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Optimal embedding results for fractional Sobolev spaces

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Abstract

This paper deals with the fractional Sobolev spaces $W^{s,p}(\Omega)$, with $s \in (0,1]$ and $p \in [1,+\infty]$. Here, we use the interpolation results in [4] to provide suitable conditions on the exponents s and p so that the spaces $W^{s,p}(\Omega)$ realize a continuous embedding when either $\Omega = \mathbb{R}^N$ or Ω is any open and bounded domain with Lipschitz boundary. Our results enhance the classical continuous embedding and, when Ω is any open bounded domain with Lipschitz boundary, we also improve the classical compact embeddings. All the results stated here are proved to be optimal. Also, our strategy does not require the use of Besov or other interpolation spaces.

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1 Introduction and statement of the main results

In this paper we prove several results concerning the continuous and compact embeddings of fractional Sobolev spaces $W^{s,p}(\Omega)$, where $s \in (0, 1]$, $p \in [1, +\infty]$ and

$$\text{either } \Omega = \mathbb{R}^N \text{ or } \Omega \subset \mathbb{R}^N \text{ is an open and bounded domain with Lipschitz boundary.} \tag{1.1}$$

All of the embeddings provided here are also shown to be optimal. In particular, our work completes the classification of the fractional Sobolev embeddings by effectively proposing (depending on the interaction between the exponents s, p and N) a series of optimal embedding results.

We remark that all the proofs that we present here are self-contained. Moreover, we do not make use of Besov spaces, but rather we use the interpolation results provided by Brezis and Mironescu in [4] (we also refer the reader interested in Besov and Triebel-Lizorkin spaces to [12, 13, 18, 19]).

Besides, we point out that the present work refines some of the results on continuous embeddings obtained in [1] and [6]. For this, we provide some specific results of compact embeddings for the spaces $W^{s,p}(\Omega)$ which go beyond the classical ones e.g. in [7] and [15, Theorem 16.1].

The embedding results that we present here can be of particular interest in the study of problems involving fractional p-Laplacian operators. Indeed, they may enrich the literature and open up new scenarios. To make an example, one could be interested in reconsidering the results in [11] in light of the embeddings presented here and possibly choose some less restrictive assumptions.

To establish our main results, we recall (1.1) and we introduce the following notation. For $s \in [0, 1]$, $p \in [1, +\infty]$[1] and for any $u : \Omega \to \mathbb{R}$ measurable, we define

$$[u]_{W^{s,p}(\Omega)} := \begin{cases} \|u\|_{L^p(\Omega)} & \text{if } s = 0, \\ \left(c_{N,s,p} \iint_{\Omega\times\Omega} \dfrac{|u(x) - u(y)|^p}{|x - y|^{N+sp}}\, dx\, dy \right)^{1/p} & \text{if } s \in (0, 1), \\ \|\nabla u\|_{L^p(\Omega)} & \text{if } s = 1. \end{cases}$$

Here, $c_{N,s,p}$ denotes a normalizing constant, given explicitly by

$$c_{N,s,p} := \frac{s\, 2^{2s}\, \Gamma\left(\frac{ps+p+N-2}{2}\right)}{\pi^{N/2}\, \Gamma(1 - s)},$$

see e.g. [20, page 130] and the references therein.

The normalizing constant is determined in such a way that, for any u in the Schwartz space of rapidly decaying functions,

$$\lim_{s\searrow 0} [u]_{W^{s,p}(\Omega)} = [u]_{W^{0,p}(\Omega)} = \|u\|_{L^p(\Omega)} \quad \text{and} \quad \lim_{s\nearrow 1} [u]_{W^{s,p}(\Omega)} = [u]_{W^{1,p}(\Omega)} = \|\nabla u\|_{L^p(\Omega)},$$

[1] If $p = +\infty$ and $s \in (0, 1)$, then $[u]_{W^{s,p}(\Omega)} = [u]_{C^{0,s}(\Omega)}$.

see [2].

As customary, we define the fractional Sobolev space

$$W^{s,p}(\Omega) := \left\{ u \in L^p(\Omega) \text{ such that } [u]_{W^{s,p}(\Omega)} < +\infty \right\}$$

endowed with the norm

$$\|u\|_{W^{s,p}(\Omega)} := \begin{cases} \|u\|_{L^p(\Omega)} & \text{if } s = 0, \\ \left([u]^p_{W^{s,p}(\Omega)} + \|u\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}} & \text{if } s \in (0,1), \\ \left(\|\nabla u\|^p_{L^p(\Omega)} + \|u\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}} & \text{if } s = 1. \end{cases} \tag{1.2}$$

In the following, we will present our continuous and compact embedding results in detail, distinguishing between the different cases $sp < N$, $sp = N$ and $sp > n$. However, all the forthcoming results can be collected into a comprehensive statement:

Theorem 1.1 *Let $s \in [0,1]$ and $p \in [1, +\infty]$, with $s \neq p$. Let $\widetilde{s} \in [0, s)$ and $\widetilde{p} \in [1, +\infty]$. Then, the space $W^{s,p}(\Omega)$ is continuously embedded into $W^{\widetilde{s},\widetilde{p}}(\Omega)$ if and only if*

$$\frac{1}{\widetilde{p}} - \frac{\widetilde{s}}{N} \geqslant \frac{1}{p} - \frac{s}{N} \quad \text{and} \quad \text{either } \widetilde{p} \geqslant p \text{ or } \Omega \text{ is bounded.}$$

Moreover, the space $W^{s,p}(\Omega)$ is compactly embedded into $W^{\widetilde{s},\widetilde{p}}(\Omega)$ if and only if

$$\frac{1}{\widetilde{p}} - \frac{\widetilde{s}}{N} > \frac{1}{p} - \frac{s}{N} \quad \text{and } \Omega \text{ is bounded.}$$

We present our main results distinguishing three different cases.

1.1 The case $sp < N$

In this setting, when $\Omega = \mathbb{R}^N$, our main result reads as follows:

Theorem 1.2 *Let $s \in [0,1]$ and $p \in [1, +\infty)$ be such that $sp < N$ and $s \neq p$. Let $\widetilde{s}$ and $\widetilde{p}$ satisfy*

$$\begin{cases} 0 \leqslant \widetilde{s} \leqslant s, \\ p \leqslant \widetilde{p} \leqslant \frac{Np}{N-(s-\widetilde{s})p}. \end{cases} \tag{1.3}$$

Then, there exists a positive constant $C = C(N, s, p, \widetilde{s}, \widetilde{p})$ such that, for any $u \in W^{s,p}(\mathbb{R}^N)$,

$$\|u\|_{W^{\widetilde{s},\widetilde{p}}(\mathbb{R}^N)} \leqslant C \|u\|_{W^{s,p}(\mathbb{R}^N)}, \tag{1.4}$$

namely, the space $W^{s,p}(\mathbb{R}^N)$ is continuously embedded in $W^{\widetilde{s},\widetilde{p}}(\mathbb{R}^N)$.

Moreover, the curve $\gamma : [0,1] \to \mathbb{R}^2$ defined as

$$\gamma(\theta) := (s_\theta, p_\theta) = \left(\theta\widetilde{s} + (1-\theta)s, \frac{p\widetilde{p}}{\widetilde{p} + \theta(p - \widetilde{p})} \right) \tag{1.5}$$

is such that, if $0 \leqslant \theta_1 \leqslant \theta_2 \leqslant 1$, then the space $W^{s_{\theta_1}, p_{\theta_1}}(\mathbb{R}^N)$ is continuously embedded,[2] in $W^{s_{\theta_2}, p_{\theta_2}}(\mathbb{R}^N)$.

[2] In the spirit of [4, formula (1.5)] we point out that the same result holds replacing $\mathbb{R}^N$ with a halfspace.

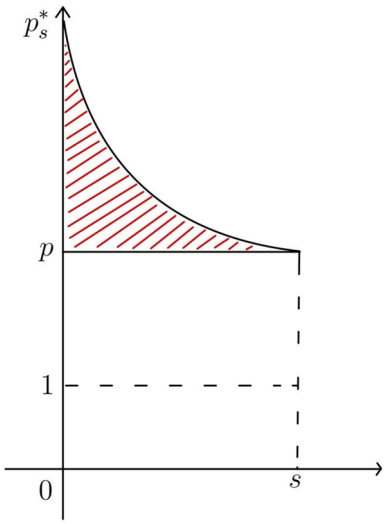


Fig. 1 The set of points satisfying (1.3)

The set of points satisfying (1.3) and realizing the embedding stated in Theorem 1.2 is illustrated in Fig. 1. Here p_s^* denotes the fractional Sobolev exponent and is defined as

$$p_s^* := \frac{Np}{N - sp}.$$

We point out that in Theorem 1.2 and in the forthcoming results, we will assume $s \neq p \neq 1$. When $s = p = 1$, the continuous embedding can be proved to hold if and only if either $1/\widetilde{p} - \widetilde{s}/N \geqslant 1 - 1/p$ and $N \geqslant 2$ or $1/\widetilde{p} - \widetilde{s}/N > 1 - 1/p$ and $N = 1$ (see [3, 6] [14, Theorem 7.31]).

When Ω is an open and bounded domain with Lipschitz boundary, we have the following:

Theorem 1.3 *Let $s \in (0, 1]$ and $p \in [1, +\infty)$ be such that $sp < N$ and $s \neq p$. Let $\Omega \subset \mathbb{R}^N$ be an open and bounded domain with Lipschitz boundary. Let $\widetilde{s}$ and $\widetilde{p}$ satisfy*[3]

$$\begin{cases} 0 \leqslant \widetilde{s} < s, \\ 1 \leqslant \widetilde{p} \leqslant \frac{Np}{N-(s-\widetilde{s})p}. \end{cases} \tag{1.6}$$

Then, there exists a positive constant $C = C(N, s, p, \widetilde{s}, \widetilde{p}, \Omega)$ such that, for any $u \in W^{s,p}(\Omega)$,

$$\|u\|_{W^{\widetilde{s},\widetilde{p}}(\Omega)} \leqslant C\|u\|_{W^{s,p}(\Omega)}, \tag{1.7}$$

namely, the space $W^{s,p}(\Omega)$ is continuously embedded in $W^{\widetilde{s},\widetilde{p}}(\Omega)$.

Moreover, the curve γ defined in (1.5) is such that, if $0 \leqslant \theta_1 \leqslant \theta_2 \leqslant 1$, then the space $W^{s_{\theta_1},p_{\theta_1}}(\Omega)$ is continuously embedded in $W^{s_{\theta_2},p_{\theta_2}}(\Omega)$.

We point out that Theorem 1.3 has been already proved in the case $\widetilde{p} = p = 2$ in [10, Lemma 2.1], and in the case $\widetilde{p} = p$ for any $p \in (1, N)$ in [9, Theorem 3.2].

The set of points satisfying (1.6) and realizing the embedding stated in Theorem 1.3 is visualized in Fig. 2.

[3] We point out that, if $\widetilde{s} = s$ and $\widetilde{p} = p$, then the result is trivial. Moreover, if $\widetilde{s} = s = 1$, then the desired result holds true for $1 \leqslant \widetilde{p} \leqslant p$.

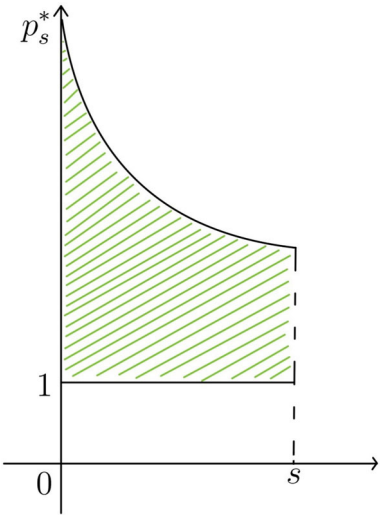


Fig. 2 The set of points satisfying (1.6)

Let us now provide the compact embeddings. On this matter, we have the following results:

Theorem 1.4 *Let $s \in (0, 1]$ and $p \in [1, +\infty)$ be such that $sp < N$ and $s \neq p$. Let $\Omega \subset \mathbb{R}^N$ be an open and bounded domain with Lipschitz boundary. Let $\widetilde{s}$ and $\widetilde{p}$ satisfy*

$$\begin{cases} 0 \leqslant \widetilde{s} < s, \\ \dfrac{sp}{sp - (p-1)\widetilde{s}} \leqslant \widetilde{p} < \frac{Np}{N-(s-\widetilde{s})p}. \end{cases} \tag{1.8}$$

Then, the space $W^{s,p}(\Omega)$ is compactly embedded in $W^{\widetilde{s},\widetilde{p}}(\Omega)$.

Moreover, the curve γ defined in (1.5) is such that, if $0 < \theta_1 < \theta_2 \leqslant 1$, then the space $W^{s_{\theta_1},p_{\theta_1}}(\Omega)$ is compactly embedded in $W^{s_{\theta_2},p_{\theta_2}}(\Omega)$.

Corollary 1.5 *Let $s \in (0, 1]$ and $p \in [1, +\infty)$ be such that $sp < N$ and $s \neq p$. Let $\Omega \subset \mathbb{R}^N$ be an open and bounded domain with Lipschitz boundary. Let $\widetilde{s}$ and $\widetilde{p}$ satisfy*

$$\begin{cases} 0 \leqslant \widetilde{s} < s, \\ 1 \leqslant \widetilde{p} < \frac{Np}{N-(s-\widetilde{s})p}. \end{cases} \tag{1.9}$$

Then, the space $W^{s,p}(\Omega)$ is compactly embedded in $W^{\widetilde{s},\widetilde{p}}(\Omega)$.

Moreover, the curve γ defined in (1.5) is such that, if $0 < \theta_1 < \theta_2 \leqslant 1$, then the space $W^{s_{\theta_1},p_{\theta_1}}(\Omega)$ is compactly embedded in $W^{s_{\theta_2},p_{\theta_2}}(\Omega)$.

The sets of points satisfying (1.8) and (1.9) are depicted in Fig. 3.

1.2 The case $sp = N$

In this case, if $\Omega = \mathbb{R}^N$ we have the following continuous embedding:

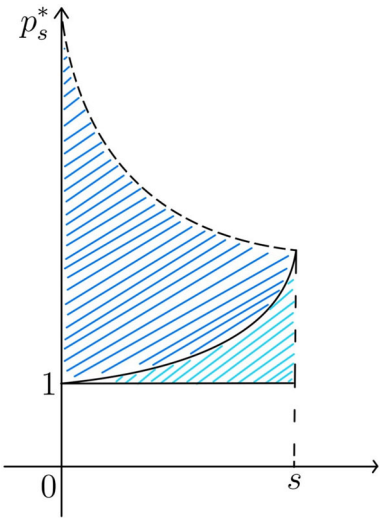


Fig. 3 The set of points satisfying (1.9). In dark blue, the set of points satisfying (1.8)

Theorem 1.6 *Let $s \in (0,1]$ and $p \in [1,+\infty)$ be such that $sp = N$ and $s \neq p$. Let $\widetilde{s}$ and $\widetilde{p}$ satisfy*

$$\text{either} \quad \begin{cases} 0 < \widetilde{s} \leqslant s, \\ p \leqslant \widetilde{p} \leqslant \dfrac{N}{\widetilde{s}}, \end{cases} \qquad \text{or} \quad \begin{cases} \widetilde{s} = 0, \\ p \leqslant \widetilde{p} < +\infty. \end{cases} \tag{1.10}$$

Then, there exists a positive constant $C = C(N,s,p,\widetilde{s},\widetilde{p})$ such that, for any $u \in W^{s,p}(\mathbb{R}^N)$,

$$\|u\|_{W^{\widetilde{s},\widetilde{p}}(\mathbb{R}^N)} \leqslant C\|u\|_{W^{s,p}(\mathbb{R}^N)}, \tag{1.11}$$

namely, the space $W^{s,p}(\mathbb{R}^N)$ is continuously embedded in $W^{\widetilde{s},\widetilde{p}}(\mathbb{R}^N)$.

Moreover, the curve γ defined in (1.5) is such that, if $0 \leqslant \theta_1 \leqslant \theta_2 \leqslant 1$, then the space $W^{s_{\theta_1},p_{\theta_1}}(\mathbb{R}^N)$ is continuously embedded in $W^{s_{\theta_2},p_{\theta_2}}(\mathbb{R}^N)$.

In addition, if $N = 1$, then the space $W^{1,1}(\mathbb{R}^N)$ is also continuously embedded in $L^\infty(\mathbb{R}^N)$.

We point out that, in this case, we can also provide a lower bound for the norm in (1.11). Indeed, by recalling the definition of the BMO norm in [17, Section 5], we have that the following result holds true:

Corollary 1.7 *Let $s \in (0,1]$ and $p \in [1,+\infty)$ be such that $sp = N$ and $s \neq p$.*

Then, for any $\widetilde{s}$ and $\widetilde{p}$ satisfying $\widetilde{s}\,\widetilde{p} = N$, there exist two positive constants $C_1 = C_1(N,\widetilde{s},\widetilde{p})$ and $C_2 = C_2(N,s,p,\widetilde{s},\widetilde{p})$ such that, for any $u \in W^{s,p}(\mathbb{R}^N)$,

$$\|u\|_{BMO} \leqslant C_1\|u\|_{W^{\widetilde{s},\widetilde{p}}(\mathbb{R}^N)} \leqslant C_2\|u\|_{W^{s,p}(\mathbb{R}^N)}. \tag{1.12}$$

In the case in which $\Omega \subset \mathbb{R}^N$ is an open and bounded domain with Lipschitz boundary, our main result reads as follows:

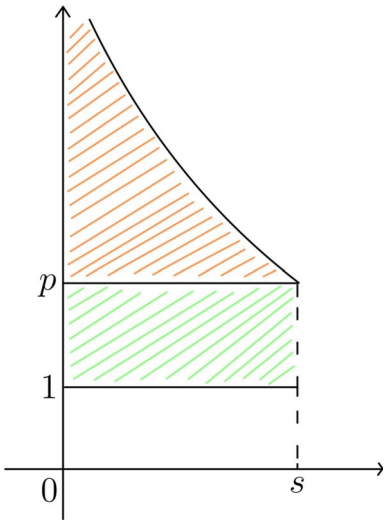


Fig. 4 In orange the set of points satisfying (1.10). In both orange and green the set of points satisfying (1.13). The set of point satisfying (1.20) coincides with the set of points satisfying (1.13) except for the curve $\widetilde{s}\,\widetilde{p} = N$

Theorem 1.8 *Let $s \in (0, 1]$ and $p \in [1, +\infty)$ be such that $sp = N$ and $s \neq p$. Let $\Omega \subset \mathbb{R}^N$ be an open and bounded domain with Lipschitz boundary. Let $\widetilde{s}$ and $\widetilde{p}$ satisfy*[4]

$$\text{either} \quad \begin{cases} 0 < \widetilde{s} < s, \\ 1 \leqslant \widetilde{p} \leqslant \dfrac{N}{\widetilde{s}}, \end{cases} \qquad \text{or} \quad \begin{cases} \widetilde{s} = 0, \\ 1 \leqslant \widetilde{p} < +\infty, \end{cases} \tag{1.13}$$

Then, there exists a positive constant $C = C(N, s, p, \widetilde{s}, \widetilde{p}, \Omega)$ such that, for any $u \in W^{s,p}(\Omega)$,

$$\|u\|_{W^{\widetilde{s},\widetilde{p}}(\Omega)} \leqslant C \|u\|_{W^{s,p}(\Omega)}, \tag{1.14}$$

namely, the space $W^{s,p}(\Omega)$ is continuously embedded in $W^{\widetilde{s},\widetilde{p}}(\Omega)$.

Moreover, the curve γ defined in (1.5) is such that, if $0 \leqslant \theta_1 \leqslant \theta_2 \leqslant 1$, then the space $W^{s_{\theta_1},p_{\theta_1}}(\Omega)$ is continuously embedded in $W^{s_{\theta_2},p_{\theta_2}}(\Omega)$.

In addition, if $N = 1$, then the space $W^{1,1}(\Omega)$ is also continuously embedded in $L^\infty(\Omega)$.

Moreover, the following compact embeddings hold true:

Theorem 1.9 *Let $s \in (0, 1]$ and $p \in [1, +\infty)$ be such that $sp = N$ and $s \neq p$. Let $\Omega \subset \mathbb{R}^N$ be an open and bounded domain with Lipschitz boundary. Let $\widetilde{s}$ and $\widetilde{p}$ satisfy*

$$\text{either} \quad \begin{cases} 0 < \widetilde{s} < s, \\ 1 \leqslant \widetilde{p} < \dfrac{N}{\widetilde{s}}, \end{cases} \qquad \text{or} \quad \begin{cases} \widetilde{s} = 0, \\ 1 \leqslant \widetilde{p} < +\infty. \end{cases} \tag{1.15}$$

Then, the space $W^{s,p}(\Omega)$ is compactly embedded in $W^{\widetilde{s},\widetilde{p}}(\Omega)$.

