting our attention to the case where marginal utility from trading is equally split among the two agents. The Pareto improvement from trading is defined in Eq. (3), so we are requiring that:

$$\mu_{1,t} \cdot f_1\left(\mu_{1,t}, \mu_{2,t}\right) = \mu_{2,t} \cdot f_2\left(\mu_{1,t}, \mu_{2,t}\right).$$

By the zero sum property this is satisfied if

$$\left(\mu_{1,t} + \mu_{2,t}\right) \cdot f_1\left(\mu_{1,t}, \mu_{2,t}\right) = 0$$

which simply means that marginal trade has to be orthogonal to the sum of marginal utilities.
There is a full sub-space of dimension $m - 1$ that is orthogonal to the sum of the two marginal utilities. Here we consider a single element that lies in the sub-plane generated by $\mu_{1,t}$ and $\mu_{2,t}$. We assume that trade for agent 1, f_1, is the orthogonal part of $\mu_{1,t}$ with respect to $\mu_{1,t} + \mu_{2,t}$ (or the vector rejection of $\mu_{1,t}$ from $\mu_{1,t} + \mu_{2,t}$). In formulas it is

$$f_1\left(\mu_{1,t}, \mu_{2,t}\right) = \mu_{1,t} - \frac{\mu_{1,t} \cdot \left(\mu_{1,t} + \mu_{2,t}\right)}{|\mu_{1,t} + \mu_{2,t}|^2}\left(\mu_{1,t} + \mu_{2,t}\right) \quad , \tag{5}$$

where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^m$. Generalizing Eq. (5) we call $f_i(\mu_{i,t}, \mu_{j,t})$ *fair trading* between agent i and j.

Proposition 2 *The fair trading mechanism between two agents defined in Equation (5) satisfies zero sum, trade and positive gradient.*

Proof Fair trading specified in (5) satisfies *zero-sum*, as the instantaneous trade of one agent is equal to the additive inverse of the instantaneous trade of the other agent:

$$f_2\left(\mu_{1,t}, \mu_{2,t}\right) = \mu_{2,t} - \frac{\mu_{2,t} \cdot \left(\mu_{1,t} + \mu_{2,t}\right)}{|\mu_{1,t} + \mu_{2,t}|^2}\left(\mu_{1,t} + \mu_{2,t}\right) = -f_1\left(\mu_{1,t}, \mu_{2,t}\right)$$

because

$$f_1\left(\mu_{1,t}, \mu_{2,t}\right) + f_2\left(\mu_{1,t}, \mu_{2,t}\right) = \left(\mu_{1,t} + \mu_{2,t}\right) - \frac{|\mu_{1,t} + \mu_{2,t}|^2}{|\mu_{1,t} + \mu_{2,t}|^2}\left(\mu_{1,t} + \mu_{2,t}\right) = 0.$$

To check that the *trade* condition is satisfied note that $f_1\left(\mu_{1,t}, \mu_{2,t}\right) = 0$ only if $\mu_{1,t} = k\mu_{2,t}$ for some $k \in \mathbb{R}$, that is when $\mu_{1,t}$ and $\mu_{2,t}$ are linearly dependent.

Positive gradient requires that:

$$\mu_{1,t} \cdot \left(\mu_{1,t} - \frac{\mu_{1,t} \cdot (\mu_{1,t} + \mu_{2,t})}{|\mu_{1,t} + \mu_{2,t}|^2} (\mu_{1,t} + \mu_{2,t}) \right) \geq 0.$$

For the above inequality to be satisfied it suffices that $|\mu_{1,t} \cdot (\mu_{1,t} + \mu_{2,t})| \leq |\mu_{1,t}||\mu_{1,t} + \mu_{2,t}|$. The latter always holds as it is the Cauchy–Schwarz inequality. As long as $\mu_{1,t}$ and $\mu_{2,t}$ are linearly independent $|\mu_{1,t} \cdot (\mu_{1,t} + \mu_{2,t})| < |\mu_{1,t}||\mu_{1,t} + \mu_{2,t}|$ and so $\mu_{1,t} f_1 (\mu_{1,t}, \mu_{2,t}) > 0$ as long as there is trade. When $\mu_{1,t}$ and $\mu_{2,t}$ are linearly dependent $|\mu_{1,t} \cdot (\mu_{1,t} + \mu_{2,t})| = |\mu_{1,t}||\mu_{1,t} + \mu_{2,t}|$, so $\mu_{1,t} f_1 (\mu_{1,t}, \mu_{2,t}) = 0$ when there is no trade. □

Note that *zero-sum*, *trade* and *positive gradient* would be satisfied for any $\alpha f_1 (\mu_{1,t}, \mu_{2,t})$, with $\alpha > 0$, where the parameter α represents the *speed* at which the dynamical system is moving, so there will be no loss in generality in assuming it equal to 1.
So, the fair trading mechanism is a bilateral pure exchange mechanism satisfying the required three assumptions. The two agents trade over $m \geq 2$ goods, starting from some initial allocation $\mathbf{X}_0 \in \mathbb{R}^{m \times 2}$ and evolving according to the following system of differential equations in matrix form, based on Eqs. (2) and (5):

$$\frac{d\mathbf{X}_t}{dt} = \left(\mu_{1,t} - \frac{\mu_{1,t} \cdot (\mu_{1,t} + \mu_{2,t})}{|\mu_{1,t} + \mu_{2,t}|^2} (\mu_{1,t} + \mu_{2,t}), \mu_{2,t} - \frac{\mu_{2,t} \cdot (\mu_{1,t} + \mu_{2,t})}{|\mu_{1,t} + \mu_{2,t}|^2} (\mu_{1,t} + \mu_{2,t}) \right). \tag{6}$$

This dynamical system is well defined, as $\mu_{1,t}$ and $\mu_{2,t}$ are defined in $\mathbf{X}_t$, and are based on the utilities U_1 and U_2. However, this system is not linear in $\mathbf{M}_t$.

To have a graphical intuition for our approach, consider Fig. 1, where we have $m = 2$ (adapted from Smith and Foley 2008). In the left panel we represent allocations of the two goods, while in the right panel we represent utilities of the two agents. The red and the blue lines (in both panels) are the boundaries of the Pareto improving allocations. The yellow curve is the Walrasian map from initial endowments to the Walrasian equilibrium allocation: it is a straight line in the Edgeworth box (left), but not necessarily in the space of utilities (right). The green line is the path obtained with fair trading: it is a straight line with 45° inclination in the right panel. The above example is just illustrative, and we are not claiming that a the limit points of the Fair Trade dynamics cannot coincide with the Walrasian allocation, even if this is generally not true.

4 The network environment

What happens if $m \leq n+1$, if for instance there are only 2 goods and many agents? In this case we consider a market mechanism based on a *weighted, undirected network G* that allows for distinct couples to match and trade according to the unique fair trading

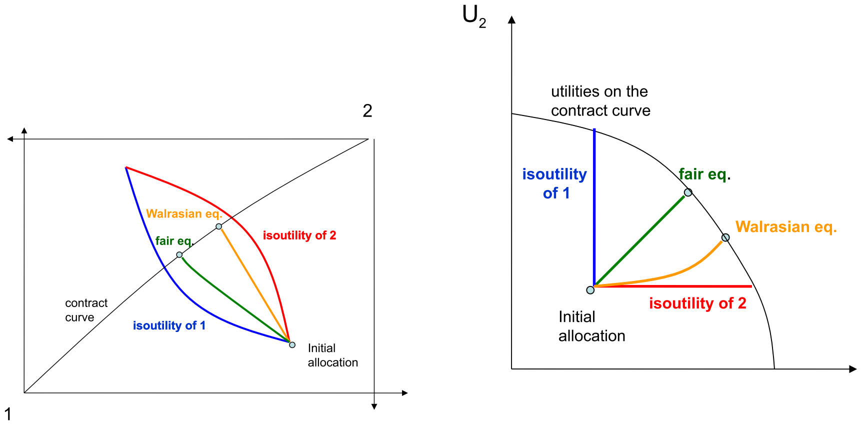

Fig. 1 Example of the difference between a Walrasian equilibrium and a fair equilibrium in the Edgeworth box and in the space of utilities

mechanism defined in Sect. 3.[1] The trade network G is identified by the symmetric adjacency matrix $\mathbf{W} = (w_{ij})$, where $w_{ii} = 0$ and $w_{ij} \in [0, 1]$ and we assume that the sum of all edge weights in G is equal to 1, $\sum_{i>j} w_{ij} = 1$. The importance of a node i in terms of the total weight of their connections is given by the *strength*, defined as $s_i = \sum_j w_{ij}$ (Barrat et al. 2004). Note that, given our assumption on the sum of edge weights $\sum_i s_i = 2$. The weight of each connection represents the fraction of trade opportunities between two agents.
Bilateral trade between agents i and j is given by $f_i(\mu_{i,t}, \mu_{j,t})$ as in Equation (5), and trade for agent i on the network G is defined as $f_i^G = \sum_{j \in N \setminus \{i\}} w_{ij} f_i(\mu_{i,t}, \mu_{j,t})$, namely the weighted sum of i's bilateral trades with all the agents connected with i. The resulting dynamical system is[2]

$$\frac{d\mathbf{X}_t}{dt} = f^G(\mathbf{X}(t), \mathbf{W}) = \left[\sum_j w_{ij} f_i(\mu_{i,t}, \mu_{j,t}) \right]_i . \tag{7}$$

As for the case of Eq. (6), this system is not linear. We stress that the trading mechanism in Eq. (7) has the property of *anonimity*: it only requires agents to know the utility gradient of each of their connections, no other information is necessary for trading. We assume that agents truthfully reveal their utility gradients, without focusing on strategic behaviour whereby agents could misrepresent their preferences to improve their situation.

Proposition 3 *The fair trading mechanism on a network satisfies zero sum, trade and positive gradient properties.*

[1] In Appendix A we show that, instead of assuming a network, we could increase the number of goods, if we want to extend the definition of *fair trading*.

[2] Here we consistently define that $f_i\left(\mu_{i,t}, \mu_{i,t}\right) = 0$.

