Quasivarieties of Wajsberg hoops

Paolo Aglianò

DIISM, Università di Siena, Italy

Abstract

In this paper we deal with quasivarieties of residuated structures which form the equivalent algebraic semantics of a positive fragment of some substructural logic. Our focus is mainly on varieties and quasivarieties of Wajsberg hoops, which are the equivalent algebraic semantics of the positive fragment of Łukasiewicz many-valued logic. In particular we study the lattice of subquasivarieties of Wajsberg hoops and we describe completely all the subvarieties of Wajsberg hoops that are primitive. Though the treatment is mostly algebraic in nature, there are obvious connections with the underlying logics.
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1. Introduction

With the birth of *Abstract Algebraic Logic*, which can be traced back to the seminal monograph by W. Blok and D. Pigozzi [17], the connections between a logic (and its extensions) and its class of algebraic models have become a fact of life and a continuing source of inspiration. In fact, formulating such a deep, but somewhat transparent and user-friendly connection has enabled the discovery of many *bridge theorems* between the two fields. Loosely speaking formulating a bridge theorem consists in stating a "logical" result in a totally algebraic fashion (or viceversa). This implies that certain logical concepts can be studied and investigated using the machinery of general algebra; even more, some logical concepts, when translated into algebra, can acquire a life on their own and be investigated *per se* independently of their logical origin.

The main topic of this paper is an example; a logic L is *structurally complete* if every admissible rule of L is derivable in L. Classical logic is structurally complete but intuitionistic logic is not: a famous example is Harrop's rule

$$\{\neg p \to (q \lor r)\} \implies \{(\neg p \to q) \lor (\neg p \to r)\}$$

which is admissible but not derivable. A logic L is *hereditary structurally complete* if L and all its extensions are structurally complete. The first to realize that these two concepts have an interesting algebraic counterpart (even before the Blok-Pigozzi connection was established) was C. Bergman [13]. The Blok-Pigozzi algebraization machinery

associates to a logic with enough structure (an *algebraizable logic*) a quasivariety of algebras, called its *equivalent algebraic semantics*. The following can be proved in a very standard way:

Theorem 1.1. *Let L be a logic and* Q *its equivalent algebraic semantics; then*

1. *L is structurally complete if and only if no proper subquasivariety of* Q *can generate the same variety as* Q;
2. *L is hereditary structurally complete if and only if every subquasivariety of Q is equational relative to* Q *(i.e. is axiomatized relative to* Q *by a set of equations).*

In this paper we investigate the algebraic properties suggested by the above theorem in quasivarieties of Wajsberg hoops. Hoops are residuated monoids, introduced in an unpublished manuscript by Büchi and Owens, grounded on the work of Bosbach on partially ordered monoids [19]. Hoops are commutative semilattice ordered monoids and are in fact residuated; the monoidal operation $\cdot$ has a residuum $\rightarrow$ which makes the underlying ordering the *inverse divisibility ordering* (i.e. $a \leq b$ if and only if there is a c with $a = bc$. Hoops have been first studied systematically by I.M.A. Ferreirim in her PhD thesis [27] and later in [16].

Wajsberg hoops are hoops that satisfy *Tanaka's equation*

$$(x \rightarrow y) \rightarrow y \approx (y \rightarrow x) \rightarrow x$$

and play a very important role in mathematical fuzzy logic. The connection between hoops and many-valued logic has been first investigated in [7]; however the real impact of Wajsberg hoops was made clear in [8]. In that paper it was shown that one cannot really understand BL-algebras, i.e. the equivalent algebraic semantics of *Hájek Basic Logic*, without understanding Wajsberg hoops; this happens because every subdirectly irreducible BL-algebra can be constructed as an *ordinal sum* of a Wajsberg algebra and Wajsberg hoops.

Wajsberg hoops are also connected with ℓ-groups: the totally ordered Wajsberg hoops generate the variety of Wajsberg hoops and they are either isomorphic to negative cones of an abelian ℓ-group or intervals in abelian ℓ-groups [9]. In terms that are perhaps more familiar to the reader versed in many-valued logic, totally ordered Wajsberg hoops are either cancellative hoops (hoops in which the underlying monoid is cancellative), or (term equivalent to) MV-algebras, whose representation in terms of intervals of abelian ℓ- groups with strong unit is actually a categorical equivalence via the Mundici's functor [42]. From the logical point of view Wajsberg hoops are (term equivalent to) zero-free subreducts of MV-algebras, the equivalent algebraic semantics of *Łukasiewicz many-valued logic*; in fact they are the equivalent algebraic semantics of the positive fragment of that logic.

