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An algebraic approach to the reconstruction of uniform hypergraphs from their degree sequence ☆

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ABSTRACT

The reconstruction of a (hyper)graph starting from its degree sequence is one of the most relevant among the inverse problems investigated in the field of graph theory. In case of graphs, a feasible solution can be quickly reached, while in case of hypergraphs Deza et al. (2018) proved that the problem is NP-hard even in the simple case of 3-uniform ones. This result opened a new research line consisting in the detection of instances for which a solution can be computed in polynomial time. In this work we deal with 3-uniform hypergraphs, and we study them from a different perspective, exploiting a connection of these objects with partially ordered sets. More precisely, we introduce a simple partially ordered set, whose ideals are in bijection with a subclass of 3-uniform hypergraphs. We completely characterize their degree sequences in case of principal ideals (here a simple fast reconstruction strategy follows), and we furthermore carry on a complete analysis of those degree sequences related to the ideals with two generators. We also consider unique hypergraphs in $\mathcal{D}^{ext}$, i.e., those hypergraphs that do not share their degree sequence with other non-isomorphic ones. We show that uniqueness holds in case of hypergraphs associated to principal ideals, and we provide some examples of hypergraphs in $\mathcal{D}^{ext}$ where this property is lost.

1. Introduction

The retrieval of information about a combinatorial object from quantitative information about its shape and topology is one of the main topics in the wide area of inverse problems. In case the investigated objects are graphs, the problem becomes even more interesting due to their wide range modeling capability. In particular, starting from a well known characterization by Erdős and Gallai [7] in 1960, subsequent research concentrated on exploiting the information about (simple) graphs that can be retrieved from their degree sequences. Several other characterizations of graphs' degree sequences appeared in literature: in [13], seven of them are listed and proved to be equivalent, leading to a constructive proof of the Erdős-Gallai's theorem. Moving from these results, various fast algorithms have been provided to reconstruct one of the graphs associated to a given degree sequence. On the other hand, it remains a fascinating problem to determine the number of different graphs (up to isomorphism) sharing the same degree sequence.

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In particular, it is of primary relevance to study those properties that guarantee uniqueness of the degree sequences, so detecting subsets of degree sequences that fully characterize isomorphic graphs.

As one can easily realize, when we pass from graphs to hypergraphs the task of characterizing the degree sequences and the related reconstruction problem becomes much harder. Surprisingly, while many necessary conditions have been provided for an integer sequence to be the degree sequence of a k-uniform hypergraph (k-uniform meaning that each hyperedge has cardinality k), most of them generalizing the Erdős and Gallai's theorem, few sufficient conditions have been determined. As an example, relying on two well known theorems by Havel and Hakimi ([11,12]), the authors of [4] set a lower bound on the length of a sequence in order to be k-uniform, according to its maximum value and to the maximum difference between its elements. This result has been later extended using a different perspective [5,9,10], providing a polynomial time strategy to determine one of the hypergraphs of the related instances.

The problems of characterizing and reconstructing k-uniform sequences remained open until 2018, when Deza et al. proved them to be NP-hard [6] even in the simplest case of 3-uniform hypergraphs.

As a consequence of this result, the research moved its focus on the investigation of relevant subclasses of k-uniform degree sequences that can be efficiently detected and that gain back the uniqueness property. These studies open broad perspectives not only to define ad hoc P-time reconstruction strategies for subclasses of k-uniform degree sequences, but also to generate and enumerate them.

In this work we frame the target by continuing the study of a specific class of unique 3-uniform degree sequences, indicated as $\mathcal{D}$, first defined in [2] and then extended in [3]. Specifically, we go deeper in the investigation of the link between the elements of $\mathcal{D}$ and the ideals of the poset $\mathcal{T}_n$ obtained by triplets of integers with the coordinate-wise ordering. This identification also allows to extend the class $\mathcal{D}$ and find new properties of the involved hypergraphs. So, we provide a complete characterization of those sequences that correspond to the principal ideals of the poset and, as an immediate consequence, we obtain a fast reconstruction algorithm for them. Then, we shift to the study of the ideals with two generators: also in this case, we carry on a complete study and we spot the first cases of ideals that do not correspond to any element of $\mathcal{D}$. In these cases also the uniqueness property is not guaranteed any more, and we provide some sufficient conditions leading to such situations.

The paper is organized as follows: in Section 2, we briefly recall some basic notions of graph theory and order theory, and we introduce the degree sequences object of our study. In Section 3, we consider the degree sequences of hypergraphs associated to the principal ideals of the poset $\mathcal{T}_n$. We show their uniqueness and provide a fast reconstruction strategy for the corresponding realization. Section 4 is dedicated to the study of those hypergraphs that are in bijection with the ideals of the poset $\mathcal{T}_n$ having two generators. We provide several combinatorial properties on these ideals and we show that in this case we can find hypergraphs which are not in $\mathcal{D}$, thus losing the uniqueness property. Therefore, we investigate some necessary and/or sufficient conditions to ensure that the hypergraphs associated with two generators ideals have the uniqueness property. Finally, we conclude showing possible future developments of our research in Section 5.

2. Basic notions and definitions

In this paper we consider a specific class of 3-uniform hypergraphs that can be read as the class of ideals of a partially ordered set. For this reason, we start this section introducing some basic concepts and definitions in both the fields of graph and order theory, and we fix the notation we are going to use.

A hypergraph H is defined as a pair of sets $H = (V, E)$, with $V = \{v_1, \dots, v_n\}$ the set of vertices and $E \subset \mathcal{P}(V) \setminus \{\emptyset\}$ the multiset of hyperedges (from now on edges), where $\mathcal{P}(V)$ denotes the power set of V. A hypergraph is called *k-uniform* if every edge contains exactly k vertices, counted with repetitions, and it is said to be *simple* if there are no repeated edges and if the vertices in each edge are all distinct. In other words, no parallel edges and no loops occur in the set E. From now on we only consider those hypergraphs that are simple and k-uniform, and with no isolated vertices. We further fix the value $k = 3$.

For each vertex $v \in V$, we define its degree as the number of edges that contain v. Considering the list of the degrees of all vertices, arranged in non increasing order, the *degree sequence* of the hypergraph is defined. We denote such integer sequence as $\pi = (\pi_1, \dots, \pi_n)$, and we say that a hypergraph H realizes π if π is its degree sequence. We further adopt the standard exponential notation $\pi = (\pi_1^{\alpha_1}, \pi_2^{\alpha_2}, \dots, \pi_m^{\alpha_m})$ to denote the α_i-times repetition of the value π_i, with $i \in \{1, \dots, m\}$.

We now recall some basic notions of order theory. For a complete overview the reader is addressed to [14]. Given a set of elements A and $\preceq$ a reflexive, antisymmetric, and transitive relation (i.e., an order relation), the couple $\mathcal{P} = (A, \preceq)$ is defined as a *partially ordered set*, briefly poset. We underline that, in general, two elements in A can be non comparable w.r.t. $\preceq$. A subset of A such that each couple of its elements is not comparable is called an *antichain*. An order *ideal* is a down-set I defined as follows: whenever $x \in I$, if $y \in A$ is such that $y \preceq x$, then $y \in I$ too. Given an element $x \in A$, it generates the *principal ideal* $\downarrow\{x\} = \{y \in A \text{ s.t. } y \preceq x\}$, and x is defined as its *generator*. If A is a finite set, any union of ideals of A is still an ideal, and can always be obtained as the finite union of principal ideals, $I = \downarrow\{x_1, \dots, x_m\} = \downarrow\{x_1\} \cup \dots \cup \downarrow\{x_m\}$ for some $x_1, \dots, x_m$. An element $m \in I$ is *maximal* if there is no $a \in I$ such that $m \prec a$. The maximal elements of an ideal I constitute an antichain, and it is known that the antichain of its maximal elements generates an ideal I.

We now introduce a particular poset that allows us to define a bijection between its elements and the hyperedges of a 3-uniform hypergraph. Then, starting from its ideals, we define a subclass of hypergraphs with appealing properties and structure, already introduced in [3].

Given an integer $n \geq 3$, we define Ω_n as the set of triplets (a_1, a_2, a_3) with $a_i \in \{1, \dots, n\}$ and $1 \leq a_1 < a_2 < a_3 \leq n$. If we consider a set of vertices $V = \{v_1, \dots, v_n\}$, it is clear that there exists a bijection between the elements of the set Ω_n and the edges of a 3-

uniform hypergraph defined on the vertices in V: the triplet (a_1,a_2,a_3) can be interpreted as the hyperedge composed by the vertices $(v_{a_1},v_{a_2},v_{a_3})$, and vice versa. Then, any subset of Ω_n is a simple 3-hypergraph. We define an order relation on Ω_n as follows:

$$(a_1,a_2,a_3) \preceq (b_1,b_2,b_3) \text{ if and only if } a_i \leq b_i \text{ with } i \in \{1,2,3\},$$

and call $\mathcal{T}_n = (\Omega_n, \preceq)$ the poset of triplets thus defined. Borrowing the notation from [3], we call $\mathcal{I}_n$ the set of the ideals of $\mathcal{T}_n$, and $\mathcal{D}_n^{ext}$ the set of degree sequences realized by the hypergraphs in $\mathcal{I}_n$, that turns out to be the set of degree sequences that is the object of our study. Next section is devoted to the study of the peculiarities of this class.