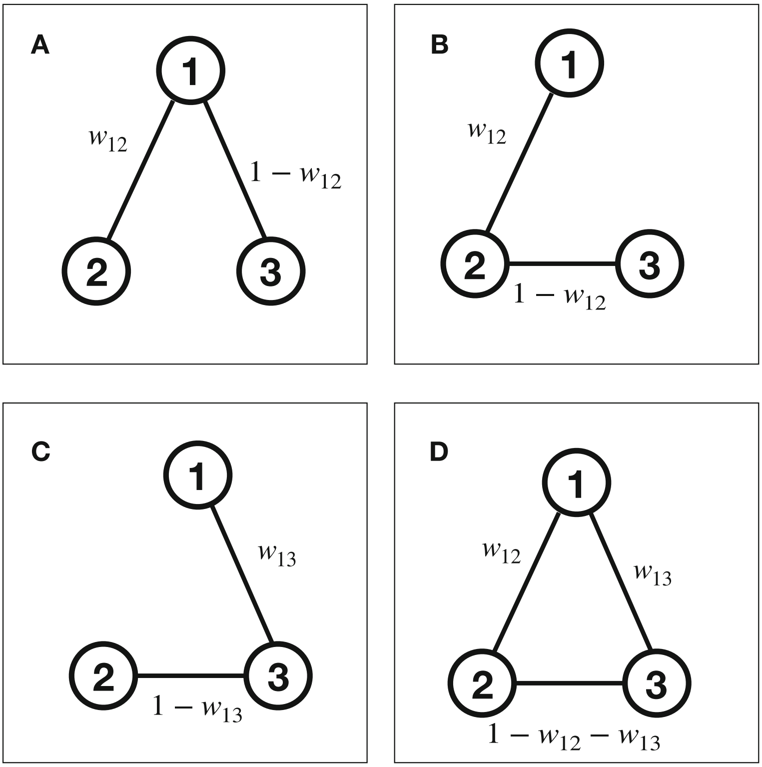

Fig. 2 Example of weighted networks with three agents. **Box A**: agent 1 trades a fraction w_{12} of her time with agent 2 and the remaining $1 - w_{12}$ with agent 3. **Box B**: agent 2 trades a fraction w_{12} of her time with agent 1 and the remaining $1 - w_{12}$ with agent 3. **Box C**: agent 3 trades a fraction w_{13} of her time with agent 1 and the remaining $1 - w_{13}$ with agent 2. **Box D**: agent 1 trades a fraction w_{12} of her time with agent 2 and the remaining $1 - w_{12}$ with agent 3

Proof Zero sum holds as for every couple i and j, which is matched with weight w_{ij}, $f_i = -f_j$ by construction, as discussed in Sect. 3.
Trade property also holds: for every couple i and j such that μ_i and μ_j are linearly independent, consider trader k such that $w_{ik} > 0$ and $w_{jk} > 0$ so that both i and j trade with k. If μ_i and μ_j are linearly independent, then at least one of them is linearly independent with μ_k, suppose it is μ_j. From fair trading between two agents, as discussed in Sect. 3, we have that the marginal utility of trader j from that matching is strictly increasing. Then, as no other trading can generate negative marginal utilities, it means that the overall marginal utility of trader j from all matchings is strictly increasing. And this can happen only if there is trade, i.e.

$$f_j^G = \sum_{i \in N} w_{ij} f_j \left(\mu_{j,t}, \mu_{i,t}\right) \neq 0.$$

Finally, positive gradient comes from the fact that f_i^G is a linear combination of f_is, so that

$$\mu_{i,t} \cdot f_i^G = \sum_{j \in N} w_{ij} \mu_{i,t} \cdot f_i \left(\mu_{i,t}, \mu_{j,t}\right)$$

which is strictly positive as long as there is trading. □

5 An analogue of the second welfare theorem for networks

In this section we prove that, given initial allocations of goods, there is a one to one mapping between the network conditions and the state of the system at any point in time, including in the limit points of the dynamics, where the corresponding allocations are in a subset of the Pareto set. Also, we will prove that this map has no holes (is simply connected). Furthermore, we relate our results to the literature on decentralized planning procedures, in particular to the work of Cornet (1983). At the end of the section we present an illustrative example and discuss the computational complexity of our mechanism. We start by providing some definitions and recalling some classical results.

Recall that resources are fixed in the economy at a point $w \in \mathbb{R}^m$, where the kth coordinate is the total quantity of good k in the economy. Given the initial resources, each possible initial allocation $\mathbf{X}_0$ is in the set $E = \{x \in \mathbb{R}_+^{m \times n} : \sum_{i \in N} x_{ik} = w_k \quad \forall k \in M\}$.

Definition 3 A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is locally Lipschitz continuous on $\mathbb{R}^n$ if for every $R > 0$ there exists a constant L such that:

$$|f(x) - f(y)| \leq L|x - y| \;\forall x, y \in \mathbb{R}^n \text{ such that } |x|, |y| \leq R. \tag{8}$$

Lemma 4 *If a function is C^1 then it is locally Lipschitz*

Theorem 1 *If f is continuously differentiable, then there exists a unique solution of the dynamical system $\frac{dx}{dt} = f(x)$ satisfying the initial condition $x(0) = x_0$.*

Proof See for example Hirsch and Smale (1974). □

In what follows we refer to $f_i^G : \mathbb{R}_+^{n \times m} \to \mathbb{R}^m$ as the function defining the dynamics of trade for agent i in Eq. (7), that is, for each agent $i = 1, \ldots, n$, f_i^G is a vector of m components, and each component, for $k = 1, \ldots, m$ is:

$$f_{ik}^G = \sum_{i \neq j} w_{ij} \left(\mu_{ik} - \frac{\sum_{k=1}^m \mu_{ik}(\mu_{ik} + \mu_{jk})}{\sum_{k=1}^m (\mu_{ik} + \mu_{jk})^2} (\mu_{ik} + \mu_{jk}) \right). \tag{9}$$

Where $\mu_{ik} = \frac{\partial U_i}{\partial x_k}(x_1, \ldots, x_k)$. For the sake of readability we drop time dependency and we refer to the matrix of good quantities for each agent simply as $\mathbf{X} \in \mathbb{R}_+^{n \times m}$ and to the adjacency matrix that identifies the network as $\mathbf{W} \in \mathcal{A} \subset \mathbb{R}^{n \times n}$, where $\mathcal{A}$ is the set of $n \times n$ symmetric matrices such that each entry is a number between 0 and 1, all the diagonal entries are equal to 0 and the sum of all entries of the matrix is equal to 2.

Definition 4 Define as $\mathcal{W}^G(\mathbf{X}_0)$ the set of limit points of the fair trade dynamics on networks, that is the set of limit points of (7) for a given initial condition $\mathbf{X}_0 \in E$.

From Sect. 2.2 we know that $\mathcal{W}^G(\mathbf{X}_0)$ is a subset of the Pareto set $\mathcal{W}$ defined in Eq. (1). We now show that keeping the initial allocation of goods fixed, for each Pareto optimum of the fair trade dynamics there exists a network configuration that implements it.

Theorem 2 *Consider the fair trade mechanism of pure exchange on networks where agents have continuously differentiable utility functions. For each initial allocation $\mathbf{X}_0$ constant, any $\mathbf{X}^*$ in the set of the limit points of the fair trade dynamics $\mathcal{W}^G(\mathbf{X}_0) \subset \mathcal{W}$ can be reached through a path of trades for some weighted network G. Moreover, each connected weighted network G leads to a different limit point.*

Proof The fact that any point $\mathbf{X}^*$ in the set of limit points of the fair trade dynamics can be reached for some weighted network G follows from the definition of $\mathcal{W}^G(\mathbf{X}_0)$. To prove that each weighted network G leads to a different limit point in $\mathcal{W}^G(\mathbf{X}_0)$, we prove that the map between the initial conditions (both endowments and network configuration) and the limit points of the trade dynamics is a homeomorphism, that is one-to-one and onto, continuous and with continuous inverse. In order to do that we first transform the parameter $\mathbf{W}$ into initial conditions, and then we show that the trade dynamics is Lipschitz continuous with respect to both $\mathbf{X}$ and $\mathbf{W}$.

Let us start by noting that the dynamical system identified by Eq. (7) has a unique solution given the initial allocation $\mathbf{X}_0$, for a fixed network $\mathbf{W}$. This is because each $f_i^G : \mathbb{R}_+^{n\times m} \to \mathbb{R}^m$ is continuously differentiable, as we assumed the utility function to be twice continuously differentiable. This can be easily verified checking each of the k components of f_i^G as per Eq. (9). It follows that $f^G(\mathbf{X}, \mathbf{W})$ is at least C^1 and by Theorem 1 the dynamical system has unique solution.

Call the solution map $\phi(t, \mathbf{X}_0, \mathbf{W})$, unique for each initial condition $\mathbf{X}_0 \in E \subset \mathbb{R}_+^{n\times m}$. This map is a homeomorphism, that is continuous, one-to-one and with continuous inverse (Hirsch and Smale 1974). We can transform the parameter $\mathbf{W}$ into initial conditions by introducing a new variable $\mathbf{S} \in \mathcal{A} \subset \mathbb{R}^{n\times n}$ and imposing that it does not change in time, so that $\mathbf{S}(t) = \mathbf{W}$ for all t. The system has now variable $\hat{\mathbf{X}}_t = [\mathbf{X}_t, \mathbf{W}]$ and the initial condition is $\hat{\mathbf{X}}_0 = [\mathbf{X}_0, \mathbf{W}]$ We now show that the function $\hat{f}^G(\hat{\mathbf{X}})$ is Lipschitz in $\hat{\mathbf{X}}$, so it has a unique solution given initial condition $\hat{\mathbf{X}}_0$. As f^G is Lipschitz in $\mathbf{X}$, $\hat{f}^G$ is Lipschitz in $\mathbf{X}$, so we just need to show that it is Lipschitz in $\mathbf{W}$ as well. This is straightforward as $\hat{f}^G$ is linear in $\mathbf{W}$.
So $\hat{f}^G$ is Lipschitz continuous both in $\mathbf{X}$ and in $\mathbf{W}$, which implies that for each initial condition $\hat{\mathbf{X}}(t_0) = [\mathbf{X}_0, \mathbf{W}]$ there is a unique solution, and given our ODE system is autonomous, distinct solutions never cross. Call $\hat{\phi}(t, \mathbf{X}_0, \mathbf{W})$ the solution map of $\hat{f}^G$, for standard arguments this map is a homeomorphism (Hirsch and Smale 1974). This implies that, given $\mathbf{X}_0$, changing the weighted network G we reach a different point in $\mathcal{W}^G(\mathbf{X}_0)$, which concludes our proof. □

Theorem 3 *If f is Lipschitz continuous in* $\mathbf{X}, \mathbf{W}$ *then the solution* $\phi(t, \mathbf{X}_0, \mathbf{W})$ *is Lipschitz continuous.*

Proof This is a classic result on the continuous dependence of solutions on parameters and initial conditions. See for example Hirsch and Smale (1974). □

Note that both if we change the network G and keep the initial endowments $\mathbf{X}_0$ fixed, and if we keep the network G fixed and we redistribute initial endowments, the limit points of the dynamic will converge to a point that is in the Pareto set. Obviously, given that the solutions are unique, changing the network or the initial allocations (or both), we will reach distinct points in the Pareto set. In our model we can redistribute initial allocations, or redistribute network connections, or both: in every case the limit allocation is Pareto efficient.

