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Dynamics of atoms in a time orbiting potential trap: consequences of the classical description

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Abstract

The classical model that describes the motion of an atom in a magnetic trap is solved in order to investigate the relationship between the failure of the usual adiabatic approximation assumption and the physical parameters of the trap. This allows to evaluate the effect that reversing of the bias field rotation produces on the vertical position of the atomic orbit, a displacement that is closely related to the adiabatic character of the trap motion. The present investigation has been motivated by a similar experimental test previously carried out in the actual magnetic time orbiting potential trap. We find that the non-adiabatic effects provided by the classical model are extremely small. Thus, we conclude that the theoretical explanation of the experimental measures, requires a quantum description of the dynamics in magnetic traps.

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I. INTRODUCTION

In the realization of Bose-Einstein condensates (BECs) in alkali atoms, the gas is confined and cooled in magnetic traps. Among the different kinds of setups, in many experiments have been used time-orbiting-potential (TOP) traps [1], which are realized with two magnetic components: a static inhomogeneous magnetic-field and an uniform rotating one. The static inhomogeneous component originates a potential well for the atoms, around a local minimum of the magnetic field. The smaller this minimum is, the tighter the atomic confinement is. Nevertheless, at zero magnetic field, atoms can leak from the trapping potential as a consequence of Majorana transitions. In order to avoid these atomic losses, in the TOP traps is added the second magnetic component, that eliminates the zero of the magnetic field at the trap center. The tight and stable confinement of the atomic clouds, achieved in the TOP traps, makes these systems well-suited for the Bose-Einstein condensation as well as for quantum statistical/dynamical studies. With respect to the theoretical investigation of the dynamics of atoms in TOP traps, the *adiabatic approximation*, in which the atomic magnetic moment is assumed to be constantly anti-aligned with respect to the magnetic field, is usually assumed. A further possible mathematical simplification, comes from the *time-average approximation*, in which one assumes as potential for the atoms the time average of the true one. Under these two approximations, the potential seen by an atom in a TOP trap results harmonic and then the system Hamiltonian has trivial quantum eigenstates. However, such systems display a residual *micromotion* due to the fact that the trapping potential is, indeed, time-dependent. This phenomenon has been theoretically predicted in Ref. [2] on the base of a classical model that prescind from the adiabatic approximation, and it has been experimentally observed in Refs. [3, 4]. These latter papers, have recently met the interest of several authors [5–10]. Other theoretical investigations on the quantum dynamics of atoms (condensates) in TOP traps with the adiabatic approximation [11, 12] ([13]), or without this approximation [14], agree with the experimental observations about this phenomenon.

A second surprising phenomenon has been observed in Ref. [3], in this experiment, and for the first time, this team has been able to reveal an *anomalous shift* in the vertical position of the atomic cloud, as the rotating uniform magnetic field reverses its rotation. In the same paper it has been hypothesized that the failure of the adiabatic approximation could be

at the origin of this phenomenon. To our knowledge, this anomalous shift has never been explained from a theoretical point of view. Thus, the main purpose of the present paper is to establish if, within a classical description of the dynamics of an atom in a TOP trap, such phenomenon can be explained. The *definite* outcome of the present paper is that the non-adiabatic effects in the classical description of this system are extremely small. Thus, such anomalous shift, cannot be explained within the classical scenario commonly used to give a simple explanation of the mechanism behind cloud stability in the magnetic traps. Consequently, we conclude that, the theoretical explanation of this phenomenon, requires a quantum description of the dynamics of atoms in a TOP trap.

This work is organized as follows. In section II and III we review some results and general ideas concerning the atomic motion in a magnetic trap. In particular in section II A are introduced the classical equations of motion describing the dynamics of an atom in a magnetic trap, whereas section II B is devoted to a general discussion about the adiabatic approximation that simplifies drastically the resolution of the equations of motion for this kind of systems. In section III, we derive several periodic orbits that take place in a magnetic trap that has a cylindric symmetry around the vertical axis. In section IV we discuss the periodic motion in a triaxial TOP trap similar to that used in the experimental investigation of Ref. [3] and we put a special emphasis on the reversing bias field rotation test. Therefore, in Sec. IV A we derive the periodic orbit that takes place in a triaxial trap for a very small asymmetry. In Sec. IV B we perform a perturbative calculation that, by using the result of App. A, allows us to evaluate the effect that the asymmetry produces on the dipole orientation in space and the consequent reposition of the atomic trajectory along the vertical axis. Finally, in Sec IV B, we evaluate the consequent shift in the vertical position of the atomic trajectory as the bias field reverses its rotation. In Section V we review some fundamental point concerning the above experimental results, and the connected numerical simulation. Thus we analyze the perspective emerging from the fact that the results provided by the classical model do not agree with what is observed experimentally.

II. ATOMIC MOTION IN A MAGNETIC TRAP

The confinement in space of a neutral particle carrying a magnetic moment, can be obtained by means of the gradient magnetic-field forces experienced by a magnetic dipole,

in the presence of an inhomogeneous magnetic field. This mechanism is at the basis of the atomic traps used by many experimental groups working on the subject of Bose-Einstein condensation [1]. In these traps, the atoms are first collected by means of laser forces in a limited region of the space where such inhomogeneous magnetic field plays its active role, and, afterwards, cooled down by an evaporative mechanism so as to make the transition to a Bose-Einstein condensate possible. So far the physics of the confinement by magnetic field has been explained mainly by means of a set of equations in which the particle is seen as a classical point-like magnetic dipole that obeys to the classical equations of motion [2]. Some quantum mechanical calculations exists for the more simple configuration without the rotating bias field [15]. Nevertheless, this rotating field plays an important role in the TOP traps and makes the physics of these systems more intriguing and complicated. In fact, an important matter inherent this motion is the validity of the adiabatic approximation. In the case where this approximation holds, the magnetic dipole is constantly anti-aligned with respect to the magnetic field and, thus, the magnetic energy of the atoms assumes the role of a space confining potential. Existing quantum mechanical approaches [11–13] of atoms (or Bose-Einstein condensates) within a magnetic trap, assumes from the start the fulfillment of the adiabatic condition which is, actually, classical in nature. Indeed the question concerning its range of validity has been tested experimentally in [3]. From these investigations it turns out that such an approximation could become less stringent in the proximity of the marginal stability configuration of a TOP trap. The critical test performed in [3] concerns the observation of an anomalous shift in the vertical position of the atomic cloud as the bias radio-frequency field reverses its rotation. The issue of adiabatic motion has also been theoretically investigated for a symmetric trap, from which the triaxial trap used in [3] is obtained by setting the symmetry axis in the rotating bias field horizontal plane. The dynamics in the cylindric trap has been studied by means of a variational technique approach to the full quantum problem in [14]. Whereas a semi-classical equations including quantum atomic correlation has been used to investigate the more general problem of the motion in a inhomogeneous magnetic field in [16].