Moreover, the curve γ defined in (1.5) is such that, if $0 < \theta_1 < \theta_2 \leqslant 1$, then the space $W^{s_{\theta_1},p_{\theta_1}}(\Omega)$ is compactly embedded in $W^{s_{\theta_2},p_{\theta_2}}(\Omega)$.

The set of point satisfying (1.10), (1.13) and (1.15) is illustrated in Fig. 4.

[4] See also footnote 3.

1.3 The case $sp > N$

When $\Omega = \mathbb{R}^N$, our main result reads as follows:

Theorem 1.10 *Let $s \in (0,1]$ and $p \in [1,+\infty]$ be such that $sp > N$. Let $\widetilde{s}$ and $\widetilde{p}$ satisfy*[5]

$$\text{either} \quad \begin{cases} 0 \leqslant \widetilde{s} \leqslant \dfrac{sp-N}{p}, \\ p \leqslant \widetilde{p} \leqslant +\infty, \end{cases} \qquad \text{or} \quad \begin{cases} \dfrac{sp-N}{p} < \widetilde{s} \leqslant s, \\ p \leqslant \widetilde{p} \leqslant \dfrac{Np}{N-(s-\widetilde{s})p}. \end{cases} \tag{1.16}$$

Then, there exists a positive constant $C = C(N,s,p,\widetilde{s},\widetilde{p})$ such that, for any $u \in W^{s,p}(\mathbb{R}^N)$,

$$\|u\|_{W^{\widetilde{s},\widetilde{p}}(\mathbb{R}^N)} \leqslant C\|u\|_{W^{s,p}(\mathbb{R}^N)}, \tag{1.17}$$

namely, the space $W^{s,p}(\mathbb{R}^N)$ is continuously embedded in $W^{\widetilde{s},\widetilde{p}}(\mathbb{R}^N)$.

Moreover, the curve γ defined in (1.5) is such that, if $0 \leqslant \theta_1 \leqslant \theta_2 \leqslant 1$, then the space $W^{s_{\theta_1},p_{\theta_1}}(\mathbb{R}^N)$ is continuously embedded in $W^{s_{\theta_2},p_{\theta_2}}(\mathbb{R}^N)$.

When $\Omega \subset \mathbb{R}^N$ is an open and bounded domain with Lipschitz boundary, we have the following:

Theorem 1.11 *Let $s \in (0,1]$ and $p \in [1,+\infty]$ be such that $sp > N$. Let $\Omega \subset \mathbb{R}^N$ be an open and bounded domain with Lipschitz boundary. Let $\widetilde{s}$ and $\widetilde{p}$ satisfy*[6]

$$\text{either} \quad \begin{cases} 0 \leqslant \widetilde{s} \leqslant \dfrac{sp-N}{p}, \\ 1 \leqslant \widetilde{p} \leqslant +\infty, \end{cases} \qquad \text{or} \quad \begin{cases} \dfrac{sp-N}{p} < \widetilde{s} < s, \\ 1 \leqslant \widetilde{p} \leqslant \dfrac{Np}{N-(s-\widetilde{s})p}. \end{cases} \tag{1.18}$$

Then, there exists a positive constant $C = C(N,s,p,\widetilde{s},\widetilde{p},\Omega)$ such that, for any $u \in W^{s,p}(\Omega)$,

$$\|u\|_{W^{\widetilde{s},\widetilde{p}}(\Omega)} \leqslant C\|u\|_{W^{s,p}(\Omega)}, \tag{1.19}$$

namely, the space $W^{s,p}(\Omega)$ is continuously embedded in $W^{\widetilde{s},\widetilde{p}}(\Omega)$.

Moreover, the curve γ defined in (1.5) is such that, if $0 \leqslant \theta_1 \leqslant \theta_2 \leqslant 1$, then the space $W^{s_{\theta_1},p_{\theta_1}}(\Omega)$ is continuously embedded in $W^{s_{\theta_2},p_{\theta_2}}(\Omega)$.

The following compact embedding holds true:

Theorem 1.12 *Let $s \in (0,1]$ and $p \in [1,+\infty]$ be such that $sp > N$. Let $\Omega \subset \mathbb{R}^N$ be an open and bounded domain with Lipschitz boundary. Let $\widetilde{s}$ and $\widetilde{p}$ satisfy*[7]

$$\text{either} \quad \begin{cases} 0 \leqslant \widetilde{s} \leqslant \dfrac{sp-N}{p}, \\ 1 \leqslant \widetilde{p} \leqslant +\infty, \end{cases} \qquad \text{or} \quad \begin{cases} \dfrac{sp-N}{p} < \widetilde{s} < s, \\ 1 \leqslant \widetilde{p} < \dfrac{Np}{N-(s-\widetilde{s})p}. \end{cases} \tag{1.20}$$

Then, the space $W^{s,p}(\Omega)$ is compactly embedded in $W^{\widetilde{s},\widetilde{p}}(\Omega)$.

Moreover, the curve γ defined in (1.5) is such that, if $0 < \theta_1 < \theta_2 \leqslant 1$, then the space $W^{s_{\theta_1},p_{\theta_1}}(\Omega)$ is compactly embedded in $W^{s_{\theta_2},p_{\theta_2}}(\Omega)$.

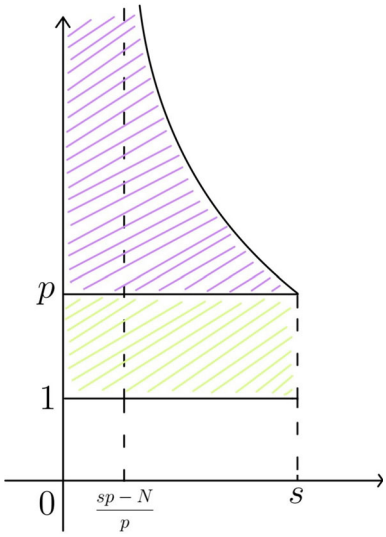


Fig. 5 In purple the set of points satisfying (1.16). In both purple and yellow the set of points satisfying (1.18). The set of points satisfying (1.15) coincides with the set of points satisfying (1.18) except for the curve $\widetilde{p} = \frac{Np}{N-(s-\widetilde{s})p}$. When $p = +\infty$, the set of points satisfying (1.16) reduces to the line $\widetilde{p} = +\infty$ and $0 \leqslant \widetilde{s} \leqslant s$

The set of point satisfying (1.10), (1.18) and (1.20) is illustrated in Fig. 5.

For the sake of completeness, we point out that some of the results we prove here can be deduced from the theory of Triebel-Lizorkin spaces. Indeed, when $\Omega = \mathbb{R}^N$, if $s \in (0, 1)$ and $\overline{s} \in (0, s]$ satisfies

$$\frac{1}{\widetilde{p}} - \frac{\overline{s}}{N} = \frac{1}{p} - \frac{s}{N},$$

then the continuous embedding of $W^{s,p}(\mathbb{R}^N)$ into $W^{\overline{s},\widetilde{p}}(\mathbb{R}^N)$ can be derived from the embeddings between Besov spaces stated in [19, Theorem 2.8.1 (a)] and [18, Theorem 2.7/1 (1)]. When $s = 1$, the continuous embedding of $W^{1,p}(\mathbb{R}^N)$ into $W^{\overline{s},\widetilde{p}}(\mathbb{R}^N)$ can be deduced from the embeddings between Triebel-Lizorkin spaces and the identity between Triebel-Lizorkin spaces and the fractional Sobolev spaces (see [18]). Moreover, if $\overline{s} \in (\widetilde{s}, s)$, then the continuous embedding of $W^{\widetilde{s},\widetilde{p}}(\mathbb{R}^N)$ into $W^{\widetilde{s},\widetilde{p}}(\mathbb{R}^N)$ can be derived from the embeddings between Besov spaces stated in [18, Proposition 2.3/2 (5)]. We also refer the interested reader to [14, Section 7].

In addition, when Ω is bounded with Lipschitz boundary, the continuous embeddings can be inferred by [19, Theorem 3.3.1 and Theorem 3.4.2].

The compact embeddings are also discussed in [12, 13].

However, the strength of our approach lies in the fact that each result we present here is fully self-contained and no prior definitions or cross-referencing with Besov and/or Triebel-

[5] We allow $\widetilde{p} = +\infty$ in (1.16), meaning that $W^{\widetilde{s},\infty}(\mathbb{R}^N) = C^{0,\widetilde{s}}(\mathbb{R}^N)$ (see the comment after Remark 8.3 in [7]). Moreover, when $p = +\infty$, we have $\widetilde{p} = +\infty$ and $0 \leqslant \widetilde{s} \leqslant s$.

[6] See also footnote 3. Moreover, when $p = +\infty$, we have $1 \leqslant \widetilde{p} \leqslant +\infty$ and $0 \leqslant \widetilde{s} \leqslant s$.

[7] When $p = +\infty$, we have $1 \leqslant \widetilde{p} \leqslant +\infty$ and $0 \leqslant \widetilde{s} < s$.

Lizorkin spaces is necessary. We believe that this is a significant advantage for anyone seeking a comprehensive understanding of the embedding between fractional Sobolev spaces.

In addition, our analysis is not limited to necessary and sufficient conditions on the exponents s, p, $\widetilde{s}$, $\widetilde{p}$ to have that the space $W^{s,p}(\Omega)$ is continuously (compactly) embedded into $W^{\widetilde{s},\widetilde{p}}(\Omega)$. Indeed, we also provide a curve of exponents s_θ, p_θ (see the definition in (1.5)) such that, if $\theta_1 \leqslant \theta_2$, then the space $W^{s_{\theta_1},p_{\theta_1}}(\Omega)$ is continuously (compactly) embedded in $W^{s_{\theta_2},p_{\theta_2}}(\Omega)$. This means that between $W^{s,p}(\Omega)$ and $W^{\widetilde{s},\widetilde{p}}(\Omega)$ there is a family of fractional Sobolev spaces $W^{s_\theta,p_\theta}(\Omega)$, increasing in θ in the sense of the embeddings, connecting them in a "continuous way". This establishes an interesting insight on how fractional spaces are linked the one to the other.

1.4 Organization of the paper

The paper is organized as follows. In Section 2 we discuss the assumptions in Theorems 1.2, 1.3, 1.6, 1.8, 1.9, 1.10, 1.11 and 1.12 and Corollary 1.5, showing that they are optimal. In Sect. 3 we provide some auxiliary results needed for the proofs of the continuous and compact embeddings. Finally, in Sect. 4 we prove our embedding results by addressing separately the cases $sp < N$, $sp = N$ and $sp > N$.

2 Optimality of the embeddings

In this section we discuss the optimality of the results stated in this paper. To this end, we will present some preliminary results which show that, with appropriate choices of $\widetilde{s}$ and $\widetilde{p}$, the space $W^{s,p}(\Omega)$ is not continuously embedded in $W^{\widetilde{s},\widetilde{p}}(\Omega)$, being Ω as in (1.1).

Throughout the rest of the section, we recall that we use the notations $W^{s,\infty}(\Omega) = C^{0,s}(\Omega)$ and $W^{0,\infty}(\Omega) = L^\infty(\Omega)$.

2.1 Some scaling properties

To start with, we provide some useful scaling properties:

Lemma 2.1 *Let $u : \mathbb{R}^N \to \mathbb{R}$ be any measurable function and γ, $\beta \in \mathbb{R}$. For any $\varepsilon \in (0,1)$, let $v_{\varepsilon,\gamma,\beta}(x) := \varepsilon^\gamma u(\varepsilon^\beta x)$.*

Then, for any $s \in (0,1]$ and $p \in [1,+\infty)$,

$$\|v_{\varepsilon,\gamma,\beta}\|_{W^{s,p}(\mathbb{R}^N)} = \varepsilon^{\gamma-\frac{\beta N}{p}+\beta s}\left(\varepsilon^{-\beta sp}\|u\|^p_{L^p(\mathbb{R}^N)} + [u]^p_{W^{s,p}(\mathbb{R}^N)}\right)^{\frac{1}{p}}. \tag{2.1}$$

Furthermore, if $p = +\infty$, for any $s \in (0,1)$,

$$\|v_{\varepsilon,\gamma,\beta}\|_{W^{s,p}(\mathbb{R}^N)} = \|v_{\varepsilon,\gamma,\beta}\|_{C^{0,s}(\mathbb{R}^N)} = \varepsilon^\gamma \|u\|_{L^\infty(\mathbb{R}^N)} + \varepsilon^{\gamma+\beta s} \sup_{\substack{x,y\in\mathbb{R}^N \\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^s}. \tag{2.2}$$

Proof We first prove (2.1). A suitable change of variables leads to

$$\|v_{\varepsilon,\gamma,\beta}\|^p_{L^p(\mathbb{R}^N)} = \varepsilon^{\gamma p}\int_{\mathbb{R}^N} |u(\varepsilon^\beta x)|^p\,dx = \varepsilon^{\gamma p-\beta N}\|u\|^p_{L^p(\mathbb{R}^N)}$$

and

$$[v_{\varepsilon,\gamma,\beta}]^p_{W^{s,p}(\mathbb{R}^N)} = c_{N,s,p}\ \varepsilon^{\gamma p} \iint_{\mathbb{R}^{2N}} \frac{|u(\varepsilon^\beta x) - u(\varepsilon^\beta y)|^p}{|x-y|^{N+sp}}\, dx\, dy = \varepsilon^{\gamma p - \beta N + \beta s p} [u]^p_{W^{s,p}(\mathbb{R}^N)}. \tag{2.3}$$

Recalling the definition in (1.2), we get (2.1), as desired.

In order to prove (2.2), we observe that

$$\|v_{\varepsilon,\gamma,\beta}\|_{L^\infty(\mathbb{R}^N)} = \varepsilon^\gamma \|u\|_{L^\infty(\mathbb{R}^N)} \tag{2.4}$$

and

$$\begin{aligned} \sup_{\substack{x,y\in\mathbb{R}^N \\ x\neq y}} \frac{|v_{\varepsilon,\gamma,\beta}(x) - v_{\varepsilon,\gamma,\beta}(y)|}{|x-y|^s} &= \varepsilon^{\gamma+\beta s} \sup_{\substack{x,y\in\mathbb{R}^N \\ x\neq y}} \frac{|u(\varepsilon^\beta x) - u(\varepsilon^\beta y)|}{|\varepsilon^\beta x - \varepsilon^\beta y|^s} \\ &= \varepsilon^{\gamma+\beta s} \sup_{\substack{x,y\in\mathbb{R}^N \\ x\neq y}} \frac{|u(x) - u(y)|}{|x-y|^s}, \end{aligned} \tag{2.5}$$

as desired. The result in the case $s = 1$ comes from a direct computation. □

Lemma 2.2 *Let $\Omega \subset \mathbb{R}^N$ be an open and bounded domain with Lipschitz boundary. Let $x_0 \in \Omega$ and $R > 0$ be such that $B_R(x_0) \subset \Omega$. Let $\gamma \in \mathbb{R}$.*

Let $u : \mathbb{R}^N \to \mathbb{R}$ be any measurable function such that $u = 0$ in $\mathbb{R}^N \setminus B_R(x_0)$.

For any $\varepsilon \in (0, 1)$, let

$$v_{\varepsilon,\gamma}(x) := \varepsilon^\gamma u\left(x_0 + \frac{x}{\varepsilon}\right).$$

Then, for any $s \in (0, 1]$ and $p \in [1, +\infty)$,

$$\varepsilon^{\gamma p + N - sp} [u]^p_{W^{s,p}(B_R(x_0))} \leqslant [v_{\varepsilon,\gamma}]^p_{W^{s,p}(\Omega)} \leqslant \varepsilon^{\gamma p + N - sp} [u]^p_{W^{s,p}(\mathbb{R}^N)} \tag{2.6}$$

and

$$\begin{aligned} \varepsilon^{\gamma + \frac{N}{p} - s} \left(\varepsilon^{sp} \|u\|^p_{L^p(\Omega)} + [u]^p_{W^{s,p}(B_R(x_0))}\right)^{\frac{1}{p}} &\leqslant \|v_{\varepsilon,\gamma}\|_{W^{s,p}(\Omega)} \\ &\leqslant \varepsilon^{\gamma + \frac{N}{p} - s} \left(\varepsilon^{sp} \|u\|^p_{L^p(\Omega)} + [u]^p_{W^{s,p}(\mathbb{R}^N)}\right)^{\frac{1}{p}}. \end{aligned} \tag{2.7}$$

Furthermore, if $p = +\infty$, for any $s \in (0, 1)$,

$$\varepsilon^{\gamma - s} \sup_{\substack{x,y\in B_R(x_0) \\ x\neq y}} \frac{|u(x) - u(y)|}{|x-y|^s} \leqslant \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|v_{\varepsilon,\gamma}(x) - v_{\varepsilon,\gamma}(y)|}{|x-y|^s} \leqslant \varepsilon^{\gamma - s} \sup_{\substack{x,y\in\mathbb{R}^N \\ x\neq y}} \frac{|u(x) - u(y)|}{|x-y|^s} \tag{2.8}$$

and

$$\begin{aligned} &\varepsilon^\gamma \|u\|_{L^\infty(\Omega)} + \varepsilon^{\gamma - s} \sup_{\substack{x,y\in B_R(x_0) \\ x\neq y}} \frac{|u(x) - u(y)|}{|x-y|^s} \\ &\leqslant \|v_{\varepsilon,\gamma}\|_{W^{s,p}(\Omega)} = \|v_{\varepsilon,\gamma}\|_{C^{0,s}(\Omega)} \\ &\leqslant \varepsilon^\gamma \|u\|_{L^\infty(\Omega)} + \varepsilon^{\gamma - s} \sup_{\substack{x,y\in\mathbb{R}^N \\ x\neq y}} \frac{|u(x) - u(y)|}{|x-y|^s}. \end{aligned} \tag{2.9}$$

Proof Up to a translation, we can suppose that $x_0 = 0$. With this, we have that

$$\operatorname{supp}(v_{\varepsilon,\gamma}) \subseteq B_{\varepsilon R} \subset B_R \subset \Omega.$$

Therefore,

$$\begin{aligned} \|v_{\varepsilon,\gamma}\|^p_{L^p(\Omega)} &= \int_{B_{\varepsilon R}} |v_{\varepsilon,\gamma}(x)|^p\,dx = \varepsilon^{\gamma p} \int_{B_{\varepsilon R}} \left|u\left(\frac{x}{\varepsilon}\right)\right|^p\,dx \\ &= \varepsilon^{\gamma p+N} \int_{B_R} |u(y)|^p\,dy = \varepsilon^{\gamma p+N} \|u\|^p_{L^p(\Omega)}. \end{aligned} \tag{2.10}$$

Moreover,

$$\begin{aligned} [v_{\varepsilon,\gamma}]^p_{W^{s,p}(\Omega)} &= c_{N,s,p} \iint_{\Omega\times\Omega} \frac{|v_{\varepsilon,\gamma}(x) - v_{\varepsilon,\gamma}(y)|^p}{|x-y|^{N+sp}}\,dx\,dy \\ &= c_{N,s,p} \iint_{B_{\varepsilon R}\times B_{\varepsilon R}} \frac{|v_{\varepsilon,\gamma}(x) - v_{\varepsilon,\gamma}(y)|^p}{|x-y|^{N+sp}}\,dx\,dy \\ &\quad + 2c_{N,s,p} \iint_{B_{\varepsilon R}\times(\Omega\setminus B_{\varepsilon R})} \frac{|v_{\varepsilon,\gamma}(x)|^p}{|x-y|^{N+sp}}\,dx\,dy \\ &\geqslant c_{N,s,p}\ \varepsilon^{\gamma p} \iint_{B_{\varepsilon R}\times B_{\varepsilon R}} \frac{\left|u\left(\frac{x}{\varepsilon}\right) - u\left(\frac{y}{\varepsilon}\right)\right|^p}{|x-y|^{N+sp}}\,dx\,dy \\ &= \varepsilon^{\gamma p+N-sp} [u]^p_{W^{s,p}(B_R)}. \end{aligned} \tag{2.11}$$

Also, by (2.3), used here with $\beta := -1$, we get

$$[v_{\varepsilon,\gamma}]^p_{W^{s,p}(\Omega)} \leqslant [v_{\varepsilon,\gamma}]^p_{W^{s,p}(\mathbb{R}^N)} = \varepsilon^{\gamma p+N-sp} [u]^p_{W^{s,p}(\mathbb{R}^N)}.$$

Combining this with (2.11) we obtain (2.6).