Remark 1. From the point of view of the poset $\mathcal{T}_n$, if the hypergraph H is represented by the ideal I_H, then the entries of its degree sequence $\pi = (\pi_1,\dots,\pi_n)$ are exactly the number of occurrences of the elements $1,\dots,n$ in the triplets of I_H.

2.1. Representing ideals of $\mathcal{T}_n$ as 3-uniform hypergraphs

The definition of the class $\mathcal{D}_n^{ext}$ actually starts from the NP-completeness proof in [6], where the reduction of the NP-complete 3-partition problem to the problem of deciding if an integer sequence is realized by a hypergraph, leads to the definition of a binary matrix with a very nice structure, that is interpreted as the incidence matrix of a hypergraph. In 2021, a generalization of this class of hypergraphs led to the definition of the class $\mathcal{D}_n$ [2], here recalled. Starting from a non-increasing integer sequence $s = (s_1,\dots,s_n)$, with $n \geq 3$ and composed both of positive and negative numbers, it is possible to define a 3-uniform and simple hypergraph H as follows: the edge (v_i,v_j,v_k) is in H if and only if $s_i + s_j + s_k > 0$. We call $\pi^{(s)}$ the degree sequence realized by H, and $\mathcal{D}_n$ the set of all the hypergraphs that are obtained through this construction.

Example 1. The degree sequence $\pi^{(s)} = (50,44,37,31,31,26,26,20,20,15,15,9)$ can be obtained starting from the integer sequence $s = (4,3,2,1,1,0,0,-1,-1,-2,-2,-3)$.

The class $\mathcal{D} = \bigcup_{n\geq 3} \mathcal{D}_n$ has been studied in depth in [1–3,8], and the following properties have been detected:

Theorem 1 *([2]). Given a degree sequence $\pi^{(s)} = (\pi_1^{(s)},\dots,\pi_n^{(s)})$ obtained from an integer sequence s, if there exists an index $i < n$ s.t. $\pi_i = \pi_{i+1}$, then there exists an integer sequence s' s.t. $s'_i = s'_{i+1}$ and $\pi^{(s')} = \pi^{(s)}$.*

Theorem 2 *([2]). If $\pi^{(s)}$ is a degree sequence obtained from an integer sequence s, then there exists one only 3-uniform hypergraph (up to isomorphism) realizing $\pi^{(s)}$.*

As a consequence, we use the notation $\mathcal{D}$ to indicate both the set of the degree sequences and the set of the related hypergraphs.

Property 1 *([2,3]).* Given a degree sequence $\pi^{(s)}$ and H the (unique) hypergraph realizing it, if the edge $(v_i,v_j,v_k) \in H$, then $(v_i,v_j,v_{k'}) \in H$ for all $j+1 \leq k' \leq k$.

Proposition 1 *([3]). Let H be a hypergraph in $\mathcal{D}_n$ and E its edge set. Then, E is an ideal in $\mathcal{T}_n$.*

The previous results can be summarized as follows: the degree sequences of the class $\mathcal{D}$ are unique (Theorem 2), and the edges of the unique hypergraph realizing them, seen as triplets of the poset $\mathcal{T}_n$, are a downward closed set with respect to the chosen order, namely, they are ideals of $\mathcal{T}_n$ (Property 1 and Proposition 1). In particular, $\mathcal{D}_n \subsetneq \mathcal{D}_n^{ext}$ [3]. We underline that the inclusion is strict, as well as the uniqueness property can be lost when moving from $\mathcal{D}$ to $\mathcal{D}^{ext}$ (see Examples 2 and 3).

Example 2. The degree sequence $\pi = (49,49,34,34,30,30,24,24,18,18,13,13)$ is realized by the ideal $I = \downarrow\{(4,6,10),(2,8,12)\}$. However, no integer sequence s exists such that $\pi = \pi^{(s)}$ (see the proof given in Example 9).

Example 3. The degree sequence $\pi = (30,24,18,18,15,13,10,9,8,5)$ is realized by both the ideals $I_1 = \downarrow\{(1,6,10),(2,5,9),(2,6,7),(3,4,8)\}$ and $I_2 = \downarrow\{(1,7,9),(2,4,10),(2,5,8),(4,5,6)\}$, but the corresponding 3-hypergraphs are not isomorphic. Therefore, the uniqueness property is lost for π.

From now on, we focus our attention on the wider class $\mathcal{D}_n^{ext}$, whose elements are the degree sequences of the ideals of $\mathcal{T}_n$. Our final goal is to study combinatorial properties of degree sequences realized by ideals, when the number of generators of the ideal is fixed, and to determine a possible characterization of those sequences that preserve or lose the uniqueness property when moving from $\mathcal{D}$ to $\mathcal{D}^{ext}$.

3. Principal ideals of $\mathcal{T}_n$ and their combinatorial properties

As a starting point, we consider the easiest case of principal ideals, i.e. those ideals generated by a single element of $\mathcal{T}_n$, and we study the degree sequences of the related hypergraphs. The structure of these down-sets is very simple, and a complete analysis can be carried on. From now on, the number of vertices of the hypergraph, n, will be fixed. We denote by $g = (a, b, c) \in \Omega_n$ the unique generator of the ideal $I_g = \downarrow\{(a, b, c)\}$, and π_g the degree sequence of the associated hypergraph.

Property 2. If $g = (a, b, c)$ is the generator of a hypergraph $H \in \mathcal{I}_n$ on n vertices, then $c = n$.

This is trivial from the definition of ideals as down-sets and the condition of having exactly n no isolated vertices in H.

Proposition 2. *Let $g = (a, b, n) \in \Omega_n$. The cardinality of the related principal ideal I_g is*

$$|I_g| = \frac{a}{6}(a^2 + 3a - 3an - 3b^2 + 6bn - 3b - 3n + 2).$$

It easily follows from the definition of principal ideal and the computation of $|\downarrow\{g\}| = \sum_{i=1}^{a} \sum_{j=i+1}^{b} \sum_{k=j+1}^{n} 1$.

Given the generator g, we can now compute all the elements of the ideal $I_g = \downarrow\{g\}$, as well as the occurrences of the numbers $1, \ldots, n$ in its triplets. In other words, we can immediately retrieve the degree sequence of the hypergraph H_g associated to I_g (see Remark 1).

For each number $i \in \{1, \ldots, n\}$, we count the number of occurrences of i in the triplets of I_g in first, second and third position, respectively. Summing up, we get the value π_i for each vertex of the hypergraph.

Lemma 1. *Let $g = (a, b, n) \in \Omega_n$. The number of occurrences of i, with $i \in \{1, \ldots, n\}$, as first element of a triplet in I_g is*

$$O_1^i = \begin{cases} \frac{(i-b)(i+b-2n+1)}{2} & \text{if } 1 \le i \le a, \\ 0 & \text{if } a+1 \le i \le n. \end{cases}$$

Proof. By definition of $\downarrow\{(a, b, n)\}$, it is clear that any number greater than a cannot be the first element of any triplet in I_g. If $i \le a$, the number of occurrences as first element is given by the number of triplets of kind (i, s, t) with $i + 1 \le s \le b$ and $s + 1 \le t \le n$, i.e.,

$$O_1^i = \sum_{s=i+1}^{b} \sum_{t=s+1}^{n} 1,$$

from which the thesis follows. $\square$

Lemma 2. *Let $g = (a, b, n) \in \Omega_n$. The number of occurrences of i, with $i \in \{1, \ldots, n\}$, as second element of a triplet in I_g is*

$$O_2^i = \begin{cases} (i-1)(n-i) & \text{if } 1 \le i \le a, \\ a(n-i) & \text{if } a+1 \le i \le b, \\ 0 & \text{if } b+1 \le i \le n. \end{cases}$$

Proof. By definition of $\downarrow\{(a, b, n)\}$, it is clear that any number greater than b cannot be the second element of any triplet in I_g. Given $i \in \{1, \ldots, n\}$, the number of occurrences as second element is given by the number of triplets (s, i, t), with $1 \le s \le \min\{i-1, a\}$ and $i + 1 \le t \le n$, i.e.,

$$O_2^i = \sum_{s=1}^{\min\{i-1,a\}} \sum_{t=i+1}^{n} 1.$$

We have to distinguish two cases in the first sum, when $i \le a$ or not. Replacing and solving the calculations lead to the thesis. $\square$

Lemma 3. *Let $g = (a, b, n) \in \Omega_n$. The number of occurrences of i, with $i \in \{1, \ldots, n\}$, as third element of a triplet in I_g is*

$$O_3^i = \begin{cases} \frac{1}{2}i^2 - \frac{3}{2}i + 1 & \text{if } 1 \le i \le a, \\ \frac{a(2i-a-3)}{2} & \text{if } a+1 \le i \le b, \\ \frac{a(2b-a-1)}{2} & \text{if } b+1 \le i \le n. \end{cases}$$

Proof. Given $i \in \{1, \dots, n\}$, the number of occurrences of i as third element is given by the number of triplets of kind (s, t, i), with $1 \le s \le \min\{i-2, a\}$ and $s+1 \le t \le \min\{i-1, b\}$, i.e.,

$$O_3 = \sum_{s=1}^{\min\{i-2,a\}} \sum_{t=s+1}^{\min\{i-1,b\}} 1.$$

We have to distinguish more cases to correctly choose the indices of the two sums: if $i \le a+1$, then it follows $i-1 < b$, since $a < b$ by definition. Then the sums stop with indices $s = i-2$ and $t = i-1$, respectively. If $a+1 < i < b+1$, then $a \le i-2$ and the sums stop with indices $s = a$ and $t = i-1$, respectively. Finally, if $i \ge b+1$ we reach $s = a$ and $t = b$. Replacing and solving the calculations lead to the thesis.