In the decentralized planning literature, a procedure with n agents is defined as a dynamical system over the space of admissible allocations, governed by a parameter δ that lies in the simplex of $\mathbb{R}^n$, where at each point in time agents truthfully reveal their marginal rates of substitutions, and the planner uses this information to revise allocations according to the equations of the dynamical system (Cornet 1983). These procedures are said to be neutral (Champsaur 1976) when any element of the Pareto set can be reached by an appropriate choice of the parameter δ, which can be interpreted as the weight of each agent.

Our model shares several characteristics with Cornet (1983): first, the dynamics depend only on the marginal utilities of individuals and on the network, which is an element of the simplex of $\mathbb{R}^n$, as each edge is non-negative and the sum of all edges is equal to one. The decentralized trading mechanism of equation (7) satisfies all the assumptions of Cornet's neutrality theorem (Theorem 2.1 in Cornet 1983): (i) the system is governed by a continuous function which codomain is a non-empty, compact subset of $\mathbb{R}^n$; (ii) utility functions are continuous and the set of allocations such that utility is larger than or equal to the the initial allocation is compact; for every network and at any point in time: (iii) every solution lies in E (the space of feasible allocations) and (iv) utility is non-decreasing for all agents and for all solutions; (v) for each network the limit points of the dynamics are in the Pareto set and (vi) if an agent is not connected to anybody else (which corresponds to having zero weight in Cornet's setting) then her utility is equal to her initial endowment's utility. Hence, by Cornet's neutrality theorem, we can claim that any point in the Pareto set $\mathcal{W}$ can be reached by the appropriate choice of the network G. Theorem 2 and the neutrality theorem imply that for any point $\mathbf{X}^P$ in the Pareto set $\mathcal{W}$, given the initial allocation $\mathbf{X}_0$, there is one and only one network G such that the limit point of the trading mechanism in (7) is $\mathbf{X}^P$. This result can be seen as an analog of the Second Welfare Theorem for networks, and draws a connection between the Accessibility of Pareto Optima in the decentralized planning literature (Bottazzi 1994; Schecter 1977) and the economics of networks.

Proposition 5 (Second welfare theorem for exchange on networks) *Any Pareto Optimal allocation can be achieved through the fair trading mechanism with the appropriate choice of the trading network.*

Lemma 6 *The set of the limit point of the dynamics* $\mathcal{W}^G(\mathbf{X}_0)$ *is simply connected.*

Proof Consider the solution map $\phi(t, \mathbf{X}_0, \mathbf{W}) : E \times \mathcal{A} \rightarrow \mathcal{W}^G(\mathbf{X}_0) \times \mathbf{W}$. $E \times \mathcal{A}$ is a convex subset of $\mathbb{R}^{n \times m + n \times n}$ as a product of two convex subset of $\mathbb{R}_+^{n \times m}$ and $\mathbb{R}^{n \times n}$ respectively, so $E \times \mathcal{A}$ is simply connected. $E \times \mathcal{A}$ and $\mathcal{W}^G(\mathbf{X}_0) \times \mathbf{W}$ are homoeomorphic, so the fact that $E \times \mathcal{A}$ is simply connected is a necessary and sufficient condition for $\mathcal{W}^G(\mathbf{X}_0) \times \mathbf{W}$ to be simply connected, and so $\mathcal{W}^G(\mathbf{X}_0)$ is simply connected □

We can characterise the set of limit points of the fair trading dynamics for any $\mathbf{X}_0 \in E$:

Proposition 7 *The set of the limit points of a fair trading dynamics on networks, $\mathcal{W}^G(\mathbf{X}_0)$, for any $\mathbf{X}_0 \in E$, is a subset of the Pareto set $\mathcal{W}$ which is homeomorphic to a closed $(n-1)$ simplex.*

Proof $\mathcal{W}^G(\mathbf{X}_0)$ is a strict subset of $\mathcal{W}$ as the stable point of the trading dynamics are Pareto Optima and all those allocations in $\mathcal{W}$ where agents are worse off than their initial allocation in the dynamics are not in $\mathcal{W}^G(\mathbf{X}_0)$. Recall that the solution map $\phi(t, \mathbf{X}_0, \mathbf{W})$ is continuous in both $\mathbf{X}_0$ and $\mathbf{W}$. Given that the set $\mathcal{W}$ is homeomorphic to a $(n-1)$ simplex and that $\mathcal{W}^G(\mathbf{X}_0)$ is simply connected, $\mathcal{W}^G(\mathbf{X}_0)$ is also homeomorphic to a $(n-1)$ simplex. □

It is worth considering an alternative proof of the homeomorphism result in Proposition 7, which provides a more intuitive understanding. Suppose that we have at least three agents, i, j and k, and that we start from three different star networks: one with i, one with j, and one with k in the core. Unless we start from an allocation that is already Pareto optimal, the three points that we would reach adopting these three networks, and starting from the same initial allocation, cannot coincide. That is because in a star network, since agents use the fair trading rule, in the limiting point in $\mathcal{W}^G(\mathbf{X}_0)$ the central agent will obtain a marginal utility that is equal to the sum of the marginal utilities of all the other agents. So, it is impossible that we reach a unique allocation in which half of the overall marginal utilities is given at the same time to each one of the three agents i, j and k. Figure 3 provides a graphical intuition of this argument in the projection of the space of marginal utilities with respect to the initial allocation.
The main implication of our result is that we can evaluate the impact of the network structure on the final allocation, as there exists a one-to-one map between each weighted, connected network and the solutions of (7) once we keep initial allocations fixed.

For each agent, given initial allocations, each network will then correspond to a potentially different utility level in equilibrium. So an agent will prefer the network for which her utility is maximised in equilibrium. We can show that for any possible initial allocation each agent has a preferred network, which is the star where said agent is in the core.

Proposition 8 *For any initial allocation of goods any agents strictly prefers the star network with themselves in the core to any other network.*

Proof Consider that agent i final utility as a function of the network can be written as:

$$U_i(\mathbf{W}) = U_i(x_i(0)) + \int_0^{\infty} \sum_{j \neq i} w_{ij} \mu_i(\phi_i(t, \mathbf{X}_0, \mathbf{W})) \cdot f_{i,j}(\phi_i(t, \mathbf{X}_0, \mathbf{W}), \phi_j(t, \mathbf{X}_0, \mathbf{W}))dt \quad (10)$$

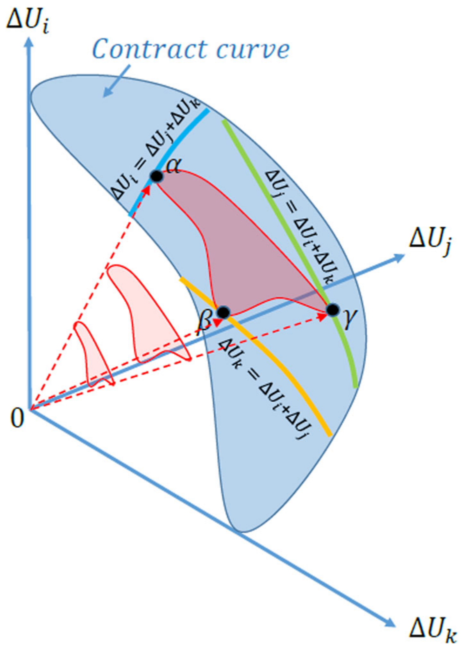

Fig. 3 Graphical intuition for Proposition 7, representing the projection for three agents of the space of marginal utilities with respect to the initial allocation. The light-blue curve represents the Pareto optimal points in $\mathcal{W}$, the red region is the set $\mathcal{W}^G$ of those points that can be reached with our trading mechanism. If we start with the three distinct stars that have one of those three agents as centers, we will end up in distinct points of $\mathcal{W}$, and hence $\mathcal{W}^G$ has the same dimension of $\mathcal{W}$

where $\phi_i(t, \mathbf{X}_0, \mathbf{W})$ is agent i's solution path as a function of the network. Given that $\mu_i(\phi_i(t, \mathbf{X}_0, \mathbf{W})) \cdot f_{i,j}(\phi_i(t, \mathbf{X}_0, \mathbf{W}), \phi_j(t, \mathbf{X}_0, \mathbf{W})) \geq 0$ for any i and j along the solution path and that $s_i = \sum_j w_{ij} = 1 - \sum_{k \neq i} \sum_{j \neq i} w_{kj}$ agent i will maximise $U_i(\mathbf{W})$ when $s_i = 1$, this completes the proof.

An alternative proof is the following.
Write $h_{ij}(t, \mathbf{X}_0, \mathbf{W}) = \mu_i(\phi_i(t, \mathbf{X}_0, \mathbf{W})) \cdot f_{i,j}(\phi_i(t, \mathbf{X}_0, \mathbf{W}), \phi_j(t, \mathbf{X}_0, \mathbf{W}))$ and consider that

$$\frac{dU_i(\mathbf{W})}{ds_i} = \int_0^\infty \sum_{j \neq i} h_{ij}(t, \mathbf{X}_0, \mathbf{W}) dt = \int_0^\infty \sum_{j \neq i} h_{ij}(t, \mathbf{X}_0, \mathbf{W}) dt \geq 0 \tag{11}$$

with the above being strictly positive unless we are already in a Pareto optimal allocation. In other words for any initial allocation each agent will prefer to have maximum strength, that is $s_i = 1$. Given that the sum of all edge weights is equal to 1, when the strength of agent i is 1 then the network is a star and agent i is in the core.

