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Frame definability in finitely valued modal logics

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In this paper we study frame definability in finitely valued modal logics and establish two main results via suitable translations: (1) in finitely valued modal logics one cannot define more classes of frames than are already definable in classical modal logic (cf. [27, Thm. 8]), and (2) a large family of finitely valued modal logics define exactly the same classes of frames as classical modal logic (including modal logics based on finite Heyting and MV-algebras, or even BL-algebras). In this way one may observe, for example, that the celebrated Goldblatt–Thomason theorem applies immediately to these logics. In particular, we obtain the central result from [26] with a much simpler proof and answer one of the open questions left in that paper. Moreover, the proposed translations allow us to determine the computational complexity of a big class of finitely valued modal logics.

1. Introduction

Propositional modal logic is, at the level of frames, famously incomparable (in terms of expressive power) with first-order logic. Indeed, there are classes of frames definable in first-order logic that are not modally definable and viceversa. An example of the former phenomenon is the class of frames axiomatizable by the first-order sentence $\forall x\, \exists y\, (Rxy \wedge Ryy)$, and an example of the latter is the class of frames axiomatized by "Löb's formula" $\Box(\Box p \to p) \to \Box p$ which defines frames with transitive relations where the converse relation is well-founded [31, pp. 33–34].

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The original Goldblatt–Thomason theorem [14, Thm. 8] provides a model-theoretic characterization of modal axiomatizability for elementary classes of frames in terms of closure under taking generated subframes, disjoint unions, bounded morphic images, and reflection of ultrafilter extensions. Even though the proof in [14] is algebraic in spirit (with a detour via the usual extension of Stone duality), there are well-known ways of obtaining the result by pure model-theoretic methods [29]. Each closure condition in the theorem is necessary (see [29, p. 6]). Furthermore, the condition of elementarity of the class can be relaxed to closure under ultrapowers. There are also results, already contained in [14], using more complicated constructions which characterize any modally axiomatizable class of frames.

Many-valued modal logics, i.e. expansions with modalities of non-classical propositional logics with an intended many-valued semantics, have been around at least since Kristen Segerberg studied three-valued modal logics in [23]. The topic gained momentum with Melvin Fitting's work when he axiomatized in [12,13] the relational semantics for these logics based on Kripke frames with finitely valued propositional evaluations in each world (possibly, also, finitely valued accessibility relations between worlds), and proposed a natural interpretation of modalities capitalizing on the lattice structure of the semantics of the base logic. This proposal inspired, in particular, a long and nowadays quite lively stream of research in fuzzy modal logics (see e.g. [2,4,5,10,16–18,21,32,33]). As a part of this research, Bruno Teheux [26] has established an analogue of the Goldblatt–Thomason theorem for modal Łukasiewicz logics determined by finitely valued Kripke models over crisp frames (i.e. with a two-valued accessibility relation).

The goal of the present paper is to investigate frame definability in the many-valued context in a general approach that encompasses Teheux's results. We prove two main results: (1) each class of crisp frames definable in a finitely valued modal logic is already definable in classical modal logic, and (2) for a large family of finitely valued modal logics, the converse inclusion also holds, that is, their definable crisp frames *coincide* with those definable in classical modal logic. We proceed via translations of the many-valued modal logic into classical modal logic and back (inspired by the Kolmogorov–Glivenko translation) which preserve crisp frames. Furthermore, for these finitely valued modal logics, our translations ensure that their computational complexity coincides with that of their two-valued counterparts.

The first result generalizes a (little known) work by Steven K. Thomason [27] in which he translated finitely valued modal logics into two-valued modal logics. His approach mostly stayed at the level of frames and did not provide an explicit recursive definition. Moreover, his result was restricted to a class of logics in a language with standard connectives and truth constants. We propose a translation close to Thomason's, but based on models, with an explicit recursive definition, and free from the mentioned syntactical restrictions. The converse inclusion that we present in our second main result has not yet been considered in the literature, as far as we know, and it applies to any modal logic based on a finite lattice algebra that can interpret a Boolean algebra (thus, including modal logics based on finite Heyting and MV-algebras, and even BL-algebras). Therefore, even though Teheux's result had been presented as a generalization of the original theorem from [14], it actually *follows* from the classical Goldblatt–Thomason theorem and our results. This also answers, as we will see, an open problem left in [26]. Incidentally, we can also obtain the extension of the Goldblatt–Thomason theorem for predicate finitely valued modal logic using the work from [34] for the classical setting.

The paper is arranged as follows: §2 succinctly presents the necessary preliminaries regarding the syntactical and semantical setting of the paper. §3 introduces our translation from finitely valued to classical modal logic, shows that it has the intended semantic behavior. §4 contains the mentioned two main results of the paper: in §4.1 we prove that many-valued modal definability implies classical modal definability and §4.2 shows that the implication can be reversed whenever the many-valued modal logic is based on a finite lattice algebra that interprets a Boolean algebra. §5 uses the results in the previous sections to prove that the problems of validity and consequence from finite sets of premises in the considered many-valued modal logics have exactly the same computational complexity as their classical counterparts. Finally, §6 offers some concluding remarks and lines for further research.

2. Preliminaries

Let $\boldsymbol{A} = \langle A, \wedge^{\boldsymbol{A}}, \vee^{\boldsymbol{A}}, \dots \rangle$ be an arbitrary *finite* (henced bounded) lattice possibly expanded with further operations (which from now on we will call a *lattice algebra*). When convenient, we will denote the top and the bottom element of $\boldsymbol{A}$ respectively as $1^{\boldsymbol{A}}$ and $0^{\boldsymbol{A}}$, although they need not be part of the signature of the algebra. For the sake of lighter notation, we will drop the superindex $\boldsymbol{A}$ in the operations when the algebra is clear from the context. In particular, we will refer to the following instances:

- A *Boolean algebra* is a lattice algebra $\boldsymbol{A} = \langle A, \wedge, \vee, \neg \rangle$ in which the lattice is distributive and $\neg$ is the complement operation (that is, for each element $a \in A$, $a \vee \neg a = 1$ and $a \wedge \neg a = 0$).
- A *pseudocomplemented lattice* is a lattice algebra $\boldsymbol{A} = \langle A, \wedge, \vee, \neg \rangle$ in which $\neg a = \max\{b \in A \mid a \wedge b = 0\}$ for each $a \in A$.
- A *Stone algebra* is a pseudocomplemented lattice $\boldsymbol{A} = \langle A, \wedge, \vee, \neg \rangle$ in which the lattice is distributive and $\neg a \vee \neg\neg a = 1$ for each $a \in A$.
- A *Heyting algebra* is a lattice algebra $\boldsymbol{A} = \langle A, \wedge, \vee, \to \rangle$ such that $a \to b = \max\{c \in A \mid a \wedge c \leq b\}$ for each $a, b \in A$.
- An MV-*algebra* is a lattice algebra $\boldsymbol{A} = \langle A, \wedge, \vee, \&, \to \rangle$ such that
  - & is commutative, monotonic w.r.t. the lattice order, and has 1 as neutral element,
  - for each $a, b \in A$, $a \to b = \max\{c \in A \mid a \,\&\, c \leq b\}$,
  - for each $a, b \in A$, $(a \to b) \vee (b \to a) = 1$,
  - and for each $a, b \in A$, $a \vee b = (a \to b) \to b$.