However, as said above, genuine quantum approaches to the study of the dynamics of atoms in the TOP trap of Ref. [3] does not exists. These would have the merit of shedding light on the condition of validity of the adiabatic approximation in the presence of the spatial motion that a magnetic dipole confined in a magnetic trap undergoes. These questions can

also be related to the existence of a Berry's phases phenomenology [17]. While acknowledging the need of a more detailed quantum investigation we will discuss here some features of the magnetic confinement in a magnetic trap entirely from the classical point of view and in particular we will focus on the test of the reversal of the bias field rotation. We believe that this has to be carried out considering the importance and the intuitive support that this model provides in explaining the trapping mechanism in the BEC experiments. Thus before of proceeding with a quantum computation it is necessary to assess clearly the scope of validity of the already existing classical-model. This will be done here by checking the validity of this model against the experimental results in [3]. In particular when we will refer generically to the experimental work or to the experimental measurements we will mean experimental work and experimental measurements as done in Ref. [3].

A. Trap classical equations

As a basic ingredient of a magnetic trap there is a magnetic field configuration having a minimum of its intensity at a point in the space generally named trap center. This is obtained with a static quadrupole magnetic field whose components along the Cartesian axes are

$$b_x = b x, \quad b_y = b y, \quad b_z = -2 b z. \quad (1)$$

Moreover, in order to prevent the spin flip transition that would occur when the atom approaches the trap center, a rotating radio-frequency uniform magnetic-field, usually known as bias field, is superimposed to the static inhomogeneous one. In such a way the atom trajectory is kept quite far away from this point. Let B_b be the bias field intensity and ω its angular velocity in the $x - y$ plane. A different configuration for the inhomogeneous field can be obtained by setting the symmetry axis to coincide with the x axis and by inverting the currents, this leads to

$$b_x = 2 b x, \quad b_y = -b y, \quad b_z = -b z, \quad (2)$$

which also provide a stable confinement. We denote by \vec{B} the total magnetic field experienced by an atom which is given by the sum of the inhomogeneous magnetic field and the bias rotating field. The configuration where the inhomogeneous magnetic field is given by Eqs. (1), is known as symmetric or cylindric TOP trap, while the one corresponding to the

configuration of Eqs. (2) is named triaxial trap and is the one used in the experimental setup of Ref. [3].

Within the classical realm, the dynamics for an atom in a magnetic trap is given by the equations of motion

$$\begin{aligned} m\ddot{\vec{r}} &= \vec{F} = (\vec{\mu} \cdot \vec{\nabla}) \vec{B} + m\vec{g}, \\ \dot{\vec{\mu}} &= \gamma \vec{\mu} \times \vec{B}, \end{aligned} \quad (3)$$

where γ connects the magnetic moment to the angular momentum of the particle, i.e. $\vec{\mu} = g' \frac{e}{2m_e} \vec{L} = \gamma \vec{L} := \mu \vec{n}$, here m_e is the electron mass, g' is the gyro-magnetic factor and \vec{n} is the unit vector parallel to the magnetic moment. In the discussion that follows we will use normalized units for the quantities involved in the trap dynamics. First of all we will assign a unit value to the period of the rotating bias field and we adopt a length unit which coincide with the characteristic length of the trap. Furthermore, in order to lift the constant in the second of Eqs. (3), we will include the γ constant in the magnetic field \vec{B} , that will be measured in angular frequency units. Consistently we will absorb the factor $1/\gamma$ in constant μ/m . For a typical TOP trap, we have a bias field frequency of the order of 10kHz and a characteristic length of the order of $1\mu\text{m}$. Thus the time unit will correspond to $T_0 = 10^{-4}\text{s}$, the length unit to $L_0 = 10^{-6}\text{m}$, and the magnetic field intensity in the new unit, will be obtained by multiplying the old value for the factor γT_0 . E.g., a magnetic field of 10.Gauss (order of magnitude of the bias field used in [3]) transforms to a field of about $10^{-3} \frac{1.6 \times 10^{-19}}{9.1 \times 10^{-31}} 10^{-4} = 1.8 \times 10^4$, a field gradient b of 100.Gauss cm^{-1} transform to $\frac{10^{-2} \times 1.8 \times 10^{11} \times 10^{-4}}{10^{-2} \times 10^6} = 18$. The parameter μ/m of the first of Eqs. (4), in the new unit is obtained by multiplying the original value for the factor $T_0/(L_0^2 \gamma)$. Thus, by doing so for a trap, as the one of Ref. [3], operating with ^{87}Rb atoms we have the following numerical values $\frac{\mu}{m} = 0.036$, which is independent from the unit chosen, $g = 0.0981$, $\omega = 2\pi$, $B_b \approx 10^4$ and a value of the field gradient b ranging from 0.1 up to a value of 10. We will refer to these values in the following as the standard trap parameters. From Eqs. (3) we finally obtain the normalized evolution equations

$$\begin{aligned} \ddot{\vec{r}} &= \frac{\mu}{m} (\vec{n} \cdot \vec{\nabla}) \vec{B} + \vec{g}, \\ \dot{\vec{n}} &= \vec{n} \times \vec{B}, \end{aligned} \quad (4)$$

where \vec{n} is the unit vector corresponding to the atomic magnetic dipole.

B. Adiabatic approximation

In the present section we summarize the conditions of validity of the adiabatic approximation and its consequences about the dynamics of atoms in magnetic traps. The adiabatic approximation, that greatly simplifies the analysis of the atomic motion, comes, pretty naturally, from an analysis of equations (4) and, in particular, from the second one. By indicating with n_{\parallel} the projection of the vector \vec{n} on the magnetic field \vec{B} , i.e. $n_{\parallel} = \vec{n} \cdot \frac{\vec{B}}{B}$, and with \vec{n}_{\perp} the component of \vec{n} perpendicular to this same field, that is $\vec{n}_{\perp} = \vec{n} - (\vec{n} \cdot \frac{\vec{B}}{B}) \frac{\vec{B}}{B}$, it is easy to show, by using the second of Eqs. (4), that the following equality holds true

$$\dot{n}_{\parallel} = \vec{n}_{\perp} \cdot \frac{\dot{\vec{B}}}{B}. \quad (5)$$