□

While Proposition 5 establishes that a social planner can redistribute network connections to achieve the desired point in the Pareto set, because the fair trade dynamics is not analytically solvable even in the simplest case, we cannot establish a general relationship between the agents' weights in the social planner's welfare function and the network. In Appendix B we explore via simulations the relationship between network's characteristics and the equilibria of the fair trade dynamics, stressing how this

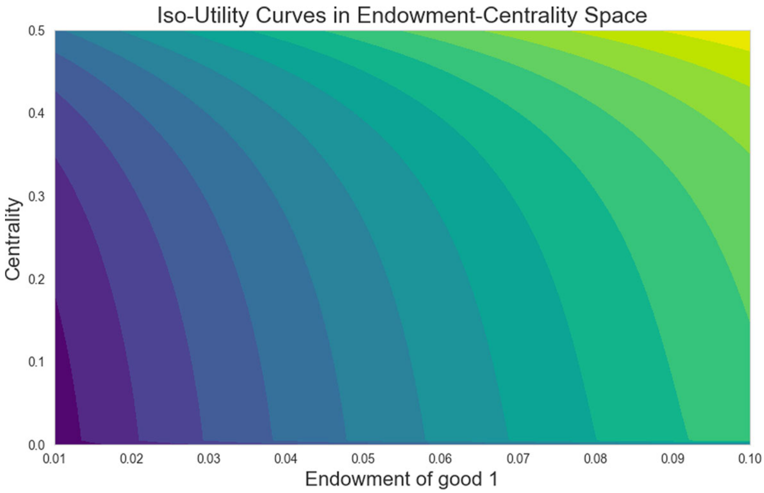

Fig. 4 Iso-utilities for agent 1 in the space of centrality and endowment of good 1 for agent 1. Centrality corresponds to $w_{12} = w_{13}$. The interpolation is done on the data produced by 5,010 experiments

is determined by a complex interaction between the network structure and the endowments. For example, while inequality in network connections is in general positively associated with inequality in the final allocation (for given endowments), it can be negatively associated if the most disadvantaged individuals in terms of endowments are those in central position in the network (see Remarks 2, 3).

As an illustration, consider a simple example with three agents and two goods. Assume each agent $i \in \{1, 2, 3\}$ has preferences defined by the Cobb–Douglas utility function $U(x_{i1}, x_{i2}) = x_{i1}^{0.5} x_{i2}^{0.5}$. In order to highlight the trade-off between network centrality and initial endowments, in all experiments we consider only networks in which 1 trades an equal fraction of time with 2 and 3, that is $w_{12} = w_{13}$, and we keep all the allocations fixed across experiments, except for the allocation of x_{11}. The initial allocations for agents 2 and 3 is $x_{21} = 0.05$, $x_{22} = 0.02$, $x_{31} = 0.01$, $x_{32} = 0.3$, while for agent 1 we fix the quantity of good 2 at $x_{12} = 0.02$ and let x_{11} vary between 0.001 and 0.01. The value on each of w_{12} and w_{13} can vary from zero, when agent 1 does not trade with anyone, to 0.5, when agent 1 trades half the time with agent 2 and half the time with agent 3, who do not trade with each other. Figure 4 shows the trade-off between centrality for agent 1, measured as $w_{12} = w_{13}$ and the initial endowment of good 1: utility levels increase by increasing both the centrality and the endowment of good 1. In this case, the network position and initial endowments are complements, meaning that centrality matters the most when the initial allocations are high, so that better opportunities of trade can be exploited with more goods to trade. Moreover, we note that a policy based exclusively on redistribution of network weights appears to be less effective than a policy exclusively based on redistribution of endowments.

Given the initial allocations and the network, the limit points of the dynamics in (7) can be computed via numerical integration. It is a legit question to ask how the computation time is affected by scaling up the number of agents and goods in the economy. Here we provide some results on the computational complexity of our mechanism.

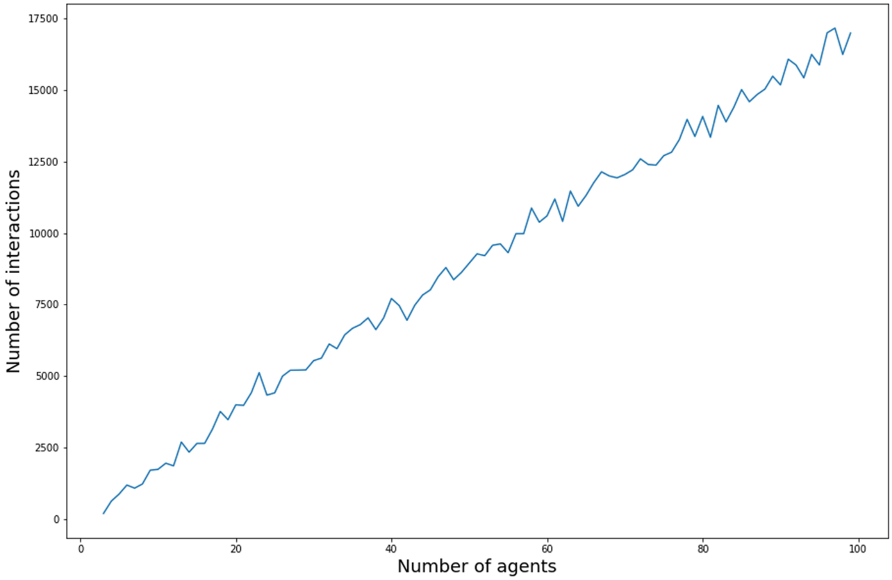

Fig. 5 Number of interactions required for convergence as a function of the number of agents, termination when $\epsilon = 0.00001$

Axtell (2005) proves that for decentralized exchange processes where groups of agents trade, provided that trade is individually rational so that the sum of utilities increases monotonically as long as there is trade, computational complexity is P (the number of interactions is bounded above by a polynomial of the number of agents and commodities). Moreover, analyzing the case of individually rational bilateral exchanges where couples of agents with Cobb–Douglas preferences are randomly matched to trade, Axtell (2005) finds that the number of interactions required to reach convergence to the equilibrium is linear in the number of agents, and increasing the number of commodities just increases the number of interactions needed without changing the linear dependency with the number of agents. Our model can be seen as an instance of this type of bilateral exchange, where instead of a random pairing of agents we have a probability distribution on the couples, represented by a weighted network with sum of weights equal to 1. On the basis of this result, if we consider our model of decentralized exchange with Cobb–Douglas utility, we expect a linear relationship between the number of agents and the number of interactions required for convergence. In order to illustrate this we computed the convergence times for our process with 2 goods and homogeneous Cobb–Douglas utility functions, letting the number of agents vary from 3 to 100. For each experiment initial allocations of endowments were randomly chosen, and the network considered is a star network with a random agent in the core. Each process is stopped at step T if the difference between the amount of goods that each agent has at T and $T - 1$ is less than $\epsilon = 0.00001$. As Fig. 5 shows, the relation between the number of agents and the number of interactions needed for convergence appears to be linear. Notice that this linear relationship is independent of the network, as it holds for any probability distribution over couples of agents, so for any weighted network where the sum of weights is one. To conclude, based on

(Axtell 2005) we can affirm that the complexity of our exchange process is P, and that in the case of Cobb–Douglas preferences the number of interactions is linear in the number of agents. While not computing convergence times for other specifications of preferences, Axtell (2005) affirms that the linear relationship holds for CES utilities as well, and we would expect this to hold for our model as well.

6 Conclusions

This paper studies an Edgeworth process on weighted graphs, where agents can continuously exchange their endowments with their neighbours, driven by their utility functions. Considering cardinal utilities, we define a family of trade dynamics which fixed points coincide with the Pareto set, and choose a specific mechanism in this family, according to which individuals equally split the utility gain of every trade. This choice is without loss of generality as the results obtained hold for all trade mechanisms that satisfy *zero sum, trade* and *positive gradient*. Under usual assumptions on the structure of preferences we prove a version of the Second Welfare Theorem on networks: for any weighted network, there exists a path of Pareto improving trades which ends in the Pareto set. Assuming Cobb–Douglas preferences, we build numerical examples of the mapping between the network topology and the final allocation in the Pareto set, and provide a brief analysis of the impact of the topology on the final allocation (in Appendices B and C). We believe that the relationship between the network and inequality should be further analysed, to understand the link between deprivation in endowments and deprivation in opportunities determined by the position in the network.

Data availability The codes for the numerical simulations are available at https://github.com/danielecassese/TradeonNetworks/tree/master.

Appendix A: More agents

Here we build on the definition of *fair trading* in Sect. 3, to show that, if there are more agents, and if every agent can trade with anyone else, we need to increase the number of goods if we want to extend the definition.

Suppose now that there are more than two agents, so that $n \geq 3$. Trade is always bilateral, and *fair trading* implies that for every trade the marginal utility from trading has to be equally split among the parts:

$$\left(\mu_{i,t}+\mu_{j,t}\right)\cdot f_i\left(\mu_{i,t},\mu_{j,t}\right)=0 \quad \forall i,j\in N, i\neq j. \tag{12}$$

This must hold for all of the $n-1$ possible couples where trader i is involved, so that individual i's instantaneous trade f_i lies in a sub-space of dimension $m-n+1$, if it exists. This clearly imposes a first constraint on the minimal possible amount m of goods.

Moreover, by the zero sum property, we need that the sum of all the instantaneous trades cancels out, $\sum f_i = 0$. This is an additional constraint, that will be satisfied only if the dimension of the sub-space where f_i lies is more than one. So the minimum number of goods that guarantees the existence of fair trading is such that $m-n+1\geq 2$, or that $m\geq n+1$.

Proposition A *If $n\geq 3$ then fair trading mechanism exists if and only if $m\geq n+1$*

Example A (3 traders) Suppose that for a certain allocation all the three vectors of marginal utilities of the traders are linearly independent. Say $\mu_1=(2,1,1)$, $\mu_2=(1,2,1)$ and $\mu_3=(1,1,2)$. f_1 has to be orthogonal to both $\mu_1+\mu_2=(3,3,2)$ and $\mu_1+\mu_3=(3,2,3)$, so that it will be of the form $f_1=k(5,-3,-3)$, for some $k\in\mathbb{R}$. Similarly we will have $f_2=h(-3,5,-3)$, for some $h\in\mathbb{R}$, and $f_3=\ell(-3,-3,5)$, for some $\ell\in\mathbb{R}$.
To balance trading we need also that $f_1+f_2+f_3=(0,0,0)$, but as they are linearly independent vectors, this is possible only for $k=h=\ell=0$, which means no trading, even if marginal utilities are not proportional. □

Remark 1 If the fair trading is between two traders ($n=2$) then two goods ($m\geq 2$) are sufficient to guarantee the existence of trade

The above can be easily verified, with two traders each trade f_1 and f_2 by construction is orthogonal to the same vector f_1+f_2, so that they will never be linearly independent. The previous example shows that if $m\leq n$, and $m\geq 3$, then fair trading is not possible. If the number of goods where instead $m=n+1$, then every candidate f_i would lie on a plane, and there would always exist a non–trivial solution for the zero sum property because we would have a homogeneous system of linear equations with n linear equations in $n+1$ variables. If m is even greater, then existence would result *a fortiori*.