Notice that the limiting cases $i = a+1$ (or $i = a+2$, equivalently) and $i = b+1$ can be arbitrarily included in the intervals we analyzed, since they describe the cases in which we get $\min\{a, a\}$ and $\min\{b, b\}$. $\square$

We are now able to compute the degree sequence of the hypergraph I_g, simply summing up the values O_1^i, O_2^i and O_3^i given in Lemmas 1, 2 and 3, with $i \in \{1, \dots, n\}$.

Theorem 3. *Let $g = (a, b, n) \in \Omega_n$. The i-th entry π_i of the degree sequence $\pi_g = (\pi_1, \dots, \pi_n)$ of the hypergraph defined by I_g is*

$$\pi_i = \begin{cases} \frac{(b-1)(2n-b-2)}{2} & \text{if } 1 \le i \le a, \\ \frac{a(2n-a-3)}{2} & \text{if } a+1 \le i \le b, \\ \frac{a(2b-a-1)}{2} & \text{if } b+1 \le i \le n. \end{cases} \tag{1}$$

Proof. The proof easily follows from Lemmas 1, 2 and 3, since to get each value π_i it is sufficient to sum the occurrences of the number i in the triplets of I_g, that is $\pi_i = O_1^i + O_2^i + O_3^i$, for all $i \in \{1, \dots, n\}$. We have to distinguish three cases according to the different values of i:

i) If $1 \le i \le a$, then $\pi_i = \frac{(i-b)(i+b-2n+1)}{2} + (i-1)(n-i) + \frac{1}{2}i^2 - \frac{3}{2}i + 1 = \frac{(b-1)(2n-b-2)}{2}$.
ii) If $a+1 \le i \le b$, then $\pi_i = 0 + a(n-i) + \frac{a(2i-a-3)}{2} = \frac{a(2n-a-3)}{2}$.
iii) If $b+1 \le i \le n$, then $\pi_i = 0 + 0 + \frac{a(2b-a-1)}{2} = \frac{a(2b-a-1)}{2}$.

We point out the non-trivial fact that, in all the three cases above, the value π_i does not depend on i. $\square$

Corollary 1. *The degrees of the vertices associated to a principal ideal can assume at most three distinct values.*

This is a straightforward consequence of Theorem 3. In particular, if we denote these values as $p_1 = \frac{(b-1)(2n-b-2)}{2}$, $p_2 = \frac{a(2n-a-3)}{2}$ and $p_3 = \frac{a(2b-a-1)}{2}$, four different cases can be distinguished:

i) $p_1 = p_2 = p_3$, that realizes if and only if $a = n-2$ and $b = n-1$. In this case, the degree sequence is constant and equal to $\pi = (p_1^n)$;
ii) $p_1 \ne p_2 = p_3$, that realizes if and only if $b = n-1$ and $a < n-2$. In this case, the degree sequence has two distinct entries and is equal to $\pi = (p_1^a, p_2^{n-a})$;
iii) $p_1 = p_2 \ne p_3$, that realizes if and only if $b = a+1$ and $b < n-1$. In this case, the degree sequence has two distinct entries and is equal to $\pi = (p_1^b, p_3^{n-b})$;
iv) $p_1 \ne p_2 \ne p_3$, that realizes in all the other cases, i.e., when no consecutive elements are present in the triplet of the generator. In this case, the degree sequence has three distinct entries and is equal to $\pi = (p_1^a, p_2^{b-a}, p_3^{n-b})$.

We also highlight that, given a degree sequence π with (at most) three distinct entries, we can immediately check if there exists a principal ideal realizing it. It is sufficient to solve the following system of equations to get the values a and b of a possible generator:

$$\begin{cases} p_1 = \frac{(b-1)(2n-b-2)}{2}, \\ p_2 = \frac{a(2n-a-3)}{2}, \\ 1 \le a < b < n \text{ integer numbers.} \end{cases} \tag{2}$$

Theorem 4. *If it exists, the solution of* (2) *is unique.*

Proof. First of all, we notice that the two equations in (2) can be solved independently as quadratic equations in b and a, respectively. We show the uniqueness of the solution for the value b, obtained by solving the quadratic equation

$$b^2 + (1-2n)b + 2n + 2p_1 - 2 = 0,$$

whose solutions are $b_{1,2} = \frac{2n-1\pm\sqrt{\Delta}}{2}$. By imposing the condition $b \le n-1$, we reach the inequality $\pm\sqrt{\Delta} \le -1$, that is satisfied for $-\sqrt{\Delta}$ only. It follows that the unique admissible solution, if it is an integer, is the value $b = \frac{2n-1-\sqrt{\Delta}}{2}$.

The case for the value a can be treated similarly. $\square$

Remark 2. Theorem 4 shows that there is only one *principal* ideal realizing π, but *does not* guarantee the uniqueness of the solution for the reconstruction problem. As a matter of fact, the same degree sequence could also be realized by a different ideal with a higher number of generators, or by a hypergraph that is not in the class $\mathcal{I}_n$.

To spread light on the doubt raised in the previous remark, we can actually show the uniqueness of the degree sequences realized by principal ideals. To this aim, we go back to the definition of the class $\mathcal{D}_n$, and we show that principal ideals can be always realized starting from an integer sequence s. Then, uniqueness follows from Theorem 2.

Theorem 5. *Given an integer sequence $\pi = (\pi_1, \ldots, \pi_n)$, if there exists a principal ideal $I_g \in \mathcal{I}_n$ realizing π, then π is unique.*

Proof. We simply show that the degree sequences realized by principal ideals are in the class $\mathcal{D}_n$, providing an integer sequence $s = (s_1, \ldots, s_n)$ from which it is possible to compute them. To this aim, we study three different cases, according to the number of different values in each degree sequence, and then we define the related sequences s.

1. All vertices have the same degree, $\pi = (p^n)$. In this case, π is realized by a principal ideal if and only if $p = \frac{(n-1)(n-2)}{2}$ (see Theorem 3 and Corollary 1), that is, π is the degree sequence of the complete 3-uniform hypergraph on n vertices, $H =\downarrow\{(n-2, n-1, n)\}$. The same hypergraph can be obtained starting from the integer sequence $s = (1^n)$.
2. The vertices take two distinct values, $\pi = (p^x, q^{n-x})$. Two further cases occur:
   i) $q = \frac{x(2n-x-3)}{2}$ and so $x = a$ and $p = \frac{(n-1)(n-2)}{2}$, see Corollary 1. We also remind that such a solution is unique, since we are assuming that there exists a principal ideal realizing the sequence (Theorem 4). We show that the integer sequence $s = (1^a, 0^{n-a})$ realizes π, i.e., $\pi = \pi^{(s)} = (\pi_1^{(s)}, \ldots, \pi_n^{(s)})$. It is sufficient to show that $\pi_1^{(s)} = p$ and $\pi_n^{(s)} = q$. The vertex v_1 always appears in the first position of an edge, and the only edges of the hypergraph containing it are those ones satisfying the inequality $1 + s_j + s_k > 0$. The couple (s_j, s_k) can be equal to $(1,1)$, $(1,0)$ or $(0,0)$. There are $\binom{a-1}{2} + (a-1)(n-a) + \binom{n-a}{2}$ possible choices in total, that is, $\pi_1^{(s)} = \frac{(n-1)(n-2)}{2} = p$.
   Similarly, the vertex v_n only appears as third element of an edge, in those ones satisfying the inequality $s_i + s_j + 0 > 0$. In this case, we have $\binom{a}{2} + a(n-a-1)$ possible choices for the couple (s_i, s_j), corresponding to $(1,1)$ and $(1,0)$, that is, $\pi_n^{(s)} = q$.
   ii) $p = \frac{(x-1)(2n-x-2)}{2}$, and so $x = b$ and $q = \frac{b(b-1)}{2}$ (see Corollary 1). Also in this case such a solution is unique, since we are assuming that there exists a principal ideal realizing the sequence (Theorem 4). Using the same argument of the previous point, it is possible to show that $\pi = \pi^{(s)}$ choosing $s = (2^b, -1^{n-b})$.
3. The vertices take three distinct values, $\pi = (p^x, q^y, r^{n-x-y})$. Again from Theorem 2 and Corollary 1, we immediately argue $x = a$, $y = b - a$, $p = \frac{(b-1)(2n-b-2)}{2}$, $q = \frac{a(2n-a-3)}{2}$ and $r = \frac{a(2b-a-1)}{2}$. In this case, the candidate sequence s is $s = (2^a, 0^{b-a}, -1^{n-b})$. As shown before, we can immediately verify that $\pi_1^{(s)} = p$ and $\pi_n^{(s)} = r$. The last value to be computed is then the degree of the vertex v_{a+1}, that can appear in the second or third position of an edge of the hypergraph we obtain from s. It appears in the second position if and only if the inequality $s_i + 0 + s_k > 0$ holds, i.e., when $(s_i, s_k) = (2, 0)$ or $(2, -1)$. The possibilities are $a(b-a-1) + a(n-b)$. Regarding the third position, being v_{a+1} the first vertex that corresponds to a value 0 in s, we only have to consider the couples $(s_i, s_j) = (2, 2)$, that are exactly $\binom{a}{2}$. Summing up, we get $\pi_{a+1}^{(s)} = a(n-a-1) + \binom{a}{2} = q$, that concludes the proof.