From this equality it results clear that the validity of the adiabatic regime is a consequence of the fast rotation of \vec{n}_{\perp} around \vec{B} , that takes place at an angular velocity of the order of B . In fact, if $\frac{\dot{\vec{B}}}{B}$ has the same direction of \vec{B} , i.e. \vec{B} changes only in magnitude, the quantity n_{\parallel} is exactly constant. If this is not the case, but $\dot{\vec{B}}$ is small, \dot{n}_{\parallel} is also a small quantity with zero average because of the fast rotation of \vec{n}_{\perp} around \vec{B} . Thus, under the hypothesis of fast rotation of \vec{n}_{\perp} (that is $B \gg 1$) and small $\dot{\vec{B}}$, we can conclude that n_{\parallel} is a quasi-constant of motion and the magnetic energy of a particle in the trap results to be

$$U = -\vec{\mu} \cdot \vec{B} = -n_{\parallel} \mu |\vec{B}|. \quad (6)$$

Now the magnetic energy only depends on the projection of the magnetic moment on the field \vec{B} and not from the orientation of the dipole in space, since the other degrees of freedom have been frozen as a result of the adiabatic approximation. Thus, a particle anti-aligned with the field, or more in general a particle for which n_{\parallel} is negative, experiences a confining magnetic potential in the proximity of a minimum of the magnetic field intensity.

In the case of the TOP traps, where the total magnetic field \vec{B} is time-dependent, it is more convenient to examine the torque equation (second of (4)) in a reference frame rotating around the z axis together the bias magnetic field. In this frame, the effective magnetic field experienced by the particle is given by $\vec{B}_0 = \vec{B} + \vec{\omega}$, where $\vec{\omega}$ is directed along the z axis. In the case of a cylindric trap the latter will be a time-independent quantity, whereas for the triaxial trap \vec{B}_0 remains time-dependent. Thus, in the case of the TOP traps, the adiabatic approximation has to be applied in the reference frame rotating with the bias field and,

therefore, it is the field \vec{B}_0 that plays the role of the field \vec{B} of Eqs. (5) and (6). Some questions related to the fact that a proper choice of the reference frame really improves the analysis of a dynamical system when the adiabatic approximation is to be used have been discussed in [18]. As a matter of fact it is out of doubt that in the TOP traps the frame co-rotating with the bias field is the one that has such property.

In the following we will summarize briefly the peculiarities of the motion in the symmetric and triaxial traps. In this last case, we will calculate the shift of the vertical position of the atomic cloud, as the bias field reverses its rotation. In the case of the symmetric trap this shift is shown to be exactly $\Delta z = -\frac{\omega}{b}$. In fact, a -2ω change in the magnetic field \vec{B}_0 , is exactly compensated by the z component of the inhomogeneous field in Eq. (1), as the orbit displaces itself of Δz in the vertical position. However, in the case of the triaxial trap, this argument does no more hold true because its inhomogeneous field configuration does not enjoy the rotational symmetry around the z axis. This produces a different dynamical evolution of the atomic magnetic moment, that influences also the trap vertical position beyond the simple argument given above. The calculation of this contribution, will show how this shift is connected to the failure of the adiabatic approximation, and to which extent this mechanism can be understood within the limit of this classical model.

III. CYLINDRIC TRAP MOTION

A. The periodic orbits

Stationary solutions of the classical equations (4) for the case of a cylindrical trap, have been given in Ref. [2]. In order to derive these solutions, and to find the stable orbits, Eqs. (4) are transformed, and then solved in the reference frame rotating around the z vertical axis which is opposite to the gravity vector \vec{g} . This frame is the most convenient one because here the bias field results to be time-independent as well as the fictitious magnetic field $\vec{B}_0 = \vec{B} + \vec{\omega}$. In the latter expression $\vec{\omega}$ is directed along the z axis, and \vec{B} includes both the bias field and the inhomogeneous field. Remarkably, this latter has the cylindrical symmetry around the z axis. For the first of Eqs. (4), also the apparent forces must be taken into account. There exists a stable solution in which the magnetic dipole is anti-parallel to field \vec{B}_0 , this complying with the torque equation, and the particle is at rest in this frame. Thus

the total force on the particle must be zero. The projection of the equilibrium equation along the z axis yield $F_z = -2\mu b n_z - mg = 0$, from which we derive

$$n_z = -\cos\vartheta = -\frac{mg}{2b\mu}. \quad (7)$$

Let us consider now the equilibrium in the $x-y$ plane. To fix the ideas we chose the rotating x axis coincident with the bias field (which has intensity B_b), thus $\vec{B}_0 = (B_b + bx, by, -2bz + \omega)$ and the gradient force has components in the $x-y$ plane given by $F_x = b\mu n_x$ and $F_y = b\mu n_y$. The balance with centrifugal force leads to $n_x = -\frac{m\omega^2}{b\mu}x$ and $n_y = -\frac{m\omega^2}{b\mu}y$. By combining this latter equation with the anti-alignment condition, that is $\vec{B}_0 \times \vec{n} = 0$, from $(\vec{B}_0 \times \vec{n})_z = 0$ we obtain $y = 0$. This means that the particle rotates in phase with the bias field. The above arguments lead, therefore, to

$$n_x = -\sqrt{1 - n_z^2} = -\sin\vartheta = -\frac{m\omega^2}{b\mu}r, \quad (8)$$

where $\vec{n} = (-\sin\vartheta, 0, -\cos\vartheta)$ and r being the distance of the equilibrium point from the origin, i.e. the radius of the particle orbit in the original frame at rest. This latter quantity can be derived by the condition of normalization of \vec{n} , and it results

$$r = \frac{\mu b}{m\omega^2} \sqrt{1 - \left(\frac{mg}{2\mu b}\right)^2}. \quad (9)$$

The magnetic moment configuration for the dynamical regime of this trap, and its relative position with respect to the bias field and total field are shown in Fig. 1. Finally the anti-alignment condition between \vec{n} and \vec{B}_0 leads to

$$\frac{B_{0z}}{B_{0x}} = \frac{-2bz + \omega}{B_b + br} = \frac{1}{\tan\vartheta}. \quad (10)$$

By combining this latter equation and the one derived from the Eqs. (7) and (8), we can obtain the z equilibrium position. From Eqs. (9) and (10) we can observe that if ω changes its sign, by keeping fixed all the other parameters of the trap, ϑ does not undergo any change, as can be gather from Eqs. (7) and (8), and a displacement in the z position of amplitude $-\frac{\omega}{b}$ will therefore occur.