The formulas of the modal language $Fm_{\boldsymbol{A}}^{\Diamond\Box}(\tau)$ are built from a denumerable set of propositional variables τ by means of the binary connectives $\wedge$ and $\vee$, an n-ary connective for each additional n-ary operation of $\boldsymbol{A}$, and the unary connectives (modalities) $\Diamond$ and $\Box$.

In particular, if $\mathbf{2}$ is the two-element Boolean algebra, we may think of the formulas of $Fm_{\mathbf{2}}^{\Diamond\Box}(\tau)$ as the usual classical modal language.

We will work with *crisp* frames $\mathfrak{F} = \langle W, R \rangle$ as in classical modal logic, i.e. W is a non-empty set (whose elements are called *worlds*) and $R \subseteq W^2$ is a binary relation (called *accessibility relation*). A Kripke $\boldsymbol{A}$-valued model $\mathfrak{M}$ is defined as a pair $\langle \mathfrak{F}, V \rangle$, where $\mathfrak{F} = \langle W, R \rangle$ is a frame and $V \colon \tau \times W \longrightarrow A$ is mapping called a *valuation*; we say that $\mathfrak{M}$ is *based on* $\mathfrak{F}$. Given such a model, for each $w \in W$ and each formula $\varphi \in Fm_{\boldsymbol{A}}^{\Diamond\Box}(\tau)$, we inductively define the truth-value $\|\varphi\|_w^{\mathfrak{M}}$ as:

$$
\begin{aligned}
\|p\|_w^{\mathfrak{M}} &= V(p, w), && \text{if } p \in \tau \\
\|\circ(\psi_1, \dots, \psi_n)\|_w^{\mathfrak{M}} &= \circ^{\boldsymbol{A}}(\|\psi_1\|_w^{\mathfrak{M}}, \dots, \|\psi_n\|_w^{\mathfrak{M}}), && \text{for each } n\text{-ary connective } \circ, \\
\|\Diamond\psi\|_w^{\mathfrak{M}} &= \sup_{\leq_{\boldsymbol{A}}}\{\|\psi\|_v^{\mathfrak{M}} \mid Rwv\}, \\
\|\Box\psi\|_w^{\mathfrak{M}} &= \inf_{\leq_{\boldsymbol{A}}}\{\|\psi\|_v^{\mathfrak{M}} \mid Rwv\}.
\end{aligned}
$$

A formula φ from $Fm_{\boldsymbol{A}}^{\Diamond\Box}(\tau)$ is said to be $\boldsymbol{A}$-*valid* in a frame $\mathfrak{F} = \langle W, R \rangle$ (in symbols, $\mathfrak{F} \models_{\boldsymbol{A}} \varphi$) if for any $\boldsymbol{A}$-valued model $\mathfrak{M}$ based on $\mathfrak{F}$, $\|\varphi\|_w^{\mathfrak{M}} = 1^{\boldsymbol{A}}$ for every world $w \in W$ (alternatively, we say that φ is *globally true* in $\mathfrak{M}$). Furthermore, a set of formulas Φ from $Fm_{\boldsymbol{A}}^{\Diamond\Box}(\tau)$ *modally* $\boldsymbol{A}$-*defines* a frame class $\mathbb{F}$ if $\mathbb{F}$ contains exactly those frames where every $\varphi \in \Phi$ is $\boldsymbol{A}$-valid (if $\Phi = \{\varphi\}$, we say that φ *modally* $\boldsymbol{A}$-*defines* $\mathbb{F}$); cf. [26, Definition 2.2 and 2.3]. Similarly, we may say that a class $\mathbb{F}$ of frames is *modally* $\boldsymbol{A}$-*definable* if there is a set of formulas Φ that modally $\boldsymbol{A}$-defines $\mathbb{F}$. Finally, given a frame class $\mathbb{F}$, we define a consequence relation in the following way: for each $\Gamma \cup \{\varphi\} \subseteq Fm_{\boldsymbol{A}}^{\Diamond\Box}(\tau)$, we write $\Gamma \vdash_{\mathrm{Log}(\mathbb{F}, \boldsymbol{A}, \tau)} \varphi$ iff for each $\mathfrak{F} \in \mathbb{F}$ and each $\boldsymbol{A}$-valued model $\mathfrak{M}$ based on $\mathfrak{F}$ we have that φ is globally true in $\mathfrak{M}$ whenever all formulas from Γ are globally true in $\mathfrak{M}$. Thus, $\mathrm{Log}(\mathbb{F}, \boldsymbol{A}, \tau)$ can be called the *global modal* $\boldsymbol{A}$-*valued logic given by* $\mathbb{F}$.

Observe that in the case $\boldsymbol{A} \cong \mathbf{2}$ we retrieve the standard definitions from classical modal logic. In this case, we use the standard notation $\langle \mathfrak{M}, w\rangle \models \varphi$ to signify that $\|\varphi\|_w^{\mathfrak{M}} = 1^{\boldsymbol{A}}$.

3. Translating finitely valued modal logics into classical modal logics

In this section, we provide a translation of formulas of a many-valued modal logic into formulas of standard two-valued modal logic. In contrast to [27], we give an explicit inductive definition of the translation already at the level of models.

Given a finite lattice algebra $\boldsymbol{A}$ and a denumerable set of variables $\tau = \{p_1, p_2, \ldots\}$, we define $\tau^* = \bigcup_{i \geq 1} \{p_i^a \mid a \in A\}$. Now, we define translations $T^{\boldsymbol{a}}$ from $Fm_{\boldsymbol{A}}^{\Diamond\Box}(\tau)$ into $Fm_{\mathbf{2}}^{\Diamond\Box}(\tau^*)$, for each element $a \in A$, by simultaneous induction as follows:

$$
\begin{aligned}
T^a(p_i) &= p_i^a \quad (i \geq 1) \\
T^a(\circ(\psi_1, \ldots, \psi_n)) &= \bigvee_{\substack{b_1,\ldots,b_n \in A \\ \circ^{\boldsymbol{A}}(b_1,\ldots,b_n)=a}} (T^{b_1}(\psi_1) \wedge \ldots \wedge T^{b_n}(\psi_n)) \\
T^a(\Diamond\psi) &= (\bigvee_{\substack{k \leq |A| \\ b_1 \ldots b_k \in A \\ b_1 \vee^{\boldsymbol{A}} \ldots \vee^{\boldsymbol{A}} b_k = a}} \bigwedge_{i=1}^{k} \Diamond T^{b_i}(\psi)) \wedge \Box(\bigvee_{\substack{b \in A \\ b \leq a}} T^b(\psi)) \\
T^a(\Box\psi) &= (\bigvee_{\substack{k \leq |A| \\ b_1,\ldots,b_k \in A \\ b_1 \wedge^{\boldsymbol{A}} \ldots \wedge^{\boldsymbol{A}} b_k = a}} \bigwedge_{i=1}^{k} \Diamond T^{b_i}(\psi)) \wedge \Box(\bigvee_{\substack{b \in A \\ a \leq b}} T^b(\psi)).
\end{aligned}
$$