Since in this paper we use (sometimes sophisticated) techniques in general algebra, in order to make it understandable to a wider audience we felt it necessary to provide a reasonably thorough introduction to the algebraic properties of varieties and quasivarieties, as well as the main algebraic results that are needed. Some of these results are already known but we wanted to present them in a form that we could use in the remainder of the paper. We do this in Sections 2, 3 and 4. In Section 5 we investigate that lattice of subquasivarieties of Wajsberg hoops with special attention to quasivarieties generated by chains. In Section 6 we tackle the problem of characterizing the primitive and structurally complete subvarieties and subquasivarieties of Wajsberg hoops. In Section 7 we look at subquasivarieties not generated by chains and we try to highlight the complexity of the lattice of subquasivarieties. Finally in Section 8 we investigate the connections (or the lack thereof) between the lattice of subquasivarieties of Wajsberg hoops and the lattice of subquasivarieties of MV-algebras.

2. Quasivarieties

For general results in universal algebras, as well as for all the unexplained basic notions, we refer the reader to [20] or [41]; we will be constantly using the *class operators* **I**, **H**, **S**, **P**, $\mathbf{P}_u$ that, applied to a class K of algebras, give the class of isomorphic images, homomorphic images, subalgebras, direct products and ultraproducts of (families of) algebras in K. A class of algebras is a **variety** if it is closed under **H**, **S** and **P**; if $\mathbf{V} = \mathbf{HSP}$, then $\mathbf{V}(\mathsf{K})$ is a variety and it is the smallest variety containing all algebras in K. The subvarieties of a variety V form a complete lattice under inclusion, that we denote by $\Lambda(\mathsf{V})$.

There are two fundamental results that we will be using many times and deserve a spotlight. Let $(\mathbf{A}_i)_{i \in I}$ be a family of algebras; we say that **B embeds** in $\prod_{i \in I} \mathbf{A}_i$ if $\mathbf{B} \in \mathbf{IS}(\prod_{i \in I} \mathbf{A}_i)$. Let p_i be the i-th projection (better the

composition of the isomorphism and the i-th projection) from $\mathbf{B}$ to $\mathbf{A}_i$; the embedding is **subdirect** if for all $i \in I$, $p_i(\mathbf{B}) = \mathbf{A}_i$ and in this case we will write

$$\mathbf{B} \leq_{sd} \prod_{i \in I} \mathbf{A}_i.$$

An algebra $\mathbf{B}$ is **subdirectly irreducible** if it is nontrivial and for any subdirect embedding

$$\mathbf{B} \leq_{sd} \prod_{i \in I} \mathbf{A}_i$$

there is an $i \in I$ such that $\mathbf{B}$ and $\mathbf{A}_i$ are isomorphic under p_i If V is a variety we denote by $\mathrm{si}(\mathsf{V})$ the class of subdirectly irreducible algebras in V.

Theorem 2.1. *(Birkhoff [15]) Every algebra can be subdirectly embedded in a product of subdirectly irreducible algebras. So if $\mathbf{A} \in \mathsf{V}$, then $\mathbf{A}$ can be subdirectly embedded in a product of members of* $\mathrm{si}(\mathsf{V})$.

A variety V is congruence distributive if the congruence lattices of all algebras in V are distributive.

Theorem 2.2. *(Jónsson's Lemma [36]) Suppose that K is a class of algebras such that $\mathsf{V}(\mathsf{K})$ is congruence distributive. Then*

1. $\mathrm{si}(\mathsf{V}) \subseteq \mathbf{HSP}_u(\mathsf{K})$;
2. *if $\mathsf{V}_1, \ldots, \mathsf{V}_n$ are subvarieties of V, then*

$$\mathrm{si}(\mathsf{V}_1 \vee \cdots \vee \mathsf{V}_n) = \mathrm{si}(\mathsf{V}_1) \cup \cdots \cup \mathrm{si}(\mathsf{V}_n)$$

where the join is taken in the lattice of subvarieties of V.

A **quasivariety** is a class of algebras defined by a set of quasiidentities; a quasiidentity is an implication whose premise is a finite join of equations and whose conclusion is a single equation. Given a class K of algebras the **quasiequational theory** of K, denoted by $\mathrm{Th}_q(\mathsf{K})$ is the set of quasiidentities holding in all algebras in K; given a set Σ of quasiidentities $\mathrm{Mod}(\Sigma)$ is the class of algebras in which every quasiidentity in Σ holds. A.I. Mal'cev showed first that for any class K of algebras $\mathrm{Mod}(\mathrm{Th}_q(\mathsf{K}))$ is a quasivariety and

$$\mathbf{ISPP}_u(\mathsf{K}) = \mathrm{Mod}(\mathrm{Th}_q(\mathsf{K})).$$

Therefore, if K is a class of algebras, then $\mathbf{Q}(\mathsf{K}) = \mathbf{ISPP}_u(\mathsf{K})$ is the quasivariety generated by K. While in the western world doing general algebra mostly meant dealing with varieties of algebras, quasivarieties were vigorously pursued in Russia, under the impulse of A.I. Mal'cev. An extensive account of the results of the Russian school can be found in [34].