B. The pseudo-periodic orbits

We derive in the present subsection a particular kind of orbits that are important for the discussion we will make in the following about the results contained in Ref. [3]. In the case

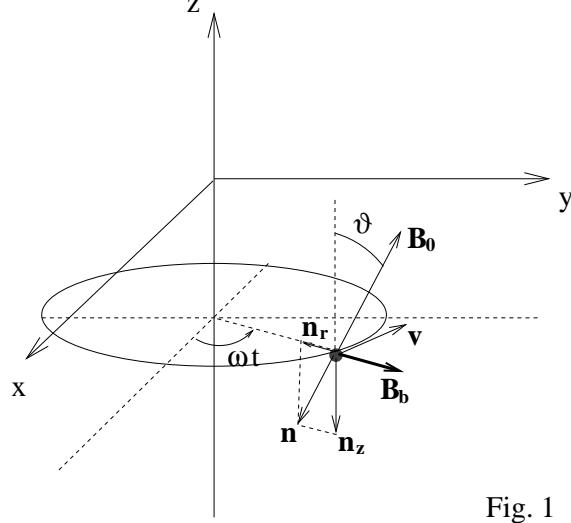


Fig. 1

FIG. 1: The relative position of the magnetic moment, the bias field for a symmetrical trap configuration.

in which the bias field B_b is much greater of its rotation frequency ω , a class of orbits which are almost periodic do exist, even if the magnetic moment is not exactly anti-aligned with total field \vec{B}_0 . As we have recalled in the discussion about the adiabatic approximation, whatever the initial angle between the dipole and the field is, it will remain unchanged since the dipole performs a very fast precessional motion around the magnetic field. As far as the orbital motion is concerned, the fast oscillation of the dipole, that happens on shorter time scales, will be averaged out, and, consequently, the force on the dipole will depend only on its time averaged value. When a motion similar to that described above will take place, the average vector $\langle \vec{n} \rangle$ replaces the original vector \vec{n} , in the expressions (7)-(10) for the periodic motion given in the previous section. Thus, $\langle \vec{n} \rangle$ will be anti-aligned with respect to \vec{B}_0 , and will have intensity $|\langle \vec{n} \rangle| = |\cos \psi|$ where ψ is the initial angle between the dipole and the field. Thus, from Eqs. (7), (8) and (10) we deduce

$$\begin{aligned} \langle n_z \rangle &= -|\cos \psi| \cos \vartheta = -\frac{mg}{2b\mu}, \\ \langle n_x \rangle &= -|\cos \psi| \sin \vartheta = -\frac{m\omega^2}{b\mu}r, \end{aligned} \tag{11}$$

that combined with the equation

$$\frac{-2bz + \omega}{B_b + br} = \frac{1}{\tan \vartheta} \tag{12}$$

determine the orbits height. It is worth to pointing out that, in this case, the orbit z coordinate depends also on the dipole-field angle ψ as also ϑ does, in virtue of Eqs. (11). In other words this orbit z -coordinate depends on the dipole initial conditions.

One may ask if, such an orbit, has any relation with the trapping mechanism operating in the BEC experiments. In this respect it should be remarked that the only classical orbits which correspond to a stationary quantum state are periodic orbits. For these trajectories, the wave associated to the orbital motion undergoes constructive interference as it returns in the same position in space, after that a period has elapsed. This condition gives account for the old Sommerfeld's quantization rule. On the contrary, in the quasi-periodic orbit, the magnetic dipole evolution, is quite independent of the orbital evolution and the particle returns in the same spatial position with its dipole state differing from that of the previous passage through that point. This would lead to destructive interference of the wave associated to the particle. Therefore, the pseudo-periodic orbits can hardly be associated to any stationary quantum state of the system.

C. Oscillations in the vertical position

In the presence of a strong adiabatic field, we can easily describe the oscillations which take place when the initial z -position of the trapped particle is different from the z position given by Eq. (12), in the case of a pseudo-periodic trajectory with initial angle ψ . Indeed, the potential energy in the case of strong adiabatic motion is given by

$$U = mgz + \mu |\cos \psi| \sqrt{(B_b + br)^2 + (-2bz + \omega)^2} \quad (13)$$

to which it corresponds the force

$$F_z = -mg - 2b\mu \langle n_z \rangle = -mg + 2b\mu |\cos \psi| \frac{(-2bz + \omega)}{\sqrt{(B_b + br)^2 + (-2bz + \omega)^2}}. \quad (14)$$

Thus, when r is small such as br can be neglected with respect to B_b , this force depends only on z , and the resulting motion is an oscillatory one around the z equilibrium position derived by Eq. (12). This oscillating regime is important for the discussion that will follow. In fact the z equilibrium position of a BEC cloud is experimentally evaluated just by measuring the average value of these oscillations.

IV. TRIAXIAL TRAP MOTION

In the case of a triaxial trap it is no more possible to obtain an exact solution for the periodic orbit as done above. Indeed, the rotational symmetry about the vertical axis is lost and, consequently, the magnetic field \vec{B}_0 in the frame rotating with the bias field, remains a time-dependent quantity. In order to overcome this problem, we will perform a perturbative calculation in which we will take as a basis the periodic orbit that takes place in this trap, for a very small b parameter. Therefore, in Sec. IV A, we will derive such an orbit, then, by using the result of calculation of App. A, we will evaluate the effect that the small inhomogeneous static magnetic field, seen as a varying one by the dipole during its motion, produces on the dipole orientation in space. We shall show that in general, the dipole orientation, after that the particle returns in its initial position in space, will be changed.

Thus, an additional rotation that brings the magnetic dipole back to its initial state, can be obtained by a static magnetic field. This magnetic field compensation will be obtained by a vertical shift of the orbit in space. Of course, such an adjustment is not needed for the symmetric trap, because the inhomogeneous field is seen as constant one during the orbital motion. It is noteworthy that, such a repositioning of the orbit, will turn out to be different for a bias field rotating in the positive or in the negative sense. This latter point will be shown in Sec. IV B

A. Zero order solution

In this section, we derive the periodic orbit that takes place for a very small b parameter. In order to accomplish this, we neglect the inhomogeneous field components in the $x - y$ plane, only in the torque equation. Thus we write

$$\dot{\vec{n}} = \vec{n} \times (B_b \cos(\omega t), B_b \sin(\omega t), -bz).$$