Appendix B: Effect of network with Cobb–Douglas preferences

In this section we investigate *via* numerical simulations how the network structure affects the limit points of the dynamics. We consider Cobb–Douglas preferences over two goods, and we study networks up to 7 nodes. By constructing the contract curve numerically, we provide a graphical example for the 3 nodes network, showing that the stable points of the fair trading mechanism are homeomorphic to a $(n-1)$ simplex. Moreover, we illustrate the relationship between inequality in network connections and inequality in equilibrium after trade. As we stressed previously in this paper, we are assuming cardinal utilities, and this is what allows to compute out-of-equilibrium

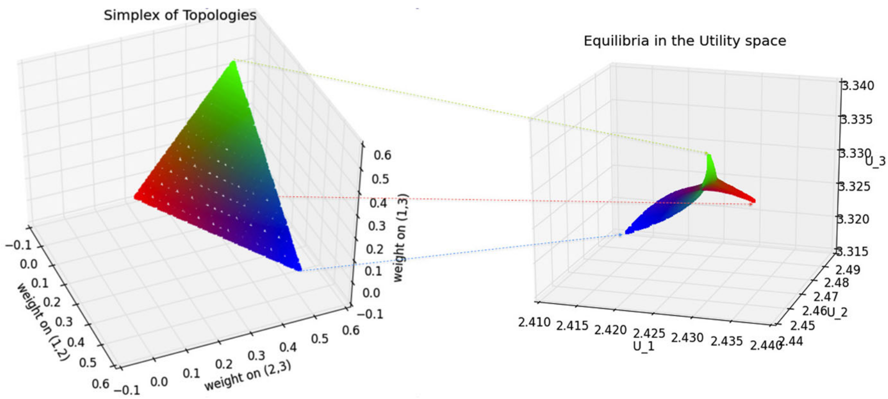

Fig. 6 Mapping between simplex of representing the space of network configurations and the corresponding equilibria. Only the three vertices are shown, map is according to colours

dynamics and to analyse the effect of network metrics on the utility image of the limit points of the dynamics. In interpreting the results of this section, it is important to keep in mind that these are not invariant under monotonic transformations of the utility functions, which is of course a limitation, as ideally we would like to learn how the network structures affect trade *independently* of the specific utility function.

As we proved in Theorem 2, each network, given an initial point in the commodity space, can be mapped into a solution which, in the limit, converges to a point on the contract curve. Here we provide a graphical illustration of this result.
Suppose that agents have a Cobb–Douglas utility function with constant return to scale: $U_i(x) = x_{i,1}^{\alpha_i} x_{i,2}^{1-\alpha_i}$. This implies that the functions are concave, and that the Pareto set is a curved $(n-1)$ simplex (Lovison and Pecci 2014).
The leftmost simplex in Fig. 6 represents the space of network configurations, each point in that space represents a weighted graph over three nodes: the barycentric coordinates of a point in the simplex correspond to the weight of the networkf's edges. Each network is then mapped to the corresponding equilibrium of the dynamical system defined by the fair trading mechanism, represented in the space of utilities on the right-hand side of the figure. The map between the two spaces is visualised through colours, a point on the simplex on the left (network) reaches the equilibrium level of utility represented by a point of the same color in the space of utilities. The figure on the right, that is the set of utilities in equilibrium, is a curved 2-simplex, with the vertices of the simplex of networks that are mapped to the vertices of the set of utilities each agent maximises her utility when is the core of a star, as we showed in Proposition (8).

In the case represented in Fig. 6 utility functions are determined by $\alpha_1 = 0.5$, $\alpha_2 = 0.4$, $\alpha_3 = 0.6$, while the initial allocations are such that agent three has the highest endowment of both goods, agent two has the lowest endowment of good 1 and endowment of good 2 higher than agent 1 that is $x_{3,1} > x_{1,1} > x_{2,1}$ and $x_{3,2} > x_{2,2} > x_{1,2}$.
The numerical examples provide illustration of our theoretical results: the map between the networks and the set of equilibria is continuous, and there is a homeomomorphism

Table 1 Simple connected non-isomorphic graphs

# of nodes	# of SCnI graphs	# of SnCnI graphs	# of networks in the experiment
3	2	0	6
4	6	1	70
5	21	2	345
6	112	10	5002
7	853	35	55,944
Total	994	48	61,367

between the simplex of topologies and the set of equilibria: each initial network is continuously mapped through our dynamical process described in Eq. (7) into a point of the curved simplex representing the set of limit points of the dynamics.

In order to investigate how the network structure affects the equilibrium we compute the trading dynamics for 61,367 networks, letting the number of nodes vary from 3 to 7, and we explore the impact of standard network metrics on the utility gain in equilibrium. Of the $2^{n(n-1)/2}$ possible graphs on n nodes we consider simple non-isomorphic ones, both connected and not connected. We include in our computations all simple connected non isomorphic (SCnI) graphs and the subset of simple non-connected isomorphic graphs (SnCnI) such that there are no isolated nodes. For each network we assume equal edge weights, setting the weight to 1 divided the number of edges. We assume that all players have the same preferences over two goods x_1, x_2, represented by a Cobb–Douglas function $U(x_1, x_2) = x_1^{0.5} x_2^{0.5}$ and that initial endowment can be either $e_1 = (1, 2)$ or $e_2 = (2, 1)$. Consider that, because of the assumption of homogeneous preferences, if all agents have the same endowments no trade will happen, so we exclude this scenario in our experiments. For each network on n nodes we let the number of agents who have endowment e_1 vary from 1 to $n/2$ if n even and $n/2 - 1$ if odd, and we compute the trade dynamics for all permutations of endowments. This assumption implies that in all networks with an odd number of nodes there is one good that is relatively scarcer in the economy, in the sense that the sum of all agents' endowment of that good is less then the sum of all endowments of the remaining good. In what follows we refer to this as the *scarce* good. Table 1 reports the number of simple connected non-isomorphic graphs when the number of nodes varies from 3 to 7 and the total number of networks after permuting for initial endowments.

To explore the role of the network structure on the trade dynamics we consider standard network and node metrics, that we define here. As a measure of the number of connections in the network we use density, which is defined as the number of edges m over the total number of possible edges between n nodes,

$$d = \frac{m}{n(n-1)} \tag{13}$$

To measure transitivity in a network a common metric is the clustering coefficient (Watts and Strogatz 2002), which measures the fraction of triangles over the total number of triads in the network. We adopt a slightly different definition of clustering coefficient, namely the fraction of triangles where one node has a different endowment than the other two over the total number of triads in the network.

Given there are two types of endowments, a connection can be either between nodes with the same endowment or between nodes with different endowments. We measure the similarity of connections with respect to initial endowments using assortativity (Newman 2002), defined as:

$$r = \frac{\mathrm{Tr}(M) - ||M^2||}{1 - ||M^2||} \quad , \tag{14}$$

where M is the mixing matrix of endowments and $||M^2||$ is the sum of all elements in the matrix M^2. Assortativity ranges from -1 (all connections between dissimilar nodes) to 1 (all connections between similar nodes).

Under the assumption of homogeneous preferences the number of initial endowments of different type e_1 and e_2 affects the trading opportunities, so as a control variable we define an endowments similarity index equal to the number of the most scarce endowment type divided the number of the less scarce one. It ranges from 1 (same number of both types of endowment) to 1/6 (highest dissimilarity in the case of 7 agents).

We consider two standard node centrality measures: node strength and betweenness centrality. Node strength is defined as the total weight of node's connections $s_i = \sum_j w_{ij}$ (Barrat et al. 2004), where $w_{ij} \in [0, 1]$ is the weight of edge ij. Betweenness (Newman 2001) of node i is defined as the number of shortest paths between pairs of nodes that pass through node i and measures the importance of nodes in connecting different parts of the network. In addition we construct two further indices: *neighbourhood disassortativity* measures the fraction of neighbours of a node which start with a different endowment than the one of the node, and the *scarcity index* is computed as the fraction of the number of endowments of the same type of node's endowment and the total number of endowments. The lower the value of the index, the less common the endowment of that agent is and so the more trade opportunity there are for that agent.

The position in the network determines the trade opportunities each agent has, and as a consequence affects the distribution of the gains from trade in equilibrium. Under the assumption of the fair trade rule, at each instantaneous trade individuals equally split the gain in utility: the star is the most unequal network as the core takes half of the total gain in utility given initial allocations, while nodes in the peripheries only get $1/2(n-1)$ of the total gain in utility, where n is the number of agents. On the other hand the most equal network is the complete network, where each node trades with each other an equal fraction of time, and where each agent gets $1/n$ of the total gain in utility.

We can measure network inequality as the Gini coefficient of the strength (or weighted degree) distribution, using:

$$G_s = \frac{\sum_i^n (2i - n - 1) s_i}{n \sum_i^n s_i} \quad , \tag{15}$$

Table 2 Effect of network metrics on utility gain

Aggregate utility gain	
Intercept	−0.5030***
	(0.006)
Clustering (dissimilar triangles only)	0.0231***
	(0.001)
Assortativity	−0.0300***
	(0.001)
Number of nodes	0.1030***
	(0.001)
Connected	0.0925***
	(0.002)
Endowments dissimilarity	0.3635***
	(0.001)
Observations	61,367
R-squared	0.818
Joint significance (p value F statistics)	0.00

Standard error in parentheses
$^{*}p < 0.10$, $^{**}p < 0.05$, $^{***}p < 0.01$

where s_i is the strength, n is the number of nodes and i is the rank of the strength in ascending order. The strength distribution of the most unequal network, the star over n nodes, is such that the node in the core (call it c) has strength $s_c = 1$ and the nodes in the periphery all have strengths strictly less than 1 and such that $\sum s_{j \neq c} = 1$.
Note that higher positional inequality does not necessarily imply higher inequality in utilities after trade. For example in a network with three agents, ranked according to their initial endowments, it could be that the inequality of the equilibrium distribution of utilities is minimised when we have the poorest agent in the core of a star: if the initial distribution of endowments is highly unequal, stars may promote redistribution, as we will show in the numerical example.
We measure inequality in final utility levels as the Gini coefficient of the individuals' utilities in equilibrium, U_i^*:

$$G_u = \frac{\sum_i^n (2i - n - 1)U_i^*}{n \sum_i^n U_i^*} \quad . \tag{16}$$

We can use these inequality measures to investigate the relation between positional inequality, endowments inequality and redistribution of welfare, keeping in mind that G_u is not independent of the specific utility function chosen. In principle, because of Theorem 2, given initial endowments we can find the inequality level in equilibrium for each network configuration, hence a social planner interested in minimising inequality could either redistribute endowments or change the interaction network. Clearly the dependency between the network and inequality in equilibrium is not trivial, as we will show in the numerical exercise.