Since $\pi \in \mathcal{D}_n$ in all cases, we conclude from Theorem 2 that such sequences have a unique realization. $\square$

A straightforward consequence of this uniqueness result is the definition of a fast strategy for the reconstruction of the degree sequences associated to principal ideals: given an integer sequence with (at most) three distinct entries, $\pi = (p^x, q^y, r^z)$, from the values x, y and z we are able, if possible, to find the candidate generator, $g = (x, x+y, x+y+z)$, and then compare the degree sequence of $I_g =\downarrow\{g\}$ with the input sequence π, see Example 4.

Example 4. Let us consider the integer sequence $\pi = (49^2, 19^6, 13^4)$ of length $n = 12$. Looking at its repetitions, namely the values $x = 2$, $y = 6$ and $z = 4$, we argue that, if π is associated to a principal ideal, its generator must be $g = (2, 8, 12)$. Indeed, the value $x = 2$ coincides with the number of the first equal degrees, while $y = 6 = b - a$ allows to deduce $b = 8$. Finally, the length $n = 12$ of the sequence trivially gives information about the last entry of the candidate generator. Since the hypergraph associated to $I_g =\downarrow \{(2, 8, 12)\}$ has degree sequence equal to π, the (unique) solution to the reconstruction problem is given.

4. Ideals generated by two (uncomparable) elements of Ω_n

In this section, we consider those hypergraphs associated to ideals obtained from two generators, $g_1 = (a, b, c)$ and $g_2 = (d, e, f)$, where we assume that:

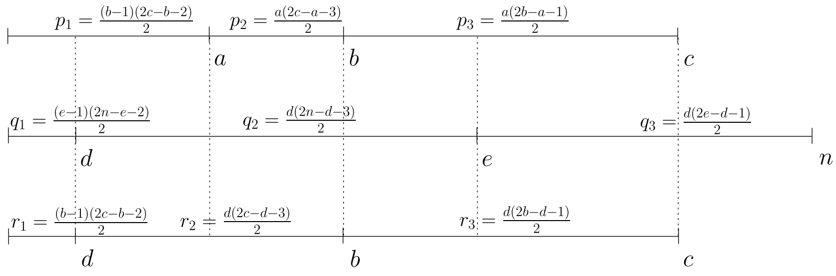

Fig. 1. The figure schematically shows the positions of the elements of g_1, g_2 and g_{min} in case $d < a < b < e < c < n$, as well as the computation of the values of π in the distinct intervals.

i) $1 \le a < b < c \le n$ and $1 \le d < e < f \le n$,
ii) at least one between c and f, or both, is equal to n. We assume w.l.g. $f = n$,
iii) g_1 and g_2 are uncomparable in $\mathcal{T}_n$.

Proposition 3. *Let $g_1 = (a, b, c)$ and $g_2 = (d, e, f)$ be two uncomparable elements of Ω_n, $I_{g_1,g_2} = \downarrow\{g_1, g_2\}$ and π_{g_1,g_2} the associated degree sequence. It holds*

$$\pi_{g_1,g_2} = \pi_{g_1} + \pi_{g_2} - \pi_{\min\{g_1,g_2\}},$$

where $\min\{g_1, g_2\} = (\min\{a, d\}, \min\{b, e\}, \min\{c, f\})$.

Proof. By definition, the principal ideal generated by the triplet $\min\{g_1, g_2\}$ corresponds to the intersection $I_{g_1} \cap I_{g_2}$. Then, by the inclusion-exclusion principle, we immediately get $I_{g_1,g_2} = I_{g_1} \cup (I_{g_2} \setminus I_{\min\{g_1,g_2\}})$, and so the thesis. $\square$

Then, degree sequences associated to ideals with two generators can be studied using the results we obtained for principal ideals, in particular Eq. (1), as well as the inclusion-exclusion principle.

Differently from the case of principal ideals, we now have to consider the mutual order of the elements of g_1 and g_2 to compute their minimum, $g_{min} = \min\{g_1, g_2\}$. We provide a complete analysis of all the possible cases, as well as the corresponding degree sequence π_{g_1,g_2}, through a graphic representation of the triplets of generators. Referring to Eq. (1), we denote with p_1, p_2, p_3 the entries of π_{g_1}, with q_1, q_2, q_3 the entries of π_{g_2}, and with r_1, r_2, r_3 the entries of $\pi_{g_{min}}$. Finally, for the sake of simplicity, we first assume a, b, c, d, e, n to be all distinct; the case in which repetitions are present will be detailed later.

Theorem 6. *Given $g_1, g_2 \in \Omega_n$, we can explicitly compute the degree sequence π_{g_1,g_2} realized by $I_{g_1,g_2} = \downarrow\{g_1, g_2\}$.*

Proof. Let be $g_1 = (a, b, c)$ and $g_2 = (d, e, n)$. We have to consider the mutual order of the elements $a, b, c, d, e \in \{1, \dots, n\}$, that has to be chosen such that g_1 and g_2 are uncomparable as elements of the poset $\mathcal{T}_n$. Altogether, five cases may occur, and for each of them we provide the related degree sequence $\pi_{g_1,g_2} = (\pi_1, \dots, \pi_n)$. The proof directly follows from Proposition 3 and Theorem 3. The latter allows to compute the degree sequences π_1, π_2 and $\pi_{\min\{g_1,g_2\}}$ starting from g_1 and g_2. In particular, when applying Theorem 3, the first generator is considered as an element of $\mathcal{T}_c$, as well as the triplet $\min\{g_1, g_2\}$. It is sufficient to compute the value $p_i + q_j - r_k$ in each interval, properly choosing the indices i, j, k, for each case. A visual representation of such computation is shown in Figs. 1, 2, 3, 4 and 5, one for each possible case.

1. Case $d < a < b < e < c < n$. The degree sequence is

$$\pi_i = \begin{cases} \frac{(e-1)(2n-e-2)}{2} & \text{if } 1 \le i \le d, \\ \frac{(b-1)(2c-b-2)+2d(n-c)}{2} & \text{if } d+1 \le i \le a, \\ \frac{a(2c-a-3)+2d(n-c)}{2} & \text{if } a+1 \le i \le b, \\ \frac{a(2b-a-1)+2d(n-b-1)}{2} & \text{if } b+1 \le i \le e, \\ \frac{a(2b-a-1)+2d(e-b)}{2} & \text{if } e+1 \le i \le c, \\ \frac{d(2e-d-1)}{2} & \text{if } c+1 \le i \le n. \end{cases}$$

Its computation is shown in Fig. 1.

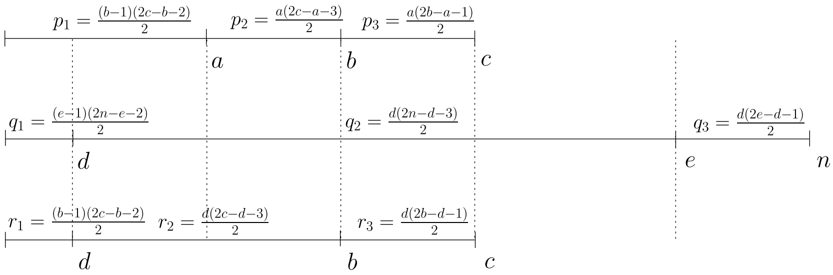

Fig. 2. The figure schematically shows the positions of the elements of g_1, g_2 and g_{min} in case $d < a < b < c < e < n$, as well as the computation of the values of π in the distinct intervals.