We can assume the adiabatic approximation i.e that \vec{n} is anti-aligned with respect to the approximate magnetic field, which is given by the bias field and by the z component of the inhomogeneous field. This condition allows us to write, in the laboratory frame, $n_x = -n_\perp \cos \omega t$ and $n_y = -n_\perp \sin \omega t$, where n_\perp indicates the intensity of the component of the dipole unit vector in the plane $x - y$. Let n_z be the intensity of the z component of the

dipole unit vector. Thus, the anti-alignment condition gives the equation

$$\frac{n_{\perp}}{n_z} = \frac{B_b}{(bz - \omega)}. \quad (15)$$

The forces on the particle are

$$\begin{aligned} F_x &= -2\mu b n_{\perp} \cos(\omega t), \\ F_y &= \mu b n_{\perp} \sin(\omega t), \\ F_z &= -\mu b n_z - m g, \end{aligned} \quad (16)$$

by solving the Newton's equations in the $x - y$ plain, we get the time law for the position coordinates x and y in the laboratory frame

$$x(t) = \frac{2\mu b}{m\omega^2} n_{\perp} \cos(\omega t), \text{ and } y(t) = -\frac{\mu b}{m\omega^2} n_{\perp} \sin(\omega t), \quad (17)$$

where n_{\perp} is still an unknown quantity. Thus, for small b the orbit takes place entirely in a plane parallel to the $x - y$ one on an elliptic trajectory, and is counter-rotating with respect to the bias field. The force equation along the z axis provides the angle by which \vec{n} must be tilted with respect to this axis, and the value of n_{\perp} . Indeed, the equation $F_z = 0$ gives, from the third of Eqs. (16),

$$-n_z = \frac{m g}{\mu b} := \cos \vartheta, \text{ and } n_{\perp} = \sin \vartheta. \quad (18)$$

By combining the latter equations with the one in (15), we get

$$\frac{-bz + \omega}{B_b} = \frac{1}{\tan \vartheta}, \quad (19)$$

which with (18) makes it possible to determines the orbit's z level.

In order to include in the torque equation the contribution given by inhomogeneous field in the plane $x - y$, we have to compute the magnetic field $\vec{b}'(t)$, in the $x - y$ plane, that the dipole experiences along its trajectory for effect of the inhomogeneous field. In the laboratory frame this is given by $\vec{b}'(t) = (2bx(t), -by(t))$, where $x(t)$ and $y(t)$ are given in Eq. (17). It is convenient to refer this field respect to the system frame rotating with the bias field, it is not difficult to find

$$\begin{aligned} b'_x &= \frac{3}{2} \frac{\mu b^2}{m\omega^2} n_{\perp} \cos 2\omega t + \frac{5}{2} \frac{\mu b^2}{m\omega^2} n_{\perp}, \\ b'_y &= -\frac{3}{2} \frac{\mu b^2}{m\omega^2} n_{\perp} \sin 2\omega t. \end{aligned} \quad (20)$$

Thus, the inhomogeneous field component on the dipole results to be the sum of a constant field and a small time varying component. Apart from the constant term which resets the value of the bias field, as occurred also in the previous case concerning the symmetric trap, the sinusoidal components produces deviations from the anti-alignment condition. In order to evaluate these deviations, and how they can be compensated, we will solve the torque equation under the presence of a constant field complemented with two small oscillating components.

B. Difference of height between ω and $-\omega$

In the present section we will apply the results found in appendix IV A, making explicit Eq. (A6), in the case of the motion in the triaxial trap. Before making these equations explicit for the case of the triaxial trap, we evaluate the order of magnitude of the deviation from the adiabatic approximation, a fair indication of this being the angle between \vec{n}_0 and \vec{B}_0 . From Eq. (A6) is clear that this quantity increases as \vec{B}_0 gets closer to ω . However, in the usual experimental setups, this condition is carefully avoided because it is known to lead to instabilities in the trap. Indeed, reasonable experimental values for the quantities that appear in (A6) are $B_0 \approx B_b$ of the order of 10^4 and $\omega = 2\pi$ in our units. Thus, if we assume the condition in which $\omega \ll B_0$, the second term of Eq. (A6) dominates over the first one. Its intensity is of the order of $\frac{b_0^2}{B_0^2}$ with b_0 standing for the order of magnitude of $|\vec{b}_1|$ and $|\vec{b}_2|$ of (A1). Now, for typical experimental parameters, $b_0 = \frac{3}{2} \frac{\mu b^2}{m\omega^2} n_\perp \approx \frac{\mu b^2}{m\omega^2} = 0.92$ (here we have used $b = 10$). Thus the amplitude of the corrections due to the second term in Eq. (A6) results of the order of 10^{-8} . The first term of the same equation, is instead of the order of $\frac{\omega}{B_0} \cdot \frac{b_0^2}{B_0^2}$, and then, it results to be smaller. However, its dependence on ω is important and it may perhaps become more relevant for trap where the adiabatic condition is not so strongly enforced, as in the case of the experimental setup being considered here. We will derive explicitly the shift entailed from this term in the following of the section. It must be noticed also that, given the relative size of the two terms in the above equation, the adiabatic approximation is broken earlier than the effect of the reversed rotation can be observed.

In order to use the result of appendix IV A, we adopt a frame rotating with the bias field

and we set

$$b_0 = \frac{3}{2} \frac{\mu b^2}{m \omega^2} n_\perp. \quad (21)$$

The only components of the fields \vec{b}_1 and \vec{b}_2 of Eq. (A1), different from zero in this frame are $b_{1,x} = b_0$ and $b_{2,y} = -b_0$, as they can be read from Eq. (20). Moreover we have $\vec{B}_0 = (B_b + \frac{5}{3}b_0, 0, -bz + \omega)$ and $\vec{n}_0 = (-\sin \vartheta, 0, -\cos \vartheta)$, where the angle ϑ is that one corresponding to the unperturbed motion and given by Eq. (18). Since initially \vec{n}_0 and \vec{B}_0 are opposite to each other, Eq. (A6) prescribes how one of them should be modified. However since \vec{n}_0 , and consequently the angle ϑ is already determined by the equilibrium equation along the z axis, only \vec{B}_0 can be modified. This modification amounts to a small time-independent contribution obtained, in the specific case of the triaxial trap, by a shift of the whole orbit in the z direction. In order to calculate the angle between \vec{n}_0 and \vec{B}_0 we note that the only component of $\vec{n}_0 \times \vec{B}_0$ different from zero is given by

$$(\vec{n}_0 \times \vec{B}_0)_y = \frac{1}{2} \left(\frac{b_0}{\Delta} \right)^2 (-2\omega \sin \vartheta + B_0 \sin \vartheta \cos \vartheta), \quad (22)$$

where one should remind that the angular frequency of the oscillating field component is 2ω instead of ω (Cf. Eqs. (20) and (A1)). The above equation makes evident that \vec{n}_0 still lies in the $x - z$ plane, and that the tilting angle of \vec{B}_0 with respect to \vec{n}_0 , whose sign is determined by the y axis, is now given by

$$\Delta \vartheta = -\frac{1}{2} \left(\frac{b_0}{\Delta} \right)^2 \left(-\frac{2\omega}{B_0} \sin \vartheta + \sin \vartheta \cos \vartheta \right), \quad (23)$$

where B_0 is the intensity of the total magnetic fields and $\Delta^2 = B_0^2 - 4\omega^2$. The new component $B_{0,z}$ must now satisfy the condition $\frac{B_{0,z}}{B_b + \frac{5}{2}b_0} = \frac{1}{\tan(\vartheta + \Delta \vartheta)}$. By using the approximation $\frac{1}{\tan(\vartheta + \Delta \vartheta)} = \frac{1}{\tan \vartheta} - \frac{\Delta \vartheta}{\sin^2 \vartheta}$ we obtain

$$z = z_0 + \frac{1}{2} \left(\frac{b_0}{\Delta} \right)^2 \left(z_0 + \frac{\omega}{b} \right), \quad (24)$$

where z_0 is the z position of the orbit in the zero approximation. From this expression it results that the shift in z , caused by the change $\omega \rightarrow -\omega$, for which $\Delta z_0 = -\frac{2\omega}{b}$ and $\Delta \omega = -2\omega$, is given by

$$\Delta z = -\frac{2\omega}{b} \left[1 + \left(\frac{b_0}{\Delta} \right)^2 \right]. \quad (25)$$