Furthermore, given any $\boldsymbol{A}$-valued model $\mathfrak{M} = \langle \mathfrak{F}, V\rangle$ based on a frame $\mathfrak{F} = \langle W, R\rangle$ for $Fm_{\boldsymbol{A}}^{\Diamond\Box}(\tau)$, we define a $\mathbf{2}$-valued model $\mathfrak{M}^*$ for $Fm_{\mathbf{2}}^{\Diamond\Box}(\tau^*)$ based on the same frame and with a valuation V^* defined as follows:

$$V^*(p_i^a, w) = 1 \text{ iff } V(p_i, w) = a, \text{ for each } i \geq 1, \text{ each } w \in W, \text{ and each } a \in A.$$

Given this, it is not hard to see that the translation $T^a(\varphi)$ simply rewrites in the classical modal language the conditions for φ to take the value a; more precisely, by induction on the complexity of formulas, we can easily prove the following lemma:

Lemma 1 *(Switch Lemma). Let $\boldsymbol{A}$ be a finite lattice algebra, τ a denumerable set of variables, and $\mathfrak{M}$ an $\boldsymbol{A}$-valued model based on a frame $\mathfrak{F}$. For each formula $\varphi \in Fm_{\boldsymbol{A}}^{\Diamond\Box}(\tau)$, and each world w, we have: $\|\varphi\|_w^{\mathfrak{M}} = a$ iff $\langle \mathfrak{M}^*, w\rangle \models T^a(\varphi)$.*

We can easily obtain the following axiomatization result for classical models of the form $\mathfrak{M}^*$:

Lemma 2. *Let $\boldsymbol{A}$ be a finite lattice algebra and τ a set of variables. Consider the following set $T^*(\tau) \subseteq Fm_{\mathbf{2}}^{\Diamond\Box}(\tau^*)$ of (modality-free) formulas:*

$$\bigvee_{a \in A} p_i^a, \ \neg(p_i^a \wedge p_i^b) \qquad (a, b \in A, a \neq b, p_i \in \tau).$$

Then:

1. *If $\mathfrak{M}$ is an $\boldsymbol{A}$-valued model, then the formulas of $T^*(\tau)$ are true in every world of the $\mathbf{2}$-valued model $\mathfrak{M}^*$.*
2. *If $\mathfrak{N}$ is a $\mathbf{2}$-valued model for the language $Fm_{\mathbf{2}}^{\Diamond\Box}(\tau^*)$ that satisfies in each world all the formulas of $T^*(\tau)$, we define a model $\mathfrak{M}$ for the language $Fm_{\boldsymbol{A}}^{\Diamond\Box}(\tau)$ as follows:*
   - *$\mathfrak{M}$ is based on the same frame as $\mathfrak{N}$,*
   - *$V^{\mathfrak{M}}(p_i, w) = a$ iff $\langle \mathfrak{N}, w\rangle \models p_i^a$.*

   Then, $\mathfrak{N} = \mathfrak{M}^*$.

Observe that if τ is finite, then $T^*(\tau)$ in Lemma 2 is finite as well.

Remark 3. It should be clear that the translation we have presented in this section allow us to give very quick proofs of certain properties of many-valued modal logics on crisp frames. For example, both compactness and the finite model property are inherited from two-valued modal logic.

4. Modal frame definability

In this section, we will establish our main results. First, we will see that in an $\boldsymbol{A}$-valued modal logic we cannot define more classes of crisp frames than are already definable in classical modal logic. Second, for a wide class of algebras, the converse also follows, namely, any class of frames which is definable in two-valued modal logic will be modally $\boldsymbol{A}$-definable.

4.1. Modal $\boldsymbol{A}$-definability implies modal $\mathbf{2}$-definability

Recall that, for a modal formula φ of $Fm_{\mathbf{2}}^{\Diamond\Box}(\tau^*)$, we have the classical property [22, Prop. 4.3] that the truth of φ in any pointed model $\langle \mathfrak{M}, w\rangle$ depends only on the restriction (denoted by $\mathfrak{M}|n$) of $\langle \mathfrak{M}, w\rangle$ to worlds that can be reached from w through R in at most n steps, where $n = \mathrm{rank}(\varphi)$, i.e., the modal rank of φ ([22, Def. 4.2]). Observe that $\mathrm{rank}(\varphi) = \mathrm{rank}(T^a(\varphi))$, since our translation does not increase the modal rank. Then, for such a formula, the Switch Lemma can be extended to:

$$\|\varphi\|_w^{\mathfrak{M}} = a \text{ iff } \langle \mathfrak{M}^*, w\rangle \models T^a(\varphi) \text{ iff } \langle \mathfrak{M}^*|n, w\rangle \models T^a(\varphi).$$

From this we may obtain the following version of [27, Theorem 8]:

Theorem 4. *Let $\boldsymbol{A}$ be a finite lattice algebra, let φ be a formula from $Fm_{\boldsymbol{A}}^{\Diamond\Box}(\tau)$ (assume w.l.o.g. that $\tau = \{p_1, \ldots, p_n\}$ is the finite set of variables that appear in the formula), and let $\mathbb{F}$ be a class of frames. Then, φ modally $\boldsymbol{A}$-defines $\mathbb{F}$ iff $\mathbb{F}$ is modally $\mathbf{2}$-defined in $Fm_{\mathbf{2}}^{\Diamond\Box}(\tau^*)$ by*

$$\varphi^* := \Big(\bigvee_{m \leq \mathrm{rank}(T^1(\varphi))} \neg\Box^m\big(\bigwedge T^*(\tau)\big)\Big) \vee T^1(\varphi).$$

Proof. ($\Rightarrow$): Assume first that φ modally $\boldsymbol{A}$-defines the frame class $\mathbb{F}$. Our goal is then to show the following two claims:

(1) the formula φ^* is $\mathbf{2}$-valid in every frame from $\mathbb{F}$, and
(2) every frame where φ^* is $\mathbf{2}$-valid belongs to the class $\mathbb{F}$.