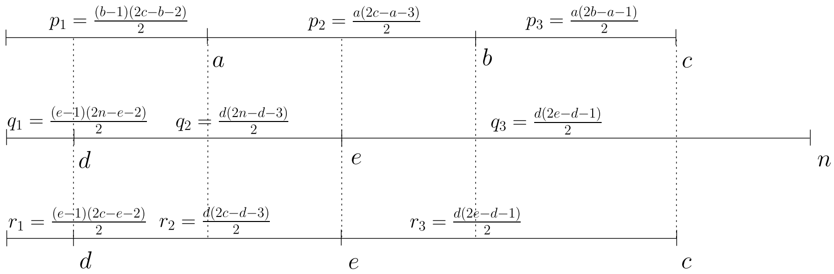

Fig. 3. The figure schematically shows the positions of the elements of g_1, g_2 and g_{min} in case $d < a < e < b < c < n$, as well as the computation of the values of π in the distinct intervals.

2. Case $d < a < b < c < e < n$. The degree sequence is

$$\pi_i = \begin{cases} \frac{(e-1)(2n-e-2)}{2} & \text{if } 1 \le i \le d, \\ \frac{(b-1)(2c-b-2)+2d(n-c)}{2} & \text{if } d+1 \le i \le a, \\ \frac{a(2c-a-3)+2d(n-c)}{2} & \text{if } a+1 \le i \le b, \\ \frac{a(2b-a-1)+2d(n-b-1)}{2} & \text{if } b+1 \le i \le c, \\ \frac{d(2n-d-3)}{2} & \text{if } c+1 \le i \le e, \\ \frac{d(2e-d-1)}{2} & \text{if } e+1 \le i \le n. \end{cases}$$

Its computation is shown in Fig. 2.

3. Case $d < a < e < b < c < n$. The degree sequence is

$$\pi_i = \begin{cases} \frac{(b-1)(2c-b-2)+2(e-1)(n-c)}{2} & \text{if } 1 \le i \le d, \\ \frac{(b-1)(2c-b-2)+2d(n-c)}{2} & \text{if } d+1 \le i \le a, \\ \frac{a(2c-a-3)+2d(n-c)}{2} & \text{if } a+1 \le i \le e, \\ \frac{a(2c-a-3)}{2} & \text{if } e+1 \le i \le b, \\ \frac{a(2b-a-1)}{2} & \text{if } b+1 \le i \le c, \\ \frac{d(2e-d-1)}{2} & \text{if } c+1 \le i \le n. \end{cases}$$

Its computation is shown in Fig. 3.

4. Case $d < e < a < b < c < n$. The degree sequence is

$$\pi_i = \begin{cases} \frac{(b-1)(2c-b-2)+2(e-1)(n-c)}{2} & \text{if } 1 \le i \le d, \\ \frac{(b-1)(2c-b-2)+2d(n-c)}{2} & \text{if } d+1 \le i \le e, \\ \frac{(b-1)(2c-b-2)}{2} & \text{if } e+1 \le i \le a, \\ \frac{a(2c-a-3)}{2} & \text{if } a+1 \le i \le b, \\ \frac{a(2b-a-1)}{2} & \text{if } b+1 \le i \le c, \\ \frac{d(2e-d-1)}{2} & \text{if } c+1 \le i \le n. \end{cases}$$

Its computation is shown in Fig. 4.

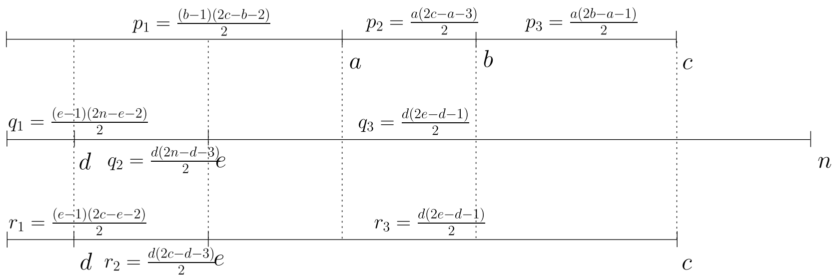

Fig. 4. The figure schematically shows the positions of the elements of g_1, g_2 and g_{min} in case $d < e < a < b < c < n$, as well as the computation of the values of π in the distinct intervals.

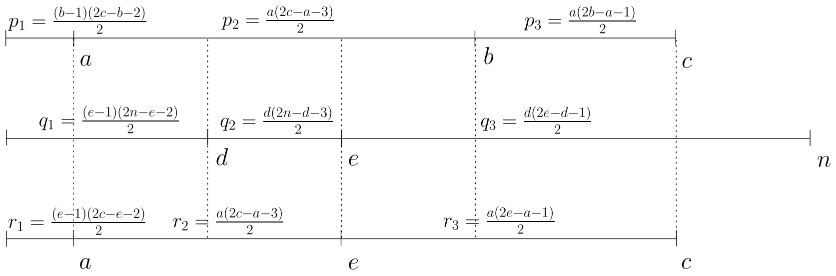

Fig. 5. The figure schematically shows the positions of the elements of g_1, g_2 and g_{min} in case $a < d < e < b < c < n$, as well as the computation of the values of π in the distinct intervals.

5. Case $a < d < e < b < c < n$. The degree sequence is

$$\pi_i = \begin{cases} \frac{(b-1)(2c-b-2)+2(e-1)(n-c)}{2} & \text{if } 1 \le i \le a, \\ \frac{(e-1)(2n-e-2)}{2} & \text{if } a+1 \le i \le d, \\ \frac{d(2n-d-3)}{2} & \text{if } d+1 \le i \le e, \\ \frac{2a(c-e-1)+d(2e-d-1)}{2} & \text{if } e+1 \le i \le b, \\ \frac{2a(b-e)+d(2e-d-1)}{2} & \text{if } b+1 \le i \le c, \\ \frac{d(2e-d-1)}{2} & \text{if } c+1 \le i \le n. \end{cases}$$

Its computation is shown in Fig. 5. □

Corollary 2. *The degrees of the vertices associated with an ideal with two generators can assume at most six distinct values.*

The result is clear since we explicitly computed π_{g_1,g_2} for all possible cases in Theorem 6. Actually, the number of distinct values can decrease to five, four or three, depending on the values of a, b, c, d and e, particularly if there are repetitions or consecutive elements. This can be clearly seen in the equation that defines π_i, where the number of cases may decrease.

Remark 3. A result similar to Corollary 2 also holds in case of principal ideals, where the number of distinct degrees can decrease to two, if two entries of the generator are consecutive numbers, or to one, if the triplet is composed by three consecutive numbers.

Example 5. Let I be the ideal generated by the two triplets $g_1 = (1, 7, 9)$ and $g_2 = (2, 6, 10)$ (case $a < d < e < b < c$). The associated degree sequence is $\pi_{g_1,g_2} = (32, 30, 15, 15, 15, 15, 11, 10, 10, 9)$, with six distinct entries. If we slightly modify the triplets of generators, by allowing consecutive elements, for example choosing $g_1' = (1, 7, 8)$ and $g_2' = (2, 3, 10)$, we get an ideal I' whose degree sequence turns out to be $(25, 15, 15, 7, 7, 7, 7, 7, 3, 3)$, where the number of distinct entries is lowered to four.

Remark 4. In case of two generators, the number of distinct elements in π cannot be less than three. This directly follows from the uncomparability of g_1 and g_2 in $\mathcal{T}_n$.

We now proceed in the study of the sequences given by two generators, and, in particular, we determine the cases where the related degree sequences preserve uniqueness.

4.1. Three distinct values degree sequences

In order to have a reduced number of distinct values in the degree sequence π_{g_1,g_2}, we need to reduce the number of different intervals in $[1,n]$ where the value π_i changes. To this aim, it is sufficient to choose uncomparable triplets g_1 and g_2 where some entries are repeated and/or consecutive. Considering all the cases listed in Theorem 6, we obtain the following:

Theorem 7. *The degree sequence π_{g_1,g_2} has exactly three distinct entries if and only if g_1 and g_2 are chosen as follows:*

i) $g_1 = (a, n-1, n)$ *and* $g_2 = (d, d+1, n)$*, with* $a < d$*, or*
ii) $g_1 = (a, a+1, a+2)$ *and* $g_2 = (d, n-1, n)$*, with* $d < a$*, or*
iii) $g_1 = (a, a+1, a+2)$ *and* $g_2 = (d, d+1, n)$*, with* $d < a$*.*

Proof. According to the computations in the proof of Theorem 6, to reduce the number of distinct degrees in π_{g_1,g_2} it is sufficient to add consecutive or equal elements in the triplets of generators. An exhaustive search of all possible choices of uncomparable generators leads to the thesis. □

Theorem 8. *Let π be an integer sequence with exactly three distinct entries. If π is realized by an ideal of $\mathcal{T}_n$ with two generators, then π is unique.*

Proof. The proof follows the same argument used in Theorem 5. We provide the sequence s for all the possible cases listed in Theorem 7:

i) The sequence $s = (3^a, 2^{d+1-a}, -1^{n-d-1})$ realizes the degree sequence in case $H = \downarrow\{(a, n-1, n), (d, d+1, n)\}$.
ii) The sequence $s = (5^d, 1^{a+2-d}, -2^{n-a-2})$ realizes the degree sequence in case $H = \downarrow\{(a, a+1, a+2), (d, n-1, n)\}$.
iii) The sequence $s = (2^{d+1}, 1^{a-d+1}, -3^{n-a-2})$ realizes the degree sequence in case $H = \downarrow\{(a, a+1, a+2), (d, d+1, n)\}$.