This expression depends on b but, considered the small size of the perturbing field $b_0 = \frac{3}{2} \frac{\mu b^2}{m \omega^2} \sin \vartheta$, it remains very small anyway even if B_0 becomes close to the bias field frequency

ω where, on the other hand, the perturbative calculation presented here would no longer apply. It is important to point out that the correction found above depends, quite naturally, on the ratio between the strength of the average inhomogeneous field experienced by the particle during its motion and the total static field B_0 . Then, it is approximatively given by the ration between b_0 , and B_b . The predicted shift is shown in Fig. 2 and compared with the zero order compensation shift $\Delta z_0 = -\frac{2\omega}{b}$. As it is expected the difference between the

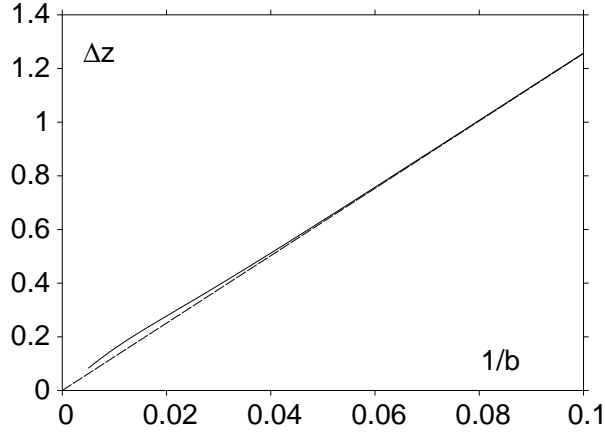


FIG. 2: The shift obtained with the theoretical calculations (continuous line) and the zero order compensation shift (dotted line).

two is really a very small one even for the parameter used in a real trap of Fig. 2.

C. Non-adiabatic regime and bi-stability of the trap

In order to compare the above theoretical derivation with the results obtained by numerically integrating the equations of motion (4), we have determined, by numeric computations, the periodic orbits for this system. The method, which is customarily used in investigations of this kind, assumes that a periodic, and then closed, orbit, named reference orbit, is known for certain values of the system parameters. A small change of one of these parameters opens in space, by a slight amount, the reference orbit when evolved over a period of the time-dependent force. Thus, a small variation of the initial condition could recover the periodicity, and close the orbit in space again. A linear map connecting initial and final values, for all orbits that lay close to the reference one, can thus be determined by performing evolutions of the system with initial conditions very close to those of the reference orbit. This map does

not appreciably change as the parameters undergo small changes. Thus the new periodic orbit can be found as the fixed point of this map by inverting numerically the related matrix. In our case the initial reference orbit is that corresponding to a small value of the parameter b . Starting from this orbit, we can determine numerically the periodic orbit of the system for any given parameter value b^* by means of many small changes of the b parameter, until the final value b^* is reached.

First of all, we have used this numerical procedure to compare the shift in the z position of the orbit with that derived in the previous section. More precisely we have numerically calculated averaged value of this quantity, and compared it with Eq. (25). As reported in

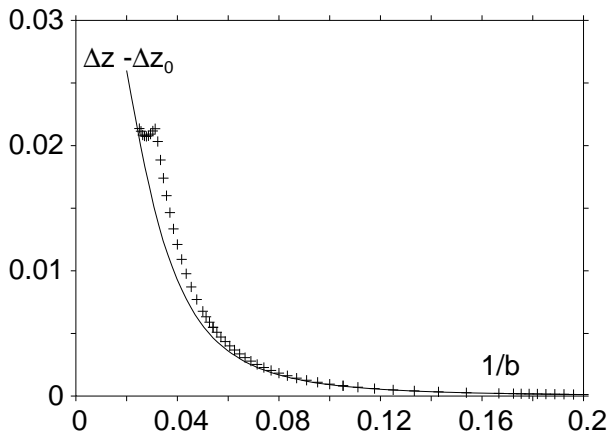


FIG. 3: Comparison between the theoretical shift (continuous line) and the numerical one (crosses). In order to make more evident this comparison the zero order shift $\Delta z_0 = \frac{2\omega}{b}$ has been subtracted to both of these two quantities. At small values of the $\frac{1}{b}$ the regularity of this dependence is somewhat broken as shown by the little terminating wiggle. This corresponds also to orbits which are on the point of becoming unstable. The parameters used for this figure are $B_r = 50.$, $\frac{\mu}{m}=0.036$, $g = 0.1$.

Fig. 3, we observe that the agreement is quite good even for values of b in a more wide range of those commonly encountered in experimental setups. The calculated shift is extremely small such as being, at the present, out of the reach of experimental verification, and certainly far away from the values observed in the experiment of Ref. [3].

By using the numerical routines that we have designed for the analysis of the periodic orbits, we have performed a coarse examination of the unstable dynamical regimes of the

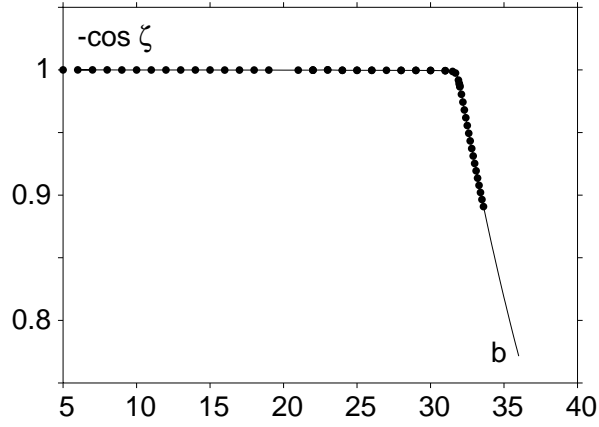


FIG. 4: The cosine of the angle between the magnetic field and the magnetic dipole at time $t = 0$ of the periodic orbit, as a function of the b parameter for both σ^+ (continuous line) and σ_- (big dots) polarization of the bias field.