Thus, (2.7) follows from (2.6) and (2.10). In the case $s = 1$, the proof of (2.6) and (2.7) comes from a direct computation.

Furthermore,

$$\begin{aligned} \sup_{\substack{x,y\in\Omega\\ x\neq y}} \frac{|v_{\varepsilon,\gamma}(x) - v_{\varepsilon,\gamma}(y)|}{|x-y|^s} &= \varepsilon^{\gamma} \sup_{\substack{x,y\in\Omega\\ x\neq y}} \frac{\left|u\left(\frac{x}{\varepsilon}\right) - u\left(\frac{y}{\varepsilon}\right)\right|}{|x-y|^s} \geqslant \varepsilon^{\gamma} \sup_{\substack{x,y\in B_{\varepsilon R}\\ x\neq y}} \frac{\left|u\left(\frac{x}{\varepsilon}\right) - u\left(\frac{y}{\varepsilon}\right)\right|}{|x-y|^s} \\ &= \varepsilon^{\gamma-s} \sup_{\substack{x,y\in B_R\\ x\neq y}} \frac{|u(x) - u(y)|}{|x-y|^s}. \end{aligned} \tag{2.12}$$

Also, from (2.5) with $\beta := -1$, we get

$$\sup_{\substack{x,y\in\Omega\\ x\neq y}} \frac{|v_{\varepsilon,\gamma}(x) - v_{\varepsilon,\gamma}(y)|}{|x-y|^s} \leqslant \sup_{\substack{x,y\in\mathbb{R}^N\\ x\neq y}} \frac{|v_{\varepsilon,\gamma}(x) - v_{\varepsilon,\gamma}(y)|}{|x-y|^s} = \varepsilon^{\gamma-s} \sup_{\substack{x,y\in\mathbb{R}^N\\ x\neq y}} \frac{|u(x) - u(y)|}{|x-y|^s}.$$

From this and (2.12), we obtain (2.8).

Also, (2.9) follows from (2.4) and (2.8). □

2.2 Towards the optimality statements

With this preliminary work, we can now establish some optimality results.

Lemma 2.3 *Let $s \in (0,1]$ and $p \in [1,+\infty)$. Let $\widetilde{s}$ and $\widetilde{p}$ satisfy*

$$\begin{cases} 0 \leqslant \widetilde{s} \leqslant 1, \\ 1 \leqslant \widetilde{p} < p. \end{cases} \tag{2.13}$$

Then, the space $W^{s,p}(\mathbb{R}^N)$ is not continuously embedded in $W^{\widetilde{s},\widetilde{p}}(\mathbb{R}^N)$.

Proof Suppose by contradiction that there exists $C > 0$ such that, for all $u \in W^{s,p}(\mathbb{R}^N)$,

$$\|u\|_{W^{\widetilde{s},\widetilde{p}}(\mathbb{R}^N)} \leqslant C\|u\|_{W^{s,p}(\mathbb{R}^N)}. \tag{2.14}$$

Let $u \in W^{s,p}(\mathbb{R}^N) \setminus \{0\}$ and, for any $\varepsilon \in (0,1)$, define the function

$$v_\varepsilon(x) := \varepsilon^{\frac{N}{p}} u(\varepsilon x).$$

By (2.1) (used here with $\gamma := N/p$ and $\beta := 1$), we get that

$$\|v_\varepsilon\|_{W^{s,p}(\mathbb{R}^N)} = \left(\|u\|^p_{L^p(\mathbb{R}^N)} + \varepsilon^{sp}\,[u]^p_{W^{s,p}(\mathbb{R}^N)}\right)^{\frac{1}{p}}$$

and

$$\|v_\varepsilon\|_{W^{\widetilde{s},\widetilde{p}}(\mathbb{R}^N)} = \varepsilon^{-N\left(\frac{1}{\widetilde{p}}-\frac{1}{p}\right)}\left(\|u\|^{\widetilde{p}}_{L^{\widetilde{p}}(\mathbb{R}^N)} + \varepsilon^{\widetilde{s}\widetilde{p}}\,[u]^{\widetilde{p}}_{W^{\widetilde{s},\widetilde{p}}(\mathbb{R}^N)}\right)^{\frac{1}{\widetilde{p}}}.$$

Thus, plugging v_ε into (2.14), we obtain that

$$\varepsilon^{-N\left(\frac{1}{\widetilde{p}}-\frac{1}{p}\right)}\left(\|u\|^{\widetilde{p}}_{L^{\widetilde{p}}(\mathbb{R}^N)} + \varepsilon^{\widetilde{s}\widetilde{p}}\,[u]^{\widetilde{p}}_{W^{\widetilde{s},\widetilde{p}}(\mathbb{R}^N)}\right)^{\frac{1}{\widetilde{p}}} \leqslant C\left(\|u\|^p_{L^p(\mathbb{R}^N)} + \varepsilon^{sp}\,[u]^p_{W^{s,p}(\mathbb{R}^N)}\right)^{\frac{1}{p}}.$$

Taking the limit as $\varepsilon \searrow 0$ and recalling that $\widetilde{p} < p$, we obtain the desired contradiction. □

We now address separately the cases $sp < N$, $sp = N$ and $sp > N$. When $sp < N$ and Ω is as in (1.1), we have the following:

Lemma 2.4 *Let $s \in (0,1]$ and $p \in [1,+\infty)$ be such that $sp < N$. Let Ω be as in (1.1) and assume that $\widetilde{s}$ and $\widetilde{p}$ satisfy*

$$\text{either} \quad \begin{cases} s < \widetilde{s} \leqslant 1, \\ p \leqslant \widetilde{p} \leqslant +\infty, \end{cases} \qquad \text{or} \quad \begin{cases} 0 \leqslant \widetilde{s} \leqslant s, \\ \frac{Np}{N-(s-\widetilde{s})p} < \widetilde{p} \leqslant +\infty. \end{cases} \tag{2.15}$$

Then, the space $W^{s,p}(\Omega)$ is not continuously embedded in $W^{\widetilde{s},\widetilde{p}}(\Omega)$.

Proof We first deal with the case $\Omega = \mathbb{R}^N$. Also, we first consider the case $\widetilde{p} = +\infty$. Suppose by contradiction that there exists $C > 0$ such that, for all $u \in W^{s,p}(\mathbb{R}^N)$,

$$\|u\|_{C^{0,\widetilde{s}}(\mathbb{R}^N)} \leqslant C\|u\|_{W^{s,p}(\mathbb{R}^N)}. \tag{2.16}$$

Let $u \in W^{s,p}(\mathbb{R}^N) \setminus \{0\}$ and, for any $\varepsilon \in (0,1)$, define the function

$$v_\varepsilon(x) := \varepsilon^{-\frac{N-sp}{p}} u\left(\frac{x}{\varepsilon}\right). \tag{2.17}$$

Then, by (2.1) and (2.2) (used here with $\gamma := -(N-sp)/p$ and $\beta := -1$), we have that

$$\|v_\varepsilon\|_{W^{s,p}(\mathbb{R}^N)} = \left(\varepsilon^{sp}\,\|u\|^p_{L^p(\mathbb{R}^N)} + [u]^p_{W^{s,p}(\mathbb{R}^N)}\right)^{\frac{1}{p}} \tag{2.18}$$

and

$$\|v_\varepsilon\|_{C^{0,\widetilde{s}}(\mathbb{R}^N)} = \varepsilon^{-\frac{N-sp}{p}} \|u\|_{L^\infty(\mathbb{R}^N)} + \varepsilon^{-\frac{N-sp}{p}-\widetilde{s}} \sup_{\substack{x,y\in\mathbb{R}^N \\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^{\widetilde{s}}}.$$

As a result, plugging v_ε into (2.16), we obtain that

$$\varepsilon^{-\frac{N-sp}{p}} \left(\|u\|_{L^\infty(\mathbb{R}^N)} + \varepsilon^{-\widetilde{s}} \sup_{\substack{x,y\in\mathbb{R}^N \\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^{\widetilde{s}}} \right) \leqslant C \left(\varepsilon^{sp} \|u\|^p_{L^p(\mathbb{R}^N)} + [u]^p_{W^{s,p}(\mathbb{R}^N)} \right)^{\frac{1}{p}}.$$

Thus, taking the limit as $\varepsilon \searrow 0$ we get a contradiction.

Now, we assume $\widetilde{p} \neq +\infty$. Suppose by contradiction that there exists $C > 0$ such that, for all $u \in W^{s,p}(\mathbb{R}^N)$,

$$\|u\|_{W^{\widetilde{s},\widetilde{p}}(\mathbb{R}^N)} \leqslant C \|u\|_{W^{s,p}(\mathbb{R}^N)}. \tag{2.19}$$

Let $u \in W^{s,p}(\mathbb{R}^N) \setminus \{0\}$ and, for any $\varepsilon \in (0,1)$, define the function v_ε as in (2.17). By (2.1) (used here $\gamma := -(N-sp)/p$ and $\beta := -1$), we infer that

$$\|v_\varepsilon\|_{W^{\widetilde{s},\widetilde{p}}(\mathbb{R}^N)} = \varepsilon^{-\frac{N}{p}+\frac{N}{\widetilde{p}}+s-\widetilde{s}} \left(\varepsilon^{\widetilde{s}\widetilde{p}} \|u\|^{\widetilde{p}}_{L^{\widetilde{p}}(\mathbb{R}^N)} + [u]^{\widetilde{p}}_{W^{\widetilde{s},\widetilde{p}}(\mathbb{R}^N)} \right)^{\frac{1}{\widetilde{p}}}.$$

Thus, plugging v_ε into (2.19) and recalling also (2.18),

$$\varepsilon^{-\frac{N}{p}+\frac{N}{\widetilde{p}}+s-\widetilde{s}} \left(\varepsilon^{\widetilde{s}\widetilde{p}} \|u\|^{\widetilde{p}}_{L^{\widetilde{p}}(\mathbb{R}^N)} + [u]^{\widetilde{p}}_{W^{\widetilde{s},\widetilde{p}}(\mathbb{R}^N)} \right)^{\frac{1}{\widetilde{p}}} \leqslant C \left(\varepsilon^{sp} \|u\|^p_{L^p(\mathbb{R}^N)} + [u]^p_{W^{s,p}(\mathbb{R}^N)} \right)^{\frac{1}{p}}.$$

We notice that, in both the cases in (2.15),

$$-\frac{N}{p} + \frac{N}{\widetilde{p}} + s - \widetilde{s} < 0. \tag{2.20}$$

Hence, taking the limit as $\varepsilon \searrow 0$, we obtain the desired contradiction.

Let now $\Omega \subset \mathbb{R}^N$ be an open bounded domain with Lipschitz boundary. Up to a translation, we can assume that $0 \in \Omega$. Let $R > 0$ be such that $B_R \subset \Omega$. We also take $u : \mathbb{R}^N \to \mathbb{R}$ measurable such that $u = 0$ in $\mathbb{R}^N \setminus B_R$, u does not vanish identically and $[u]_{W^{s,p}(\mathbb{R}^N)} < +\infty$. For any $\varepsilon \in (0,1)$, we define

$$v_\varepsilon(x) := \varepsilon^{-(N-sp)/p} u\left(\frac{x}{\varepsilon}\right).$$

Suppose by contradiction that there exists $C > 0$ such that, for all $u \in W^{s,p}(\Omega)$,

$$\|u\|_{W^{\widetilde{s},\widetilde{p}}(\Omega)} \leqslant C \|u\|_{W^{s,p}(\Omega)}.$$

We exploit Lemma 2.2 with $\gamma := -(N-sp)/p$. More precisely, if $\widetilde{p} \neq +\infty$, we use (2.7) to find that

$$\|v_\varepsilon\|_{W^{\widetilde{s},\widetilde{p}}(\Omega)} \geqslant \varepsilon^{-\frac{N}{p}+\frac{N}{\widetilde{p}}+s-\widetilde{s}} \left(\varepsilon^{\widetilde{s}\widetilde{p}} \|u\|^{\widetilde{p}}_{L^{\widetilde{p}}(\Omega)} + [u]^{\widetilde{p}}_{W^{\widetilde{s},\widetilde{p}}(B_R)} \right)^{\frac{1}{\widetilde{p}}}$$

and

$$\|v_\varepsilon\|_{W^{\widetilde{s},\widetilde{p}}(\Omega)} \leqslant C \|v_\varepsilon\|_{W^{s,p}(\Omega)} \leqslant C \left(\varepsilon^{sp} \|u\|^p_{L^p(\Omega)} + [u]^p_{W^{s,p}(\mathbb{R}^N)} \right)^{\frac{1}{p}}.$$

The last two displays entail that

$$\varepsilon^{-\frac{N}{\widetilde{p}}+\frac{N}{p}+s-\widetilde{s}}\left(\varepsilon^{\widetilde{s}\widetilde{p}}\,\|u\|^{\widetilde{p}}_{L^{\widetilde{p}}(\Omega)}+[u]^{\widetilde{p}}_{W^{\widetilde{s},\widetilde{p}}(B_R)}\right)^{\frac{1}{\widetilde{p}}}\leqslant C\left(\varepsilon^{sp}\,\|u\|^{p}_{L^{p}(\Omega)}+[u]^{p}_{W^{s,p}(\mathbb{R}^N)}\right)^{\frac{1}{p}}.$$

Recalling (2.20) and taking the limit as $\varepsilon\searrow 0$, we get the desired contradiction.

If instead $\widetilde{p}=+\infty$, we exploit (2.7) and (2.9) and see that

$$\varepsilon^{-\frac{N-sp}{p}}\|u\|_{L^\infty(\Omega)}+\varepsilon^{-\frac{N-sp}{p}-\widetilde{s}}\sup_{\substack{x,y\in B_R\\ x\neq y}}\frac{|u(x)-u(y)|}{|x-y|^{\widetilde{s}}}\leqslant\|v_\varepsilon\|_{C^{0,\widetilde{s}}(\Omega)}$$

$$\leqslant C\|v_\varepsilon\|_{W^{s,p}(\Omega)}\leqslant C\left(\varepsilon^{sp}\,\|u\|^{p}_{L^{p}(\Omega)}+[u]^{p}_{W^{s,p}(\mathbb{R}^N)}\right)^{\frac{1}{p}}.$$

By taking the limit as $\varepsilon\searrow 0$, we get the desired contradiction also in the case $\widetilde{p}=+\infty$. □

We now consider the case $sp=N$. In this case, the following result holds true:

Lemma 2.5 *Let $s\in(0,1]$ and $p\in[1,+\infty)$ be such that $sp=N$. Let Ω be as in (1.1).*

If $N\geqslant 2$, assume that $\widetilde{s}$ and $\widetilde{p}$ satisfy[8]

$$\text{either}\quad\begin{cases}s<\widetilde{s}\leqslant 1,\\ p\leqslant\widetilde{p}\leqslant+\infty,\end{cases}\qquad\text{or}\qquad\begin{cases}0\leqslant\widetilde{s}\leqslant s,\\ \dfrac{N}{\widetilde{s}}<\widetilde{p}\leqslant+\infty.\end{cases}\tag{2.21}$$

If $N=1$, assume that $\widetilde{s}$ and $\widetilde{p}$ satisfy

$$\text{either}\quad\begin{cases}s<\widetilde{s}\leqslant 1,\\ p\leqslant\widetilde{p}\leqslant+\infty,\end{cases}\qquad\text{or}\qquad\begin{cases}0<\widetilde{s}\leqslant s,\\ \dfrac{N}{\widetilde{s}}<\widetilde{p}\leqslant+\infty.\end{cases}\tag{2.22}$$

Then, the space $W^{s,p}(\Omega)$ is not continuously embedded in $W^{\widetilde{s},\widetilde{p}}(\Omega)$.

We point out that the only difference between the cases $N\geqslant 2$ in (2.21) and $N=1$ in (2.22) is that in the latter we do not allow $\widetilde{s}=0$. The reason for this is that when $N=1$ and $\widetilde{s}=0$, from (2.22) we would have that $\widetilde{p}=+\infty$, namely we look at the embedding of $W^{s,p}(\Omega)$ into $L^\infty(\Omega)$ when $sp=1$. In this case, when $p\in(1,+\infty)$ we know that this embedding is false by [6, formulas (1.9)–(1.10) in Theorem B], while when $p=1$ there is a continuous embedding of $W^{1,1}(\Omega)$ into $L^\infty(\Omega)$, in light of [6, formula (1.6) in Theorem B].

Proof of Lemma 2.5 We first consider the case $\widetilde{p}=+\infty$. We recall that the choice of $\widetilde{s}=0$ can occur only in (2.21) and leads to $W^{\widetilde{s},\widetilde{p}}(\Omega)=W^{0,\infty}(\Omega)=L^\infty(\Omega)$. In this case, we have that if $s\in(0,1]$ and $p\in(1,+\infty)$ with $sp=N$, then

$$\text{if } N\geqslant 2,\text{ then } W^{s,p}(\Omega)\text{ is not continuously embedded in } L^\infty(\Omega),$$

as established in [6, formulas (1.9)–(1.10) in Theorem B] (see also Appendix A of [8] for additional explicit examples).

Hence, from now on, we suppose that $\widetilde{s}>0$.

We first address the case $\Omega=\mathbb{R}^N$. If $\widetilde{p}=+\infty$, we suppose by contradiction that there exists $C>0$ such that, for all $u\in W^{s,p}(\mathbb{R}^N)$,

$$\|u\|_{C^{0,\widetilde{s}}(\mathbb{R}^N)}\leqslant C\|u\|_{W^{s,p}(\mathbb{R}^N)}.\tag{2.23}$$

[8] With a slight abuse of notation, when $\widetilde{s}=0$ in the second possibility in (2.21), we allow $\widetilde{p}=+\infty$.

For any $\varepsilon \in (0,1)$ and for any u satisfiying (2.23) we define the function

$$v_\varepsilon(x) := u\left(\frac{x}{\varepsilon}\right). \tag{2.24}$$

By (2.1) and (2.2) (used here with $\gamma := 0$ and $\beta := -1$), we have that

$$\|v_\varepsilon\|_{W^{s,p}(\mathbb{R}^N)} = \left(\varepsilon^{sp}\,\|u\|^p_{L^p(\mathbb{R}^N)} + [u]^p_{W^{s,p}(\mathbb{R}^N)}\right)^{\frac{1}{p}} \tag{2.25}$$

and

$$\|v_\varepsilon\|_{C^{0,\widetilde{s}}(\mathbb{R}^N)} = \|u\|_{L^\infty(\mathbb{R}^N)} + \varepsilon^{-\widetilde{s}} \sup_{\substack{x,y\in\mathbb{R}^N\\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^{\widetilde{s}}}.$$

Thus, plugging v_ε into (2.23), we get

$$\|u\|_{L^\infty(\mathbb{R}^N)} + \varepsilon^{-\widetilde{s}} \sup_{\substack{x,y\in\mathbb{R}^N\\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^{\widetilde{s}}} \leqslant C\left(\varepsilon^{sp}\,\|u\|^p_{L^p(\mathbb{R}^N)} + [u]^p_{W^{s,p}(\mathbb{R}^N)}\right)^{\frac{1}{p}}.$$

Since $\widetilde{s} > 0$, by taking the limit as $\varepsilon \searrow 0$, we get a contradiction with (2.23).

We now assume that $\widetilde{p} \neq +\infty$. Suppose by contradiction that there exists $C > 0$ such that, for all $u \in W^{s,p}(\mathbb{R}^N)$,

$$\|u\|_{W^{\widetilde{s},\widetilde{p}}(\mathbb{R}^N)} \leqslant C\|u\|_{W^{s,p}(\mathbb{R}^N)}. \tag{2.26}$$

For any $\varepsilon \in (0,1)$ we consider the function v_ε as in (2.24). By (2.1) (used here with $\gamma := 0$ and $\beta := -1$), we infer that (2.25) holds and

$$\|v_\varepsilon\|_{W^{\widetilde{s},\widetilde{p}}(\mathbb{R}^N)} = \varepsilon^{-\left(\widetilde{s}-\frac{N}{\widetilde{p}}\right)}\left(\varepsilon^{\widetilde{s}\widetilde{p}}\,\|u\|^{\widetilde{p}}_{L^{\widetilde{p}}(\mathbb{R}^N)} + [u]^{\widetilde{p}}_{W^{\widetilde{s},\widetilde{p}}(\mathbb{R}^N)}\right)^{\frac{1}{\widetilde{p}}}.$$

Thus, plugging v_ε into (2.26), we get

$$\varepsilon^{-\left(\widetilde{s}-\frac{N}{\widetilde{p}}\right)}\left(\varepsilon^{\widetilde{s}\widetilde{p}}\,\|u\|^{\widetilde{p}}_{L^{\widetilde{p}}(\mathbb{R}^N)} + [u]^{\widetilde{p}}_{W^{\widetilde{s},\widetilde{p}}(\mathbb{R}^N)}\right)^{\frac{1}{\widetilde{p}}} \leqslant C\left(\varepsilon^{sp}\,\|u\|^p_{L^p(\mathbb{R}^N)} + [u]^p_{W^{s,p}(\mathbb{R}^N)}\right)^{\frac{1}{p}}. \tag{2.27}$$

Now, in both the cases in (2.21) when $N \geqslant 2$ and in (2.22) when $N = 1$, we have that

$$\widetilde{s} - \frac{N}{\widetilde{p}} > 0. \tag{2.28}$$

Hence, taking the limit as $\varepsilon \searrow 0$ in (2.27), we obtain the desired contradiction.