If Q is a quasivariety and $\mathbf{A} \in \mathsf{Q}$, a **relative congruence** of $\mathbf{A}$ is a congruence θ such that $\mathbf{A}/\theta \in \mathsf{Q}$; relative congruences form an algebraic lattice $\mathrm{Con}_{\mathsf{Q}}(\mathbf{A})$ and for any congruence lattice property P we say that $\mathbf{A} \in \mathsf{Q}$ is **relatively** P if $\mathrm{Con}_{\mathsf{Q}}(\mathbf{A})$ satisfies P. So for instance $\mathbf{A}$ is **relatively subdirectly irreducible** if $\mathrm{Con}_{\mathsf{Q}}(\mathbf{A})$ has a unique minimal element; since clearly $\mathrm{Con}_{\mathsf{Q}}(\mathbf{A})$ is a meet subsemilattice of $\mathrm{Con}(\mathbf{A})$, any subdirectly irreducible algebra is relatively subdirectly irreducible for any quasivariety to which it belongs. For a quasivariety Q we denote by Q_{rsi} the class of relatively subdirectly irreducible algebras in Q.

We have the equivalent of Birkhoff's and Jónsson's results for quasivarieties:

Theorem 2.3. *Let Q be any quasivariety.*

1. *(Mal'cev [39]) Every $\mathbf{A} \in \mathsf{Q}$ is a subdirectly embeddable in a product of algebras in Q_{rsi}.*
2. *(Czelakowski-Dziobiak [24]) If $\mathsf{Q} = \mathbf{Q}(\mathsf{K})$, then $\mathsf{Q}_{rsi} \subseteq \mathbf{ISP}_u(\mathsf{K})$.*

The class of all subquasivarieties of a given quasivariety V is a lattice under inclusion, called the **lattice of subquasivarieties** of Q and denoted by $\Lambda_q(\mathsf{Q})$ Lattices of subquasivarieties are in general very complex. A quasivariety

Q is **Q-universal** [45] if for any other quasivariety Q' of finite type, $\Lambda_q(Q')$ is a homomorphic image of a sublattice of $\Lambda_q(Q)$.

Lemma 2.4. *For every Q-universal quasivariety* Q

- *the free lattice on ω generators is embeddable in $\Lambda_q(Q)$;*
- $|\Lambda_q(Q)| = 2^{\aleph_0}$.

So the lattice of subquasivarieties of a Q-universal quasivariety is horribly complex and unfortunately Q-universal quasivarieties are ubiquitous. First clearly Q-universality is upward hereditary: if Q is Q-universal and $Q \subseteq Q'$, then Q' is universal as well. Second in [1] the authors gave a sufficient condition for a quasivariety to be Q-universal, condition that is satisfied in many cases. Here is a *dumbed-down* version of the condition that works well in our case (see [1], Corollary 3.4).

Lemma 2.5. *Let* Q *be a quasivariety such that* $\mathbf{V}(Q)$ *is congruence distributive and has the congruence extension property. If* Q *contains and infinite family of simple algebras, such that none is embeddable in any other, then* Q *is Q-universal.*

A quasivariety Q is locally finite if every finitely generated algebra in Q is finite, and it is finitely generated if it is generated by finitely many finite algebras. The following facts are easy to check:

- for any quasivariety Q, $\mathbf{H}(Q) = \mathbf{V}(Q)$;
- Q is locally finite if and only if $\mathbf{V}(Q)$ is such;
- for any quasivariety Q and any subquasivariety Q' of Q, $\mathbf{V}(Q')$ is the smallest variety V such that $Q' \subseteq V \cap Q$.

3. Structurally complete and primitive quasivarieties

Because of the results in [17], one may argue that quasivarieties represent the real algebraic counterparts of logics understood as consequence relations (as opposed to varieties, that are counterparts of logics viewed as a set of theorems). In fact there are some interesting algebraic properties of quasivarieties that have been considered only because of their connection with logic: to a logic with certain characteristics one can associate a quasivariety of algebras called to *equivalent algebraic semantics*. Conversely, given a quasivariety Q with certain algebraic properties one can find a logic such that Q is its equivalent. The procedure is algorithmic and it allows to pass definitions from one side to the other. However, once a logical property is transformed into an algebraic one, than it can be applied to any quasivariety of algebras, irregardless of its "logicizability". The properties we are introducing in this section have both a logical origin that has been discussed in Section 1; more information can be found in [13].