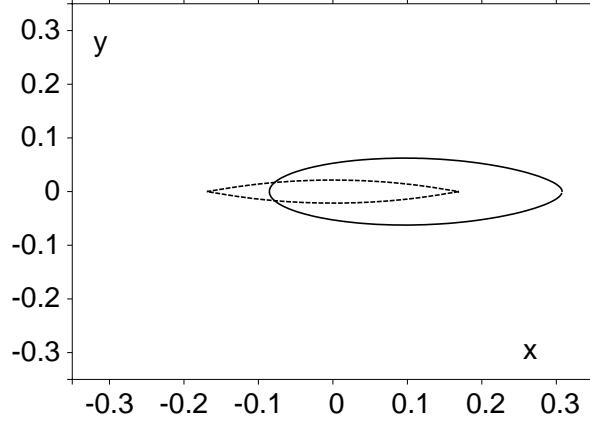


FIG. 5: Two bistable orbits corresponding to trap values of $b = 10.$, $B_r = 11.36$ and σ^+ (anti-clockwise) bias field.

trap. It is found that signatures of instability appears, quite naturally, as the intensity of the inhomogeneous field experienced by the particle becomes of the same order than that of the bias field. For these values the peculiar scenario leading to unstable behavior with orbit bifurcations, orbit bi-stability and eventually chaotic behavior will emerge. However the instability is always connected, at least for the range of the parameters observed here, to the loss of the system adiabatic condition. We have numerically determined the angle between

the effective field \vec{B}_0 and the dipole vector at time $t = 0$, i.e. when the bias field crosses the positive x axis. This is shown in Fig. 4 for both directions of bias field rotation. As it can be seen here the adiabatic approximation is quite resilient even if the trap parameters are rather loose, in terms of the adiabatic condition enforcement, with respect to those normally used in a TOP trap. For the same range of values of the b parameter where an appreciable dis-alignment occurs we have observed a bi-stability regime. The two orbits involved in this regime are shown in Fig. 5. The analysis of the unstable regimes of this system would require a systematic study that is out of the purpose of the present work.

V. COMPARISON WITH THE EXPERIMENTAL RESULTS

There is an evident discrepancy between the results obtained here and the experimental ones reported in Ref. [3]. One of the hypothesis that we can follow to deal with this situation is to assume that a classical model is not an adequate one to describe the dynamics of an atom in the triaxial TOP trap and that a quantum calculation is in order. Before to assume the above hypothesis as the final one, let us to try an alternative analysis that take in account of a procedure for the bias-field reversal, similar to that one actually performed in the experiment of [3]. In the reasoning which follow we will refer to a symmetric trap considered that the perturbation to the magnetic dipole motion produced by the inhomogeneous field of the triaxial trap are really negligible for the experimental parameters. Thus the effect that we are going to describe will occur indifferently for the symmetric and triaxial trap. Let us suppose that the system moves actually on a quasi-periodic orbit with an angle ψ between dipole and field. We remind that this angle does determine the z position of the orbit according to Eq. (11) and Eq. (12). Now, if the sense of rotation of the bias field is abruptly changed the system will experience the oscillations along the z axis discussed in Section III C, because the z coordinate corresponding to the new pseudo-periodic orbit will be different from the one in which the system is actually in. In the experiment, the averaging of these oscillations is assumed as the z coordinate of the BEC stationary state corresponding to the new bias field rotation sense. On the other hands, the classical equations will lead to a shift that coincide with a zero order compensation shift only if the angle ψ , and by consequence the equilibrium ϑ , does not change. On the contrary, if this does not occur an anomalous contribution should be added to the standard compensation shift. Indeed, this

would be the case if we assume that during the switching between the two rotation senses the dipole orientation in space does not change. If this is the case, the angle ψ will change uniquely as a result of the change of the direction of \vec{B}_0 in space, as a consequence of the change of its z projection by an amount of $\Delta\vec{B}_{0,z} = \Delta\omega = -2\omega$. In appendix B, we have calculated the shift obtained under these conditions, assuming that both the magnetic dipole and the field lies in the plane $x - y$, and that ψ is the initial angle between the two. This latter quantity can, of course, assume positive or negative values. After a bit of algebra, reported in appendix B we have found

$$\Delta z = -\frac{2\omega}{b} \left(1 + \frac{mg}{2b\mu} \frac{1}{\sin\vartheta} \frac{\sin\psi}{\cos^2\psi} \right). \quad (26)$$

This expression evidences how by choosing appropriately the angle ψ , the trend of the experimental data, according to which the deviation increases as b approaches its minimum value, could be reproduced. We notice that in this case, also ϑ approaches the zero value, thus amplifying even further the expected effect. The above expression indicates that the anomalous shift effect disappear in absence of gravity, a particularity which coincides also with the conclusions reported in the experimental work. Here we are presuming that the switching between positive and negative ω does occur so fast that the dipole has no time to adapt to the new field \vec{B}_0 , present after the switch has taken place. Although this is rather unlikely to occur, an alternative mechanism wouldn't make sense. In fact, it would be possible to imagine that a loss of the adiabaticity could take place during the switching process, if at certain point the intensity of the magnetic field would get close to zero, in such a way that the angle between the field and the dipole emerges out the switching process changed. This, of course, would depend on how technically this switching is carried out. If this would be the case, the new oscillation point would be displaced by a quantity that contains a further anomalous contribution. However, if it would be so, the anomalous Δz_a would be only a fictitious one being dependent just on how the switching process is carried out. Moreover the question of why the system selects a particular value of $\psi \neq 0$ and, in addition, of how a pseudo-periodic orbit can be put in correspondence with a quantum state, would still remain open ones. In this respect we believe that although our arguments, based on periodic orbits, are particular sound it might still be possible that this is not yet sufficient to properly describe quantum stationary states of the BEC cloud and quantum calculation should be in order. In any case further evidences in support of some of the possibilities

expressed here should also be searched for.

VI. CONCLUSION

In this work, we have used the classical equations of motion to investigate some properties of the magnetic confinement of a neutral atom in the TOP trap. Our initial purpose was that of understanding the origin of an anomalous shift in the vertical position of a BEC cloud that has been observed experimentally in Ref. [3]. We have calculated analytically this shift and found that this is not in agreement with these measurements. In spite of this, our calculations show that, in principle, there is no reason for this shift being limited only to the standard compensation shift $\frac{2\omega}{b}$, as it is the case in the symmetrical trap. We have shown that an anomalous shift exists in any case in the triaxial trap. In fact, the anomalous shift calculated here is produced by the interplay between the particle space dynamics, and that of its magnetic moment driven by the torque equation. However its magnitude results to be extremely small for the parameters used in the normal trap. On the contrary, if such a shift were to exist in line with the calculations made in [3], of which account has been given here, it would be independent of the kind of trap used, i.e. symmetrical or triaxial one, and would depend basically by the peculiarity of the switching precess. In addition we have observed a bi-stability regime in the range of values of the b parameter where the adiabatic regime is lost. The systematic study of this bi-stability regime and of the consequent unstable-dynamics, deserves a further investigation.