We investigate the effect of network metrics on the aggregate gain in utility for the network, that is the sum for all nodes of their gain in utility after trade (utility of

Table 3 Effect of network metrics on utility gain

Aggregate utility gain	
Intercept	−0.5038***
	(0.007)
Density	0.0055***
	(0.001)
Assortativity	−0.0309***
	(0.001)
Number of nodes	0.1028***
	(0.001)
Connected	0.0932***
	(0.002)
Endowments dissimilarity	0.3654***
	(0.001)
Observations	61,367
R-squared	0.817
Joint significance (p value F-statistics)	0.00

standard error in parentheses
$^{*}p < 0.10$, $^{**}p < 0.05$, $^{***}p < 0.01$

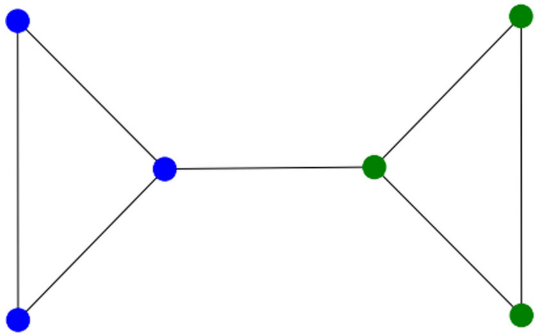

Fig. 7 Highly assortative networks generating high utility gain

equilibrium endowments—utility of initial endowments). Tables 2 and 3 report the results of the OLS regressions. In both cases we control for the number of nodes, the dissimilarity in initial endowments and wether the network is connected or not. As density and clustering are highly correlated (Pearson's correlation coefficient 0.83) in order to avoid multicollinearity we drop one of the two alternatively. Comparing Tables 2 and 3 we can see that while both density and clustering have a significative positive impact on utility gain, clustering shows a stronger correlation than density. Both regressions show that a more disassortative network brings higher utility gains, even if the magnitude of the coefficient is less than we expected.

A possible explanation is illustrated in the example in Fig. 7, showing a highly assortative network which generates a high aggregate utility gain: the two groups with different endowments manage to profitably trade thanks to the two agents bridging them, who are going to extract the highest utility gain from trade. So while assortativity

Table 4 Effect of network assortativity on inequality

Gini of post-trade utility	
Intercept	0.0150***
	(0.001)
Assortativity	0.0185***
	(0.001)
Gini scarcity index	0.0283***
	(0.007)
Observations	41
R-squared	0.828
Joint significance (p value F-statistics)	0.00

standard error in parentheses
$^{*}p < 0.10$, $^{**}p < 0.05$, $^{***}p < 0.01$

has little impact on aggregate utility gain, it has a stronger effect on the distribution of this gain. To see this consider a simple exercise: we take all possible permutations of initial endowments of the network in Fig. 7, and we check the relationship between the Gini coefficient of utility after trading and the assortativity index, controlling for endowments similarity. Table 4 shows that the more assortative the network the higher the inequality post-trade, and we can check that most of the variance in inequality is explained by assortativity. This is not necessarily true if the network is more densly connected and there are no nodes which have a clear advantage because of their position.

The relation between the networks and inequality has been explored in Borondo et al. (2014), who find a relation between the network structure and meritocracy: when the network is sparse then individuals' compensations depend on the position in the network instead of their ability to produce value. In a different setting (Bowles et al. 2011) study the impact of networks on inequality where agent play a coalitional game. They find a connection between network sparseness and inequality by studying how the extremal Lorenz distribution changes under different networks. We investigate the impact of the network on the distribution of welfare at equilibrium estimating the dependence of inequality in post-trade utility on positional inequality, measured as the Gini coefficient of strength distribution. We include in the OLS regression assortativity and density and we control for the Gini coefficient of the scarcity index. Table 5 shows that higher Gini index of strength is associated with higher inequality in utility post-trade, and the effect is significative and quite strong. More assortative networks also lead to a more unequal distribution of utility after trade, as we illustrated with the example of the 6-node network above, while on the contrary a more dense network leads to more equal final distribution. It is important to note that in our experiment there never is high inequality in terms of initial endowments: there are only two possible endowments which have the same value in utility term for all agents, and an agent can have an initial advantage only if they own a relatively scarce endowment. The magnitude of this is not large, as the maximum value of the Gini coefficient of the scarcity index is 0.15. In Appendix C we discuss experiments with larger inequality in initial allocations.

Table 5 Effect of network metrics on inequality

Gini of post-trade utility	
Intercept	0.099***
	(0.000)
Gini strength	0.0254***
	(0.000)
Assortativity	0.0087***
	(0.000)
Density	−0.0084***
	(0.000)
Gini scarcity index	0.0458***
	(0.001)
Observations	61,637
R-squared	0.615
Joint significance (p value F-statistics)	0.00

Standard error in parentheses
$^{*}p < 0.10$, $^{**}p < 0.05$, $^{***}p < 0.01$

Remark 2 A more equal allocation in equilibrium can be implemented by redistributing network strength from high strength agents to low strength ones.

To see how the position of an agent in the network affects her own utility, we regress the utility gain from trade on node strength and node betweenness centrality, controlling for how assortative the immediate neighbourhood of the node is and how scarce is the endowment of the agent. The results of the OLS regression are reported in Table 6. Node strength and betweenness centrality are both positively correlated with the utility gain in equilibrium; the higher the fraction of agent's neighbours, the larger the utility gain. Moreover agents endowed with larger quantities of the relatively scarce good in the economy are able to extract more utility from trade. All effects are significative.

Appendix C: Experiment with large inequality in initial endowments

If we allow inequality in initial endowments to be larger we expect that when agents who are disadvantaged in endowments are also disadvantaged in terms of network connections, inequality in network strength will increase inequality in utilities, and viceversa. To verify this hypothesis we consider a different exercise, allowing much larger variation in endowments and edge weights. To limit the computational burden, instead of considering all simple non isomorphic graphs we restrict our attention to a specific family of weighted connected graphs that can be seen as a linear combination of stars, where the minimally connected network is a star and the maximally connected network is a complete one. This class of networks is a weighted analogous to *nested split graphs* (König et al. 2014), that are graphs with a nested neighbourhood structure, where the set of neighbours of lower degree nodes is contained in the set of neighbours of higher degree ones. Except for the limiting case of the complete graph, the nodes in

Table 6 Effect of node metrics on post-trade utility

Individual utility gain	
Intercept	0.0505***
	(0.000)
Strength	0.1246***
	(0.000)
Betweenness	0.0702***
	(0.000)
Neighbourhood disassortativity	0.0583***
	(0.000)
Scarcity index	− 0.0875***
	(0.000)
Observations	423,643
R-squared	0.832
Joint significance (p value F-statistics)	0.00

standard error in parentheses
$^{*}p < 0.10$, $^{**}p < 0.05$, $^{***}p < 0.01$

our networks can be divided in two partitions according to their degree: nodes in the core are connected between each other and with all the nodes in the periphery, while nodes in the periphery are connected only to nodes in the core, giving a *multi-hub* network. We generate networks in this class with 3, 5 and 7 nodes and we let initial endowments of the two goods in the economy vary such that the total quantity of each good is constant across all experiments, $e_1 = \sum_n x_{i,1} = 30$ and $e_2 = \sum_n x_{i,2} = 18$ respectively. All agents have the same preferences over the two goods, represented by a Cobb–Douglas function $U(x_1, x_2) = x_1^{0.5} x_2^{0.5}$. To initialise each experiment with n agents, we generate a set of different initial endowments such that the sum of good 1 is 30 and the sum of good 2 is 18 and we compute the limit points of the trading dynamics for each network and endowments, generating 107,484 experiments with 3 agents, 76,433 experiments with 5 agents and 10,762 with 7. We then split the obtained dataset according to the following rule: we rank agents in terms of initial endowments and of network strength: the agent with the largest endowments gets the highest endowments ranking and the agent with the largest strength gets the highest strength ranking. The ranking for initial endowments is found by evaluating utility at initial conditions for each agent.[3] Then we put in one group, (*different rank*) all those experiments for which the most disadvantaged agent in terms of endowments is advantaged in terms of connections. This means all the experiments where the poorest agent is at most second in the network strength ranking with 3 agents, first with 5 and 7 agents. Conversely we collect in the other group (*similar rank*) the remaining experiments. Results are summarized in Table 7, showing a positive and significant relationship between inequality in endowments and inequality in equilibrium utility in the *same rank* group and a negative significant relationship in the *different rank* both

[3] Alternative ranking measures have been evaluated, namely the sum of initial endowments and the Euclidean norm of the vector of initial endowments, and they do not change our results.

Table 7 Inequality dependence on endowments and strength

OLS	Same rank	Different rank
Intercept	0.0186***	0.0163***
	(0.000)	(0.000)
Gini Endowments	0.9521***	0.9604***
	(0.001)	(0.001)
Gini strength	0.0510***	−0.0072***
	(0.002)	(0.003)
Observations	99,974	72,397
R-squared	0.821	0.845
Joint significance	0.00	0.00

Standard error in parentheses
$^{*}p < 0.10$, $^{**}p < 0.05$, $^{***}p < 0.01$

with homogeneous and non homogeneous preferences. On the basis of this result we can say:

Remark 3 A social planner interested in implementing a more equal allocation in equilibrium may decide to redistribute in two ways: either redistributing endowments from agents who have high network strength to those who have low network strength, or redistributing network strength (changing the network) from agents with high initial endowments to agents with low initial endowments.

Appendix D: Effect of higher order structures

Higher order structures often capture important properties of the network and of the dynamical process (Salnikov et al. 2019). We already took into account the role of triangles elsewhere in this paper, here we focus on two types higher order structures on 4 nodes: tetrahedra where only one node has endowment e_1 (e_2) while the remaining nodes have e_2 (e_1) and tetrahedra where two nodes have endowment e_1 (e_2) and the other two have e_2 (e_1). For each we compute an index analogous to the clustering coefficient, which gives the fraction of higher order structures in the network over the possible number of high order structures of that dimension. Given that the two measures are correlated (Pearson's 0.59), we include each of them separately in the regression

The results reported in Tables 8 and 9 show that both types of higher order structure are positively and significantly correlated with total utility gain, and that there is effectively no difference if we include one or the other in the regression. Similarly with the clustering coefficient for dissimilar triangles we can check that both higher order structures decrease inequality after trade, but even in this case density explains more of the variance than both higher order indices.