A quasivariety Q is **structurally complete** if all its proper subquasivarieties generate proper subvarieties of $\mathbf{H}(Q)$; we have the following characterization.

Theorem 3.1. *[21] For a quasivariety* Q *the following are equivalent:*

1. Q *is structurally complete;*
2. *for all quasivarieties* $Q' \subseteq Q$ *if* $\mathbf{H}(Q') = \mathbf{H}(Q)$, *then* $Q = Q'$;
3. *for all* $\mathbf{A} \in Q$ *if* $\mathbf{V}(\mathbf{A}) = \mathbf{H}(Q)$, *then* $\mathbf{Q}(\mathbf{A}) = Q$;
4. $Q = \mathbf{Q}(\mathbf{F}_Q(\omega))$.

Proof. That (1) and (2) are equivalent is obvious. If (2) holds, and $\mathbf{V}(\mathbf{A}) = \mathbf{H}(\mathbf{Q}(\mathbf{A})) = \mathbf{H}(Q)$, then $\mathbf{Q}(\mathbf{A}) = Q$ and (3) trivially implies (4). Finally assume(4) and let $Q' \subseteq Q$ such that $\mathbf{H}(Q') = \mathbf{H}(Q)$. Then $\mathbf{F}_{Q'}(\omega) = \mathbf{F}_Q(\omega)$ and thus

$$Q = \mathbf{Q}(\mathbf{F}_Q(\omega)) = \mathbf{Q}(\mathbf{F}_{Q'}(\omega)) \subseteq Q'.$$

Thus $Q = Q'$ and (2) holds. □

For any quasivariety Q, we define the **least Q-quasivariety** as the smallest $Q' \subseteq Q$ such that $\mathbf{H}(Q) = \mathbf{H}(Q')$. This concept has been introduced by J. Gispert in [32] for MV-algebras and it is very useful since:

Corollary 3.2. *For any quasivariety* Q, $\mathbf{Q}(\mathsf{F}_Q(\omega))$ *is structurally complete and moreover it is the least* Q-*quasivariety.*

It follows at once that a quasivariety Q is structurally complete if and only if it coincides with its least Q-quasivariety. As a consequence the structurally complete subvarieties of a quasivariety Q are exactly those that are the least Q'-quasivarieties for some $Q' \subseteq Q$; even more, since $\mathbf{H}(Q)$ is a variety, the structurally complete subquasivarieties of a variety V are exactly the least V'-quasivarieties for some subvariety V' of V. This is not as good as it seems; in general describing the least V-quasivariety is not an easy task, since it requires knowledge of the free countably generated algebra in V.

To get more information we need some definitions: let $\mathbf{A}$ be an algebra and K a class of algebras of the same type as $\mathbf{A}$. We say that

- $\mathbf{A}$ is **projective** in K if for all $\mathbf{B} \in K$ if $f : \mathbf{B} \longrightarrow \mathbf{A}$ is a surjective epimorphism, then there is an embedding $g : \mathbf{A} \longrightarrow \mathbf{B}$ with $gf = id_\mathbf{A}$;
- $\mathbf{A}$ is **weakly projective** in K if for all $\mathbf{B} \in K$ if $\mathbf{A} \in \mathbf{H}(\mathbf{B})$, then $\mathbf{A} \in \mathbf{S}(\mathbf{B})$.

It is clear that if $\mathbf{A}$ is projective in K, then $\mathbf{A}$ is weakly projective in K. An algebra $\mathbf{A}$ is **finitely presented** in Q if it is nontrivial and can be defined by a finite set of generators and relations in Q. This means that there is a finite set X and a compact congruence θ of $\mathbf{F}_Q(X)$ such that $\mathbf{F}_Q(X)/\theta \cong \mathbf{A}$.