Let now $\Omega \subset \mathbb{R}^N$ be an open and bounded domain with Lipschitz boundary. Up to a translation, we can assume that $0 \in \Omega$. Let $R > 0$ be such that $B_R \subset \Omega$ and take $u : \mathbb{R}^N \to \mathbb{R}$ measurable such that $u = 0$ in $\mathbb{R}^N \setminus B_R$, u does not vanish identically and $[u]_{W^{s,p}(\mathbb{R}^N)} < +\infty$. For any $\varepsilon \in (0,1)$, we define

$$v_\varepsilon(x) := u\left(\frac{x}{\varepsilon}\right).$$

Let $\widetilde{s} > 0$ and suppose by contradiction that there exists $C > 0$ such that, for all $u \in W^{s,p}(\Omega)$,

$$\|u\|_{W^{\widetilde{s},\widetilde{p}}(\Omega)} \leqslant C\|u\|_{W^{s,p}(\Omega)}.$$

We exploit Lemma 2.2 with $\gamma := 0$. In this way, if $\widetilde{p} \neq +\infty$, using (2.7) we have that

$$\|v_\varepsilon\|_{W^{\widetilde{s},\widetilde{p}}(\Omega)} \geqslant \varepsilon^{-\left(\widetilde{s}-\frac{N}{\widetilde{p}}\right)}\left(\varepsilon^{\widetilde{s}\widetilde{p}}\,\|u\|^{\widetilde{p}}_{L^{\widetilde{p}}(\Omega)} + [u]^{\widetilde{p}}_{W^{\widetilde{s},\widetilde{p}}(B_R)}\right)^{\frac{1}{\widetilde{p}}}$$

and

$$\|v_\varepsilon\|_{W^{\widetilde{s},\widetilde{p}}(\Omega)} \leqslant C\|v_\varepsilon\|_{W^{s,p}(\Omega)} \leqslant C\left(\varepsilon^{sp}\|u\|^p_{L^p(\Omega)} + [u]^p_{W^{s,p}(\mathbb{R}^N)}\right)^{\frac{1}{p}}.$$

The last two displays show that

$$\varepsilon^{-\left(\widetilde{s}-\frac{N}{\widetilde{p}}\right)}\left(\varepsilon^{\widetilde{s}\widetilde{p}}\|u\|^{\widetilde{p}}_{L^{\widetilde{p}}(\Omega)} + [u]^{\widetilde{p}}_{W^{\widetilde{s},\widetilde{p}}(B_R)}\right)^{\frac{1}{\widetilde{p}}} \leqslant C\left(\varepsilon^{sp}\|u\|^p_{L^p(\Omega)} + [u]^p_{W^{s,p}(\mathbb{R}^N)}\right)^{\frac{1}{p}}.$$

Recalling (2.28) and taking the limit as $\varepsilon \searrow 0$, we get the desired contradiction.

If instead $\widetilde{p} = +\infty$, we use (2.9) and we find that

$$\|u\|_{L^\infty(\Omega)} + \varepsilon^{-\widetilde{s}} \sup_{\substack{x,y\in B_R\\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^{\widetilde{s}}} \leqslant \|v_\varepsilon\|_{C^{0,\widetilde{s}}(\Omega)}$$

$$\leqslant C\|v_\varepsilon\|_{W^{s,p}(\Omega)} \leqslant C\left(\varepsilon^{sp}\|u\|^p_{L^p(\Omega)} + [u]^p_{W^{s,p}(\mathbb{R}^N)}\right)^{\frac{1}{p}}.$$

Taking the limit as $\varepsilon \searrow 0$, we get the desired contradiction. □

For $sp > N$, we provide the following:

Lemma 2.6 *Let $s \in (0,1]$ and $p \in [1,+\infty]$ be such that $sp > N$. Let Ω be as in (1.1) and assume that $\widetilde{s}$ and $\widetilde{p}$ satisfy*

$$\text{either} \quad \begin{cases} s < \widetilde{s} \leqslant 1, \\ p \leqslant \widetilde{p} \leqslant +\infty, \end{cases} \qquad \text{or} \quad \begin{cases} \dfrac{sp-N}{p} < \widetilde{s} \leqslant s, \\ \dfrac{Np}{N-(s-\widetilde{s})p} < \widetilde{p} \leqslant +\infty. \end{cases} \tag{2.29}$$

Then, the space $W^{s,p}(\Omega)$ is not continuously embedded in $W^{\widetilde{s},\widetilde{p}}(\Omega)$.

Proof We first address the case $\Omega = \mathbb{R}^N$. If $\widetilde{p} \neq +\infty$, we assume by contradiction that there exists $C > 0$ such that, for all $u \in W^{s,p}(\mathbb{R}^N)$,

$$\|u\|_{W^{\widetilde{s},\widetilde{p}}(\mathbb{R}^N)} \leqslant C\|u\|_{W^{s,p}(\mathbb{R}^N)}. \tag{2.30}$$

For any $\varepsilon \in (0,1)$ and for any u satisfiying (2.30) we define the function

$$v_\varepsilon(x) := \varepsilon^{s-\frac{N}{p}} u\left(\frac{x}{\varepsilon}\right). \tag{2.31}$$

By (2.1) (used here with $\gamma := s - N/p$ and $\beta := -1$), we infer that

$$\|v_\varepsilon\|_{W^{s,p}(\mathbb{R}^N)} = \left(\varepsilon^{sp}\|u\|^p_{L^p(\mathbb{R}^N)} + [u]^p_{W^{s,p}(\mathbb{R}^N)}\right)^{\frac{1}{p}}$$

and

$$\|v_\varepsilon\|_{W^{\widetilde{s},\widetilde{p}}(\mathbb{R}^N)} = \varepsilon^{-\frac{N}{p}+\frac{N}{\widetilde{p}}+s-\widetilde{s}}\left(\varepsilon^{\widetilde{s}\widetilde{p}}\|u\|^{\widetilde{p}}_{L^{\widetilde{p}}(\mathbb{R}^N)} + [u]^{\widetilde{p}}_{W^{\widetilde{s},\widetilde{p}}(\mathbb{R}^N)}\right)^{\frac{1}{\widetilde{p}}}.$$

Thus, plugging v_ε into (2.30), we get

$$\varepsilon^{-\frac{N}{p}+\frac{N}{\widetilde{p}}+s-\widetilde{s}}\left(\varepsilon^{\widetilde{s}\widetilde{p}}\|u\|^{\widetilde{p}}_{L^{\widetilde{p}}(\mathbb{R}^N)} + [u]^{\widetilde{p}}_{W^{\widetilde{s},\widetilde{p}}(\mathbb{R}^N)}\right)^{\frac{1}{\widetilde{p}}} \leqslant C\left(\varepsilon^{sp}\|u\|^p_{L^p(\mathbb{R}^N)} + [u]^p_{W^{s,p}(\mathbb{R}^N)}\right)^{\frac{1}{p}}.$$

We notice that, in both the cases in (2.29),

$$-\frac{N}{p} + \frac{N}{\widetilde{p}} + s - \widetilde{s} < 0. \tag{2.32}$$

Hence, taking the limit as $\varepsilon \searrow 0$, we obtain the desired contradiction.

We now address the case $\widetilde{p} = +\infty$. Assume by contradiction that there exists $C > 0$ such that, for all $u \in W^{s,p}(\mathbb{R}^N)$,

$$\|u\|_{C^{0,\widetilde{s}}(\mathbb{R}^N)} \leqslant C\|u\|_{W^{s,p}(\mathbb{R}^N)}. \tag{2.33}$$

For any $\varepsilon \in (0,1)$ and for any u satisfiying (2.33), we recall the definition in (2.31) and we exploit (2.2) (with $\gamma := s - N/p$ and $\beta := -1$) to see that

$$\|v_\varepsilon\|_{C^{0,\widetilde{s}}(\mathbb{R}^N)} = \varepsilon^{s-\frac{N}{p}}\|u\|_{L^\infty(\mathbb{R}^N)} + \varepsilon^{s-\widetilde{s}-\frac{N}{p}} \sup_{\substack{x,y\in\mathbb{R}^N \\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^{\widetilde{s}}}.$$

Thus, using also (2.1) and plugging v_ε into (2.33), we obtain that

$$\varepsilon^{s-\frac{N}{p}}\|u\|_{L^\infty(\mathbb{R}^N)} + \varepsilon^{s-\widetilde{s}-\frac{N}{p}} \sup_{\substack{x,y\in\mathbb{R}^N \\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^{\widetilde{s}}} \leqslant C\left(\varepsilon^{sp}\|u\|^p_{L^p(\mathbb{R}^N)} + [u]^p_{W^{s,p}(\mathbb{R}^N)}\right)^{\frac{1}{p}}.$$

We notice that, in both the cases in (2.29),

$$\widetilde{s} > s - \frac{N}{p}. \tag{2.34}$$

Hence, taking the limit as $\varepsilon \searrow 0$, we obtain the desired contradiction.

When $p = +\infty$, we have that $C^{0,s}(\mathbb{R}^N)$ is not continuously embedded in $C^{0,\widetilde{s}}(\mathbb{R}^N)$. Then, the desired resut is proved also for the case $p = +\infty$.

Now, let $\Omega \subset \mathbb{R}^N$ be any open bounded domain with Lipschitz boundary. Up to a translation, we can assume $0 \in \Omega$. Let $R > 0$ be such that $B_R \subset \Omega$ and take $u : \mathbb{R}^N \to \mathbb{R}$ measurable such that $u = 0$ in $\mathbb{R}^N \setminus B_R$, u does not vanish identically and $[u]_{W^{s,p}(\mathbb{R}^N)} < +\infty$. For any $\varepsilon \in (0,1)$, let

$$v_\varepsilon(x) := \varepsilon^{s-\frac{N}{p}} u\left(\frac{x}{\varepsilon}\right).$$

Suppose by contradiction that there exists $C > 0$ such that, for all $u \in W^{s,p}(\Omega)$,

$$\|u\|_{W^{\widetilde{s},\widetilde{p}}(\Omega)} \leqslant C\|u\|_{W^{s,p}(\Omega)}.$$

We exploit Lemma 2.2 with $\gamma := s - \frac{N}{p}$ and $\beta := -1$. Hence, if $\widetilde{p} \neq +\infty$,

$$\|v_\varepsilon\|_{W^{\widetilde{s},\widetilde{p}}(\Omega)} \geqslant \varepsilon^{-\frac{N}{p}+\frac{N}{\widetilde{p}}+s-\widetilde{s}}\left(\varepsilon^{\widetilde{s}\widetilde{p}}\|u\|^{\widetilde{p}}_{L^{\widetilde{p}}(\Omega)} + [u]^{\widetilde{p}}_{W^{\widetilde{s},\widetilde{p}}(B_R)}\right)^{\frac{1}{\widetilde{p}}}$$

and

$$\|v_\varepsilon\|_{W^{\widetilde{s},\widetilde{p}}(\Omega)} \leqslant C\|v_\varepsilon\|_{W^{s,p}(\Omega)} \leqslant C\left(\varepsilon^{sp}\|u\|^p_{L^p(\Omega)} + [u]^p_{W^{s,p}(\mathbb{R}^N)}\right)^{\frac{1}{p}}.$$

The last two displays show that

$$\varepsilon^{-\frac{N}{p}+\frac{N}{\widetilde{p}}+s-\widetilde{s}}\left(\varepsilon^{\widetilde{s}\widetilde{p}}\|u\|^{\widetilde{p}}_{L^{\widetilde{p}}(\Omega)} + [u]^{\widetilde{p}}_{W^{\widetilde{s},\widetilde{p}}(B_R)}\right)^{\frac{1}{\widetilde{p}}} \leqslant C\left(\varepsilon^{sp}\|u\|^p_{L^p(\Omega)} + [u]^p_{W^{s,p}(\mathbb{R}^N)}\right)^{\frac{1}{p}}.$$

Recalling (2.32) and taking the limit as $\varepsilon \searrow 0$, we get the desired contradiction.

If instead $\widetilde{p} = +\infty$,

$$\varepsilon^{s-\frac{N}{p}} \|u\|_{L^\infty(\Omega)} + \varepsilon^{s-\widetilde{s}-\frac{N}{p}} \sup_{\substack{x,y\in B_R \\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^{\widetilde{s}}} \leqslant \|v_\varepsilon\|_{C^{0,\widetilde{s}}(\Omega)}$$

$$\leqslant C\|v_\varepsilon\|_{W^{s,p}(\Omega)} \leqslant C\left(\varepsilon^{sp}\|u\|^p_{L^p(\Omega)} + [u]^p_{W^{s,p}(\mathbb{R}^N)}\right)^{\frac{1}{p}}.$$

Recalling (2.34) and taking the limit as $\varepsilon \searrow 0$, we get the desired contradiction in this case as well.

When $p = +\infty$, we have that $C^{0,s}(\Omega)$ is not continuously embedded in $C^{0,\widetilde{s}}(\Omega)$. Then, the desired resut is proved also for the case $p = +\infty$. □

2.3 A useful compact embedding

We now show that, when $p \in [1, +\infty)$ and $\widetilde{s} \in (0, s)$, the embedding of $W^{s,p}(\Omega)$ in $W^{\widetilde{s},p}(\Omega)$ is compact. This result will be the core of the proof of Theorem 1.9 but it will be also crucial in the proof of the optimality statements for Theorems 1.3, 1.8 and 1.11.

Lemma 2.7 *Let $s \in (0, 1]$ and $p \in [1, +\infty)$. Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain. Let $\widetilde{s} \in [0, s)$.*

Then, the space $W^{s,p}(\Omega)$ is compactly embedded in $W^{\widetilde{s},p}(\Omega)$.

Proof If $\widetilde{s} = 0$, then $W^{s,p}(\Omega)$ is compactly embedded in $L^p(\Omega)$, thanks to [7, Theorem 7.1]. This is the desired claim in this case.

If $\widetilde{s} \in (0, s)$, there exists $\widetilde{\theta} \in (0, 1)$ such that $\widetilde{s} = (1-\widetilde{\theta})s$. Hence, we can apply [4, Theorem 1] (recalled in the forthcoming Theorem 3.1) with $s_1 = 0$, $s_2 = s$ and $p_1 = p_2 = p$ to see that there exists a constant $C > 0$ such that, for any $u \in W^{s,p}(\Omega)$,

$$\|u\|_{W^{\widetilde{s},p}(\Omega)} \leqslant C\|u\|^{\widetilde{\theta}}_{L^p(\Omega)}\|u\|^{1-\widetilde{\theta}}_{W^{s,p}(\Omega)}. \tag{2.35}$$

Moreover, we know that the space $W^{s,p}(\Omega)$ is compactly embedded in $L^p(\Omega)$ (see e.g. [7, Theorem 7.1]). Namely, for any bounded sequence u_n in $W^{s,p}(\Omega)$, we have that u_n converges weakly to some u in $W^{s,p}(\Omega)$, and therefore u_n converges strongly to u in $L^p(\Omega)$, as $n \to +\infty$.

Therefore, from (2.35) we infer that

$$\|u_n - u\|_{W^{\widetilde{s},p}(\Omega)} \leqslant C\|u_n - u\|^{\widetilde{\theta}}_{L^p(\Omega)}\|u_n - u\|^{1-\widetilde{\theta}}_{W^{s,p}(\Omega)}.$$

This implies that

$$\lim_{n\to+\infty} \|u_n - u\|_{W^{\widetilde{s},p}(\Omega)} = 0,$$

which proves the desired compact embedding of $W^{s,p}(\Omega)$ in $W^{\widetilde{s},p}(\Omega)$. □

2.4 Optimality of Theorems 1.2 and 1.3 and Corollary 1.5

We are now in the position to consider the case $sp < N$ and to show that Theorems 1.2 and 1.3 and Corollary 1.5 are optimal.

Proposition 2.8 *Theorem 1.2 is optimal, namely if $\widetilde{s}$ and $\widetilde{p}$ do not satisfy (1.3), then the space $W^{s,p}(\mathbb{R}^N)$ is not continuously embedded in $W^{\widetilde{s},\widetilde{p}}(\mathbb{R}^N)$.*

Proof We notice that if $\widetilde{s}$ and $\widetilde{p}$ do not satisfy (1.3), then one of the following holds:

$$\begin{cases} 0 \leqslant \widetilde{s} \leqslant 1, \\ 1 \leqslant \widetilde{p} < p, \end{cases} \quad \text{or} \quad \begin{cases} s < \widetilde{s} \leqslant 1, \\ p \leqslant \widetilde{p} \leqslant +\infty, \end{cases} \quad \text{or} \quad \begin{cases} 0 \leqslant \widetilde{s} \leqslant s, \\ \frac{Np}{N-(s-\widetilde{s})p} < \widetilde{p} \leqslant +\infty, \end{cases}$$

namely either (2.13) or (2.15) holds true.

Now, if (2.13) holds, then the desired result follows from Lemma 2.3, while if (2.15) is verified, then Lemma 2.4 in the case $\Omega = \mathbb{R}^N$ yields the desired result. □

Proposition 2.9 *Theorem 1.3 is optimal, namely if $\widetilde{s}$ and $\widetilde{p}$ do not satisfy (1.6), then the space $W^{s,p}(\Omega)$ is not continuously embedded in $W^{\widetilde{s},\widetilde{p}}(\Omega)$.*

Proof We notice that, if $\widetilde{s}$ and $\widetilde{p}$ do not satisfy (1.6), then one of the following holds:

$$\begin{cases} s \leqslant \widetilde{s} \leqslant 1, \\ 1 \leqslant \widetilde{p} < p, \end{cases} \quad \text{or} \quad \begin{cases} s < \widetilde{s} \leqslant 1, \\ p \leqslant \widetilde{p} \leqslant +\infty, \end{cases} \quad \text{or} \quad \begin{cases} 0 \leqslant \widetilde{s} \leqslant s, \\ \frac{Np}{N-(s-\widetilde{s})p} < \widetilde{p} \leqslant +\infty. \end{cases} \tag{2.36}$$

We notice that the second and third cases in (2.36) coincide with (2.15). Hence, if this is the case, the desired result follows by Lemma 2.4.

If not, the first case in (2.36) occurs. We distinguish two cases. If $\widetilde{s} = s$, then the desired result can be deduced from [16, Theorem 1.1].

If instead $\widetilde{s} \in (s, 1]$, then we assume by contradiction that $W^{s,p}(\Omega)$ is continuously embedded in $W^{\widetilde{s},\widetilde{p}}(\Omega)$. Moreover, by Lemma 2.7, we have that $W^{\widetilde{s},\widetilde{p}}(\Omega)$ is compactly embedded in $W^{s,\widetilde{p}}(\Omega)$. Accordingly, we obtain that $W^{s,p}(\Omega)$ is continuously embedded in $W^{s,\widetilde{p}}(\Omega)$, which is in contradiction with [16, Theorem 1.1]. □

Proposition 2.10 *Corollary 1.5 is optimal, namely if $\widetilde{s}$ and $\widetilde{p}$ do not satisfy (1.9), then the space $W^{s,p}(\Omega)$ is not compactly embedded in $W^{\widetilde{s},\widetilde{p}}(\Omega)$.*

Proof In light of Proposition 2.9, if $\widetilde{s}$ and $\widetilde{p}$ do not satisfy (1.6), then $W^{s,p}(\Omega)$ is not compactly embedded in $W^{\widetilde{s},\widetilde{p}}(\Omega)$. Thus, it only remains to show that the points which belong to the critical curve are such that the compact embedding stated in Corollary 1.5 does not hold.