Table 8 Effect of higher order structures on post-trade utility

	Total utility gain
Intercept	−0.5019***
	(0.006)
Tetrahedra (1 different endowment)	0.0162***
	(0.004)
Assortativity	−0.0308***
	(0.001)
Connected	0.0941***
	(0.002)
Endowments similarity	0.3656***
	(0.001)
Number of nodes	0.1028***
	(0.001)
Observations	61,367
R-squared	0.817
Joint significance (p value F-statistics)	0.00

Standard error in parentheses
$^{*}p < 0.10$, $^{**}p < 0.05$, $^{***}p < 0.01$

Table 9 Effect of higher order structures on post-trade utility

	Total utility gain
Intercept	−0.5017***
	(0.006)
Tetrahedra (2 different endowments)	0.0255***
	(0.004)
Assortativity	−0.0308***
	(0.001)
Connected	0.0941***
	(0.002)
Endowments similarity	0.3648***
	(0.001)
Number of nodes	0.1028***
	(0.001)
Observations	61,367
R-squared	0.817
Joint significance (p value F-statistics)	0.00

Standard error in parentheses
$^{*}p < 0.10$, $^{**}p < 0.05$, $^{***}p < 0.01$

Appendix E: Details of the simulations

In Sect. 1 we consider as starting point to generate the weighted networks in our experiments the set of simple non isomorphic graphs on $n \in [3, 7]$ nodes, and then we compute all permutations of endowments for each graph in this set. We did this operation for the sake of computational speed, but one shortcoming of this approach is that some of these permutations are redundant as they are equivalent in terms of initial configuration of the trade dynamics. To clarify this, recall that nodes in a graph can be grouped into orbits with respect to the graph automorphisms. Orbits identify the "role" of nodes in the graph: for example in a star, there are 2 roles, the hub and the periphery. Figure 5 shows two connected graphs on 4 nodes, one showing 3 node orbits the other 2. Node orbits are important because help us identify those networks which are equivalent in terms of trade dynamics. Take as example the leftmost graph in Fig. 8, and consider the case in which 1 agent has endowment $(1, 2)$ and 3 agents have endowment $(2, 1)$. There are only three different trade configurations: one in which either of the two green nodes has endowment (1,2), one in which the red node has (1,2) and finally one in which the blue node has endowment (1,2), as can be seen in Fig. 9.

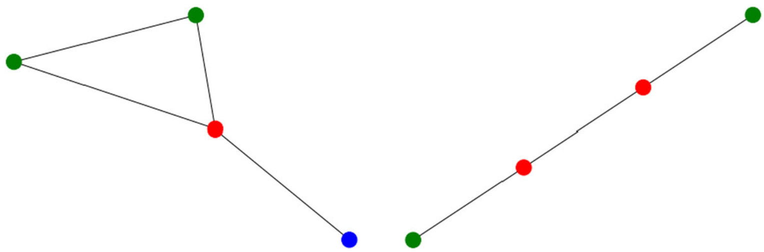

Fig. 8 Two examples of connected non-isomorphic graphs on 4 nodes where nodes with the same orbit have the same colour

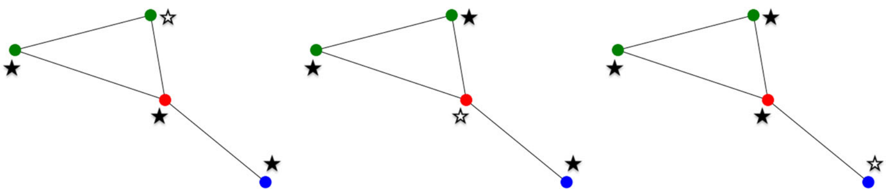

Fig. 9 The three different trade configurations for the graph with 3 different node roles when there is one different endowment. Endowment types are represented by filled or empty stars. Note that in the leftmost graph, permuting the endowment between the two green nodes makes no difference for the trade dynamics

Appendix F: Further numerical examples

In this section we discuss some additional numerical examples on networks with 3 nodes. We have three agents with Cobb–Douglas utility function with constant return to scale: $U_i(x) = x_1^{\alpha_i} x_2^{1-\alpha_i}$. Call α_i the exponent of the utility function for agent i, and $x_i(0) = (x_{i,1}, x_{i,2})$ the initial allocation for agent i. The network is represented by a unitary 2-simplex, where the barycentric coordinates of a point represent the weight of each edge.

Let us start from the example in Fig. 6. For the same case, Fig. 10 shows the projections of the set of equilibria on the planes of the utility of two agents respectively, and makes the homeomorphism more evident. Agent 3 has the highest initial endowment, and ends up having the highest level of utility in all the possible cases, ranging from 3.315 in its minimum, when the network is a star in which agent 2 is the core (blue vertex), to 3.330 in its maximum (when agent 3 is the core of the star). From this we can infer that the trade with agent 1 is the most advantageous for agent 3, as well as for agent 2, as also her utility hits the minimum point when she cannot trade with 3, and then increases when they trade on networks in which most of the interactions are between 2 and 3 (there is higher weight on this edge, as represented in the blue area). Clearly there is an asymptote in the growth of agent 1 utility moving towards a star in which agent 3 is the core (green area) and viceversa for agent 3 moving towards a star for which 1 is the core (red area). Looking at Fig. 10, utility of agent 1 is represented on the x axis, and utility of agent 3 on z axis: the figure has a twist in correspondence of the green area, where the utility of 1 stabilises around 2.430 and utility of 3 steeply increases till its maximum, while in correspondence of the red area utility of 1 stabilises around 3.330 while utility of 1 reaches its maximum.

In Fig. 11 we start from a different point in the space of goods, keeping the same utility functions. The initial allocations are such that $x_{1,1} > x_{2,1} > x_{3,1}$ and $x_{2,2} > x_{1,2} > x_{3,2}$ that is agent 1 and 2 have a lot of both goods and agent 3 is the poorest in both goods. As before each agent maximises her utility gain when she is the core of a star. Agent 3 is the one who is worse off by being a peripheral node when agent

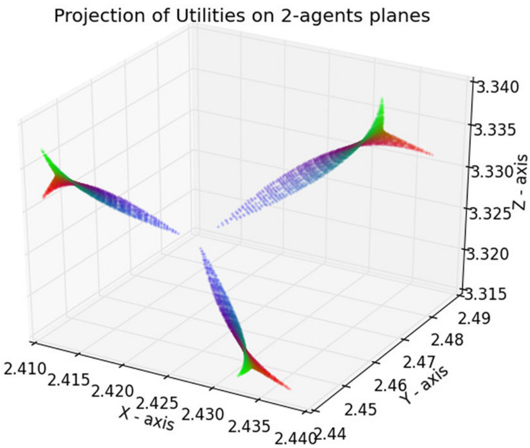

Fig. 10 Projection of equilibria in the space of utility on agents' planes

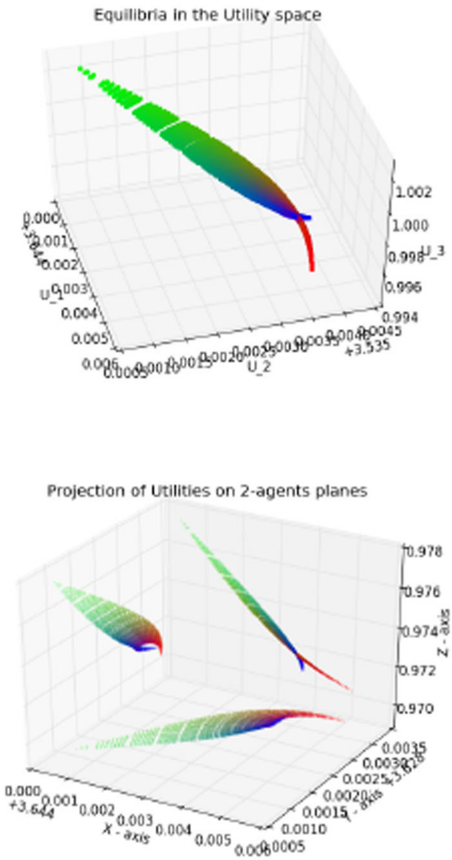

Fig. 11 Equilibria of the fair trading represented on the space of utilities for the case $\alpha_1 = \alpha_2 = \alpha_3 = 0.5$ (left) and projection on two-agents' planes

1 or agent 2 are the core. This is not surprising as she is the one with the worst initial allocation. Viceversa utilities for agents 2 and 3 hit their minimum when agent 3 is in the core. By going towards the points in which the frequency of trades is mainly between agents 1 and 2 (the networks represented by the edge between the red and blue vertices in Fig. 6) their utility is close to the maximum, meaning that both rich agents would prefer trading among themselves because they can extract more utility, instead of trading with the *poor* agent only.

In Fig. 12 it is possible to observe the shape of the equilibrium points in the space of commodity one and commodity two respectively, holding the other commodity constant. As we would expect this is also a curved simplex, with each agent getting the highest quantity of each commodity (the vertices of the curved simplex) when they are the core of a star network.