To see (1), take any **2**-valued model $\mathfrak{M}$ for $Fm_{\mathbf{2}}^{\Diamond\Box}(\tau^*)$ based on a frame $\mathfrak{F}$ from $\mathbb{F}$. The formula φ^* is a material implication, so assume that at an arbitrary world w of $\mathfrak{M}$ all the antecedents of φ^* hold. Then, let $k = \operatorname{rank}(T^1(\varphi))$ and consider the pointed model $\langle \mathfrak{M}|k, w\rangle$. The theory

$$\bigvee_{a\in A} p_i^a, \quad \bigwedge_{\substack{a,b\in A\\ a\neq b}} \neg(p_i^a \wedge p_i^b) \qquad (1 \leq i \leq n)$$

globally holds in $\langle \mathfrak{M}|k, w\rangle$ since that is what the hypothesis of all the antecedents of φ^* holding means. Now take any model based on $\mathfrak{F}$ of the form $\mathfrak{N}^*$ for $Fm_{\mathbf{2}}^{\Diamond\Box}(\tau^*)$ such that $\langle \mathfrak{M}|k, w\rangle \cong \langle \mathfrak{N}^*|k, w\rangle$ corresponding to some $\mathfrak{N}$ for $Fm_{\boldsymbol{A}}^{\Diamond\Box}(\tau)$. There is always one such model: e.g. interpret the predicates of τ^* for the worlds in $\langle \mathfrak{M}|k, w\rangle$ as in that model and in every other world let p_i^1 hold and p_i^a fail, for each $1 \leq i \leq n$ and each $a \neq 1$. Now, by hypothesis, $\|\varphi\|_w^{\mathfrak{N}} = 1$, so, by the Switch Lemma, $\langle \mathfrak{N}^*, w\rangle \models T^1(\varphi)$. Then, $\langle \mathfrak{N}^*|k, w\rangle \models T^1(\varphi)$, and so $\langle \mathfrak{M}|k, w\rangle \models T^1(\varphi)$, as desired.

In order to prove (2), suppose that φ^* is **2**-valid in a frame $\mathfrak{F}$. Take any model for $Fm_{\mathbf{2}}^{\Diamond\Box}(\tau^*)$ of the form $\mathfrak{M}^*$ based on $\mathfrak{F}$ for some corresponding model $\mathfrak{M}$ for $Fm_{\boldsymbol{A}}^{\Diamond\Box}(\tau)$ also based on $\mathfrak{F}$. Since φ^* is globally true in $\mathfrak{M}^*$ (and all the antecedents of φ^* hold at any world), we must have that $T^1(\varphi)$ is globally true in $\mathfrak{M}^*$, and by the Switch Lemma, φ is globally true in $\mathfrak{M}$. Since for any such $\mathfrak{M}$ based on $\mathfrak{F}$ there is a corresponding $\mathfrak{M}^*$, it follows that φ is $\boldsymbol{A}$-valid in $\mathfrak{F}$. Thus, $\mathfrak{F} \in \mathbb{F}$.

($\Leftarrow$): Assume now that φ^* modally **2**-defines the class $\mathbb{F}$. As before, we need to show the following two claims:

(1) the formula φ is $\boldsymbol{A}$-valid in every frame in $\mathbb{F}$, and
(2) every frame where φ is $\boldsymbol{A}$-valid is in the class $\mathbb{F}$.

To see (1), consider any $\boldsymbol{A}$-valued model $\mathfrak{M}$ based on a frame $\mathfrak{F} \in \mathbb{F}$. In the corresponding model $\mathfrak{M}^*$ for $Fm_{\mathbf{2}}^{\Diamond\Box}(\tau^*)$ the formula $T^1(\varphi)$ is globally true (as φ^* is), so by the Switch Lemma, φ is globally true in $\mathfrak{M}$, as desired. To see (2), suppose now that φ is $\boldsymbol{A}$-valid in the frame $\mathfrak{F}$. Reasoning as before, by the Switch Lemma, this means that φ^* is **2**-valid in $\mathfrak{F}$, so $\mathfrak{F} \in \mathbb{F}$. $\Box$

*4.2. When does modal **2**-definability imply modal $\boldsymbol{A}$-definability?*

The aim of this subsection is to provide sufficient conditions for recovering the modal $\boldsymbol{A}$-definability of a class of frames from its modal **2**-definability. Whether the conditions we provide are necessary or not is left as an open problem. In Theorem 4 we have seen that, given a finite lattice algebra $\boldsymbol{A}$, any class of (crisp) frames definable by a formula of $Fm_{\boldsymbol{A}}^{\Diamond\Box}$ in the $\boldsymbol{A}$-valued associated modal logic is definable by a formula of classical modal logic. To have the reciprocal, one would expect the algebra $\boldsymbol{A}$ to interpret classical logic in some sense.

Assume for the rest of this section that $\boldsymbol{A}$ has the following non-trivial lattice reduct[1]

$$\operatorname{Red}(\boldsymbol{A}) = \langle A, \wedge^{\boldsymbol{A}}, \vee^{\boldsymbol{A}}, \neg^{\boldsymbol{A}}\rangle$$

enriched with $\neg^{\boldsymbol{A}} x = u^{\boldsymbol{A}}(x)$, where $u(x)$ is a distinguished unary term. An example is when $\boldsymbol{A}$ is a bounded residuated lattice and $u(x) = x \to 0$. However, in general, we do not require $\boldsymbol{A}$ to be residuated.

In this context, we provide a sufficient algebraic condition for formulas in the language $Fm_{\boldsymbol{A}}^{\Diamond\Box}$ to $\boldsymbol{A}$-define any class of frames definable in classical modal logic. We start by introducing a useful definition:

[1] For the sake of lighter notation, in this section, we often drop the superindex $\boldsymbol{A}$ in the operations.

Definition 1. A $\{\vee, \wedge, 1, \neg\}$-algebra $\boldsymbol{B}$ is said to be *interpretable in* $\boldsymbol{A}$ *via a unary* $Fm_{\boldsymbol{A}}$*-term* $t(x)$ if

- $Eq(t^{\boldsymbol{A}}) = \{\langle a, b\rangle \in A^2 \mid t^{\boldsymbol{A}}(a) = t^{\boldsymbol{A}}(b)\}$ is a congruence of $\mathrm{Red}(\boldsymbol{A})$,
- $t^{\boldsymbol{A}}(1^{\boldsymbol{A}}) = 1^{\boldsymbol{A}}$, and
- $Red(\boldsymbol{A})/Eq(t^{\boldsymbol{A}})$ is isomorphic to $\boldsymbol{B}$.

