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Petr Cintula, Petr Hájek and Carles Noguera, eds.

Volume 38
Handbook of Mathematical Fuzzy Logic. Volume 2
Petr Cintula, Petr Hájek and Carles Noguera, eds.

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Volume 1

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Volume 1

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CONTENTS

Preface	v
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Volume 1

The list of authors of Volume 1	xi
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I Introduction to Mathematical Fuzzy Logic

Libor Běhounek, Petr Cintula, and Petr Hájek	1
1 Propositional logics of continuous t-norms	2
1.1 Standard semantics of t-norm fuzzy logics	2
1.2 Axiomatic systems for logics of continuous t-norms	13
1.3 Algebraic semantics	21
2 Variations of basic propositional fuzzy logics	26
2.1 Discarding axioms or rules	28
2.2 Adding new connectives	36
2.3 Discarding connectives	43
2.4 Adding axioms or rules	49
3 Families of fuzzy logics in the logical landscape	55
3.1 Fuzzy logics among substructural logics	56
3.2 Core and Δ -core fuzzy logics	59
3.3 Fuzzy logics as algebraically implicative semilinear logics	64
4 Metamathematics of propositional fuzzy logics	68
4.1 Completeness theorems	69
4.2 Functional representation	72
4.3 Proof theory	74
4.4 Computational complexity	77
5 Predicate fuzzy logics	78
5.1 Syntax	78
5.2 Semantics	82
5.3 The identity and sorts of objects	87
5.4 Axiomatic theories over fuzzy logic	89
5.5 Higher-order fuzzy logic	92

II A General Framework for Mathematical Fuzzy Logic	
Petr Cintula and Carles Noguera	103
1 Introduction	103
2 Weakly implicative logics	105
2.1 Basic notions and a first completeness theorem	106
2.2 Weakly implicative logics and a second completeness theorem	112
2.3 Advanced semantics and a third completeness theorem	116
2.4 Algebraically implicative logics	121
2.5 Substructural logics	125
2.6 Deduction theorems and proof by cases in substructural logics	136
2.7 Generalized disjunctions	142
3 Semilinear logics	153
3.1 Basic definitions, properties, and examples	154
3.2 Disjunction and semilinearity	160
3.3 Strengthening completeness: densely ordered chains	165
3.4 Strengthening completeness: arbitrary classes of chains	170
4 First-order predicate semilinear logics	176
4.1 Basic syntactic and semantic notions	177
4.2 Axiomatic systems and soundness	182
4.3 Predicate substructural logics	186
4.4 Completeness theorem	190
4.5 \exists -Henkin theories, Skolemization, and witnessed semantics	195
5 Historical remarks and further reading	200
III Proof Theory for Mathematical Fuzzy Logic	
George Metcalfe	209
1 Introduction	209
2 A Gentzen systems primer	210
2.1 Rules, systems, derivations	211
2.2 A proof system for lattices	211
2.3 A sequent calculus for classical logic	215
2.4 Substructural logics	218
3 From sequents to hypersequents	219
3.1 The core set of rules	220
3.2 Adding structural rules	223
3.3 Soundness and completeness	227
4 Eliminations and applications	230
4.1 Cut elimination	232
4.2 Density elimination	239
5 The fundamental fuzzy logics	245
5.1 Gödel logic	245
5.2 Łukasiewicz logic and Giles's game	250

5.3 Product logic and related systems	257
5.4 Uniform systems	259
6 Quantifiers and modalities	260
6.1 First-order quantifiers	261
6.2 Propositional quantifiers	269
6.3 Modalities	272
7 Historical remarks and further reading	277

IV Algebraic Semantics: Semilinear FL-Algebras

Rostislav Horčík 283

1 Substructural logics	283
2 Algebraic preliminaries	287
2.1 Universal algebra	287
2.2 Residuated maps and Galois connections	290
3 FL-algebras	293
3.1 Examples of FL-algebras	298
3.2 Involutive FL-algebras	299
3.3 A bottom and a top element	300
3.4 Algebraizability	300
3.5 Congruences and filters	302
3.6 Semilinear varieties of FL-algebras	303
3.7 Nuclei and conuclei	307
3.8 Dedekind-MacNeille completion	312
4 Completeness with respect to distinguished semantics	318
4.1 Integral semilinear varieties	320
4.2 Non-integral semilinear varieties	332
5 Subvariety lattice	336
5.1 General facts	337
5.2 Finitely many atoms	339
5.3 Continuum many atoms	344
6 Historical remarks and further reading	350

V Hájek's Logic BL and BL-Algebras

Manuela Busaniche and Franco Montagna 355

1 Introduction	355
2 Definitions and basic properties of BL-algebras	357
2.1 Continuous t-norms	358
2.2 Hoops and BL-algebras	359
2.3 BL-algebras as residuated lattices	361
2.4 BL-chains and completeness	363
2.5 Important subvarieties of BL-algebras	365