Lemma 3.3. *Let* Q *be a quasivariety and let* $K \subseteq Q$ *such that every* $\mathbf{A} \in K$ *is weakly projective in* $\mathbf{Q}(K)$. *Then* $\mathbf{Q}(K)$ *is structurally complete.*

Proof. Let $Q \subseteq \mathbf{Q}(K)$ with $\mathbf{H}(Q) = \mathbf{H}(\mathbf{Q}(K))$; then for any $\mathbf{A} \in K$ there exists a $\mathbf{B} \in Q$ with $\mathbf{A} \in \mathbf{H}(\mathbf{B})$. As $\mathbf{A}$ is weakly projective in Q, $\mathbf{A} \in \mathbf{S}(\mathbf{B})$ and so $\mathbf{A} \in Q$. This implies $Q = \mathbf{Q}(K)$ and so $\mathbf{Q}(K)$ is structurally complete. □

It is well-known (and a standard exercise in many books) that every algebra is embeddable in an ultraproduct of its finitely generated subalgebras; it is less known but still true ([34], Proposition 2.1.18) that any quasivariety Q is generated by its finitely presented algebras. Since any quasivariety is also generated by its relative subdirectly irreducible algebras we have

Corollary 3.4. *Let* Q *be a quasivariety; if either*

1. *every finitely generated algebra in* Q *is weakly projective, or*
2. *every finitely presented algebra in* Q *is weakly projective, or*
3. *every finitely generated relative subdirectly irreducible in* Q *is weakly projective,*

then Q *is structurally complete.*

Clearly (1) implies both (2) and (3); however none of these conditions is necessary and to get a necessary one we have to consider a smaller class of quasivarieties. We say that a class K of algebras is **tame** if every finitely generated algebra in K is finitely presented. Note that the concept has content: any class K of algebras of finite type which is locally finite in the usual sense (i.e. every finitely generated algebra in K is finite) is tame since in that case finite, finitely generated and finitely presented coincide. Tame classes of algebras have been studied mainly in groups (better, in algebras in which groups are interpretable): for instance any nilpotent class of groups is tame, so abelian groups are tame (and it is an example of a tame non locally finite variety).

If $\mathbf{A} \in Q$ we define

$$[Q : \mathbf{A}] = \{\mathbf{B} \in Q : \mathbf{A} \notin \mathbf{IS}(\mathbf{B})\}.$$

It is also folklore that if $\mathbf{A}$ is finitely presented, then there is a first order formula (that in many cases can be made explicit) Ψ such that for any $\mathbf{B}$, $\mathbf{A} \in \mathbf{S}(\mathbf{B})$ if and only if $\mathbf{B} \vdash \Psi$.

Lemma 3.5. *Let* Q *be any quasivariety.*

1. *If* **A** ∈ Q *is finitely presented, then* [Q : **A**] *is closed under* **ISP**$_u$ *(i.e. it is a universal class);*
2. *if* **A** *is also relatively subdirectly irreducible in* Q*, then* [Q : **A**] *is a quasivariety;*
3. *if* **A** *is finitely presented and relatively subdirectly irreducible in* Q*, then* **A** *is weakly projective in* Q *if and only if* [Q : **A**] *is a variety.*

Proof. Let's prove (1) Consider **B** ∈ **ISP**$_u$([Q : **A**]); if **A** ∈ **IS**(**B**), then **A** ∈ **ISP**$_u$([Q : **A**]). Hence there exists a family $(\mathbf{A}_i)_{i \in I} \subseteq$ [Q : **A**] and an ultrafilter U on I such that $\mathbf{C} = \Pi_{i \in I} \mathbf{A}/U$ and **A** ∈ **IS**(**C**). So, if Ψ is the first order formula mentioned above, $\mathbf{C} \vDash \Psi$; but then by Łòs Lemma Ψ must be valid in each $\mathbf{A}_i$ ($i \in J$ for some $J \in U$), which is clearly a contradiction, since $\mathbf{A}_i \in$ [Q : **A**]. So **A** ∉ **IS**(**B**) and **B** ∈ [Q : **A**].

For (2) we proceed as in (1) using **ISPP**$_u$ up to the point in which **A** ∈ **ISPP**$_u$(**C**); but since in this case **A** is relatively subdirectly irreducible, really **A** ∈ **ISP**$_u$(**C**) and the previous argument applies.

For (3) we observe that [Q : **A**] is a quasivariety by (2). First suppose that **A** is weakly projective in Q; let **B** ∈ [Q : **A**] and let $f : \mathbf{B} \longrightarrow \mathbf{C}$ be en epimorphism. If **A** ∈ **S**(**C**) we let $\mathbf{D} = f^{-1}(\mathbf{C}$; then $\mathbf{D} \leq \mathbf{B}$ so that **D** ∈ [Q : **A**] and moreover **A** ∈ **H**(**D**). Since **A** is weakly projective, **A** ∈ **S**(**D**), a contradiction. Therefore **A** ∉ **S**(**C**) and **C** ∈ [Q : **A**].