Equivalently, $t^{\boldsymbol{A}}(a) \wedge^{\boldsymbol{B}'} t^{\boldsymbol{A}}(b) := t^{\boldsymbol{A}}(a \wedge^{\boldsymbol{A}} b)$, $t^{\boldsymbol{A}}(a) \vee^{\boldsymbol{B}'} t^{\boldsymbol{A}}(b) := t^{\boldsymbol{A}}(a \vee^{\boldsymbol{A}} b)$, $\neg^{\boldsymbol{B}'} t^{\boldsymbol{A}}(a) := t^{\boldsymbol{A}}(\neg^{\boldsymbol{A}} a)$ are well-defined operations in $t^{\boldsymbol{A}}(A)$, and $\boldsymbol{B}$ is isomorphic to the algebra

$$\boldsymbol{B}' = \langle t^{\boldsymbol{A}}[A], \wedge^{\boldsymbol{B}'}, \vee^{\boldsymbol{B}'}, \neg^{\boldsymbol{B}'}, 1^{\boldsymbol{A}}\rangle.$$

Clearly, $t^{\boldsymbol{A}} \colon A \to B'$ is an epimorphism and $\boldsymbol{B}'$ is a lattice with top element $1^{\boldsymbol{A}}$. If $\boldsymbol{A}$ is bounded, then so is $\boldsymbol{B}'$ and $0^{\boldsymbol{B}'} = t^{\boldsymbol{A}}(0^{\boldsymbol{A}})$.

Example 5. Any pseudocomplemented lattice $\boldsymbol{A}$ interprets via $t(x) = \neg\neg x$ its algebra of *regular elements* $\mathrm{Reg}(\boldsymbol{A}) = \{a \in A \mid \neg\neg a = a\}$,[2] which happens to be a Boolean algebra. To see this, notice that $\mathrm{Reg}(\boldsymbol{A}) = \{\neg\neg a \mid a \in A\}$ because $\neg\neg\neg\neg a = \neg\neg a$, and $\neg\neg 1 = 1$. Moreover, $\neg\neg a = \neg\neg b$ is a congruence since $\neg\neg a = \neg\neg a'$ and $\neg\neg b = \neg\neg b'$ imply:

$$\neg\neg(a \wedge b) = \neg\neg(\neg\neg a \wedge \neg\neg b) = \neg\neg(\neg\neg a' \wedge \neg\neg b') = \neg\neg(a' \wedge b'),$$
$$\neg\neg(a \vee b) = \neg\neg(\neg\neg a \vee \neg\neg b) = \neg\neg(\neg\neg a' \vee \neg\neg b') = \neg\neg(a' \vee b').$$

Therefore, in $B = \mathrm{Reg}(\boldsymbol{A})$ we have the following induced operations:

$$a \wedge^{\boldsymbol{B}} b := \neg\neg(a \wedge b) = \neg\neg a \wedge \neg\neg b = a \wedge b$$
$$a \vee^{\boldsymbol{B}} b := \neg\neg(a \vee b) \text{ (no further reduction is possible)}$$
$$\neg^{\boldsymbol{B}} a := \neg\neg(\neg a) = \neg a$$

and $\mathrm{Reg}(\boldsymbol{A})$ is a Boolean algebra because $a \wedge^{\boldsymbol{B}} \neg a = \neg\neg 0 = 0$ and $a \vee^{\boldsymbol{B}} \neg a = \neg\neg(a \vee \neg a) = 1$ by the density of $a \vee \neg a$ in pseudocomplemented lattices (see [1] for the identities utilized). In particular, any Heyting algebra $\boldsymbol{A}$ interprets a Boolean algebra in this way.

Remark 6. In Example 5, $\mathrm{Reg}(\boldsymbol{A})$ is not necessarily a subalgebra of $\boldsymbol{A}$ (because of disjunction), but it contains as a subalgebra its *Boolean skeleton* $\mathbf{B}(\boldsymbol{A})$. These algebras coincide if and only if $\boldsymbol{A}$ is a Stone algebra (see [6]).

Example 7. Any finite MV-algebra $\boldsymbol{A}$ interprets its Boolean skeleton $\mathbf{B}(\boldsymbol{A})$, via $t(x) = nx$ for any $n \geq |A|$, because $n\boldsymbol{A} = \mathbf{B}(\boldsymbol{A})$ and $n(\cdot) \colon \boldsymbol{A} \to \boldsymbol{A}$ is an endomorphism thanks to the validity of the following equations:

$$n(x \wedge y) \approx nx \wedge ny$$
$$n(x \vee y) \approx nx \vee ny$$
$$n(\neg x) \approx \neg nx$$
$$n1 \approx 1.$$

To see this, consider a subdirect representation $\boldsymbol{A} \subseteq \Pi_{i\in I}\boldsymbol{C}_i$, where the $\boldsymbol{C}_i$ are finite MV-chains. Each chain has length at most n and thus it is easily verified by cases that $na \in \{0^{\boldsymbol{C}_i}, 1^{\boldsymbol{C}_i}\}$ in each chain $\boldsymbol{C}_i$; therefore

[2] The formulas in this example should be read in the light of the previous definition, though we drop some superscripts for the sake of a lighter notation.

$n\boldsymbol{A} \subseteq \mathbf{B}(\boldsymbol{A})$. Moreover, if $a = \langle a_i \rangle_i \in \mathbf{B}(\boldsymbol{A})$, then $a_i \in \mathbf{B}(\boldsymbol{C}_i) = \{0^{\boldsymbol{C}_i}, 1^{\boldsymbol{C}_i}\}$. Therefore, $na_i = a_i$ for each $i \in I$, and thus $a = na \in n\boldsymbol{A}$.

Notice that if $\boldsymbol{A}$ is a Glivenko bounded residuated lattice in the sense of [9], then $\boldsymbol{A}$ interprets $\mathrm{Reg}(\boldsymbol{A})$ via $\neg\neg x$. Although this algebra is not necessarily Boolean, in some cases it is an MV-algebra and, thus, it interprets a Boolean algebra.

Example 8. By [28, Thm. 2], in every BL-algebra one can define an MV-algebra, and then, in the finite case, one can interpret a Boolean algebra as remarked before.

We are interested in term interpretations of Boolean algebras due to the following result:

Theorem 9. *Let $\boldsymbol{A}$ be a finite lattice algebra which interprets via a term t(x) a Boolean algebra, let φ be a formula from $Fm_{\mathbf{2}}^{\Diamond\Box}(\tau)$ and let $\mathbb{F}$ be a class of frames. Then, φ modally $\mathbf{2}$-defines $\mathbb{F}$ iff $t(\varphi)$ modally $\boldsymbol{A}$-defines $\mathbb{F}$.*

Then, putting this together with Theorem 4, we obtain:

Corollary 10. *Let $\boldsymbol{A}$ be a finite lattice algebra which interprets via a term $t(x)$ a Boolean algebra. Then, the class of modally $\boldsymbol{A}$-definable frames coincides with the class of modally $\mathbf{2}$-definable frames.*

Example 11. Thanks to the previous examples, Corollary 10 implies that the modal logic associated to an expansion of a finite pseudocomplemented lattice or a finite BL-algebra defines the same class of crisp frames as classical modal logic (this includes, respectively, Heyting algebras and MV-algebras). In particular, this result includes the modal extensions of Łukasiewicz finitely valued logics studied by Teheux in [26] and solves the open problem left in that paper of determining whether their definable classes of frames coincide with those definable in classical modal logic.