2.6	Implicative filters	366
2.7	Regular and dense elements	367
3	Decomposition theorems for BL-algebras	368
3.1	Ordinal sums	368
3.2	Decomposition of BL-chains into irreducible hoops	369
3.3	Decomposition into regular and dense elements	375
3.4	Poset products	375
3.5	BL-algebras as subdirect poset products	377
3.6	BL-algebras as MV-product algebras with a conucleus	379
4	Classes generating the variety of BL-algebras	381
4.1	General facts about ordinal sums	381
4.2	Generation by finite members	383
4.3	Generation by t-norm BL-algebras	384
4.4	Generation by single BL-chains	385
4.5	Generating the variety of SBL-algebras	388
4.6	Generating classes and completeness theorems	390
5	Varieties of BL-algebras	391
5.1	Varieties generated by t-norm algebras	392
5.2	Varieties generated by regular BL-chains	395
5.3	Canonical representation of regular BL-chains	398
5.4	Axiomatizing the variety generated by a finite BL-chain	404
5.5	Axiomatizing the variety generated by an arbitrary finite BL-algebra	407
6	Amalgamation, interpolation and Beth's property in BL	408
6.1	General facts about interpolation and Beth's property	408
6.2	Amalgamation	410
6.3	Deductive interpolation and Beth's property, and strong amalgamation	416
6.4	Craig interpolation and implicative Beth property	419
7	Completions of BL-algebras	421
7.1	Negative results about completions	422
7.2	k -subvarieties and classification	424
7.3	Completions in k -subvarieties	425
7.4	Canonical extensions of BL-algebras	428
7.5	Canonical extensions of k -subvarieties	431
8	Further investigations of BL-algebras	436
8.1	Introduction	436
8.2	Generalizations of the notion of BL-algebra	436
8.3	Implicative subreducts of BL-algebras and of their generalizations	439
8.4	Further results about BL-algebras	440
9	Historical remarks and further reading	442

Volume 2

The list of authors of Volume 2

v

VI Łukasiewicz Logic and MV-Algebras

Antonio Di Nola and Ioana Leuştean	469
1 Introduction	469
2 Definitions and basic properties	470
2.1 MV-algebras	470
2.2 The lattice structure of an MV-algebra	472
2.3 The implication	477
2.4 Examples of MV-algebras	481
2.5 The interval MV-algebra of an ℓu -group	483
2.6 The distance function	485
2.7 Ideals, filters, and homomorphisms	486
2.8 The Boolean center	490
2.9 The Riesz decomposition property	491
3 Ideals in MV-algebras	493
3.1 The lattice of the ideals of A	494
3.2 Congruences and quotient MV-algebras	496
3.3 Prime ideals	499
3.4 Maximal ideals	501
3.5 The radical	502
3.6 The spectral topology	505
4 Classes of MV-algebras	510
4.1 MV-chains	510
4.2 Simple and semisimple MV-algebras	513
4.3 Local and perfect MV-algebras	515
5 MV-algebras and Abelian ℓu -groups	524
5.1 The functor Γ	524
5.2 Properties of $\Gamma(G, u)$	532
5.3 Chang's completeness theorem	536
5.4 Functional representation of semisimple MV-algebras	539
5.5 Perfect MV-algebras and Abelian ℓ -groups	541
5.6 Representations by ultrapowers	545
6 Łukasiewicz ∞ -valued logic	547
6.1 The syntax of \mathfrak{L}	547
6.2 The Lindenbaum–Tarski algebra $(\mathfrak{L}(\Theta), \mathfrak{L})$	553
6.3 Consistent sets, linear sets, and deductive systems	554
6.4 The semantics of \mathfrak{L}	560

7 Varieties of MV-algebras	565
7.1 Komori’s Theorem	566
7.2 Equational characterization of varieties of MV-algebras	573
8 Historical remarks and further reading	578
VII Gödel–Dummett Logics	
Matthias Baaz and Norbert Preining	585
1 Introduction and History	585
1.1 The Gödel conditional	586
1.2 History of Gödel logics	587
1.3 Syntax and semantics for propositional Gödel logics	587
1.4 Syntax and semantics for first-order Gödel logics	588
1.5 Axioms and deduction systems for Gödel logics	590
1.6 Topology and order	591
2 Propositional Gödel logics	592
2.1 Completeness of \mathbf{H}^0 for \mathbf{LC}	593
2.2 The Delta operator	597
3 First-order Gödel logics	597
3.1 Preliminaries	597
3.2 Relationships between Gödel logics	600
3.3 Axiomatizability results	603
3.4 Relation to Kripke frames	611
3.5 Number of different Gödel logics	617
4 Further topics	618
4.1 The Delta operator	618
4.2 The Takeuti–Titani rule and quantified propositional logics	618
4.3 Fragments of Gödel logics	619
4.4 Entailment	620
4.5 Satisfiability	622
VIII Fuzzy Logics with Enriched Language	
Francesc Esteva, Lluís Godo, and Enrico Marchioni	627
1 Introduction	627
2 Expansions with truth-constants	628
2.1 Rational Pavelka logic	629
2.2 Expansions of the logic of a continuous t-norm with truth-constants	632
2.3 About the structure of real $L_*(\mathcal{C})$ -chains	634
2.4 Partial embeddability property	636
2.5 About (finite) strong real completeness	640
2.6 About canonical real completeness	642
2.7 Completeness results for evaluated formulas	645