Conversely suppose that **A** is not weakly projective in Q; then there is a **B** ∈ Q with **A** ∈ **H**(**B**) and **A** ∉ **S**(**B**). It follows that **B** ∈ [Q : **A**] but **H**(**B**) ⊄ [Q : **A**]. Therefore [Q : **A**] is not a variety and (3) holds. □

A subquasivariety Q′ of Q is **equational relative to** Q if Q′ = **H**(Q′) ∩ Q; a quasivariety Q is **primitive** if every subquasivariety of Q is equational relative to Q. Clearly primitivity is downward hereditary and a variety V is primitive if and only if every subquasivariety of V is a variety. The following is a straightforward exercise:

Lemma 3.6. *For a quasivariety* Q *the following are equivalent:*

1. Q *is primitive;*
2. *every subquasivariety of* Q *is structurally complete (i.e.* Q *is hereditarily structurally complete).*

A more interesting fact is:

Lemma 3.7. *Let* Q *be a quasivariety and* **A** ∈ Q*; then* **A** *is weakly projective in* Q *if and only if* [Q : **A**] *is equational relative to* Q.

Proof. Suppose that **A** is weakly projective in Q; we have to show that

$$[Q : \mathbf{A}] = \mathbf{V}([Q : \mathbf{A}]) \cap Q.$$

So take **B** in Q such that there is a **C** ∈ [Q : **A**] with **B** ∈ **H**(**C**) (here we are using the hypothesis that [Q : **A**] is a quasivariety). If **B** ∉ [Q : **A**] then **A** ∈ **S**(**B**); therefore **A** ∈ **SH**(**C**) ⊆ **HS**(**C**). This means that there is a subalgebra **D** of **C** with **A** ∈ **H**(**D**); since **A** is weakly projective **A** ∈ **S**(**D**) ⊆ **S**(**C**), a clear contradiction. So **B** ∈ [Q : **A**] and [Q : **A**] is equational relative to Q.

Conversely assume that [Q : **A**] is equational relative to Q and let **B** ∈ Q with **A** ∈ **H**(**B**). If **A** ∉ **S**(**B**), then **B** ∈ [Q : **A**] and since [Q : **A**] is equational we must have **A** ∈ [Q : **A**] a contradiction. So **A** ∈ **S**(**B**) and **A** is weakly projective in Q. □

The following result was obtained for varieties of lattices by Slavik [46] (but his proof works for any variety of algebras) and later generalized to quasivarieties by Gorbunov [34].

Theorem 3.8. *Let* Q *be a quasivariety of finite type. Then (1) implies (2) which is equivalent to (3); if* Q *is also tame, then they are all equivalent:*

1. Q *is primitive;*
2. *for every finitely presented relative subdirectly irreducible algebra in* **A** ∈ Q*,* [Q : **A**] *is equational relative to* Q;

3. *every finitely presented relative subdirectly irreducible algebra in Q is weakly projective in Q.*

Proof. (2) and (3) are equivalent by Lemma 3.7. Hence assume (1); if $\mathbf{A} \in Q_{rsi}$ is finitely presented, then $[Q : \mathbf{A}]$ is a subquasivariety of Q and by (1) is equational relative to Q, hence (2) holds.

Conversely assume (3), let Q be tame and let $Q' \subseteq Q$; we have to show that $Q' = \mathbf{V}(Q') \cap Q$. Suppose that

$$\mathbf{A} \in (\mathbf{V}(Q') \cap Q) \setminus Q';$$

then $\mathbf{A}$ is a subdirect product of a family $(\mathbf{A}_i)_{i \in I}$ of finitely presented algebras in Q_{rsi}. Clearly each $\mathbf{A}_i \in Q$; moreover $\mathbf{A} \in \mathbf{H}(\mathbf{B})$ for some $\mathbf{B} \in Q'$ and since $\mathbf{A}_i \in \mathbf{H}(\mathbf{A})$ for all i, it follows by (3) that $\mathbf{A}_i \in \mathbf{S}(\mathbf{B}) \in Q'$. Therefore $\mathbf{A} \in Q'$, a clear contradiction, so Q' is equational relative to Q and Q is weakly projective.   □

We have a necessary and sufficient condition if Q is tame.

Theorem 3.9. *If Q is a tame quasivariety, then the following are equivalent.*

1. Q *is primitive;*
2. *if $\mathbf{A}$ is relative subdirectly irreducible, then $[Q : \mathbf{A}]$ is a variety;*
3. *every relative subdirectly irreducible finitely presented algebra $\mathbf{A} \in Q$ is weakly projective in Q;*
4. *every relative subdirectly irreducible finitely presented algebra $\mathbf{A} \in Q$ is weakly projective in the class of finitely presented algebras in Q.*

Again this has been proved by V. Slavik for locally finite varieties of lattices [46] and extended to locally finite quasivarieties of algebras by Gorbunov ([34], Proposition 5.1.24); tameness is more general than local finiteness and the same proofs go through with trivial modifications.