We then consider the case of extreme inequality in which agent 1 starts with a lot of both goods and agents 2 and 3 have a much inferior initial allocation, more precisely $x_{1,1} > x_{3,1} > x_{2,1}$ and $x_{1,2} > x_{2,2} = x_{3,2}$, results are represented in Fig. 13 for the case of a Cobb–Douglas with $\alpha_1 = \alpha_2 = \alpha_3 = 0.5$, and in Fig. 14 for the case in which they all prefer good 2 than good 1, that is their utility functions are such that

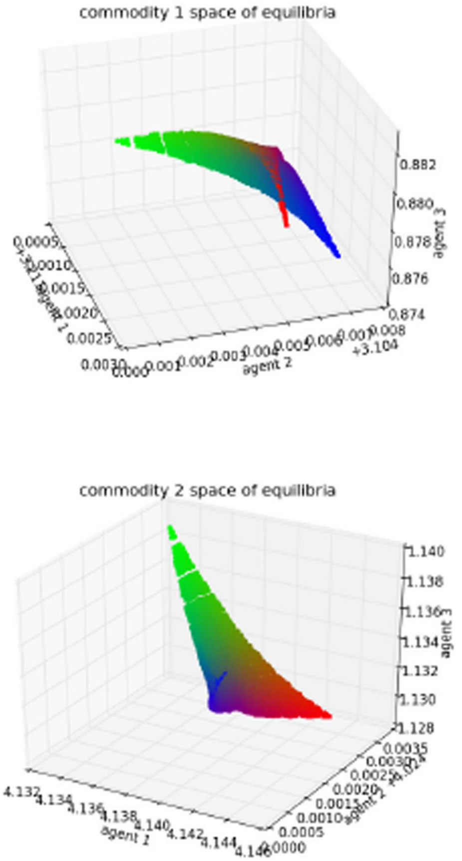

Fig. 12 Set of equilibria of a fair trading on the space of one commodity only

$\alpha_i = 0.2$ for $i = 1, 2, 3$. Given the disproportion in initial allocations, the utility of agent 1 is greater than the two "poor" agents for all possible networks, while agents 2 and 3 maximise their utility when they are the core of a weighted star, as expected. Nonetheless note that both agents 2 and 3 will prefer to be in the periphery of the star where agent 1 is the core than being in the periphery of the star where any of the other "poor" agent is in the core, even if the richest agent is maximizing her utility in this case. This is because both agents 2 and 3 prefer to have a consistent number of trades with agent 1, that is they will always prefer to trade in networks in which the weight of the edge connecting them with agent 1 is higher *ceteris paribus*, and this determines the "boomerang" shape of the set of equlibria.

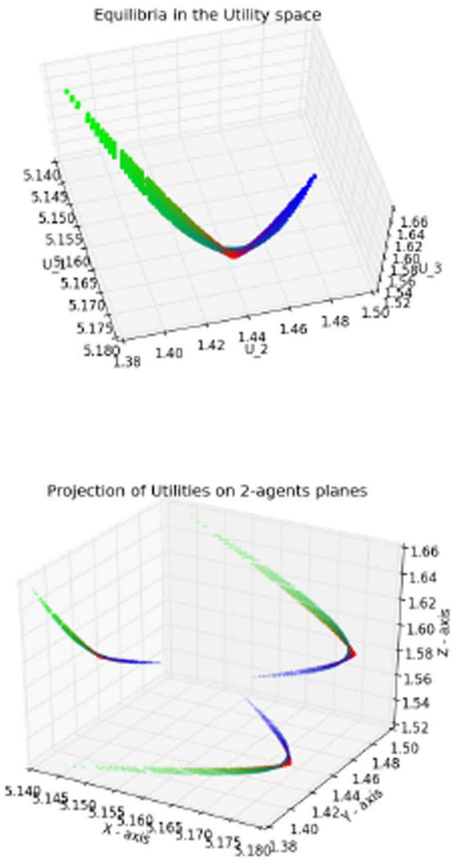

Fig. 13 $\alpha_1 = \alpha_2 = \alpha_3 = 0.5$, extreme inequality: agent 1 is rich agents 2,3 are poor

Now consider the case in which agent 1 is still the richest, but the initial allocation is much less unequal than the previous two cases. The initial allocations in this case are $x_{1,1} > x_{2,1} > x_{3,1}$ and $x_{1,2} > x_{2,2} > x_{3,2}$, so agent 3 is the poorest. The results are represented in Fig. 15, preferences are the same as before. We can see how the picture drastically changes: now agent 2 worst position is when she is a peripheral node of a star where agent 1 is the core, and the higher the frequency of trade in which agent 1 is involved, the lower agent's 2 utility. Agent 3, the most disadvantaged, is worst off when she is in the periphery of a star with 2 in the core, she would prefer agent 1 to be the core. In general her utility will decrease the higher the weight on the edge between 2 and 1.

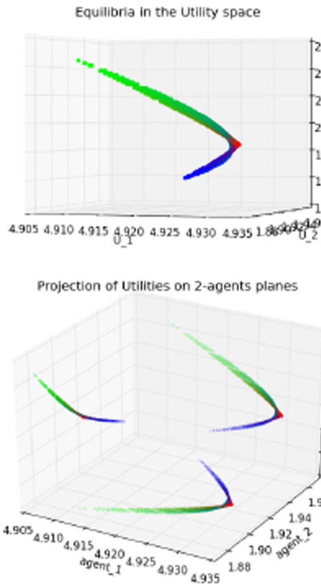

Fig. 14 $\alpha_1 = \alpha_2 = \alpha_3 = 0.2$, extreme inequality: agent 1 is rich agents 2,3 are poor

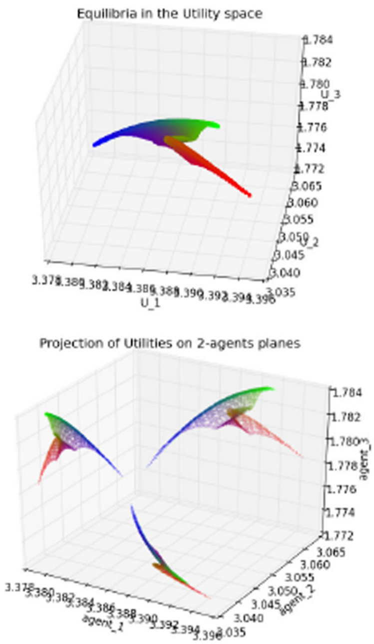

Fig. 15 $\alpha_1 = \alpha_2 = \alpha_3 = 0.2$ moderate inequality: agent 1 richer than agents 2 and 3

References

Anderson SP, Goeree JK, Holt CA (2004) Noisy directional learning and the logit equilibrium. Scand J Econ 106(3):581–602
Axtell R (2005) The complexity of exchange. Econ J 115(504):F193–F210
Barrat A, Barthélemy M, Pastor-Satorras R, Vespignani A (2004) The architecture of complex weighted networks. Proc Natl Acad Sci 101(11):3747–3752
Bervoets S, Bravo M, Faure M (2020) Learning with minimal information in continuous games. Theor Econ 15(4):1471–1508
Borondo J, Borondo F, Sickert C, Hidalgo C (2014) To each according to its degree: the meritocracy and Topocracy of embedded markets. Sci Rep. https://doi.org/10.1038/srep03784
Bottazzi J-M (1994) Accessibility of Pareto optima by Walrasian exchange processes. J Math Econ 23(6):585–603
Bowles S et al (2011) Inequality and network structure. Games Econ Behav 74:47–97
Champsaur P (1976) Neutrality of planning procedures in an economy with public goods. Rev Econ Stud 43(2):293–299
Cornet B (1983) Neutrality of planning procedures. J Math Econ 11:141–160
Cowan R, Jonard N (2004) Network structure and the diffusion of knowledge. J Econ Dyn Control 28:1557–1575
de Melo W (1976) On the structure of the pareto set of generic mappings. Bol Soc Bras Mat 7(2):121–126
Dreze J, de la Valle Poussin D (1983) A tatonnement process for public goods. Rev Econ Stud 28:133–150
Fisher F (2003) Disequilibrium and stability. In: Petri F, Hahn F (eds) General equilibrium, problems and prospects. Routledge, London
Flåm S (2019) Emergence of price-taking behavior. Econ Theory 70:847–870
Fleiner T, Jagadeesan R, Jankó Z, Teytelboym A (2019) Trading networks with frictions. Econometrica 87(5):1633–1661
Foley D (2010) What's wrong with the fundamental existence and welfare theorems? J Econ Behav Organ 75:115–131
Ghosal S, Morelli M (2004) Retrading in market games. J Econ Theory 115(1):151–181
Hahn F (1962) On the stability of a pure exchange equilibrium. Int Econ Rev 3(2):206–213
Hahn F (1982) Stability. In: Arrow K, Intriligator M (eds) Handbook of mathematical economics. North Holland, Amsterdam
Hirsch MW, Smale S (1974) Differential equations, dynamical systems, and linear algebra. Elsevier, New York
Hurwicz L, Radner R, Reiter S (1975a) A stochastic decentralized resource allocation process: part i. Econometrica 43(2):187–221
Hurwicz L, Radner R, Reiter S (1975b) A stochastic decentralized resource allocation process: part ii. Econometrica 43(3):363–393
Kalai E (1977) Proportional solutions to bargaining situations: intertemporal utility comparisons. Econometrica 45(7):1623–1630
König MD, Rogers T (2023) Endogenous technology cycles in dynamic r&d networks. Eur Econ Rev 158:104531
König MD, Tessone CJ, Zenou Y (2014) Nestedness in networks: a theoretical model and some applications. Theor Econ 9(3):695–752
Lovison A, Pecci F (2014) Hierarichical stratification of pareto sets. arXiv:1407.1755
Malinvaud E (1972) Prices for individual consumption, quantity indicators for collective consumption. Rev Econ Stud 39:385–496
Manea M (2016) 'Models of Bilateral Trade in Networks', in Yann Bramoullé, Andrea Galeotti, and Brian W. Rogers (eds), The Oxford Handbook of the Economics of Networks, Oxford Handbooks (2016; online edn, Oxford Academic, 2 June 2016), https://doi.org/10.1093/oxfordhb/9780199948277.013.15
Mas-Colell A (1990) The theory of general economic equilibrium. Cambridge University Press, Cambridge
Newman MEJ (2001) Scientific collaboration networks. II. Shortest paths, weighted networks, and centrality. Phys. Rev. E 64:016132
Newman MEJ (2002) Assortative mixing in networks. Phys. Rev. Lett. 89:208701
Petri F, Hahn EF (2003) General equilibrium, problems and prospects. Routledge, London
Robert P, Touboul J (2016) On the dynamics of random neuronal networks. J Stat Phys 165(3):545–584

Salnikov V, Cassese D, Lambiotte R (2019) Simplicial complexes and complex systems. Eur J Phys 40:014001

Schecter S (1977) Accessibility of optima in pure exchange economies. J Math Econ 4(3):197–216

Shapley, L. S. (1969) Utility comparison and the theory of games. In: La Decision: Aggregation et Dynamique des Ordres de Preference, Paris: Editions du Centre National de la Recherche Scientifique, 251–63.

Smale S (1975) Global analysis and economics I Pareto optimum and a generalization of Morse theory. Synth Math Methods Soc Sci 31(2):345–358

Smith D, Foley D (2008) Classical thermodynamics and economic general equilibrium theory? J Econ Dyn Control 32:7–65

Uzawa H (1962) On the stability of Edgeworth's barter process. Int Econ Rev 3(2):218–232

Wan Y (1978) On the structure and stability of local Pareto optima in a pure exchange economy. J Math Econ 5:255–274

Watts D, Strogatz S (2002) Collective dynamics of small-world networks. Nature 393:440–442