The rest of this subsection is devoted to proving Theorem 9. We start by showing that the two-valued global consequence of modal formulas is preserved when allowing models to take values on an arbitrary finite Boolean algebra.

Proposition 12. *Let $\mathfrak{F}$ be a frame and $\boldsymbol{B}$ be a finite Boolean algebra. Then, for each set $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathbf{2}}^{\Diamond\Box}(\tau)$ of formulas, we have:*

$$\Gamma \vDash_{\mathrm{Log}(\{\mathfrak{F}\},\boldsymbol{B},\tau)} \varphi \ \textit{ if and only if } \ \Gamma \vDash_{\mathrm{Log}(\{\mathfrak{F}\},\mathbf{2},\tau)} \varphi.$$

In particular, when $\Gamma = \emptyset$, we obtain that φ modally $\mathbf{2}$-defines and $\boldsymbol{B}$-defines the same class of frames.

Proof. Without loss of generality, we may identify $\boldsymbol{B}$ with a power $\mathbf{2}^n$ of the two-element algebra, and thus any valuation $V : \tau \times W \to B$ has the form $V(p, w) = \langle V_i(p, w)\rangle_{i=1}^n$ where $V_i : \tau \times W \to 2$. Take a model $\mathfrak{M} = \langle \mathfrak{F}, V\rangle$. An easy induction shows that for any formula ψ and any $w \in W$, $\|\psi\|_w^{\mathfrak{M}} = \langle \|\psi\|_w^{\langle \mathfrak{F}, V_i\rangle}\rangle_{i=1}^n$.

Assume that $\Gamma \vDash_{\mathrm{Log}(\{\mathfrak{F}\},\mathbf{2},\tau)} \varphi$ and Γ is globally true in a $\boldsymbol{B}$-valued model $\mathfrak{M} = \langle \mathfrak{F}, V\rangle$. That is, for each $\psi \in \Gamma$ and each $w \in W$, $\|\psi\|_w^{\mathfrak{M}} = 1^{\boldsymbol{B}} = \langle 1\rangle_{i=1}^n$, and hence $\|\psi\|_w^{\langle \mathfrak{F}, V_i\rangle} = 1$. Therefore, $\|\varphi\|_w^{\langle \mathfrak{F}, V_i\rangle} = 1$ for each $i \in \{1, \dots, n\}$ and each $w \in W$, i.e. φ is globally true in $\mathfrak{M}$ as desired. Reciprocally, assume that $\Gamma \vDash_{\mathrm{Log}(\{\mathfrak{F}\},\boldsymbol{B},\tau)} \varphi$ and Γ is globally true in a classical model $\mathfrak{M} = \langle \mathfrak{F}, V\rangle$. Then, the diagonal valuation $V'(p, w) = \langle V(p, w)\rangle_{i=1}^n$ only gives values in $\{\langle 1\rangle_{i=1}^n, \langle 0\rangle_{i=1}^n\} \subseteq B$ and is such that $\|\psi\|_w^{\langle \mathfrak{F}, V'\rangle} = \langle 1\rangle_{i=1}^n = 1^{\boldsymbol{B}}$ for each $\psi \in \Gamma$ and each $w \in W$. By hypothesis, $\|\varphi\|_w^{\langle \mathfrak{F}, V'\rangle} = \langle \|\varphi\|_w^{\langle \mathfrak{F}, V\rangle}\rangle_{i=1}^n = \langle 1\rangle_{i=1}^n$ for each $w \in W$, and thus $\|\varphi\|_w^{\mathfrak{M}} = 1$ for each $w \in W$. $\Box$

The next step consists in obtaining a similar preservation result when changing the algebra via the notion of interpretability defined above.

Proposition 13. *Let $\mathfrak{F}$ be a frame and assume that a finite lattice algebra $\boldsymbol{A}$ interprets a $\{\vee, \wedge, 1, \neg\}$-algebra $\boldsymbol{B}$ via a unary term $t(x)$. Then, for each set of formulas $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathbf{2}}^{\Diamond\Box}(\tau)$, we have:*

$$\Gamma \vDash_{\mathrm{Log}(\{\mathfrak{F}\},\boldsymbol{B},\tau)} \varphi \;\; \textit{iff} \;\; t[\Gamma] \vDash_{\mathrm{Log}(\{\mathfrak{F}\},\boldsymbol{A},\tau)} t(\varphi).$$

In particular, when $\Gamma = \emptyset$, we obtain that the class of frames modally $\boldsymbol{B}$-defined by φ coincides with the class of frames modally $\boldsymbol{A}$-defined by $t(\varphi)$.

Proof. For any valuation $V\colon \tau \times W \to A$ (forming an $\boldsymbol{A}$-valued model $\mathfrak{M}$ based on $\mathfrak{F}$), define a valuation $V^* = t^{\boldsymbol{A}} \circ V\colon \tau \times W \to B$ (forming a $\boldsymbol{B}$-valued model $\mathfrak{M}'$ based on $\mathfrak{F}$). Then, one can show by induction on the complexity of formulas $\psi \in Fm_{\mathbf{2}}^{\Diamond\Box}(\tau)$ that for any $w \in W$:

$$\|\psi\|_w^{\mathfrak{M}'} = t^{\boldsymbol{A}}(\|\psi\|_w^{\mathfrak{M}}) = \|t(\psi)\|_w^{\mathfrak{M}} \in B.$$

Assume that $\Gamma \vDash_{\mathrm{Log}(\{\mathfrak{F}\},\boldsymbol{B},\tau)} \varphi$ and $t[\Gamma]$ is globally true in an $\boldsymbol{A}$-valued model $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$. Then, by the previous observation and the fact that $t^{\boldsymbol{A}}(1^{\boldsymbol{A}}) = 1^{\boldsymbol{A}}$, Γ is globally true in the $\boldsymbol{B}$-valued model $\mathfrak{M}'$ and, hence, so is φ. Therefore, $t(\varphi)$ is globally true in $\mathfrak{M}$.

Reciprocally, assume that $t[\Gamma] \vDash_{\mathrm{Log}(\{\mathfrak{F}\},\boldsymbol{A},\tau)} t(\varphi)$. Suppose now that Γ is globally true in a $\boldsymbol{B}$-valued model $\mathfrak{N} = \langle \mathfrak{F}, V_B \rangle$. Choose an $\boldsymbol{A}$-valued $\mathfrak{M}$ with a valuation $V\colon \tau \times W \to A$ such that $V_B = t^{\boldsymbol{A}} \circ V$ and hence $\mathfrak{N} = \mathfrak{M}'$. Then, $t[\Gamma]$ is globally true in $\mathfrak{M}$ and, hence, so is $t(\varphi)$. Therefore, φ is globally true in $\mathfrak{N}$ as desired. $\Box$

Clearly, Theorem 9 follows from Propositions 12 and 13. Furthermore, since the preservation of consequence in these propositions holds at the level of a fixed frame, it also holds for the consequence given by a class of frames.