2.8	Forcing the canonical interpretation of truth-constants: two approaches .	650
2.9	Some open questions	653
2.10	First-order fuzzy logics expanded with truth-constants	654
3	Expansions with truth-stressing and truth-depressing hedges	660
3.1	The logic L_S of truth-stressing hedges	661
3.2	On the logics L_S and their associated quasivarieties	665
3.3	The case of truth-depressers	670
4	Expansions with an involutive negation	671
4.1	Expanding a Δ -core fuzzy logic with an involutive negation	672
4.2	Real completeness	675
4.3	The lattice of subvarieties of Π_{\sim} -algebras and of SBL_{\sim} -algebras	676
5	Expansions of Łukasiewicz logic	680
5.1	Rational Łukasiewicz logic	681
5.2	Expansions with the product connective	683
5.3	Completeness results	686
5.4	The expressive power of $L\Pi_{\frac{1}{2}}$	696
6	Historical remarks and further reading	701
6.1	Expansions with truth-constants	701
6.2	Expansions with truth-stressing and truth-depressing hedges	703
6.3	Expansions with an involutive negation	705
6.4	Expansions of Łukasiewicz logic	706

IX Free Algebras and Functional Representation for Fuzzy Logics

Stefano Aguzzoli, Simone Bova, and Brunella Gerla

713

1	Free algebras	713
2	MV-algebras and related structures	717
2.1	McNaughton Theorem	718
2.2	Wajsberg hoops	730
2.3	Finitely valued Łukasiewicz logics	733
2.4	Extensions of MV-algebras	734
2.5	MV-algebras with division operators	737
3	Product algebras	738
3.1	Free cancellative hoops	738
3.2	Free product algebras	739
4	Gödel algebras and related structures	742
4.1	Functional representation of Gödel algebras	742
4.2	Finite spectral duality for Gödel algebras	746
4.3	Gödel hoops	754
4.4	Finite-valued Gödel algebras	754
4.5	Nilpotent minimum algebras	755
4.6	Functional representation of free NM-algebras	759

5	BL-algebras and related structures	763
5.1	Free BL_{Δ} -algebras	763
5.2	Free BL-algebras	766
5.3	SBL-algebras and some revisitations	776
5.4	Locally finite varieties of BL-algebras	777
6	Pierce representation	782
7	Open problems	786
8	Historical remarks and further reading	786
X	Computational Complexity of Propositional Fuzzy Logics	
	Zuzana Haniková	793
1	Introduction	793
2	Notions and problems	795
2.1	Logics and algebras	795
2.2	A visit to complexity theory	801
2.3	Formulas as inputs	804
2.4	Classical logic and Boolean algebras	806
2.5	Decision problems in the reals	807
3	General results and methods	809
3.1	Basic inclusions and reductions	809
3.2	Negations	812
3.3	Eliminating compound terms	813
3.4	Lower bounds	815
4	Łukasiewicz logic	817
4.1	The standard MV-algebra	817
4.2	Axiomatic extensions of Łukasiewicz logic	820
5	Logics of standard BL-algebras	822
5.1	Gödel and product logics	822
5.2	BL and SBL	824
5.3	Other logics of standard BL-algebras	827
6	Logics in modified languages	832
6.1	Falsehood-free fragments	832
6.2	Logics with Δ	833
6.3	Logics with constants	834
6.4	Logics with an involutive negation	836
6.5	The logics $\mathbb{L}\Pi$ and $\mathbb{L}\Pi_{\frac{1}{2}}$	838
7	MTL and its axiomatic extensions	842
7.1	(Weak) nilpotent minimum logic	843
8	Overview of results and open problems	845
9	Historical remarks and further reading	847

XI Arithmetical Complexity of First-Order Fuzzy Logics

Petr Hájek, Franco Montagna, and Carles Noguera 853

1	Introduction	853
2	General results and general semantics	855
3	Complexity of standard semantics	861
4	Complexity of finite and rational-chain semantics	877
5	Further topics on arithmetical hierarchy in first-order logics	889
5.1	Semantics of witnessed models	889
5.2	Semantics of models over complete chains	892
5.3	Fragments with implication and negation	896
5.4	Complexity of hoop logics	899
5.5	Monadic BL logics	902
5.6	Extensions of Łukasiewicz logic	904
5.7	Open problems	905
6	Historical remarks and further reading	906

INDEX 909