The computation of the degree sequences from the sequences s determined above concludes the proof. □

4.2. Four distinct values degree sequences

When increasing the number of distinct degrees from three to four, similar results can be achieved.

Theorem 9. *The degree sequence π_{g_1,g_2} has exactly four distinct entries if and only if g_1 and g_2 are chosen as follows:*

i) $g_1 = (a, n-1, n)$ *and* $g_2 = (d, e, n)$*, with* $a < d < e-1$*, or*
ii) $g_1 = (a, b, n)$ *and* $g_2 = (d, d+1, n)$*, with* $a < d < b$*, or*
iii) $g_1 = (a, a+1, a+2)$ *and* $g_2 = (d, e, n)$*, with* $d < a < e$*, or*
iv) $g_1 = (a, b, b+1)$ *and* $g_2 = (d, n-1, n)$*, with* $d < a < b$*, or*
v) $g_1 = (a, b, b+1)$ *and* $g_2 = (d, d+1, n)$*, with* $d < a$*, or*
vi) $g_1 = (a, b, b+1)$ *and* $g_2 = (d, d+1, n)$*, with* $a < d$*, or*
vii) $g_1 = (a, a+1, c)$ *and* $g_2 = (d, d+1, n)$*, with* $d < a$*, or*
viii) $g_1 = (a, a+1, c)$ *and* $g_2 = (d, d+1, n)$*, with* $d < a$*, or*
ix) $g_1 = (a, b, b+1)$ *and* $g_2 = (a, a+1, n)$*, with* $a+1 < b$*.*

Proof. Also in this case, an exhaustive search on the different cases presented in the proof of Theorem 6 leads to the thesis. □

Remark 5. Differently from the case in which the distinct degrees are only three, now the triplets of generators can share some elements. In particular, $a = d$ can hold under certain assumptions on the order of the elements, as shown in case *ix)*.

Theorem 10. *Let π_{g_1,g_2} be an integer sequence with exactly four distinct entries. If π is realized by an ideal of $\mathcal{T}_n$ with two generators, then π is unique.*

Proof. Also in this case, the thesis is achieved by showing that there exists a sequence s for each possible case from i) to ix):

i) $s = (3^a, 2^{d-a}, 0^{e-d}, -1^{n-e})$.
ii) $s = (4^a, 2^{d+1-a}, -1^{b-d-1}, -2^{n-b})$.

iii) $s = (6^d, 1^{a+2-d}, -2^{e-a-2}, -3^{n-e})$.
iv) $s = (7^d, 1^{a-d}, 0^{b+1-a}, -3^{n-b-1})$.
v) $s = (3^{d+1}, 1^{a-d-1}, 0^{b+1-a}, -4^{n-b-1})$.
vi) $s = (3^a, 2^{d+1-a}, -1^{b-d}, -2^{n-b-1})$.
vii) $s = (5^d, 1^{a+1-d}, -1^{c-a-1}, -2^{n-c})$.
viii) $s = (4^{d+1}, 2^{a-d}, -2^{c-a-1}, -6^{n-c})$.
ix) $s = (4^a, 0, -1^{b-a}, -3^{n-b-1})$.

The computation of the degree sequences from the provided s shows that $\pi_{g_1,g_2} = \pi^{(s)}$ for each case, leading to the thesis. □

4.3. Five distinct values degree sequences

The analysis of the hypergraphs related to degree sequences π_{g_1,g_2} with five distinct values reveals some interesting aspects. In particular, we spot some cases for which it is not possible to find an integer sequence s yielding a hypergraph realizing π_{g_1,g_2}, and this provides the first cases of elements in $\mathcal{D}^{ext} \setminus \mathcal{D}$. Furthermore, also establishing the uniqueness of π_{g_1,g_2} becomes an issue, even if this property is not always lost, but only in some cases.

Theorem 11. *The degree sequence π_{g_1,g_2} has exactly five distinct entries if and only if g_1 and g_2 are chosen as follows:*

i) $g_1 = (a, b, n)$ *and* $g_2 = (d, e, n)$*, with* $a < d < e < b$*, or*
ii) $g_1 = (a, b, b+1)$ *and* $g_2 = (d, e, n)$*, with* $d < a < b+1 < e$*, or*
iii) $g_1 = (a, b, b+1)$ *and* $g_2 = (d, e, n)$*, with* $d < a < e < b+1$*, or*
iv) $g_1 = (a, b, b+1)$ *and* $g_2 = (d, e, n)$*, with* $d < e < a$*, or*
v) $g_1 = (a, b, b+1)$ *and* $g_2 = (d, e, n)$*, with* $a < d < e < b$*, or*
vi) $g_1 = (a, a+1, c)$ *and* $g_2 = (d, e, n)$*, with* $d < e < a$*, or*
vii) $g_1 = (a, a+1, c)$ *and* $g_2 = (d, e, n)$*, with* $d < a < e < c$*, or*
viii) $g_1 = (a, a+1, c)$ *and* $g_2 = (d, e, n)$*, with* $d < a < c < e$*, or*
ix) $g_1 = (a, b, c)$ *and* $g_2 = (d, n-1, n)$*, with* $d < a$*, or*
x) $g_1 = (a, b, c)$ *and* $g_2 = (d, d+1, n)$*, with* $d < a < b$*, or*
xi) $g_1 = (a, b, c)$ *and* $g_2 = (d, d+1, n)$*, with* $a < d < b-1$*, or*
xii) $g_1 = (a, b, c)$ *and* $g_2 = (d, b, n)$*, with* $d < a$ *and at most two consecutive elements in* g_1*, or*
xiii) $g_1 = (a, b, n)$ *and* $g_2 = (d, e, n)$*, with* $d < a < b < e$ *and no consecutive elements in* g_1 *and* g_2*, or*
xiv) $g_1 = (a, b, c)$ *and* $g_2 = (a, e, n)$*, with* $e < b$ *and no consecutive elements in* g_1 *and* g_2*.*

Proof. Again, an exhaustive search on the different cases presented in the proof of Theorem 6 leads to the thesis. □

Remark 6. Also in this case, we notice that the two generators can contain repeated entries. In particular, we can have $a = d$ or $b = e$, under certain assumptions on the order of the elements, as it is shown in cases *xii)* and *xiv)*.

Theorem 12. *Let π_{g_1,g_2} be an integer sequence with exactly five distinct entries. If the generators $g_1 = (a, b, c)$ and $g_2 = (d, e, n)$ are such that $b \neq e$, then π_{g_1,g_2} is unique.*

Proof. The proof follows the same argument used in Theorem 5. We provide the sequence s for all the possible cases $i) - xi)$ and $xiii) - xiv)$:

i) $s = (6^a, 4^{d-a}, 0^{e-d}, -2^{b-e}, -3^{n-b})$.
ii) $s = (6^d, 1^{a-d}, 0^{b+1-a}, -2^{e-b-1}, -3^{n-e})$.
iii) $s = (7^d, 3^{a-d}, 0^{e-a}, -1^{b+1-e}, -6^{n-b-1})$.
iv) $s = (5^d, 3^{e-d}, 1^{a-e}, 0^{b+1-a}, -6^{n-b-1})$.
v) $s = (7^a, 6^{d-a}, -1^{e-d}, -3^{b+1-e}, -4^{n-b-1})$.
vi) $s = (4^d, 3^{e-d}, 2^{a+1-e}, -2^{c-a-1}, -6^{n-c})$.
vii) $s = (5^d, 3^{e-d}, 2^{a+1-e}, -3^{c-a-1}, -7^{n-c})$.
viii) $s = (6^d, 1^{a+1-d}, -1^{c-a-1}, -2^{e-c}, -3^{n-e})$.
ix) $s = (9^d, 2^{a-d}, 0^{b-a}, -1^{c-b}, -4^{n-c})$.
x) $s = (4^{d+1}, 3^{a-d-1}, 0^{b-a}, -2^{c-b}, -7^{n-c})$.
xi) $s = (4^a, 2^{d+1-a}, -1^{b-d-1}, -2^{c-b}, -3^{n-c})$.
xiii) $s = (6^d, 4^{a-d}, 0^{b-a}, -2^{e-b}, -3^{n-e})$.
xiv) $s = (4^a, 0^{e-a}, -1^{b-e}, -2^{c-b}, -3^{n-c})$.