To this purpose, let $s \in (0, 1]$ and $p \in [1, +\infty)$ be such that $sp < N$. Let $\widetilde{s}$ and $\widetilde{p}$ satisfy

$$0 \leqslant \widetilde{s} \leqslant s \qquad \text{and} \qquad \frac{Np}{N - sp} = \frac{N\widetilde{p}}{N - \widetilde{s}\widetilde{p}}. \tag{2.37}$$

Notice that the second condition in (2.37) means that the fractional Sobolev exponents for the couples (s, p) and $(\widetilde{s}, \widetilde{p})$ coincide.

Up to translation, suppose that $0 \in \Omega$ and let $R > 0$ be such that $B_R \subset \Omega$. Let $u \in C_0^\infty(B_R) \setminus \{0\}$ and, for $\varepsilon \in (0, 1)$, let

$$v_\varepsilon(x) := \varepsilon^{-\frac{N-sp}{p}} u\left(\frac{x}{\varepsilon}\right).$$

We remark that, for all $x \in B_R \setminus \{0\}$, there exists $\varepsilon_0 \in (0, 1)$, depending on x, such that if $\varepsilon \in (0, \varepsilon_0]$ we have that $|x|/\varepsilon > R$, and therefore $v_\varepsilon(x) = 0$.

This says that

$$v_\varepsilon \to 0 \text{ a.e. in } \Omega \text{ as } \varepsilon \searrow 0. \tag{2.38}$$

Furthermore, formulas (2.6) and (2.10) (used here with $\gamma := -(N - sp)/p$) give that

$$\|v_\varepsilon\|_{L^p(\Omega)}^p = \varepsilon^{sp} \|u\|_{L^p(\Omega)}^p \qquad \text{and} \qquad [v_\varepsilon]_{W^{s,p}(\Omega)}^p \leqslant [u]_{W^{s,p}(\mathbb{R}^N)}^p,$$

which in turn imply that v_ε is bounded in $W^{s,p}(\Omega)$.

Thus, if the embedding of $W^{s,p}(\Omega)$ into $W^{\widetilde{s},\widetilde{p}}(\Omega)$ were compact, up to a subsequence, v_ε would converge to some v in $W^{\widetilde{s},\widetilde{p}}(\Omega)$, and therefore also pointwise in Ω. In light of (2.38), we would then have that

$$v_\varepsilon \text{ converges to } 0 \text{ in } W^{\widetilde{s},\widetilde{p}}(\Omega) \text{ as } \varepsilon \searrow 0. \tag{2.39}$$

On the other hand, formula (2.6), together with the second assumption in (2.37), gives that

$$\|v_\varepsilon\|^{\widetilde{p}}_{L^{\widetilde{p}}(\Omega)} = \varepsilon^{\widetilde{p}\widetilde{s}} \|u\|^{\widetilde{p}}_{L^{\widetilde{p}}(\Omega)} \qquad \text{and} \qquad [v_\varepsilon]^{\widetilde{p}}_{W^{\widetilde{s},\widetilde{p}}(\Omega)} \geqslant [u]^{\widetilde{p}}_{W^{\widetilde{s},\widetilde{p}}(B_R)}.$$

Hence,

$$\lim_{\varepsilon\to 0} \left(\|v_\varepsilon\|^{\widetilde{p}}_{L^{\widetilde{p}}(\Omega)} + [v_\varepsilon]^{\widetilde{p}}_{W^{\widetilde{s},\widetilde{p}}(\Omega)} \right) \geqslant [u]^{\widetilde{p}}_{W^{\widetilde{s},\widetilde{p}}(B_R)} > 0.$$

This is in contradiction with (2.39), and therefore the desired claim is established. □

2.5 Optimality of Theorems 1.6, 1.8 and 1.9

We now address the case $sp = N$ and we show that Theorems 1.6, 1.8 and 1.9 are optimal.

Proposition 2.11 *Theorem 1.6 is optimal, namely if $\widetilde{s}$ and $\widetilde{p}$ do not satisfy (1.10), then the space $W^{s,p}(\mathbb{R}^N)$ is not continuously embedded in $W^{\widetilde{s},\widetilde{p}}(\mathbb{R}^N)$.*

Proof Assume that $\widetilde{s}$ and $\widetilde{p}$ do not satisfy (1.10). Then one of the following[9] holds:

$$\begin{cases} 0 \leqslant \widetilde{s} \leqslant 1, \\ 1 \leqslant \widetilde{p} < p, \end{cases} \quad \text{or} \quad \begin{cases} s < \widetilde{s} \leqslant 1, \\ p \leqslant \widetilde{p} \leqslant +\infty, \end{cases} \quad \text{or} \quad \begin{cases} 0 \leqslant \widetilde{s} \leqslant s, \\ \dfrac{N}{\widetilde{s}} < \widetilde{p} \leqslant +\infty. \end{cases} \tag{2.40}$$

The first possibility in (2.40) coincides with (2.13), and therefore the desired result follows from Lemma 2.3.

The second and third possibilities in (2.40) give (2.21) if $N \geqslant 2$, and so Lemma 2.5 in the case $\Omega = \mathbb{R}^N$ with $N \geqslant 2$ yields the desired result.

Also, we point out that the second and third possibilities in (2.40) when $\widetilde{s} \neq 0$ give (2.22) if $N = 1$. Thus, the desired result when $\Omega = \mathbb{R}$ follows from Lemma 2.5.

In the case $N = 1$, $\widetilde{s} = 0$ and $\widetilde{p} = +\infty$, we remark that one of the assumptions in Theorem 1.6 is that $s \neq p$, and therefore, in this case, we have that $p \in (1, +\infty)$. This allows us to exploit [6, formulas (1.9)–(1.10) in Theorem B] and obtain the desired conclusion. □

Proposition 2.12 *Theorem 1.8 is optimal, namely if $\widetilde{s}$ and $\widetilde{p}$ do not satisfy (1.13), then the space $W^{s,p}(\Omega)$ is not continuously embedded in $W^{\widetilde{s},\widetilde{p}}(\Omega)$.*

Proof Suppose that $\widetilde{s}$ and $\widetilde{p}$ do not satisfy (1.13). Then one of the following holds:

$$\begin{cases} s \leqslant \widetilde{s} \leqslant 1, \\ 1 \leqslant \widetilde{p} < p, \end{cases} \quad \text{or} \quad \begin{cases} s < \widetilde{s} \leqslant 1, \\ p \leqslant \widetilde{p} \leqslant +\infty, \end{cases} \quad \text{or} \quad \begin{cases} 0 \leqslant \widetilde{s} \leqslant s, \\ \dfrac{N}{\widetilde{s}} < \widetilde{p} \leqslant +\infty, \end{cases} \tag{2.41}$$

If the first case in (2.41) occurs, we distinguish two cases. If $\widetilde{s} = s$, then the desired result can be deduced from [16, Theorem 1.1].

[9] With a slight abuse of notation, when $\widetilde{s} = 0$ in the second possibility in (2.40) and (2.41), we allow $\widetilde{p} = +\infty$.

If instead $\widetilde{s} \in (s, 1]$, we assume by contradiction that $W^{s,p}(\Omega)$ is continuously embedded in $W^{\widetilde{s},\widetilde{p}}(\Omega)$. Also, by Lemma 2.7, we have that $W^{\widetilde{s},\widetilde{p}}(\Omega)$ is compactly embedded in $W^{s,\widetilde{p}}(\Omega)$. As a result, we obtain that $W^{s,p}(\Omega)$ is continuously embedded in $W^{s,\widetilde{p}}(\Omega)$, which is in contradiction with [16, Theorem 1.1].

The second and third possibilities in (2.41) give either (2.21) if $N \geqslant 2$ or (2.22) if $N = 1$ and $\widetilde{s} \neq 0$, and therefore Lemma 2.5 in these cases yields the desired result.

It remains to check the case $N = 1$ and $\widetilde{s} = 0$. In this situation we have that $\widetilde{p} = +\infty$. Also, we observe that $s \neq p$ by the assumptions in the statement of Theorem 1.8. Thus, in this case, we have that $p \in (1, +\infty)$ and we can exploit [6, formulas (1.9)–(1.10) in Theorem B] to finish the proof. □

Proposition 2.13 *Theorem 1.9 is optimal, namely if $\widetilde{s}$ and $\widetilde{p}$ do not satisfy (1.15), then the space $W^{s,p}(\Omega)$ is not compactly embedded in $W^{\widetilde{s},\widetilde{p}}(\Omega)$.*

Proof If $\widetilde{s}$ and $\widetilde{p}$ do not satisfy (1.13), then $W^{s,p}(\Omega)$ is not compactly embedded in $W^{\widetilde{s},\widetilde{p}}(\Omega)$, thanks to Proposition 2.12. Hence, it only remains to show that the points which belong to the critical curve $\widetilde{s}\widetilde{p} = N$ are such that the compact embedding in Theorem 2.13 does not hold.

To this purpose, let $s \in (0, 1]$ and $p \in [1, +\infty)$ be such that $sp = N$ and $s \neq p$. Let $\widetilde{s}$ and $\widetilde{p}$ satisfy

$$\widetilde{s} \in (0, s] \qquad \text{and} \qquad \widetilde{s}\widetilde{p} = N. \tag{2.42}$$

Up to a translation, we suppose that $0 \in \Omega$ and we let $R > 0$ such that $B_R \subset \Omega$. Let $u \in C_0^\infty(B_R) \setminus \{0\}$ and, for any $\varepsilon \in (0, 1)$, let

$$v_\varepsilon(x) := u\left(\frac{x}{\varepsilon}\right).$$

For all $x \in B_R \setminus \{0\}$ there exists $\varepsilon_0 \in (0, 1)$ such that, for all $\varepsilon \in (0, \varepsilon_0]$, we have that $|x|/\varepsilon > R$ and thus, by construction, we have that $v_\varepsilon(x) = 0$. As a result,

$$v_\varepsilon \to 0 \text{ a.e. in } \Omega \text{ as } \varepsilon \searrow 0. \tag{2.43}$$

In addition, we exploit formulas (2.6) and (2.10) (used here with $\gamma := 0$) to see that

$$\|v_\varepsilon\|^p_{L^p(\Omega)} = \varepsilon^{sp}\|u\|^p_{L^p(\Omega)} \qquad \text{and} \qquad [v_\varepsilon]^p_{W^{s,p}(\Omega)} \leqslant [u]^p_{W^{s,p}(\mathbb{R}^N)}.$$

This gives that v_ε is a bounded sequence in $W^{s,p}(\Omega)$.

Accordingly, if the embedding of $W^{s,p}(\Omega)$ into $W^{\widetilde{s},\widetilde{p}}(\Omega)$ were compact, up to a subsequence, we would have that v_ε converges to some v in $W^{\widetilde{s},\widetilde{p}}(\Omega)$ as $\varepsilon \searrow 0$. This and (2.43) imply that

$$v_\varepsilon \text{ converges to } 0 \text{ in } W^{\widetilde{s},\widetilde{p}}(\Omega) \text{ as } \varepsilon \searrow 0. \tag{2.44}$$

Furthermore, formula (2.6) and the second assumption in (2.42) give that

$$\|v_\varepsilon\|^{\widetilde{p}}_{L^{\widetilde{p}}(\Omega)} = \varepsilon^{\widetilde{p}\widetilde{s}}\|u\|^{\widetilde{p}}_{L^{\widetilde{p}}(\Omega)} \qquad \text{and} \qquad [v_\varepsilon]^{\widetilde{p}}_{W^{\widetilde{s},\widetilde{p}}(\Omega)} \geqslant [u]^{\widetilde{p}}_{W^{\widetilde{s},\widetilde{p}}(B_R)}.$$

We thereby obtain that

$$\lim_{\varepsilon\to 0}\left(\|v_\varepsilon\|^{\widetilde{p}}_{L^{\widetilde{p}}(\Omega)} + [v_\varepsilon]^{\widetilde{p}}_{W^{\widetilde{s},\widetilde{p}}(\Omega)}\right) \geqslant [u]^{\widetilde{p}}_{W^{\widetilde{s},\widetilde{p}}(B_R)} > 0,$$

which is in contradiction with (2.44), and therefore the desired claim is established. □

2.6 Optimality of Theorems 1.10, 1.11 and 1.12

Finally, we discuss the optimality of the results stated in the case $sp > N$.

Proposition 2.14 *Theorem 1.10 is optimal, namely if $\widetilde{s}$ and $\widetilde{p}$ do not satisfy (1.16), then the space $W^{s,p}(\mathbb{R}^N)$ is not continuously embedded in $W^{\widetilde{s},\widetilde{p}}(\mathbb{R}^N)$.*

Proof Observe that, if $\widetilde{s}$ and $\widetilde{p}$ do not satisfy (1.16), then one of the following holds:

$$\begin{cases} 0 \leqslant \widetilde{s} \leqslant 1, \\ 1 \leqslant \widetilde{p} < p, \end{cases} \quad \text{or} \quad \begin{cases} s < \widetilde{s} \leqslant 1, \\ p \leqslant \widetilde{p} \leqslant +\infty, \end{cases} \quad \text{or} \quad \begin{cases} \dfrac{sp-N}{p} < \widetilde{s} \leqslant s, \\ \dfrac{Np}{N-(s-\widetilde{s})p} < \widetilde{p} \leqslant +\infty, \end{cases} \tag{2.45}$$

namely either (2.13) or (2.29) holds true.

Therefore, the desired result follows from Lemma 2.3 if (2.13) holds true and from Lemma 2.6 in the case $\Omega = \mathbb{R}^N$ if (2.29) is satisfied. □

Proposition 2.15 *Theorem 1.11 is optimal, namely if $\widetilde{s}$ and $\widetilde{p}$ do not satisfy (1.18), then the space $W^{s,p}(\Omega)$ is not continuously embedded in $W^{\widetilde{s},\widetilde{p}}(\Omega)$.*

Proof Notice that, if $\widetilde{s}$ and $\widetilde{p}$ do not satisfy (1.18), then one of the following holds:

$$\text{either} \quad \begin{cases} s \leqslant \widetilde{s} \leqslant 1, \\ 1 \leqslant \widetilde{p} < p, \end{cases} \quad \text{or} \quad \begin{cases} s < \widetilde{s} \leqslant 1, \\ p \leqslant \widetilde{p} \leqslant +\infty, \end{cases} \quad \text{or} \quad \begin{cases} \dfrac{sp-N}{p} < \widetilde{s} \leqslant s, \\ \dfrac{Np}{N-(s-\widetilde{s})p} < \widetilde{p} \leqslant +\infty. \end{cases} \tag{2.46}$$

The second and third cases in (2.46) coincide with (2.29). Hence, if this is the case, the desired result follows from Lemma 2.6.

If not, the first case in (2.46) occurs. We now distinguish two cases. If $\widetilde{s} = s$, then the desired result can be deduced from [16, Theorem 1.1].

If instead $\widetilde{s} \in (s, 1]$, we assume by contradiction that $W^{s,p}(\Omega)$ is continuously embedded in $W^{\widetilde{s},\widetilde{p}}(\Omega)$. Also, by Lemma 2.7, we have that $W^{\widetilde{s},\widetilde{p}}(\Omega)$ is continuously embedded in $W^{s,\widetilde{p}}(\Omega)$. Thus, combining these two facts, we obtain that $W^{s,p}(\Omega)$ is continuously embedded in $W^{s,\widetilde{p}}(\Omega)$, which is in contradiction with [16, Theorem 1.1].

When $p = +\infty$, we have that $C^{0,s}(\Omega)$ is not continuously embedded in $C^{0,\widetilde{s}}(\Omega)$. Then, the desired resut is proved also for the case $p = +\infty$. □

Proposition 2.16 *Theorem 1.12 is optimal, namely if $\widetilde{s}$ and $\widetilde{p}$ do not satisfy (1.20), then the space $W^{s,p}(\Omega)$ is not compactly embedded in $W^{\widetilde{s},\widetilde{p}}(\Omega)$.*

Proof In light of Proposition 2.15, if $\widetilde{s}$ and $\widetilde{p}$ do not satisfy (1.18), then $W^{s,p}(\Omega)$ is not compactly embedded in $W^{\widetilde{s},\widetilde{p}}(\Omega)$. Thus, it only remains to show that the points which belong to the curve

$$\widetilde{p} = \frac{Np}{N - p(s-\widetilde{s})}$$

are such that the compact embedding stated in Theorem 1.12 does not hold.

To this purpose, let $s \in (0, 1]$ and $p \geqslant 1$ be such that $sp > N$. Let $\widetilde{s}$ and $\widetilde{p}$ satisfy

$$\frac{sp-N}{p} < \widetilde{s} < s \quad \text{and} \quad \widetilde{s} - \frac{N}{\widetilde{p}} = s - \frac{N}{p}. \tag{2.47}$$

Without loss of generality, we suppose that $0 \in \Omega$ and we let $R > 0$ such that $B_R \subset \Omega$. Let also $u \in C_0^\infty(B_R) \setminus \{0\}$ and, for any $\varepsilon \in (0,1)$,

$$v_\varepsilon(x) := \varepsilon^{s-\frac{N}{p}} u\left(\frac{x}{\varepsilon}\right).$$

We observe that for all $x \in B_R \setminus \{0\}$ there exists $\varepsilon_0 \in (0,1)$ such that, for all $\varepsilon \in (0, \varepsilon_0]$, we have that $|x|/\varepsilon > R$, and as a result $v_\varepsilon(x) = 0$. This gives that

$$v_\varepsilon \to 0 \text{ a.e. in } \Omega \text{ as } \varepsilon \searrow 0. \tag{2.48}$$

Now, we use formulas (2.6) and (2.10) (with $\gamma := s - N/p$) to see that

$$\|v_\varepsilon\|^p_{L^p(\Omega)} = \varepsilon^{sp}\|u\|^p_{L^p(\Omega)} \qquad \text{and} \qquad [v_\varepsilon]^p_{W^{s,p}(\Omega)} \leqslant [u]^p_{W^{s,p}(\mathbb{R}^N)}.$$

As a consequence, v_ε is a bounded sequence in $W^{s,p}(\Omega)$.

Thus, if the embedding of $W^{s,p}(\Omega)$ into $W^{\widetilde{s},\widetilde{p}}(\Omega)$ were compact, we would have that v_ε converges, up to a subsequence, to some function v in $W^{\widetilde{s},\widetilde{p}}(\Omega)$ as $\varepsilon \searrow 0$. From this and (2.48) we would then have that

$$v_\varepsilon \text{ converges to } 0 \text{ in } W^{\widetilde{s},\widetilde{p}}(\Omega) \text{ as } \varepsilon \searrow 0. \tag{2.49}$$

Also, formula (2.6) and the second assumption in (2.47) give that

$$\|v_\varepsilon\|^{\widetilde{p}}_{L^{\widetilde{p}}(\Omega)} = \varepsilon^{\widetilde{p}\widetilde{s}}\|u\|^{\widetilde{p}}_{L^{\widetilde{p}}(\Omega)} \qquad \text{and} \qquad [v_\varepsilon]^{\widetilde{p}}_{W^{\widetilde{s},\widetilde{p}}(\Omega)} \geqslant [u]^{\widetilde{p}}_{W^{\widetilde{s},\widetilde{p}}(B_R)}.$$

Consequently,

$$\lim_{\varepsilon\to 0}\left(\|v_\varepsilon\|^{\widetilde{p}}_{L^{\widetilde{p}}(\Omega)} + [v_\varepsilon]^{\widetilde{p}}_{W^{\widetilde{s},\widetilde{p}}(\Omega)}\right) \geqslant [u]^{\widetilde{p}}_{W^{\widetilde{s},\widetilde{p}}(B_R)} > 0,$$

which is in contradiction with (2.49), thus proving the desired claim. □

3 Auxiliary results

In this section we provide some preliminary results that will be used to prove our main theorems.