Most results in the literature are about structurally complete and primitive *varieties* of algebras and the reason is quite obvious; first the two concepts are easier to formulate for varieties. In fact a variety is structurally complete if and only if every proper subquasivariety generates a proper subvariety and it is primitive if and only if every sub-quasivariety is a variety. Secondly being subdirectly irreducible is an absolute concept (every subdirectly irreducible algebra is relative subdirectly irreducible in any quasivariety to which it belongs) while being relative subdirectly irreducible depends essentially on the subquasivariety we are considering. Of course when a quasivariety is generated by a "simple" class (e.g. by finitely many finite algebras), then Theorem 2.3(2) gives a simple solution; but in general describing the relative subdirectly irreducible algebras in a quasivariety is not an easy task.

Let's say that a (quasi)variety is **structurally precomplete** if all its proper sub(quasi)varieties are structurally complete; it is obvious that a structurally precomplete quasivariety is primitive if and only if it is structurally complete. A little less obvious but very useful is:

Lemma 3.10. *Let V be a structurally precomplete variety; then V is primitive if and only if it is structurally complete.*

Proof. One direction is obvious. Let then Q be a proper subvariety of V; since V is structurally complete, then Q must generate a proper subvariety of V. This means that there is a proper subvariety V' of V such that $\mathbf{H}(Q) = V'$; but V' is structurally complete, hence $Q = V'$ (Theorem 3.1). So Q is a variety and V is primitive.   □

Remark 3.11. A variety is **minimal** if it does not have any proper nontrivial subvarieties; so a minimal variety is primitive if it has no proper subquasivarieties. In [14] it is shown that a locally finite minimal variety is primitive if and only if it has exactly one subdirectly irreducible algebra that is embeddable in any nontrivial member of the variety. Moreover this is always the case if the variety is congruence modular. Recently this result has been extended in two directions in [22]: the author showed that every minimal dual discriminator variety is primitive and, if the variety is also idempotent, then minimality can be dropped.

Remark 3.12. If $\mathbf{L}$ is a finite projective subdirectly irreducible lattice, then $\mathbf{V}(\mathbf{L})$ is primitive (this is obvious by Jónnson's Lemma) so for instance $\mathbf{V}(\mathbf{2})$ and $\mathbf{V}(\mathbf{N}_5)$ are primitive. By [43] every finite semidistributive lattice satisfying the Whitman condition (W) is projective, so all the (finite sets of) subdirectly irreducible ones generate a primitive

variety. On the other hand a variety V of modular lattices is primitive if and only if $\mathsf{V} = \mathbf{V}(\mathbf{M}_n)$ for some n, where $\mathbf{M}_n$ is the height 3 modular lattice with n coatoms [34]. For a thorough investigation of primitive varieties of lattices we direct the reader to [35].

4. Splittings

A **splitting** of a lattice $\mathbf{L}$ is a pair of elements $a, b \in L$ such that L is the disjoint union of the ideal generated by a and the filter generated by b; in this case a must be completely meet prime and b completely join prime [50]. Splittings in lattice of subvarieties have been extensively studied, starting from the seminal paper [40]; for residuated structures (which are the focus of this paper) we quote [3], [4], [5], [6] and [10]. On the other hand splittings in lattices of subquasivarieties has received much less attention, but the theory is not so different. Suppose that Q_1, Q_2 is a splitting in $\Lambda_q(\mathsf{Q})$; if Σ_1 is the quasiequational theory of Q_1 (i.e. all the quasiequations holding in Q_1), then

$$\mathsf{Q}_1 = \mathrm{Mod}(\Sigma_1) = \bigcap \{\mathrm{Mod}(\sigma) : \sigma \in \Sigma_1\}.$$

As Q_1 is completely meet prime it must be $\mathsf{Q}_1 = \mathrm{Mod}(\sigma_1)$ for some $\sigma_1 \in \Sigma$.

On the other hand every algebra in a quasivariety is embeddable in an ultraproduct of its finitely generated subalgebras, each of which is a subdirect product of (necessarily finitely generated) relative subdirectly irreducible algebras. It follows that

$$\mathsf{Q}_2 = \bigvee \{\mathbf{Q}(\mathbf{A}) : \mathbf{A} \in \mathsf{Q}_{rsi}, \mathbf{A} \text{ is finitely generated}\};$$

as Q_2 is completely join prime $\mathsf{Q}_2 = \mathbf{Q}(\mathbf{A})$ for some finitely generated $\mathbf{A} \in \mathsf{Q}_{rsi}$.