Corollary 14. *Let $\mathbb{F}$ be a class of frames and assume that a finite lattice algebra $\boldsymbol{A}$ interprets a Boolean algebra via a unary term $t(x)$. Then, for each set of formulas $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathbf{2}}^{\Diamond\Box}(\tau)$, we have:*

$$\Gamma \vDash_{\mathrm{Log}(\mathbb{F},\mathbf{2},\tau)} \varphi \;\; \textit{iff} \;\; t[\Gamma] \vDash_{\mathrm{Log}(\mathbb{F},\boldsymbol{A},\tau)} t(\varphi).$$

We end this subsection with two remarks regarding the algebraic character of the notion of interpretability that has been instrumental in our translation.

Remark 15. The property of interpreting a Boolean algebra in $\boldsymbol{A}$ via a term $t(x)$ is quasiequational in $\boldsymbol{A}$. Hence, if $\boldsymbol{A}$ interprets a Boolean algebra via $t(x)$, then any member of $\mathbf{Q}(\boldsymbol{A})$ interprets *some* Boolean algebra via $t(x)$ (not necessarily the same one). If $t(x)$ is idempotent (which is the case in the examples we have), this property has a simple equational characterization because the identities in $\boldsymbol{A}$:

$$t(x \vee y) \approx t(t(x) \vee t(y))$$
$$t(x \wedge y) \approx t(t(x) \wedge t(y))$$
$$t(\neg x) \approx t(\neg t(x))$$
$$t(1) \approx 1$$

imply the congruence character of $Eq(t)$. They are actually equivalent to idempotency plus congruence. Therefore, if $\boldsymbol{A}$ interprets $\boldsymbol{B}$ via an idempotent term t, then any member of $\mathbf{V}(\boldsymbol{A})$ interprets a member of $\mathbf{V}(\boldsymbol{B})$ via t.

Remark 16. The conditions for nx in Example 7 are equational, except the inclusion $\mathbf{B}(\boldsymbol{A}) \subseteq n\boldsymbol{A}$ which is given by a quasiequation. Hence, any algebra in the variety $\mathbf{V}(\boldsymbol{A})$ generated by $\boldsymbol{A}$ interprets via $t(x) = nx$ a subalgebra of its Boolean skeleton, and any algebra in the quasivariety $\mathbf{Q}(\boldsymbol{A})$ interprets the full Boolean skeleton.

According to [8], all algebras of a variety $\mathbb{V}$ of MV-algebras interpret via a term their full skeleton if and only if $\mathbb{V}$ satisfies the equation $2x^2 \approx (2x)^2$, in which case $t(x) = 2x^2$ does the job. This is the case of the variety generated by the *Chang algebra*. However, this example is orthogonal to ours because the only non-trivial finite MV-algebra satisfying this equation is $\mathbf{2}$.

4.3. Goldblatt–Thomason theorem and related results

With the main theorems in hand, we are already in a position to state a Goldblatt–Thomason theorem for a large class of many-valued modal logics as a consequence of the classical theorem itself [14, Thm. 8] and Corollary 10.

Corollary 17 *(Finitely valued Goldblatt–Thomason Theorem). Let $\boldsymbol{A}$ be a finite lattice algebra which interprets via some term a Boolean algebra. Furthermore, let $\mathbb{F}$ be an elementary class of frames. Then, $\mathbb{F}$ is modally $\boldsymbol{A}$-definable by a set of formulas in $Fm_{\boldsymbol{A}}^{\Diamond\Box}(\tau)$ iff $\mathbb{F}$ is closed under taking generated subframes, disjoint unions, and bounded morphic images, and reflects ultrafilter extensions.*

It is known that the characterizing conditions of Corollary 17 could be weakened to closure under *ultrapowers* or under *ultrafilter extensions* (see [15]). Moreover, from the two theorems obtained by Van Benthem in [30, Section 4.2], we obtain the following additional characterizations of finite (transitive) definable frames:

Corollary 18. *Let $\boldsymbol{A}$ be a finite lattice algebra which interprets via some term a Boolean algebra. Then, a class $\mathbb{F}$ of finite frames is modally $\boldsymbol{A}$-definable in $Fm_{\boldsymbol{A}}^{\Diamond\Box}(\tau)$ iff it is closed under taking generated subframes, finite disjoint unions, and local p-morphic images.*

Corollary 19. *Let $\boldsymbol{A}$ be a finite lattice algebra which interprets via some term a Boolean algebra. Then, a class $\mathbb{F}$ of finite transitive frames is modally $\boldsymbol{A}$-definable in $Fm_{\boldsymbol{A}}^{\Diamond\Box}(\tau)$ iff it is closed under taking generated subframes, finite disjoint unions, and p-morphic images.*

Observe now that the translation in §3 may be extended to the first-order modal setting rather easily (after all, the semantics of $\Diamond$ and $\Box$ is similar to that of $\exists$ and $\forall$):

$$T^a(P_i^n) = P_i^{na} \quad (i \geq 1)$$

$$T^a(\exists x\, \psi) = \Big(\bigvee_{\substack{k \leq |A| \\ b_1 \ldots b_k \in A \\ b_1 \vee^{\boldsymbol{A}} \ldots \vee^{\boldsymbol{A}} b_k = a}} \bigwedge_{i=1}^{k} \exists x\, T^{b_i}(\psi)\Big) \wedge \forall x \Big(\bigvee_{\substack{b \in A \\ b \leq a}} T^b(\psi)\Big)$$

$$T^a(\forall x\, \psi) = \Big(\bigvee_{\substack{k \leq |A| \\ b_1, \ldots, b_k \in A \\ b_1 \wedge^{\boldsymbol{A}} \ldots \wedge^{\boldsymbol{A}} b_k = a}} \bigwedge_{i=1}^{k} \exists x\, T^{b_i}(\psi)\Big) \wedge \forall x \Big(\bigvee_{\substack{b \in A \\ a \leq b}} T^b(\psi)\Big).$$

Consequently, using [34, Thm. 3.6] (a Goldblatt–Thomason Theorem for first-order modal logic) and our method we could obtain:

Corollary 20. *Let $\boldsymbol{A}$ be a finite lattice algebra which interprets via some term a Boolean algebra. Let $\mathbb{F}$ be a class of frames closed under elementary equivalence. Then, $\mathbb{F}$ is modally $\boldsymbol{A}$-definable by a set of formulas in $Fm_{\boldsymbol{A}}^{\Diamond\Box\exists\forall}(\tau)$ iff it is closed under bounded morphic images, taking generated subframes, and disjoint unions.*

5. Computational complexity of many-valued consequence

Following an idea used in [11, Section 3.3], we may rewrite our translation from §3, using additional propositional letters, in such a way that the length of the translated formula becomes polynomial on the length of the original one. This technique will allow us to easily check that the complexity of the consequence and validity problems of many-valued modal logics coincides with that of their two-valued counterparts.