3.1 An interpolation result

We start by recalling the following interpolation result due to Brezis and Mironescu (see [4, Theorem 1]). We state it in a way which is suitable for our analysis:

Theorem 3.1 *Let $0 \leqslant s_1 \leqslant s_2 \leqslant 1$ and $p_1, p_2 \in [1, +\infty]$ with $s_2 \neq p_2$. Let Ω be either $\mathbb{R}^N$ or a bounded Lipschitz domain of $\mathbb{R}^N$.*

For every $\theta \in [0,1]$, let

$$s_\theta := \theta s_1 + (1-\theta)s_2 \qquad \text{and} \qquad p_\theta := \frac{p_1\, p_2}{(1-\theta)p_1 + \theta p_2}.$$

Then, there exists a positive constant $C = C(s_1, s_2, p_1, p_2, \theta, \Omega)$ such that, for any $u \in W^{s_1,p_1}(\Omega) \cap W^{s_2,p_2}(\Omega)$,

$$\|u\|_{W^{s_\theta,p_\theta}(\Omega)} \leqslant C\|u\|^\theta_{W^{s_1,p_1}(\Omega)}\|u\|^{1-\theta}_{W^{s_2,p_2}(\Omega)}.$$

3.2 A suitable curve of points

Now, we show that any points $\widetilde{s}$ and $\widetilde{p}$ satisfying (1.3) belong to a suitable curve. For this, if $sp < N$, we define the fractional Sobolev exponent

$$p_s^* := \frac{Np}{N - sp}.$$

Lemma 3.2 *Let $s \in (0, 1]$ and $p \in [1, +\infty)$ be such that $sp < N$. Assume that $\widetilde{s}$ and $\widetilde{p}$ satisfy (1.3).*

Then, there exists $q \in [p, p_s^*]$ such that the curve $\gamma_q : [0, 1] \to \mathbb{R}^2$ defined as

$$\gamma_q(\theta) := \left((1-\theta)s, \frac{pq}{q - \theta(q-p)}\right) \tag{3.1}$$

contains the point $(\widetilde{s}, \widetilde{p})$, namely there exists $\widetilde{\theta} \in [0, 1]$ such that $\gamma_q(\widetilde{\theta}) = (\widetilde{s}, \widetilde{p})$.

Proof Notice that if $\widetilde{s} = s$, from (1.3) we see that $\widetilde{p} = p$. In this case we then have that $\gamma_q(0) = (\widetilde{s}, \widetilde{p})$ for every $q \in [p, p_s^*]$. Thus, the couple $(\widetilde{s}, \widetilde{p})$ belongs to the curve γ_q.

Suppose instead that $\widetilde{s} \in [0, s)$ and $\widetilde{p} = p$. In this case, taking $q := p$, we see that $\gamma_q(\theta) = ((1-\theta)s, p)$. Therefore, there exists $\widetilde{\theta} \in (0, 1]$ such that $\gamma_q(\widetilde{\theta}) = (\widetilde{s}, \widetilde{p})$, as desired.

Given these preliminary observations, from now on we suppose that $\widetilde{s} \neq s$ and $\widetilde{p} \neq p$.

Moreover, we remark that the condition $\widetilde{s}\widetilde{p} = sp$ is incompatible with (1.3), since

$$\widetilde{s}\widetilde{p} \leqslant \frac{Np\widetilde{s}}{N - (s-\widetilde{s})p} = \frac{\widetilde{s}spp_s^*}{sp + (p_s^* - p)\widetilde{s}} = \left(\frac{\widetilde{s}p_s^*}{\widetilde{s}p_s^* + p(s-\widetilde{s})}\right) sp < sp.$$

Therefore, from now on, we also assume that $\widetilde{s}\widetilde{p} \neq sp$.

Now, we observe that the couple $(\widetilde{s}, \widetilde{p})$ belongs to the curve γ_q if and only if it is contained in the graph of the function $f_q : [0, s] \to \mathbb{R}$ defined as

$$f_q(t) := \frac{spq}{sp + (q-p)t}.$$

We set

$$q := \frac{p\widetilde{p}(s-\widetilde{s})}{sp - \widetilde{s}\widetilde{p}}$$

and we have that

$$\widetilde{p} = \frac{spq}{sp + (q-p)\widetilde{s}} = f_q(\widetilde{s}).$$

Hence, it remains to check that

$$q \in [p, p_s^*]. \tag{3.2}$$

For this, we notice that, in light of (1.3),

$$q > \frac{p^2(s-\widetilde{s})}{p(s-\widetilde{s})} = p$$

and

$$q \leqslant \frac{\dfrac{sp^2p_s^*(s-\widetilde{s})}{sp + (p_s^* - p)\widetilde{s}}}{sp - \dfrac{\widetilde{s}spp_s^*}{sp + (p_s^* - p)\widetilde{s}}} = p_s^*.$$

These considerations proves (3.2), as desired. □

We now check that if $\widetilde{s}$ and $\widetilde{p}$ satisfy (1.8), they belong to the curve γ_q, as stated in this result:

Lemma 3.3 *Let $s \in (0,1]$ and $p \in [1,+\infty)$ be such that $sp < N$. Let $\widetilde{s}$ and $\widetilde{p}$ satisfy (1.8). Then, there exists $q \in [1, p_s^*)$ such that the curve $\gamma_q : [0,1] \to \mathbb{R}^2$ defined in (3.1) contains the point $(\widetilde{s}, \widetilde{p})$, namely there exists $\widetilde{\theta} \in (0,1]$ such that $\gamma_q(\widetilde{\theta}) = (\widetilde{s}, \widetilde{p})$.*

Proof The proof of this result is similar to the proof of Lemma 3.2. For the reader convenience, we provide all the details here.

If $\widetilde{s} \in [0, s)$ and $\widetilde{p} = p$, we take $q := p$ and we see that $\gamma_q(\theta) = ((1-\theta)s, p)$. Therefore, there exists $\widetilde{\theta} \in [0,1]$ such that $\gamma_q(\widetilde{\theta}) = (\widetilde{s}, \widetilde{p})$, as desired. Hence, we will now suppose that $\widetilde{p} \neq p$.

From (1.8) we also have that

$$\widetilde{s}\widetilde{p} \leqslant \frac{Np\widetilde{s}}{N-(s-\widetilde{s})p} = \frac{\widetilde{s}spp_s^*}{sp+(p_s^*-p)\widetilde{s}} = \left(\frac{\widetilde{s}p_s^*}{\widetilde{s}p_s^*+p(s-\widetilde{s})}\right) sp < sp.$$

Now, we observe that the point $(\widetilde{s}, \widetilde{p})$ belongs to the curve γ_q in (3.1) if and only if it is contained in the graph of the function $f_q : [0,s] \to \mathbb{R}$ defined as

$$f_q(t) := \frac{spq}{sp+(q-p)t}.$$

We set

$$q := \frac{p\widetilde{p}(s-\widetilde{s})}{sp-\widetilde{s}\widetilde{p}}$$

and we point out that

$$\widetilde{p} = \frac{spq}{sp+(q-p)\widetilde{s}} = f_q(\widetilde{s}).$$

Thus, the proof of Lemma 3.3 is complete if we show that $q \in [1, p_s^*)$. For this, one notices that, by (1.8),

$$q \geqslant \frac{\dfrac{sp^2(s-\widetilde{s})}{sp-(p-1)\widetilde{s}}}{sp-\dfrac{\widetilde{s}sp}{sp-(p-1)\widetilde{s}}} = 1$$

and

$$q < \frac{\dfrac{sp^2p_s^*(s-\widetilde{s})}{sp+(p_s^*-p)\widetilde{s}}}{sp-\dfrac{\widetilde{s}spp_s^*}{sp+(p_s^*-p)\widetilde{s}}} = p_s^*.$$

□

We now show that, under that assumption that the points $\widetilde{s}$ and $\widetilde{p}$ satisfy (1.10), a result similar to the one presented in Lemma 3.3 holds true when $sp = N$. We point out that this choice allows the point q to be selected in a larger interval.

Lemma 3.4 *Let $s \in (0,1]$ and $p \in [1,+\infty)$ be such that $sp = N$. Let $\widetilde{s}$ and $\widetilde{p}$ satisfy (1.10). Then, there exists $q \in [p,+\infty)$ such that the curve $\gamma_q : [0,1] \to \mathbb{R}^2$ defined in (3.1) contains the point $(\widetilde{s}, \widetilde{p})$, namely there exists $\widetilde{\theta} \in [0,1]$ such that $\gamma_q(\widetilde{\theta}) = (\widetilde{s}, \widetilde{p})$.*

Proof The proof of this result is similar to the proof of Lemmata 3.2 and 3.3. For the reader convenience, we provide all the details here.

If $\widetilde{s} = s$, then (1.10) and the fact that $sp = N$ give that $\widetilde{p} = p$, and therefore $\gamma_q(0) = (\widetilde{s}, \widetilde{p})$ for every $q \in [p, +\infty)$.

Moreover, if $\widetilde{s} \in [0, s)$ and $\widetilde{p} = p$, we take $q := p$ and we see that $\gamma_q(\theta) = ((1-\theta)s, p)$. Therefore, there exists $\widetilde{\theta} \in [0, 1]$ such that $\gamma_q(\widetilde{\theta}) = (\widetilde{s}, \widetilde{p})$, as desired.

Given these preliminary observations, from now on we suppose that $\widetilde{s} \neq s$ and $\widetilde{p} \neq p$.

Also, we note that if $\widetilde{s}$ and $\widetilde{p}$ satisfy condition (1.10), then $\widetilde{s}\widetilde{p} < N = sp$.

Now, we notice that the point $(\widetilde{s}, \widetilde{p})$ belongs to the curve γ_q in (3.1) if and only if it is contained in the graph of the function $f_q : [0, s] \to \mathbb{R}$ defined as

$$f_q(t) := \frac{spq}{sp + (q-p)t}.$$

Thus, we set

$$q := \frac{p\widetilde{p}(s - \widetilde{s})}{sp - \widetilde{s}\widetilde{p}}$$

and we find that

$$\widetilde{p} = \frac{spq}{sp + (q-p)\widetilde{s}} = f_q(\widetilde{s}).$$

Since $\widetilde{p} \geqslant p$, we have that

$$q \geqslant \frac{p^2(s - \widetilde{s})}{p(s - \widetilde{s})} = p,$$

which concludes the proof. □

3.3 An embedding result

We now state a preliminary embedding result which will be useful to prove Theorem 1.3. We point out that our result is similar to [1, Lemma 2.1], but we provide here a self-contained proof that does not involve the use of Besov or other interpolation spaces.

Lemma 3.5 *Let $s \in (0, 1)$ and $p \in [1, +\infty]$. Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain. Let $\widetilde{s} \in (0, s)$ and $\widetilde{p} \in [1, p]$.*

Then, there exists a positive constant $C = C(N, s, p, \widetilde{s}, \widetilde{p}, \Omega)$ such that, for any $u \in W^{s,p}(\Omega)$,

$$\|u\|_{W^{\widetilde{s},\widetilde{p}}(\Omega)} \leqslant C\|u\|_{W^{s,p}(\Omega)},$$

namely, the space $W^{s,p}(\Omega)$ is continuously embedded in $W^{\widetilde{s},\widetilde{p}}(\Omega)$.

Proof We first suppose that $p \in [1, +\infty)$ and we claim that

$$\|u\|_{L^{\widetilde{p}}(\Omega)} \leqslant C\,\|u\|_{L^p(\Omega)}. \tag{3.3}$$

Indeed, if $\widetilde{p} = p$, then the claim is obvious. If instead $\widetilde{p} \in [1, p)$, by the Hölder inequality with exponents $p/\widetilde{p}$ and $p/(p - \widetilde{p})$,

$$\|u\|_{L^{\widetilde{p}}(\Omega)} \leqslant |\Omega|^{\frac{1}{\widetilde{p}} - \frac{1}{p}}\,\|u\|_{L^p(\Omega)},$$

which gives (3.3).

We also show that

$$[u]_{W^{\widetilde{s},\widetilde{p}}(\Omega)} \leqslant C\,[u]_{W^{s,p}(\Omega)}. \tag{3.4}$$

To this end, it is convenient to write

$$\frac{|u(x)-u(y)|^{\widetilde{p}}}{|x-y|^{N+\widetilde{s}\widetilde{p}}} = \frac{|u(x)-u(y)|^{\widetilde{p}}}{|x-y|^{\frac{N\widetilde{p}}{p}+s\widetilde{p}}}\,\frac{1}{|x-y|^{\frac{N(p-\widetilde{p})}{p}-\widetilde{p}(s-\widetilde{s})}}.$$

If $\widetilde{p}=p$, then the claim in (3.4) follows from this and the fact that Ω is bounded.

If instead $\widetilde{p}\in[1,\,p)$, we use the Hölder inequality with exponents $p/\widetilde{p}$ and $p/(p-\widetilde{p})$ to find that

$$\begin{aligned}
&\iint_{\Omega\times\Omega}\frac{|u(x)-u(y)|^{\widetilde{p}}}{|x-y|^{N+\widetilde{s}\widetilde{p}}}\,dx\,dy\\
&\leqslant\left(\iint_{\Omega\times\Omega}\frac{|u(x)-u(y)|^{p}}{|x-y|^{N+sp}}\,dx\,dy\right)^{\frac{\widetilde{p}}{p}}\left(\iint_{\Omega\times\Omega}\frac{dx\,dy}{|x-y|^{N-\frac{p\widetilde{p}(s-\widetilde{s})}{p-\widetilde{p}}}}\right)^{\frac{p-\widetilde{p}}{p}}\\
&\leqslant C\left(\iint_{\Omega\times\Omega}\frac{|u(x)-u(y)|^{p}}{|x-y|^{N+sp}}\,dx\,dy\right)^{\frac{\widetilde{p}}{p}},
\end{aligned}$$

from which we obtain (3.4).

The claim of Lemma 3.5 in this case now follows from (3.3) and (3.4).

We now focus on the case $p=+\infty$. If in addition $\widetilde{p}=+\infty$, our claim becomes

$$\|u\|_{C^{0,\widetilde{s}}(\Omega)} \leqslant C\|u\|_{C^{0,s}(\Omega)},$$

which in turn boils down to

$$[u]_{C^{0,\widetilde{s}}(\Omega)} \leqslant C[u]_{C^{0,s}(\Omega)}. \tag{3.5}$$

For this, we observe that, for any $\widetilde{s}\in(0,\,s)$,

$$\frac{|u(x)-u(y)|}{|x-y|^{\widetilde{s}}} = \frac{|u(x)-u(y)|}{|x-y|^{s}}|x-y|^{s-\widetilde{s}} \leqslant [u]_{C^{0,s}(\Omega)}|x-y|^{s-\widetilde{s}}.$$

Since Ω is bounded, this implies (3.5), as desired.

If instead $\widetilde{p}\in[1,\,p)=[1,\,+\infty)$, our claim becomes

$$\|u\|_{W^{\widetilde{s},\widetilde{p}}(\Omega)} \leqslant C\|u\|_{C^{0,s}(\Omega)}. \tag{3.6}$$

In this situation, we point out that

$$\|u\|^{p}_{L^{\widetilde{p}}(\Omega)} = \int_{\Omega}|u(x)|^{\widetilde{p}}\,dx \leqslant \|u\|^{\widetilde{p}}_{L^{\infty}(\Omega)}|\Omega|.$$

Also,

$$\iint_{\Omega\times\Omega}\frac{|u(x)-u(y)|^{\widetilde{p}}}{|x-y|^{N+\widetilde{s}\widetilde{p}}}\,dx\,dy \leqslant [u]^{\widetilde{p}}_{C^{0,s}(\Omega)}\iint_{\Omega\times\Omega}\frac{|x-y|^{s\widetilde{p}}}{|x-y|^{N+\widetilde{s}\widetilde{p}}}\,dx\,dy \leqslant C[u]^{\widetilde{p}}_{C^{0,s}(\Omega)}.$$

From the last two displays, we obtain (3.6). □

4 Proof of the embedding results

In this section we prove our main embedding results.

4.1 The case $sp < N$

We first provide the following useful observation:

Lemma 4.1 *Let $s \in [0,1]$ and $p \in [1,+\infty)$ be such that $sp < N$. Let*

$$\widetilde{s} \in [0,s] \quad \text{and} \quad \widetilde{p} \in \left[1, \frac{Np}{N-(s-\widetilde{s})p}\right]. \tag{4.1}$$

Let γ be the curve defined in (1.5). Let $0 \leqslant \theta_1 < \theta_2 \leqslant 1$ and let $\gamma(\theta_1) = (s_{\theta_1}, p_{\theta_1})$ and $\gamma(\theta_2) = (s_{\theta_2}, p_{\theta_2})$.

Then,

$$0 \leqslant s_{\theta_2} \leqslant s_{\theta_1} \leqslant 1, \quad \min\{p,\widetilde{p}\} \leqslant p_{\theta_1}, p_{\theta_2} \quad \text{and} \quad p_{\theta_2} \leqslant \frac{Np_{\theta_1}}{N-(s_{\theta_1}-s_{\theta_2})p_{\theta_1}}. \tag{4.2}$$

If $p \leqslant \widetilde{p}$, then $p_{\theta_1} \leqslant p_{\theta_2}$.

Furthermore, if in addition

$$\widetilde{p} \geqslant \frac{sp}{sp-(p-1)\widetilde{s}}, \tag{4.3}$$

then

$$p_{\theta_2} \geqslant \frac{s_{\theta_1}p_{\theta_1}}{s_{\theta_1}p_{\theta_1}-(p_{\theta_1}-1)s_{\theta_2}}. \tag{4.4}$$

Moreover, $s_{\theta_1} = s_{\theta_2}$ if and only if $s = \widetilde{s}$, and $p_{\theta_1} = p_{\theta_2}$ if and only if $p = \widetilde{p}$.

Also,

$$p_{\theta_2} = \frac{Np_{\theta_1}}{N-(s_{\theta_1}-s_{\theta_2})p_{\theta_1}}$$

if and only if

$$\widetilde{p} = \frac{Np}{N-(s-\widetilde{s})p}.$$

Proof By inspection, one sees that $0 \leqslant s_{\theta_2} \leqslant s_{\theta_1} \leqslant 1$. Also, for all $\theta \in [0,1]$,

$$\frac{1}{p_\theta} = \frac{\widetilde{p}+\theta(p-\widetilde{p})}{p\widetilde{p}} = \frac{1}{p} + \theta\left(\frac{1}{\widetilde{p}}-\frac{1}{p}\right) \leqslant \max\left\{\frac{1}{p},\frac{1}{\widetilde{p}}\right\} = \frac{1}{\min\{p,\widetilde{p}\}}. \tag{4.5}$$

Hence, to complete the proof of (4.2), it remains to check that

$$p_{\theta_2} \leqslant \frac{Np_{\theta_1}}{N-(s_{\theta_1}-s_{\theta_2})p_{\theta_1}}. \tag{4.6}$$

To this end, we first observe that, thanks to (4.1),

$$\frac{N}{p}-\frac{N}{\widetilde{p}} \leqslant s-\widetilde{s}.$$

This implies that

$$\begin{aligned}\frac{1}{p_{\theta_1}}-\frac{1}{p_{\theta_2}} &= \frac{\widetilde{p}-\theta_1(\widetilde{p}-p)}{p\widetilde{p}} - \frac{\widetilde{p}-\theta_2(\widetilde{p}-p)}{p\widetilde{p}} = (\theta_2-\theta_1)\left(\frac{1}{p}-\frac{1}{\widetilde{p}}\right)\\ &\leqslant \frac{(\theta_2-\theta_1)(s-\widetilde{s})}{N} = \frac{s_{\theta_1}-s_{\theta_2}}{N}.\end{aligned}$$

From this, the desired result in (4.6) plainly follows.

If in addition $p \leqslant \widetilde{p}$, by the definition of p_θ we have that $p_{\theta_1} \leqslant p_{\theta_2}$.

We now also assume (4.3) and we prove (4.4). For this, we introduce the notation

$$\alpha := 1 - \frac{1}{p}, \qquad \widetilde{\alpha} := 1 - \frac{1}{\widetilde{p}} \qquad \text{and} \qquad \beta := s - \widetilde{s}$$

and we observe that

$$\begin{aligned}
&\left(1 - \frac{1}{p_{\theta_1}}\right) s_{\theta_2} - \left(1 - \frac{1}{p_{\theta_2}}\right) s_{\theta_1} \\
&= \big(\alpha - \theta_1(\alpha - \widetilde{\alpha})\big)(s - \theta_2\beta) - \big(\alpha - \theta_2(\alpha - \widetilde{\alpha})\big)(s - \theta_1\beta) \\
&= \beta\alpha(\theta_1 - \theta_2) + (\alpha - \widetilde{\alpha})\big(\theta_2(s - \theta_1\beta) - \theta_1(s - \theta_2\beta)\big) \\
&= \beta\alpha(\theta_1 - \theta_2) + (\alpha - \widetilde{\alpha})(\theta_2 - \theta_1)s \\
&= (\theta_2 - \theta_1)\big(\alpha(s - \beta) - \widetilde{\alpha}s\big) \\
&= (\theta_2 - \theta_1)\big(\alpha\widetilde{s} - \widetilde{\alpha}s\big) \\
&= (\theta_2 - \theta_1)\left(\left(1 - \frac{1}{p}\right)\widetilde{s} - \left(1 - \frac{1}{\widetilde{p}}\right)s\right).
\end{aligned}$$

In light of this, we have that (4.3) implies (4.4).