The computation of the degree sequences from the provided s concludes the proof. □

The case *xii)* in Theorem 11, i.e. when $b = e$, is harder to analyze. Indeed, in such situation the sequence π_{g_1,g_2} may or may not belong to $\mathcal{D}$. We present in the following example that both cases are feasible:

Example 6. The following degree sequences are realized by ideals whose generators are as in case xii) of Theorem 11, and one only belongs to the class $\mathcal{D}$:

$\pi_1 = (34, 28, 28, 23, 23, 9, 9, 9, 9, 9, 4, 4)$ is realized by the ideal $I_1 = \downarrow\{(3, 5, 10), (1, 5, 12)\}$ and can be obtained starting from $s_1 = (5, 3, 3, 2, 2, -4, -4, -4, -4, -4, -6, -6)$.

$\pi_2 = (30, 22, 22, 17, 17, 17, 12, 12, 5, 5)$ is realized by $I_2 = \downarrow\{(3, 6, 8), (1, 6, 10)\}$ and does not admit any generating sequence s.

To prove that π_2 belongs to $\mathcal{D}^{ext} \setminus \mathcal{D}$ we proceed by contradiction, assuming the existence of a generating sequence s such that $\pi_2 = \pi^{(s)}$. By Theorem 1, we can find a sequence s with the form $s = (s_1, s_3^2, s_6^3, s_8^2, s_{10}^2)$. Since the triplets $(1, 7, 8), (2, 3, 10)$ and $(4, 5, 6)$ do not belong to I_2, the following inequalities hold: $s_1 + 2s_8 \leq 0$, $2s_3 + s_{10} \leq 0$ and $3s_6 \leq 0$. Summing up, we find that $0 \geq (s_1 + s_6 + s_{10}) + 2(s_3 + s_6 + s_8)$, in contradiction with the fact that the generators of I_2 are edges of the hypergraph.

In Example 6 we show that $\pi_2 \notin \mathcal{D}$ by contradiction, making considerations about feasible inequalities in case a sequence s exists. Such inequalities can be generalized obtaining the following sufficient condition:

Lemma 4. *Let π_{g_1,g_2} be the degree sequence associated to $I = \downarrow\{g_1, g_2\}$, with $g_1 = (a, b, c)$, $g_2 = (d, b, n)$ and $d < a$. If $b \geq a+3$ and $a \geq d+2$, then $\pi_{g_1,g_2} \in \mathcal{D}^{ext} \setminus \mathcal{D}$.*

Proof. By contradiction, if a sequence s realizing π_{g_1,g_2} exists, by Theorem 1 and Theorem 6 it is of kind $s = (s_d^d, s_a^{a-d}, s_b^{b-a}, s_c^{c-b}, s_n^{n-c})$, and the following inequalities hold:

$s_d + 2s_c \leq 0$. We remind that $b < c - 1$, otherwise only four distinct values are present in π_{g_1,g_2}, because of the presence of consecutive numbers in g_1 (Theorem 9). As a consequence, the triplet $(d, c-1, c)$ corresponds in s to the elements $s_d + s_c + s_c$, whose sum is negative since the edge is not part of the hypergraph (the triplet is not comparable both with g_1 and g_2 in $\mathcal{T}_n$).

$2s_a + s_n \leq 0$. Again, the elements in s corresponding to the triplet $(a-1, a, n)$ are $s_a + s_a + s_n$, being $a \geq d+2$ by hypothesis, and the sum is negative since the triplet is not comparable with the generators in $\mathcal{T}_n$.

$3s_b \leq 0$. In this case, we refer to the triplet $(b-2, b-1, b)$, corresponding to $s_b + s_b + s_b$ since $b \geq a+3$ by hypothesis. Again, the corresponding edge is not part of the hypergraph since it is not comparable with the two generators of the ideal.

Summing the members of the three inequalities we reach a contradiction with the hypothesis that $s_a + s_b + s_c > 0$ and $s_d + s_b + s_n > 0$. $\square$

Remark 7. Actually, it is possible to show that π_{g_1,g_2} is in the class $\mathcal{D}$ in the following cases:

$b < a+3$: it is realized by the sequence $s = (5^d, 3^{a-d}, 2^{b-a}, -4^{c-b}, -6^{n-c})$.

$b \geq a+3$ and $d = a-1$: it is realized by the sequence $s = (9^d, 6, 0^{b-a}, -5^{c-b}, -7^{n-c})$.

Thus, we reach the complete characterization of the sequences with five distinct values in $\mathcal{D}$:

Theorem 13. *Let π_{g_1,g_2} be a five distinct values degree sequence realized by an ideal with two generators. Then, $\pi_{g_1,g_2} \in \mathcal{D}^{ext} \setminus \mathcal{D}$ if and only if $g_1 = (a, b, c)$ and $g_2 = (d, b, n)$, with $b \geq a+3$ and $a \geq d+2$.*

4.4. Six distinct values degree sequences

Let us now consider the general case of two generators that provide a six different values degree sequence. We start by showing two examples of degree sequences having six different values and associated with two generators: the first one which admits a sequence s and the second one which does not. Moving from the examples, we characterize the degree sequences π_{g_1,g_2} which do not admit a sequence s. For the remaining cases, we provide a general form for one of the related sequences s, so obtaining their uniqueness.

Example 7. Let us consider the six distinct values sequence $\pi_{g_1,g_2} = (14, 10, 8, 8, 7, 6, 4)$, whose generators are $g_1 = (2, 4, 6)$ and $g_2 = (1, 5, 7)$. To guess a possible sequence s such that $\pi_{g_1,g_2} = \pi^{(s)}$, we explicitly write the inequalities derived from the presence in the hypergraph I_{g_1,g_2} of the edges $(1, 5, 7)$ and $(2, 4, 6)$, and of their intersection $(1, 4, 6)$. Thus we deduce $s_1 + s_5 + s_7 > 0$, $s_2 + s_4 + s_6 > 0$ and $s_1 + s_4 + s_6 > 0$. On the other side, $(2, 4, 7)$, $(2, 5, 7)$, $(3, 4, 6)$, $(2, 5, 6)$ and $(1, 6, 7)$ are not in I_{g_1,g_2}. Thus, we compute the inequalities $s_2 + s_4 + s_7 \leq 0$, $s_2 + s_5 + s_7 \leq 0$, $s_3 + s_4 + s_6 \leq 0$, $s_2 + s_5 + s_6 \leq 0$ and $s_1 + s_6 + s_7 \leq 0$. By testing the previous inequalities, we are able to guess a generating sequence s satisfying them, i.e., $s = (5, 3, 0, 0, -1, -2, -3)$.

Example 8. Now, let $\pi_{g_1,g_2} = (41, 35, 35, 23, 23, 21, 21, 15, 15, 15, 4, 4)$ be the sequence whose generators are $g_1 = (3, 7, 10)$ and $g_2 = (1, 5, 12)$. Assuming there exists an integer sequence s such that $\pi_{g_1,g_2} = \pi^{(s)}$, we compute the two inequalities $s_1 + s_5 + s_{12} > 0$ and $s_3 + s_7 + s_{10} > 0$. In the opposite, the triplets $(1, 9, 10)$, $(1, 7, 12)$, $(2, 3, 12)$ and $(4, 5, 7)$ are not in the ideal $\downarrow\{g_1, g_2\}$. Thus, we compute the inequalities $s_1 + s_9 + s_{10} \leq 0$, $s_1 + s_7 + s_{12} \leq 0$, $s_2 + s_3 + s_{12} \leq 0$ and $s_4 + s_5 + s_7 \leq 0$. By construction of s, we have $s_1 > s_3 > s_5 > s_7 > s_9 > s_{10} > s_{12}$. This ordering implies $s_1 + s_{10} + s_{10} \leq 0$, $s_1 + s_7 + s_{12} \leq 0$, $s_3 + s_3 + s_{12} \leq 0$ and $s_5 + s_5 + s_7 \leq 0$. By summing, we get $0 \geq 2s_1 + 2s_3 + 2s_5 + 2s_7 + 2s_{10} + 2s_{12} = 2(s_1 + s_5 + s_{12}) + 2(s_3 + s_7 + s_{10})$, in contradiction with the fact that $s_1 + s_5 + s_{12} > 0$ and $s_3 + s_7 + s_{10} > 0$. In conclusion, there is no sequence s from which it is possible to compute π_{g_1,g_2}.

Remark 8. We point out a subtle difference between the entries of the two generators in Example 7 and those in Example 8: in the first case, the values of the union of the entries of the two generators are fairly close (they almost form an interval since the element 3 is missing), whereas in Example 8 the entries are at least at distance two apart.

As witnessed in Example 8, also in case of six distinct values in the degree sequence π_{g_1,g_2}, finding a generating sequence s is not always possible. Even worst, we have no explicit clue about the uniqueness property of the sequence. In the sequel, we will carry on the analysis of the six valued degree sequences obtained from two generators according to the mutual positions of the entries of the generators. By Theorem 6, we know that five cases only are possible, when the non comparability of the generators is required. In particular, we will show that for three of them the degree sequences belong to the class $\mathcal{D}$, thus obtaining their uniqueness as a direct consequence. In the remaining two cases nothing can be said, in general (see Example 9).