In conclusion, the general feeling is that, the approach based on the classical model, leaves so many unanswered questions, that we have the only possible alternative to explain the experimental results by means a quantum mechanical calculation. In fact, in the quantum mechanical approach, the periodicity condition of both the space and the internal spin variables, could be much more stringent than in the classical case thus amplifying an effect that appears otherwise to be quite a small one.

Appendix A: Dynamics of the small oscillating components

In the present appendix we consider a dipole moving in a magnetic field given by a static component with added small sinusoidal components. In particular, the present section

is aimed at derive the small deviations from the anti-alignment, due to the effect of the sinusoidal components like the ones of Eqs. (20). Let the total field acting on the magnetic dipole be

$$\vec{B}_0 + \vec{b}_1 \cos \omega t + \vec{b}_2 \sin \omega t, \quad (\text{A1})$$

and let us suppose that $|\vec{b}_1|$ and $|\vec{b}_2|$ are very small with respect to $|\vec{B}_0|$, which is a time-independent vector. We look for a periodic solution of the form

$$\vec{n} = \vec{n}_0 + \vec{n}_1 \cos \omega t + \vec{n}_2 \sin \omega t \quad (\text{A2})$$

where \vec{n}_1 and \vec{n}_2 are small deviations from the anti-alignment. Here \vec{n}_0 is the zero order solution which is anti-aligned with respect to the field \vec{B}_0 . By inserting the above expressions in the torque equation for the dipole unit vector, and by neglecting the higher order terms coming from both the product between the small rotating components of the magnetic moment and the small rotating components of the magnetic field, one obtains the following equations

$$\begin{aligned} -\omega \vec{n}_1 &= \vec{n}_2 \times \vec{B}_0 + \vec{n}_0 \times \vec{b}_2, \\ \omega \vec{n}_2 &= \vec{n}_1 \times \vec{B}_0 + \vec{n}_0 \times \vec{b}_1. \end{aligned} \quad (\text{A3})$$

These equations can be solved for \vec{n}_1 and \vec{n}_2 , thus we obtain [19]

$$\begin{aligned} \vec{n}_1 &= \frac{\omega \vec{n}_0 \times \vec{b}_2 - B_0 (\vec{b}_1)_\perp}{B_0^2 - \omega^2}, \\ \vec{n}_2 &= \frac{-\omega \vec{n}_0 \times \vec{b}_1 - B_0 (\vec{b}_2)_\perp}{B_0^2 - \omega^2}. \end{aligned} \quad (\text{A4})$$

Here the subscript \perp means perpendicular to the static magnetic field. To the smallest order in the oscillating field amplitude, the constant term in the torque equations yields

$$\vec{n}_0 \times \vec{B}_0 + \frac{1}{2} (\vec{n}_1 \times \vec{b}_1 + \vec{n}_2 \times \vec{b}_2) = 0. \quad (\text{A5})$$

This expression shows that the anti-alignment condition between \vec{n}_0 and \vec{B}_0 can no longer be satisfied. Thus we calculate the last two terms of the above equation by using the Eqs. (A4), after a bit of algebra we obtain

$$\begin{aligned} \vec{n}_0 \times \vec{B}_0 &= \omega \frac{(\vec{b}_2 \cdot \vec{n}_0) \vec{b}_1 - (\vec{b}_1 \cdot \vec{n}_0) \vec{b}_2}{2 (B_0^2 - \omega^2)} + \\ &\quad \frac{\{(\vec{b}_1 \cdot \vec{B}_0) (\vec{n}_0 \times \vec{b}_1) + (\vec{b}_2 \cdot \vec{B}_0) (\vec{n}_0 \times \vec{b}_2)\}}{2 (B_0^2 - \omega^2)}. \end{aligned} \quad (\text{A6})$$

This equation indicates how \vec{n}_0 or \vec{B}_0 must be modified in order to satisfy the torque equation.

Appendix B

We assume that the dipole and the magnetic field \vec{B}_0 lies both in the $x - y$ plane of the rotating frame and that the system moves in a quasi-periodic orbit. Let ϑ be the angle between the z axis and the field and ψ the angle formed by the dipole with the field, both measured with reference to the y axis rotations. It is not difficult to calculate the change in the vertical position of the orbit that occurs as a consequence of a change of the dipole-field angle ψ and of the bias pulsation frequency ω . Indeed from Eq. (11), noting that $|\cos \psi| = -\cos \psi$ for $\psi > \frac{\pi}{2}$, we obtain

$$\Delta\vartheta = \frac{mg}{2b\mu} \frac{\sin \psi}{\cos^2 \psi} \frac{1}{\sin \vartheta} \Delta\psi, \quad (\text{B1})$$

which is the change that the pseudo-periodic orbit angle ϑ undergoes. Thus from Eq. (12) we obtain that Δz and $\Delta\omega$ are correlated as follows

$$-2b\Delta z + \Delta\omega = -B_b \frac{\Delta\vartheta}{\sin^2 \vartheta} = -\frac{B_b}{\sin^2 \vartheta} \frac{mg}{2b\mu} \frac{\sin \psi}{\cos^2 \psi} \frac{1}{\sin \vartheta} \Delta\psi, \quad (\text{B2})$$

where the small term corresponding to the inhomogeneous field has been neglected with respect to B_b . On the other hand the change in the field direction, as a result of small changes of the z component is simply given by

$$-\frac{\Delta\vartheta}{\sin^2 \vartheta} = \frac{\Delta\omega}{B_b}, \quad (\text{B3})$$

where the fact that the vertical position of the orbit is unmodified during the phase that reverts the rotation of the bias field has been taken into account. A change in the field direction reflects in a change of the dipole-field angle and, assuming that in the mean time the dipole angle has remained fixed, we have $\Delta\psi = -\Delta\vartheta$. Thus

$$\Delta z = -\frac{2\omega}{b} \left(1 + \frac{mg}{2b\mu} \frac{1}{\sin \vartheta} \frac{\sin \psi}{\cos^2 \psi} \right). \quad (\text{B4})$$

As a consequence of the instantaneous modification of the bias field rotation, the system performs oscillations around a position which is displaced by a quantity Δz with respect to the initial position of the quasi-periodic orbit.

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