Finally, the last statements follow from the definition in (1.5). □

We now address the case in which the domain is $\mathbb{R}^N$, by proving Theorem 1.2.

Proof of Theorem 1.2 Let $\widetilde{s}$ and $\widetilde{p}$ satisfy (1.3). Then, by Lemma 3.2 there exist $q \in [p, p_s^*]$ and $\widetilde{\theta} \in [0, 1]$ such that

$$\widetilde{s} = (1 - \widetilde{\theta})s \qquad \text{and} \qquad \widetilde{p} = \frac{pq}{q - \widetilde{\theta}(q - p)} = \frac{pq}{p\widetilde{\theta} + q(1 - \widetilde{\theta})}.$$

Hence, we can exploit Theorem 3.1 with $s_1 := 0$, $s_2 := s$, $p_1 := q$ and $p_2 := p$ and obtain that there exists a positive constant $C_1 = C_1(N, s, p, \widetilde{s}, \widetilde{p})$ such that, for any $u \in L^q(\mathbb{R}^N) \cap W^{s,p}(\mathbb{R}^N)$,

$$\|u\|_{W^{\widetilde{s},\widetilde{p}}(\mathbb{R}^N)} \leqslant C_1 \|u\|^{\widetilde{\theta}}_{L^q(\mathbb{R}^N)} \|u\|^{1-\widetilde{\theta}}_{W^{s,p}(\mathbb{R}^N)}. \tag{4.7}$$

Moreover, since $q \in [p, p_s^*]$, the space $W^{s,p}(\mathbb{R}^N)$ is continuously embedded in $L^q(\mathbb{R}^N)$, namely

$$\|u\|_{L^q(\mathbb{R}^N)} \leqslant C_2 \|u\|_{W^{s,p}(\mathbb{R}^N)}$$

for some $C_2 = C_2(N, p, s) > 0$ (see e.g. [7, Theorem 6.5]).

In light of this and (4.7), for any $u \in L^q(\mathbb{R}^N) \cap W^{s,p}(\mathbb{R}^N) = W^{s,p}(\mathbb{R}^N)$,

$$\|u\|_{W^{\widetilde{s},\widetilde{p}}(\mathbb{R}^N)} \leqslant C_1 C_2^{\widetilde{\theta}} \|u\|_{W^{s,p}(\mathbb{R}^N)},$$

which is (1.4).

Having established the first claim of Theorem 1.2, we now focus on the second one. For this, let $0 \leqslant \theta_1 < \theta_2 \leqslant 1$ and consider $\gamma(\theta_1) = (s_{\theta_1}, p_{\theta_1})$ and $\gamma(\theta_2) = (s_{\theta_2}, p_{\theta_2})$, where the curve γ has been defined in (1.5). Thanks to the assumptions in (1.3) and Lemma 4.1, we are in the position of exploiting (1.4) with $s := s_{\theta_1}$, $p := p_{\theta_1}$, $\widetilde{s} := s_{\theta_2}$ and $\widetilde{p} := p_{\theta_2}$ and we obtain that $W^{s_{\theta_1}, p_{\theta_1}}(\mathbb{R}^N)$ is continuously embedded in $W^{s_{\theta_2}, p_{\theta_2}}(\mathbb{R}^N)$, as desired. This concludes the proof of Theorem 1.2. □

We now prove the continuous embedding in the case in which Ω is an open bounded Lipschitz domain and $sp < N$.

Proof of Theorem 1.3 Let $\widetilde{s}$ and $\widetilde{p}$ satisfy (1.6). Two cases can occur: either $\widetilde{p} \geqslant p$ or $\widetilde{p} < p$.

If $\widetilde{p} \geqslant p$, then $\widetilde{s}$ and $\widetilde{p}$ satisfy (1.3). Hence, by Lemma 3.2 there exist $q \in [p, p_s^*]$ and $\widetilde{\theta} \in [0, 1]$ such that

$$\widetilde{s} = (1-\widetilde{\theta})s \qquad \text{and} \qquad \widetilde{p} = \frac{pq}{q - \widetilde{\theta}(q-p)} = \frac{pq}{p\widetilde{\theta} + q(1-\widetilde{\theta})}.$$

From this and Theorem 3.1 there exists a positive constant $C_1 = C_1(N, s, p, \widetilde{s}, \widetilde{p})$ such that, for any $u \in L^q(\Omega) \cap W^{s,p}(\Omega)$,

$$\|u\|_{W^{\widetilde{s},\widetilde{p}}(\Omega)} \leqslant C_1 \|u\|^{\widetilde{\theta}}_{L^q(\Omega)} \|u\|^{1-\widetilde{\theta}}_{W^{s,p}(\Omega)}. \tag{4.8}$$

Moreover, since $q \in [p, p_s^*]$, the space $W^{s,p}(\Omega)$ is continuously embedded in $L^q(\Omega)$, namely there exists $C_2 = C_2(N, s, p, \Omega) > 0$ such that

$$\|u\|_{L^q(\Omega)} \leqslant C_2 \|u\|_{W^{s,p}(\Omega)},$$

see e.g. [7, Theorem 6.7]. Thus, using this into (4.8), we conclude that, for any $u \in L^q(\Omega) \cap W^{s,p}(\Omega) = W^{s,p}(\Omega)$,

$$\|u\|_{W^{\widetilde{s},\widetilde{p}}(\Omega)} \leqslant C_1 \, C_2^{\widetilde{\theta}} \|u\|_{W^{s,p}(\Omega)}.$$

which is (1.7).

Let us consider the case $\widetilde{p} < p$. If $\widetilde{s} \in (0, s)$, then the desired embedding is a direct consequence of Lemma 3.5. Otherwise, if $\widetilde{s} = 0$, then the desired result follows e.g. from [7, Theorem 6.7].

This completes the proof of the first part of Theorem 1.3 and we now focus on the second part. For this, let $0 \leqslant \theta_1 < \theta_2 \leqslant 1$ and consider $\gamma(\theta_1) = (s_{\theta_1}, p_{\theta_1})$ and $\gamma(\theta_2) = (s_{\theta_2}, p_{\theta_2})$, where the curve γ has been defined in (1.5). Thanks to the assumptions in (1.6) and Lemma 4.1, we can use (1.7) with $s := s_{\theta_1}$, $p := p_{\theta_1}$, $\widetilde{s} := s_{\theta_2}$ and $\widetilde{p} := p_{\theta_2}$ and we obtain that $W^{s_{\theta_1}, p_{\theta_1}}(\Omega)$ is continuously embedded in $W^{s_{\theta_2}, p_{\theta_2}}(\Omega)$, as desired. This concludes the proof of Theorem 1.3. □

We now prove the compact embeddings stated in Theorem 1.4 and Corollary 1.5.

Proof of Theorem 1.4 Let $\widetilde{s}$ and $\widetilde{p}$ satisfy (1.8). Then, from Lemma 3.3 there exist $q \in [1, p_s^*)$ and $\widetilde{\theta} \in (0, 1]$ such that

$$\widetilde{s} = (1-\widetilde{\theta})s \qquad \text{and} \qquad \widetilde{p} = \frac{pq}{q - \widetilde{\theta}(q-p)} = \frac{pq}{p\widetilde{\theta} + q(1-\widetilde{\theta})}.$$

Hence, from Theorem 3.1 there exists $C > 0$ such that, for any $u \in L^q(\Omega) \cap W^{s,p}(\Omega)$,

$$\|u\|_{W^{\widetilde{s},\widetilde{p}}(\Omega)} \leqslant C \|u\|^{\widetilde{\theta}}_{L^q(\Omega)} \|u\|^{1-\widetilde{\theta}}_{W^{s,p}(\Omega)}. \tag{4.9}$$

Moreover, since $q \in [1, p_s^*)$, the space $W^{s,p}(\Omega)$ is compactly embedded in $L^q(\Omega)$, namely for any bounded sequence u_n in $W^{s,p}(\Omega)$, we have that u_n converges weakly to some u in $W^{s,p}(\Omega)$, and therefore u_n converges strongly to u in $L^q(\Omega)$, as $n \to +\infty$ (see e.g. [7, Corollary 7.2]).

Thus, by (4.9), we infer that

$$\|u_n - u\|_{W^{\widetilde{s},\widetilde{p}}(\Omega)} \leqslant C \|u_n - u\|^{\widetilde{\theta}}_{L^q(\Omega)} \|u_n - u\|^{1-\widetilde{\theta}}_{W^{s,p}(\Omega)}.$$

This implies that

$$\lim_{n\to+\infty} \|u_n - u\|_{W^{\widetilde{s},\widetilde{p}}(\Omega)} = 0,$$

which proves the compact embedding of $W^{s,p}(\Omega)$ in $W^{\widetilde{s},\widetilde{p}}(\Omega)$.

Let now $0 < \theta_1 < \theta_2 \leqslant 1$ and consider $\gamma(\theta_1) = (s_{\theta_1}, p_{\theta_1})$ and $\gamma(\theta_2) = (s_{\theta_2}, p_{\theta_2})$, where the curve γ has been defined in (1.5). Thanks to (1.8), we are in the position of using Lemma 4.1. We thereby deduce that $W^{s_{\theta_1},p_{\theta_1}}(\Omega)$ is compactly embedded in $W^{s_{\theta_2},p_{\theta_2}}(\Omega)$, as desired. This concludes the proof of Theorem 1.4. □

Proof of Corollary 1.5 Let $\widetilde{s}$ and $\widetilde{p}$ satisfy (1.9). If they also satisfy (1.8), then the desired result follows from Theorem 1.4.

We can now assume that $\widetilde{s}$ and $\widetilde{p}$ satisfy (1.9) but do not satisfy (1.8), namely

$$0 \leqslant \widetilde{s} < s \qquad \text{and} \qquad 1 \leqslant \widetilde{p} < \frac{sp}{sp-(p-1)\widetilde{s}}.$$

In this situation, we have that

$$0 \leqslant \frac{s+\widetilde{s}}{2} < s$$

and

$$1 \leqslant \widetilde{p} < \frac{sp}{sp-(p-1)\widetilde{s}} \leqslant \frac{Np}{N-\frac{s-\widetilde{s}}{2}p}.$$

That is, $\frac{s+\widetilde{s}}{2}$ and $\widetilde{p}$ satisfy (1.6), and therefore we deduce from Theorem 1.3 that the space $W^{s,p}(\Omega)$ is continuously embedded in $W^{\frac{s+\widetilde{s}}{2},\widetilde{p}}(\Omega)$.

Thus, if u_n is a bounded sequence in $W^{s,p}(\Omega)$, then u_n converges weakly to some u in $W^{s,p}(\Omega)$, and we have that u_n converges weakly to u in $W^{\frac{s+\widetilde{s}}{2},\widetilde{p}}(\Omega)$.

Since $\widetilde{s} < s$, we are also in the position of using Theorem 1.4 and we infer that $W^{\frac{s+\widetilde{s}}{2},\widetilde{p}}(\Omega)$ is compactly embedded in $W^{\widetilde{s},\widetilde{p}}(\Omega)$. Then, we conclude that u_n converges strongly to u in $W^{\widetilde{s},\widetilde{p}}(\Omega)$, which completes the proof of the first part of Corollary 1.5.

The second part of Corollary 1.5 is a consequence of the assumptions in (1.9) and Lemma 4.1. □

4.2 The case $sp = N$

We now address the case $sp = N$. In this case, we will use the following observation:

Lemma 4.2 *Let $s \in [0,1]$ and $p \in [1,+\infty)$ be such that $sp = N$. Let*

$$\widetilde{s} \in [0,s] \qquad \text{and} \qquad \widetilde{p} \in \left[1, \frac{N}{\widetilde{s}}\right].$$

Let γ be the curve defined in (1.5). Let $0 \leqslant \theta_1 < \theta_2 \leqslant 1$ and let $\gamma(\theta_1) = (s_{\theta_1}, p_{\theta_1})$ and $\gamma(\theta_2) = (s_{\theta_2}, p_{\theta_2})$.

Then,

$$0 \leqslant s_{\theta_2} \leqslant s_{\theta_1} \leqslant 1, \qquad \min\{p, \widetilde{p}\} \leqslant p_{\theta_1}, p_{\theta_2} \qquad \text{and} \qquad p_{\theta_2} \leqslant \frac{N}{s_{\theta_2}}.$$

Also, if $p \leqslant \widetilde{p}$, then $p_{\theta_1} \leqslant p_{\theta_2}$.

Moreover, $s_{\theta_1} = s_{\theta_2}$ if and only if $s = \widetilde{s}$, and $p_{\theta_1} = p_{\theta_2}$ if and only if $p = \widetilde{p}$.

Proof By inspection, one sees that $0 \leqslant s_{\theta_2} \leqslant s_{\theta_1} \leqslant 1$. Also, the fact that $p_\theta \geqslant \min\{p, \widetilde{p}\}$ for all $\theta \in [0, 1]$ follows from the computations in (4.5).

We now check that

$$p_{\theta_2} \leqslant \frac{N}{s_{\theta_2}}. \tag{4.10}$$

For this, we observe that

$$sp = N \geqslant \widetilde{s}\widetilde{p}$$

and therefore

$$\frac{1}{p_{\theta_2}} = \frac{1}{p} + \theta_2\left(\frac{1}{\widetilde{p}} - \frac{1}{p}\right) = \frac{1-\theta_2}{p} + \frac{\theta_2}{\widetilde{p}} \geqslant \frac{(1-\theta_2)s}{N} + \frac{\theta_2\widetilde{s}}{N} = \frac{s_{\theta_2}}{N},$$

which gives (4.10).

Moreover, if $p \leqslant \widetilde{p}$, then, by the definition of p_θ it follows that $p_{\theta_1} \leqslant p_{\theta_2}$. □

We now deal with the case $\Omega = \mathbb{R}^N$.

Proof of Theorem 1.6 Let $\widetilde{s}$ and $\widetilde{p}$ satisfy (1.10). If $\widetilde{s}\widetilde{p} = N$, then the result follows from [6, formula (1.6) in Theorem B]. If not, by Lemma 3.4 there exist $q \in [p, +\infty)$ and $\widetilde{\theta} \in [0, 1]$ such that

$$\widetilde{s} = (1-\widetilde{\theta})s \qquad \text{and} \qquad \widetilde{p} = \frac{pq}{q - \widetilde{\theta}(q-p)} = \frac{pq}{p\widetilde{\theta} + q(1-\widetilde{\theta})}.$$

Thus, by Theorem 3.1, there exists a positive constant $C_1 = C_1(N, s, p, \widetilde{s}, \widetilde{p})$ such that, for any $u \in L^q(\mathbb{R}^N) \cap W^{s,p}(\mathbb{R}^N)$,

$$\|u\|_{W^{\widetilde{s},\widetilde{p}}(\mathbb{R}^N)} \leqslant C_1 \|u\|^{\widetilde{\theta}}_{L^q(\mathbb{R}^N)} \|u\|^{1-\widetilde{\theta}}_{W^{s,p}(\mathbb{R}^N)}. \tag{4.11}$$

Moreover, since $q \in [p, +\infty)$, the space $W^{s,p}(\mathbb{R}^N)$ is continuously embedded in $L^q(\mathbb{R}^N)$, namely

$$\|u\|_{L^q(\mathbb{R}^N)} \leqslant C_2 \|u\|_{W^{s,p}(\mathbb{R}^N)}$$

for some $C_2 = C_2(N, p, s) > 0$ (see e.g. [7, Theorem 6.9]). Therefore, for any $u \in L^q(\mathbb{R}^N) \cap W^{s,p}(\mathbb{R}^N) = W^{s,p}(\mathbb{R}^N)$, from this and (4.11) we deduce that

$$\|u\|_{W^{\widetilde{s},\widetilde{p}}(\mathbb{R}^N)} \leqslant C_1\, C_2^{\widetilde{\theta}} \|u\|_{W^{s,p}(\mathbb{R}^N)},$$

which gives (1.11).

Let now $0 \leqslant \theta_1 < \theta_2 \leqslant 1$ and consider $\gamma(\theta_1) = (s_{\theta_1}, p_{\theta_1})$ and $\gamma(\theta_2) = (s_{\theta_2}, p_{\theta_2})$, where the curve γ has been defined in (1.5). Then, Lemma 4.2 gives that we can use (1.11) with $s := s_{\theta_1}$, $p := p_{\theta_1}$, $\widetilde{s} := s_{\theta_2}$ and $\widetilde{p} := p_{\theta_2}$. We thereby obtain that $W^{s_{\theta_1}, p_{\theta_1}}(\mathbb{R}^N)$ is continuously embedded in $W^{s_{\theta_2}, p_{\theta_2}}(\mathbb{R}^N)$, as desired.

We stress that in the case $N = 1$, one has that the space $W^{1,1}(\mathbb{R}^N)$ is continuously embedded in $L^\infty(\mathbb{R}^N)$ (see e.g. [5, Theorem 8.8]). This concludes the proof of Theorem 1.6. □

Proof of Corollary 1.7 In light of Theorem 1.6, we only need to show that, for all $u \in W^{\widetilde{s},\widetilde{p}}(\mathbb{R}^N)$,

$$\|u\|_{BMO} \leqslant C_1 \|u\|_{W^{\widetilde{s},\widetilde{p}}(\mathbb{R}^N)}. \tag{4.12}$$

To this end, we observe that, for any $r > 0$ and for any $x \in \mathbb{R}^N$,

$$\begin{aligned}
&\iint_{B_r(x)\times B_r(x)} |u(y)-u(z)|\,dy\,dz = \iint_{B_r(x)\times B_r(x)} \frac{|u(y)-u(z)|}{|y-z|^{\frac{2N}{\widetilde{p}}}}\,|y-z|^{\frac{2N}{\widetilde{p}}}\,dy\,dz\\
&\leqslant (2r)^{\frac{2N}{\widetilde{p}}} \iint_{B_r(x)\times B_r(x)} \frac{|u(y)-u(z)|}{|y-z|^{\frac{2N}{\widetilde{p}}}}\,dy\,dz\\
&\leqslant (2r)^{\frac{2N}{\widetilde{p}}}\,|B_r|^{\frac{2\widetilde{p}-2}{\widetilde{p}}} \left(\iint_{B_r(x)\times B_r(x)} \frac{|u(y)-u(z)|^{\widetilde{p}}}{|y-z|^{2N}}\,dy\,dz\right)^{\frac{1}{\widetilde{p}}}\\
&\leqslant c_{N,s,p}^{-\frac{1}{\widetilde{p}}}(2r)^{\frac{2N}{\widetilde{p}}}\,|B_r|^{2-\frac{2}{\widetilde{p}}}\,\|u\|_{W^{\widetilde{s},\widetilde{p}}(\mathbb{R}^N)}.
\end{aligned}$$

Dividing by $|B_r|^2$ and taking the supremum over $r > 0$ and $x \in \mathbb{R}^N$, we get the desired result in (4.12). □

We now consider the case in which $sp = N$ and Ω is an open and bounded domain Lipschitz boundary.

Proof of Theorem 1.8 Let $\widetilde{s}$ and $\widetilde{p}$ satisfy (1.13). Then, either $\widetilde{p} \geqslant p$ or $\widetilde{p} < p$.

Suppose first that $\widetilde{p} \geqslant p$. If in addition $\widetilde{s}\widetilde{p} = N$, then the desired result follows from [6, formula (1.6) in Theorem B].

If instead $\widetilde{s}\widetilde{p} < N$, we observe that $\widetilde{s}$ and $\widetilde{p}$ satisfy (1.10), and therefore we are in the position of exploiting Lemma 3.4. In this way, we obtain that there exist $q \in [p, +\infty)$ and $\widetilde{\theta} \in [0, 1]$ such that

$$\widetilde{s} = (1-\widetilde{\theta})s \qquad \text{and} \qquad \widetilde{p} = \frac{pq}{q-\widetilde{\theta}(q-p)} = \frac{pq}{p\widetilde{\theta}+q(1-\widetilde{\theta})}.$$

From this and Theorem 3.1 there exists a positive constant $C_1 = C_1(N, s, p, \widetilde{s}, \widetilde{p})$ such that, for any $u \in L^q(\Omega) \cap W^{s,p}(\Omega)$,

$$\|u\|_{W^{\widetilde{s},\widetilde{p}}(\Omega)} \leqslant C\|u\|^{\widetilde{\theta}}_{L^q(\Omega)}\|u\|^{1-\widetilde{\theta}}_{W^{s,p}(\Omega)}. \tag{4.13}$$

Moreover, since $q \in [p, +\infty)$, the space $W^{s,p}(\Omega)$ is continuously embedded in $L^q(\Omega)$, namely there exists $C_2 = C_2(N, s, p, \Omega) > 0$ such that

$$\|u\|_{L^q(\Omega)} \leqslant C_2\|u\|_{W^{s,p}(\Omega)},$$

see e.g. [7, Theorem 6.10]. Thus, for any $u \in L^q(\Omega) \cap W^{s,p}(\Omega) = W^{s,p}(\Omega)$, this and (4.13) give that

$$\|u\|_{W^{\widetilde{s},\widetilde{p}}(\Omega)} \leqslant C_1\,C_2^{\widetilde{\theta}}\|u\|_{W^{s,p}(\Omega)},$$

which establishes (1.14).