Theorem 14. *Let π_{g_1,g_2} be a six distinct values degree sequence with generators $g_1 = (a, b, c)$ and $g_2 = (d, e, n)$. The uniqueness of π_{g_1,g_2} is guaranteed only if either $d < a < b < c < e$, or $d < e < a < b < c$ or, finally, $a < d < e < b < c$.*

Proof. Let $g_1 = (a, b, c)$ and $g_2 = (d, e, n)$ be two non comparable elements in $\mathcal{T}_n$ that generate a six valued degree sequence π_{g_1,g_2}. By Theorem 6, it holds that there are no consecutive numbers in each generator. Five different cases arise according to the mutual positions of the entries of g_1 and g_2, that we group in two situations. On one hand, the proof follows the same argument used in Theorem 5, and we provide the sequence s realizing π_{g_1,g_2} (this holds for three out of five cases).

i) Case $d < a < b < c < e$. We have $\pi_{g_1,g_2} = \pi^{(s)}$, with $s = (10^d, 2^{a-d}, 0^{b-a}, -1^{c-b}, -4^{e-c}, -5^{n-e})$.
ii) Case $d < e < a < b < c$. We have $\pi_{g_1,g_2} = \pi^{(s)}$, with $s = (6^d, 5^{e-d}, 4^{a-e}, 0^{b-a}, -3^{c-b}, -10^{n-c})$.
iii) Case $a < d < e < b < c$. We have $\pi_{g_1,g_2} = \pi^{(s)}$, with $s = (8^a, 6^{d-a}, 0^{e-d}, -3^{b-e}, -4^{c-b}, -5^{n-c})$.

Note that these are three out of the five cases presented in the proof of Theorem 6. For the remaining two cases, $d < a < e < b < c$ and $d < a < b < e < c$, it is not possible to establish if they are in the class $\mathcal{D}$ or $\mathcal{D}^{ext} \setminus \mathcal{D}$, see Example 9. □

Remark 9. We notice that the two cases $d < a < e < b < c$ and $d < a < b < e < c$ are similar to case *xii)* in Theorem 11, where $d < a < b = e < c$, that is exactly the case in which the associated degree sequences can both belong to the class $\mathcal{D}$ or $\mathcal{D}^{ext} \setminus \mathcal{D}$.

Example 9. We provide an example of non-unique, and so without a generating sequence s, degree sequence associated to an ideal with two generators. We are in the third case presented in Theorem 6, namely $d < a < e < b < c$, and we consider $\pi_1 = (41, 35, 35, 23, 23, 21, 21, 15, 15, 15, 4, 4)$ the degree sequence of the hypergraph $I_1 = \downarrow\{(3, 7, 10), (1, 5, 12)\}$. We can construct a second 3-hypergraph H, realizing π, removing from I_1 the edges $(1, 4, 12)$, $(1, 5, 12)$, $(2, 7, 9)$, $(3, 7, 10)$ and then adding $(1, 7, 12)$, $(2, 3, 12)$, $(1, 9, 10)$, $(4, 5, 7)$. The degree sequence does not change, since we remove and add the same number of elements in the triplets, and the two hypergraphs are not isomorphic. Indeed, H loses the structure of ideal in $\mathcal{T}_{12}$.

On the other hand, the degree sequence $\pi_2 = (11, 10, 8, 7, 5, 5, 2)$ realized by $I_2 = \downarrow\{(2, 4, 6), (1, 3, 7)\}$ is in the class $\mathcal{D}$, and can be obtained starting from the integer sequence $s = (4, 3, 1, 0, -2, -2, -4)$.

Similarly, if we move to the case $d < a < b < e < c$, the degree sequence $\pi_3 = (20, 15, 10, 10, 10, 9, 8, 5)$ associated to $I_3 = \downarrow\{(2, 5, 7), (1, 6, 8)\}$ is equal to $\pi^{(s)}$, choosing $s = (7, 4, 0, 0, 0, -2, -3, -4)$, while the sequence $\pi_4 = (30, 20, 20, 17, 17, 13, 10, 10, 5, 5)$ realized by $I_4 = \downarrow\{(3, 5, 8), (1, 6, 10)\}$ is in $\mathcal{D}_{ext} \setminus \mathcal{D}$. The proof is analogous to that one presented in Example 6: if such a sequence s exists, then $s_1 + 2s_8 \leq 0$, $2s_3 + s_{10} \leq 0$ and $2s_5 + s_6 \leq 0$, since $(1, 7, 8)$, $(2, 3, 9)$ and $(4, 5, 6)$ are not edges of the hypergraph. It follows $(s_1 + s_6 + s_{10}) + 2(s_3 + s_5 + s_8) \leq 0$, in contradiction with the values in the triplets of generators.

Similarly to case *xii)* in Theorem 11, if there is enough distance between the values in the triplets of generators we can ensure that no integer sequence s realizing the degree sequence π_{g_1,g_2} exists. The following result is a simple generalization of Lemma 4:

Theorem 15. *Let π_{g_1,g_2} be the degree sequence associated to $I = \downarrow\{g_1, g_2\}$, with $g_1 = (a, b, c)$ and $g_2 = (d, e, n)$. Then, $\pi_{g_1,g_2} \in \mathcal{D}^{ext} \setminus \mathcal{D}$ if one of the following cases occurs:*

i) *$d < a < b < e < c$ and $c \geq e + 2$, $a \geq d + 2$, $b \geq a + 2$, or*

ii) $d < a < e < b < c$ and $c \geq b + 2$, $a \geq d + 2$, $b \geq e + 2$.

Proof. The proof is analogous to the proof presented in Lemma 4. If a sequence s realizing π_{g_1,g_2} exists, we have:

i) $s_d + 2s_c \leq 0$, $2s_a + s_n \leq 0$ and $2s_b + s_e \leq 0$, whose sum contradicts the fact that $s_a + s_b + s_c > 0$ and $s_d + s_e + s_n > 0$.
ii) $s_d + 2s_c \leq 0$, $2s_a + s_n \leq 0$ and $2s_b + s_e \leq 0$, whose sum contradicts the fact that $s_a + s_b + s_c > 0$ and $s_d + s_e + s_n > 0$.

This reasoning holds each time we have enough equal elements in the sequence s. □

5. Conclusions and open problems

In this study, we consider an algebraic approach to the reconstruction of 3-uniform hypergraphs from their degree sequences. The problem is known to be NP-hard, so we consider a relevant subclass of degree sequences $\mathcal{D}$ that has been defined in [2] and then extended to $\mathcal{D}^{ext}$ in [3], with the perspective of defining on them a fast reconstruction strategy. Remarkably, the class $\mathcal{D}$ has the uniqueness property, i.e., each of its elements is associated to one single 3-uniform hypergraph (up to isomorphism).

In this paper, we show how to associate to each element of $\mathcal{D}_n^{ext}$ an ideal of the poset $\mathcal{T}_n$ of triplets on n integer numbers, and then we study the reconstruction problem according to the number of generators of the ideals. We provide a complete characterization of the degree sequences related to ideals having one or two generators. In the last case, we spot the first cases where the uniqueness property is lost.

Several open problems arise concerning both computational and combinatorial aspects of this research. A first step ahead will be the characterization of the six distinct values degree sequences that are not unique, and that belong to $\mathcal{D}^{ext} \setminus \mathcal{D}$. Moving from these first cases, we aim at characterizing the unique degree sequences related to generic ideals. The final goal would be to use the detected algebraic properties to define a fast reconstruction strategy for the elements of $\mathcal{D}^{ext}$.

On the other side, from a more combinatorial perspective, we can formulate some interesting enumeration problems concerning our investigation. For instance, given the number n, we would like to count the number of different degree sequences in $\mathcal{D}_n$ determined by the ideals of $\mathcal{T}_n$. This number is clearly less than the total number of ideals, but results are known only for small values of n or for a fixed (small) number of generators.

Furthermore, it is still unknown the range of distinct values that a degree sequence of $\mathcal{D}^{ext}$ can have when varying the number of generators of the associated ideal. So, one can think of defining a hierarchy on $\mathcal{D}^{ext}$ according to the distinct values in the degree sequences and then enumerate them according to such parameter.

CRediT authorship contribution statement

Michela Ascolese: Writing – review & editing, Writing – original draft, Investigation, Conceptualization. **Andrea Frosini:** Writing – review & editing, Writing – original draft, Investigation, Conceptualization. **Elisa Pergola:** Writing – review & editing, Investigation, Conceptualization. **Simone Rinaldi:** Writing – review & editing, Writing – original draft, Investigation, Conceptualization. **Laurent Vuillon:** Writing – review & editing, Investigation, Conceptualization.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: Michela Ascolese reports travel was provided by Francesco Severi National Institute of Higher Mathematics National Group of Scientific Calculations. If there are other authors, they declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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