Let $\tau^* = \{q^a_\varphi \mid a \in A, \varphi \in Fm_{\boldsymbol{A}}^{\Diamond\Box}(\tau)\}$ and redefine $T^*(\tau)$ as

$$\bigvee_{a\in A} q^a_{p_i},\ \neg(q^a_{p_i} \wedge q^b_{p_i}) \qquad (a, b \in A, a \neq b, p_i \in \tau).$$

Then consider the theory $T^*(\tau) \cup \{E(\varphi) \mid \varphi \in Fm_{\boldsymbol{A}}^{\Diamond\Box}(\tau)\}$:

$$E(\circ(\psi_1,\dots,\psi_n)) := q^a_{\circ(\psi_1,\dots,\psi_n)} \quad\leftrightarrow\quad \bigvee_{\substack{b_1,\dots,b_n\in A\\ \circ^{\boldsymbol{A}}(b_1,\dots,b_n)=a}} (q^{b_1}_{\psi_1} \wedge \dots \wedge q^{b_n}_{\psi_n})$$

$$E(\Diamond\psi) := q^a_{\Diamond\psi} \quad\leftrightarrow\quad (\bigvee_{\substack{k\leq|A|\\ b_1,\dots,b_k\in A\\ b_1\vee^{\boldsymbol{A}}\dots\vee^{\boldsymbol{A}}b_k=a}} \bigwedge_{i=1}^{k} \Diamond q^{b_i}_\psi) \wedge \Box(\bigvee_{\substack{b\in A\\ b\leq a}} q^b_\psi)$$

$$E(\Box\psi) := q^a_{\Box\psi} \quad\leftrightarrow\quad (\bigvee_{\substack{k\leq|A|\\ b_1,\dots,b_k\in A\\ b_1\wedge^{\boldsymbol{A}}\dots\wedge^{\boldsymbol{A}}b_k=a}} \bigwedge_{i=1}^{k} \Diamond q^{b_i}_\psi) \wedge \Box(\bigvee_{\substack{b\in A\\ a\leq b}} q^b_\psi).$$

Observe that the length of these formulas is always bounded by $c2^{|A|}|A|$ where c is a constant number.

Using this theory, we can obtain a translation from many-valued to classical consequence:

Theorem 21. *Let $\boldsymbol{A}$ be a finite lattice algebra, τ a denumerable set of variables, and $\mathbb{F}$ a class of frames. Then, for each $\Gamma \cup \{\varphi\} \subseteq Fm_{\boldsymbol{A}}^{\Diamond\Box}(\tau)$, we have:*

$$\Gamma \vDash_{\mathrm{Log}(\mathbb{F},\boldsymbol{A},\tau)} \varphi \quad \textit{iff}$$
$$\{q^1_\theta \mid \theta\in\Gamma\} \cup T^*(\tau) \cup \{E(\psi) \mid \psi \in Fm_{\boldsymbol{A}}^{\Diamond\Box}(\tau)\} \vDash_{\mathrm{Log}(\mathbb{F},\mathbf{2},\tau^*)} q^1_\varphi.$$

Furthermore, if $\Gamma \cup \{\varphi\}$ is finite, the theory $T^*(\tau) \cup \{E(\psi) \mid \psi \in Fm_{\boldsymbol{A}}^{\Diamond\Box}(\tau)\}$ can be taken to be finite as well, involving only the relevant axioms for ψ being a subformula of some formula of $\Gamma \cup \{\varphi\}$. Then, as in Theorem 4, we have that

$$\vDash_{\mathrm{Log}(\mathbb{F},\boldsymbol{A},\tau)} \varphi \quad \textit{iff}$$
$$\vDash_{\mathrm{Log}(\mathbb{F},\mathbf{2},\tau^*)} (\bigwedge_{m\leq \mathrm{rank}(\varphi)} \Box^m(\bigwedge(T^*(\tau) \cup \{E(\psi) \mid \psi \textit{ subformula of } \varphi\})) \to q^1_\varphi.$$

Note that the combined length of the set of translated formulas in the first statement is polynomial w.r.t. the combined length of formulas in $\Gamma \cup \{\varphi\}$. Similarly, the length of the translated formula in the last statement is polynomial w.r.t. the length of φ (thanks, among others, to the bounded size observed above for the formulas $E(\psi)$), and thus $\boldsymbol{A}$-valued consequence and validity are polynomially reducible to their classical counterparts. Therefore, from Theorem 21 and the results from [7] for consequence in two-valued modal logic, the problem of consequence from a finite set of premises in many-valued modal logics over all crisp frames is decidable. Moreover, is in EXPTIME for the $\boldsymbol{A}$-valued analogues of the logics K, T, and B, in PSPACE for analogues of the logic S4, and in co-NP for the analogues of the logics KD45 and S5. Similarly, using the classical results from [20], we obtain that the problem of validity in many-valued modal logics over all crisp frames is in PSPACE for the $\boldsymbol{A}$-valued analogues of the logics K, T, B, and S4, and in co-NP for the analogues of KD45 and S5. Finally, thanks to the reverse translation for the many-valued modal logics in Corollary 14, we can conclude that all these computational problems are also complete in their corresponding complexity class. In particular, we have covered the complexity results in [3] for finitely valued Łukasiewicz modal logics through a completely different proof.

6. Conclusion

In this paper we have only scratched the surface of the potential of the translation introduced in §3. We believe that the application in obtaining Corollary 17 is quite a nice illustration of the power of this translation. We did not only provide an alternative to the rather complex proof from [26], but we also generalized the result to any finite residuated lattice that interprets a Boolean algebra by a term (the case of finite MV-algebras being just one example). Moreover, we also managed to prove some new Goldblatt–Thomason style results that were not considered in [26]. Observe that, due to the duality of the modalities $\Diamond$ and $\Box$ classically, the translation can be defined in the context of unimodal systems as well where we only have one of $\Diamond$ or $\Box$.

A similar translation to that in §3 can be offered for many-valued modal logics on frames with a many-valued relation by considering a suitable polymodal classical counterpart (we leave the details to the reader; the idea is to introduce classical modalities for each value of the accessibility relation). However, due to the added complexity introduced by the many-valued accessibility relation, we are not able to obtain an analogue of our main result with the help of such translation.

The more important open problem around Theorem 9, though, is whether all the conditions we have found are actually necessary. There are more general definitions of interpretability (see [19]) because the homomorphism condition is needed only to handle the modal operators. It would be interesting to explore these more general versions.

In future work, we intend to use the techniques in this paper to study other topics in finitely valued first-order and modal logic, such as 0-1 laws. Finally, we notice that the results obtained here might be relevant for the philosophical debate [24] around the so called "Suszko's thesis" [25], namely, that many-valued logics can be reduced to two-valued logic.

Declaration of competing interest

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