Let us consider the case $\widetilde{p} < p$. If $\widetilde{s} > 0$, then the desired embedding is a direct consequence of Lemma 3.5. Otherwise, if $\widetilde{s} = 0$, the desired result is the one stated e.g. in [7, Theorem 6.10].

Let now $0 \leqslant \theta_1 < \theta_2 \leqslant 1$ and consider $\gamma(\theta_1) = (s_{\theta_1}, p_{\theta_1})$ and $\gamma(\theta_2) = (s_{\theta_2}, p_{\theta_2})$, where the curve γ has been defined in (1.5). Then, Lemma 4.2 gives that we can use (1.14) with $s := s_{\theta_1}$, $p := p_{\theta_1}$, $\widetilde{s} := s_{\theta_2}$ and $\widetilde{p} := p_{\theta_2}$. We thereby obtain that $W^{s_{\theta_1},p_{\theta_1}}(\Omega)$ is continuously embedded in $W^{s_{\theta_2},p_{\theta_2}}(\Omega)$, as desired.

We stress that in the case $N = 1$, the space $W^{1,1}(\Omega)$ is continuously embedded in $L^\infty(\Omega)$ (see e.g. [5, Theorem 8.8]). This concludes the proof of Theorem 1.8. □

Now, we prove the desired compact embedding in the case $sp = N$.

Proof of Theorem 1.12 Let $\widetilde{s}$ and $\widetilde{p}$ satisfy (1.15).

If $\widetilde{p} = p$, the desired compact embedding is a direct consequence of Lemma 2.7. Thus, we now distinguish the cases $\widetilde{p} > p$ and $\widetilde{p} < p$.

If $\widetilde{p} > p$, we set $\overline{s} := N/\widetilde{p}$ and we observe that $\overline{s} \in (0, s)$. We can therefore apply Theorem 1.8 and conclude that $W^{s,p}(\Omega)$ is continuously embedded in $W^{\overline{s},\widetilde{p}}(\Omega)$. Moreover, we have that $\widetilde{s} \in [0, \overline{s})$, and so we deduce from Lemma 2.7 that $W^{\overline{s},\widetilde{p}}(\Omega)$ is compactly embedded in $W^{\widetilde{s},\widetilde{p}}(\Omega)$. Combining these pieces of information, we obtain that the space $W^{s,p}(\Omega)$ is compactly embedded in $W^{\widetilde{s},\widetilde{p}}(\Omega)$, as desired.

Now, we deal with the case $\widetilde{p} < p$. We observe that $\frac{s+\widetilde{s}}{2} \in (0, s)$ and that

$$\widetilde{p} < p = \frac{N}{s} < \frac{2N}{s+\widetilde{s}}.$$

Thus, we can apply Theorem 1.8 to see that $W^{s,p}(\Omega)$ is continuously embedded in $W^{\frac{s+\widetilde{s}}{2},\widetilde{p}}(\Omega)$. Moreover, since $\frac{s+\widetilde{s}}{2} > \widetilde{s}$, from Lemma 2.7 we infer that $W^{\frac{s+\widetilde{s}}{2},\widetilde{p}}(\Omega)$ is compactly embedded in $W^{\widetilde{s},\widetilde{p}}(\Omega)$. From these considerations, we obtain that the space $W^{s,p}(\Omega)$ is compactly embedded in $W^{\widetilde{s},\widetilde{p}}(\Omega)$, which concludes the case $\widetilde{p} < p$.

The first statement of Theorem 1.9 is thereby proved, and we now focus on the second one.

Let $0 < \theta_1 < \theta_2 \leqslant 1$ and consider $\gamma(\theta_1) = (s_{\theta_1}, p_{\theta_1})$ and $\gamma(\theta_2) = (s_{\theta_2}, p_{\theta_2})$, where the curve γ has been defined in (1.5). Then, Lemma 4.2 gives that we can use the first statement of Theorem 1.12 with $s := s_{\theta_1}$, $p := p_{\theta_1}$, $\widetilde{s} := s_{\theta_2}$ and $\widetilde{p} := p_{\theta_2}$ and obtain that $W^{s_{\theta_1},p_{\theta_1}}(\Omega)$ is compactly embedded in $W^{s_{\theta_2},p_{\theta_2}}(\Omega)$, as desired. □

4.3 The case $sp > N$

We now consider the case $sp > N$. We provide some auxiliary statements.

Lemma 4.3 *Let $s \in (0, 1]$ and $p \in [1, +\infty]$ be such that $sp > N$. Let*

$$\overline{s} \in \left(\frac{sp - N}{p}, s\right] \quad \text{and} \quad \overline{p} := \frac{Np}{N - (s - \overline{s})p}. \tag{4.14}$$

Let Ω be either $\mathbb{R}^N$ or a bounded Lipschitz domain of $\mathbb{R}^N$.

Then, $W^{s,p}(\Omega)$ is continuously embedded in $W^{\overline{s},\overline{p}}(\Omega)$.

Proof To this end, notice that if $\overline{s} = s$ then $\overline{p} = p$, and therefore the claim is trivial.

Otherwise, we observe that, if $\overline{s}$ and $\overline{p}$ satisfy (4.14), then the point $(\overline{s}, \overline{p})$ belongs to the curve

$$\left(s - \frac{\theta N}{p}, \frac{p}{1-\theta}\right) \quad \text{with } \theta \in [0, 1].$$

This says that we are in the setting of Theorem 3.1 with $s_1 := \frac{sp-N}{p}$, $p_1 := +\infty$, $s_2 := s$ and $p_2 := p$. Therefore, we can exploit the claim of Theorem 3.1 with $s_\theta := \overline{s}$ and $p_\theta := \overline{p}$. In this way, we obtain that there exists a constant $C = C(N, s, p, \theta) > 0$ such that

$$\|u\|_{W^{\overline{s},\overline{p}}(\Omega)} \leqslant C \|u\|^{\theta}_{C^{0,\frac{sp-N}{p}}(\Omega)} \|u\|^{1-\theta}_{W^{s,p}(\Omega)}.$$

From this and the continuous embedding of $W^{s,p}(\Omega)$ in $C^{0,\frac{sp-N}{p}}(\Omega)$ (see [7, Theorem 8.2]), we thus conclude that

$$\|u\|_{W^{\bar{s},\bar{p}}(\Omega)} \leqslant C\|u\|_{W^{s,p}(\Omega)},$$

which gives the desired result. □

Lemma 4.4 *Let $s \in (0,1]$ and $p \in [1,+\infty]$. Let $\widetilde{s} \in [0,s]$. Let Ω be either $\mathbb{R}^N$ or a bounded Lipschitz domain of $\mathbb{R}^N$.*

Then, $W^{s,p}(\Omega)$ is continuously embedded in $W^{\widetilde{s},p}(\Omega)$.

Proof We notice that the point $(\widetilde{s}, p)$ belongs to the curve $\big((1-\theta)s, p\big)$, with $\theta \in (0,1]$. As a consequence, we can use Theorem 3.1 with $s_1 := 0$, $p_1 := p$, $s_2 := s$, $p_2 := p$, $s_\theta := \widetilde{s}$ and $p_\theta := p$, obtaining that there exists a constant $C = C(N,s,p,\theta) > 0$ such that

$$\|u\|_{W^{\widetilde{s},p}(\Omega)} \leqslant C\|u\|^{\theta}_{L^p(\Omega)}\|u\|^{1-\theta}_{W^{s,p}(\Omega)} \leqslant C\|u\|_{W^{s,p}(\Omega)},$$

which implies the desired claim. □

We will also need the following observation:

Lemma 4.5 *Let $s \in (0,1]$ and $p \in [1,+\infty]$ be such that $sp > N$. Let $\widetilde{s} \in [0,s]$ and $\widetilde{p} \in [1,+\infty]$.*

Let γ be the curve defined in (1.5). Let $0 \leqslant \theta_1 < \theta_2 \leqslant 1$ and let $\gamma(\theta_1) = (s_{\theta_1}, p_{\theta_1})$ and $\gamma(\theta_2) = (s_{\theta_2}, p_{\theta_2})$.

Then, we have that $0 \leqslant s_{\theta_2} \leqslant s_{\theta_1} \leqslant 1$ and that $\min\{p,\widetilde{p}\} \leqslant p_{\theta_1}, p_{\theta_2} \leqslant +\infty$.

If $p \leqslant \widetilde{p}$, then $p_{\theta_1} \leqslant p_{\theta_2}$.

Furthermore, if

$$\frac{sp-N}{p} < \widetilde{s} \leqslant s \quad \text{and} \quad 1 \leqslant \widetilde{p} \leqslant \frac{Np}{N-(s-\widetilde{s})p}, \tag{4.15}$$

then

$$\frac{s_{\theta_1}p_{\theta_1}-N}{p_{\theta_1}} < s_{\theta_2} \leqslant s_{\theta_1} \quad \text{and} \quad 1 \leqslant p_{\theta_2} \leqslant \frac{Np_{\theta_1}}{N-(s_{\theta_1}-s_{\theta_2})p_{\theta_1}}. \tag{4.16}$$

Moreover, $s_{\theta_1} = s_{\theta_2}$ if and only if $s = \widetilde{s}$, and $p_{\theta_1} = p_{\theta_2}$ if and only if $p = \widetilde{p}$.

Also, if $p \in [1,+\infty)$, then

$$p_{\theta_2} = \frac{Np_{\theta_1}}{N-(s_{\theta_1}-s_{\theta_2})p_{\theta_1}}$$

if and only if

$$\widetilde{p} = \frac{Np}{N-(s-\widetilde{s})p}.$$

Proof By inspection, one sees that $0 \leqslant s_{\theta_2} \leqslant s_{\theta_1} \leqslant 1$ and $p_{\theta_1}, p_{\theta_2} \leqslant +\infty$. Also, the fact that $p_\theta \geqslant \min\{p,\widetilde{p}\}$ for all $\theta \in [0,1]$ follows from the computations in (4.5).

If in addition $p \leqslant \widetilde{p}$, by the definition of p_θ it follows that $p_{\theta_1} \leqslant p_{\theta_2}$.

We now check that (4.15) implies (4.16). For this, we point out that

$$\begin{aligned} s_{\theta_2} - \frac{s_{\theta_1}p_{\theta_1}-N}{p_{\theta_1}} &= s_{\theta_2} - s_{\theta_1} + \frac{N}{p_{\theta_1}} = (\theta_2-\theta_1)(\widetilde{s}-s) + \frac{N}{p} + N\theta_1\left(\frac{1}{\widetilde{p}}-\frac{1}{p}\right) \\ &\geqslant (\theta_2-\theta_1)(\widetilde{s}-s) + \frac{N}{p} - N\theta_1\left(\frac{s-\widetilde{s}}{N}\right) \end{aligned}$$

$$= \theta_2(\widetilde{s} - s) + \frac{N}{p} > -\frac{N\theta_2}{p} + \frac{N}{p} \geqslant 0,$$

which gives the first claim in (4.16).

Also,

$$\frac{1}{p_{\theta_2}} - \frac{N - (s_{\theta_1} - s_{\theta_2})p_{\theta_1}}{Np_{\theta_1}} = \frac{1}{p_{\theta_2}} - \frac{1}{p_{\theta_1}} + \frac{s_{\theta_1} - s_{\theta_2}}{N}$$
$$= (\theta_2 - \theta_1)\left(\frac{1}{\widetilde{p}} - \frac{1}{p}\right) + \frac{(\theta_2 - \theta_1)(s - \widetilde{s})}{N} \geqslant (\theta_2 - \theta_1)\left(\frac{\widetilde{s} - s}{N} + \frac{s - \widetilde{s}}{N}\right) = 0$$

which gives the second claim in (4.16).

Finally, the last statement follows from the definition in (1.5). □

We now establish Theorems 1.10 and 1.11.

Proof of Theorems 1.10 and 1.11 In what follows, Ω is either $\mathbb{R}^N$ or an open and bounded domain with Lipschitz boundary. In the first case, we suppose that (1.16) is in force, while in the latter case we assume (1.18).

We provide the proof of the first statement of Theorems 1.10 and 1.11.

We first consider the case $\widetilde{p} = +\infty$. Then, we are in the case $\widetilde{s} \in [0, (sp-N)/p]$. From [7, Theorem 8.2] we know that $W^{s,p}(\Omega)$ is continuously embedded in $C^{0,\frac{sp-N}{p}}(\Omega)$. Moreover, we have that $C^{0,\frac{sp-N}{p}}(\Omega)$ is continuously embedded in $C^{0,\widetilde{s}}(\Omega)$. These considerations prove that $W^{s,p}(\Omega)$ is continuously embedded in $C^{0,\widetilde{s}}(\Omega)$, which is the desired statement when $\widetilde{p} = +\infty$.

Now, we deal with the case $\widetilde{p} \in [p, +\infty)$. We set

$$\overline{s} := s - N\left(\frac{1}{p} - \frac{1}{\widetilde{p}}\right) \tag{4.17}$$

and we point out that

$$\overline{s} \in \left(\frac{sp - N}{p}, s\right] \quad \text{and} \quad \widetilde{p} = \frac{Np}{N - (s - \overline{s})p}.$$

Thus, we are in the position of using Lemma 4.3 with $\overline{p}$ replaced by $\widetilde{p}$ and we deduce that $W^{s,p}(\Omega)$ is continuously embedded in $W^{s-N\left(\frac{1}{p}-\frac{1}{\widetilde{p}}\right),\widetilde{p}}(\Omega)$.

Also, we claim that

$$\overline{s} \in [\widetilde{s}, s]. \tag{4.18}$$

Indeed, since $p \leqslant \widetilde{p}$, we have that $\overline{s} \leqslant s$.

Moreover, if $\widetilde{s} \in [0, (sp - N)/p]$, we see that

$$\widetilde{s} \leqslant s - \frac{N}{p} < s - N\left(\frac{1}{p} - \frac{1}{\widetilde{p}}\right) = \overline{s}. \tag{4.19}$$

If instead $\widetilde{s} \in ((sp - N)/p, s]$, we use the condition on $\widetilde{p}$ in (1.16) to see that

$$\frac{1}{\widetilde{p}} \geqslant \frac{1}{p} - \frac{s - \widetilde{s}}{N}$$

and therefore

$$\widetilde{s} \leqslant \frac{N}{\widetilde{p}} - \frac{N}{p} + s = \overline{s}. \tag{4.20}$$

Gathering these observations, we obtain (4.18).

Thanks to (4.18), we can exploit Lemma 4.4 with p replaced by $\widetilde{p}$ and s replaced by $\overline{s}$. In this way, we obtain that $W^{s-N\left(\frac{1}{p}-\frac{1}{\widetilde{p}}\right),\widetilde{p}}(\Omega)$ is continuously embedded in $W^{\widetilde{s},\widetilde{p}}(\Omega)$. As a result, we conclude in this case that $W^{s,p}(\Omega)$ is continuously embedded in $W^{\widetilde{s},\widetilde{p}}(\Omega)$, as desired.

If Ω is an open and bounded domain with Lipschitz boundary, we also consider the case $\widetilde{p} \in [1, p)$. In this situation, we employ Lemma 3.5 and we deduce that $W^{s,p}(\Omega)$ is continuously embedded in $W^{\widetilde{s},\widetilde{p}}(\Omega)$.

This concludes the proof of the first statement of Theorems 1.10 and 1.11.

We now prove the second claim of Theorems 1.10 and 1.11. For this, take $0 \leqslant \theta_1 < \theta_2 \leqslant 1$ and consider $\gamma(\theta_1) = (s_{\theta_1}, p_{\theta_1})$ and $\gamma(\theta_2) = (s_{\theta_2}, p_{\theta_2})$, where the curve γ has been defined in (1.5). Lemma 4.5 allows us to use the first statement of either Theorem 1.10 or Theorem 1.11 with $s := s_{\theta_1}$, $p := p_{\theta_1}$, $\widetilde{s} := s_{\theta_2}$ and $\widetilde{p} := p_{\theta_2}$ and obtain the desired embedding. □

Now, we prove the desired compact embedding in the case $sp > N$.

Proof of Theorem 1.12 Let $\widetilde{s}$ and $\widetilde{p}$ satisfy (1.20).

If $\widetilde{p} = p$, the desired compact embedding is a direct consequence of Lemma 2.7. Thus, we now distinguish the cases $\widetilde{p} > p$ and $\widetilde{p} < p$.

If $\widetilde{p} \in (p, +\infty)$, we define $\overline{s}$ as in (4.17) and we recall that $\overline{s} \in [\widetilde{s}, s)$ (thanks to (4.18) and the fact that $\widetilde{p} \neq p$).

Also, since we know that $\overline{s} \in ((sp - N)/p, s)$, it is

$$\frac{Np}{N-(s-\overline{s})p} = \frac{Np}{N - N\left(\frac{1}{p}-\frac{1}{\widetilde{p}}\right)p} = \frac{p}{1-\left(\frac{1}{p}-\frac{1}{\widetilde{p}}\right)p} = \widetilde{p}.$$

As a result, we find that $\overline{s}$ and $\widetilde{p}$ satisfy the assumptions in (1.18). Therefore, we can apply Theorem 1.11 and conclude that $W^{s,p}(\Omega)$ is continuously embedded in $W^{\overline{s},\widetilde{p}}(\Omega)$.

Notice also that $\overline{s} > \widetilde{s}$ (recall the computations in (4.19) and (4.20) and use the fact that $\widetilde{p} < \frac{Np}{N-(s-\widetilde{s})p}$). Hence, we deduce from Lemma 2.7 that $W^{\overline{s},\widetilde{p}}(\Omega)$ is compactly embedded in $W^{\widetilde{s},\widetilde{p}}(\Omega)$. Combining these pieces of information, we infer that the space $W^{s,p}(\Omega)$ is compactly embedded in $W^{\widetilde{s},\widetilde{p}}(\Omega)$, as desired.

Now, we deal with the case $\widetilde{p} < p$. We observe that $\frac{s+\widetilde{s}}{2} \in (0, s)$. Moreover, if $\frac{s+\widetilde{s}}{2} \in ((sp - N)/p, s)$,

$$\frac{1}{\widetilde{p}} > \frac{1}{p} > \frac{1}{p} - \frac{s-\widetilde{s}}{2N}.$$

Hence, $\frac{s+\widetilde{s}}{2}$ and $\widetilde{p}$ satisfy the assumptions in (1.18). Therefore, we can apply Theorem 1.11 to see that $W^{s,p}(\Omega)$ is continuously embedded in $W^{\frac{s+\widetilde{s}}{2},\widetilde{p}}(\Omega)$.

Moreover, since $\frac{s+\widetilde{s}}{2} > \widetilde{s}$, from Lemma 2.7 we infer that $W^{\frac{s+\widetilde{s}}{2},\widetilde{p}}(\Omega)$ is compactly embedded in $W^{\widetilde{s},\widetilde{p}}(\Omega)$. Gathering these considerations, we conclude that the space $W^{s,p}(\Omega)$ is compactly embedded in $W^{\widetilde{s},\widetilde{p}}(\Omega)$, which concludes the case $\widetilde{p} < p$.

Now, let $0 < \theta_1 < \theta_2 \leqslant 1$ and consider $\gamma(\theta_1) = (s_{\theta_1}, p_{\theta_1})$ and $\gamma(\theta_2) = (s_{\theta_2}, p_{\theta_2})$, where the curve γ has been defined in (1.5). Then, Lemma 4.5 gives that we can use the first statement of Theorem 1.12 with $s := s_{\theta_1}$, $p := p_{\theta_1}$, $\widetilde{s} := s_{\theta_2}$ and $\widetilde{p} := p_{\theta_2}$ and obtain that $W^{s_{\theta_1},p_{\theta_1}}(\Omega)$ is compactly embedded in $W^{s_{\theta_2},p_{\theta_2}}(\Omega)$, as desired. □

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Data availability No data was used for the research described in the article.

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