A **splitting algebra** is a finitely generated algebra $\mathbf{A} \in \mathsf{Q}_{rsi}$ such that there is a $\mathsf{Q}_1 \subseteq \mathsf{Q}$ such that $\mathsf{Q}_1, \mathbf{Q}(\mathbf{A})$ is a splitting in $\Lambda_q(\mathsf{Q})$; in this case σ_1 is called the **splitting quasiequation** for $\mathbf{A}$. In other words $\mathbf{A}$ is splitting if there exists a largest subquasivariety Q_1 of Q, called the **conjugate quasivariety of $\mathbf{A}$** such that $\mathbf{A} \notin \mathsf{Q}_1$.

A class of algebras K has **the finite embeddability property** (FEP for short) if for all $\mathbf{A} \in \mathsf{K}$ and for all partial subalgebra $\mathbf{A}'$ of $\mathbf{A}$, there is a finite $\mathbf{B} \in \mathsf{K}$ such that $\mathbf{A}'$ is embeddable in $\mathbf{B}$. For a quasivariety Q we let Q_{fin} be the class of finite algebras in Q.

Theorem 4.1. *For a quasivariety Q the following are equivalent:*

1. Q *has the FEP;*
2. *every algebra in Q_{rsi} has the FEP;*
3. $\mathsf{Q} = \mathbf{ISPP}_u(\mathsf{Q}_{fin})$.

Proof. (1) implies (2) is obvious. Assume then (2) and let

$$\Psi = \bigwedge_{i=1}^{n} (p_i \approx q_i) \rightarrow r \approx s$$

be a quasi equation in the language of Q such that $\mathsf{Q} \nvDash \Psi$; since any algebra is a subdirect product of algebras in Q_{rsi}, there exists an $\mathbf{A} \in \mathsf{Q}_{rsi}$ such that $\mathbf{A} \nvDash \Psi$. Let $x_1, \ldots, x_n$ be the variables in Ψ; then there exists $a_1, \ldots, a_n \in A$ such that $p_i(a_1, \ldots, a_n) = q_i(a_1, \ldots, a_n)$ for all i but $r(a_1, \ldots, a_n) \neq s(a_1, \ldots, a_n)$. Let

$$A' = \{a_1, \ldots, a_n\} \cup \{t(a_1, \ldots, a_n) : t \text{ is a subterm of } \Psi\};$$

then $\mathbf{A}'$ is a finite partial subalgebra of $\mathbf{A}$ and, since Q_{rsi} has the FEP, there exists a finite $\mathbf{B} \in \mathsf{Q}_{rsi}$ such that $\mathbf{A}'$ is embeddable in $\mathbf{B}$. But clearly $\mathbf{B} \nvDash \Psi$, hence (3) holds by counterpositive.

The proof that (3) implies (1) appears in [27] and it is an easy modification of the analogous result in [26] for varieties. $\square$

As a consequence we get:

Theorem 4.2. *Let* Q *be any quasivariety with the FEP and let* $\mathsf{Q}_1, \mathsf{Q}_2$ *be a splitting in* Q*; then there exists a finite algebra* $\mathbf{A} \in \mathsf{Q}_{rsi}$ *such that* $\mathsf{Q}_2 = \mathbf{Q}(\mathbf{A})$*.*

Proof. Since Q has the FEP by Theorem 4.1 we may assume that Q is the join in $\Lambda_q(\mathsf{Q})$ of all its finitely generated subquasivarieties. Since Q_2 is completely join prime, then Q_2 is contained in one of them and hence it is itself finitely generated. Hence Q_2 is the join of a set of quasivarieties and each of one is generated by a single finite algebra that (by Theorem 2.3(2)) can be taken to be relatively subdirectly irreducible; but since Q_2 is completely join irreducible, it must be equal to one of them. This concludes the proof. $\quad\square$

Lemma 4.3. *Let* Q *be a quasivariety; then every finitely presented* $\mathbf{A} \in \mathsf{Q}_{rsi}$ *is splitting in* $\Lambda_q(\mathsf{Q})$ *with conjugate quasivariety* $[\mathsf{Q} : \mathbf{A}]$*.*

Proof. Since $\mathbf{A}$ is relative subdirectly irreducible and finitely presented, then $[\mathsf{Q} : \mathbf{A}]$ is a quasivariety. Suppose Q' is a quasivariety such that $\mathbf{A} \notin \mathsf{Q}'$; if $\mathsf{Q}' \nsubseteq [\mathsf{Q} : \mathbf{A}]$ then there is an algebra $\mathbf{B} \in \mathsf{Q}'$ with $\mathbf{A} \in \mathbf{S}(\mathbf{B}) \subseteq \mathsf{Q}'$, a contradiction. Hence $\mathsf{Q}' \subseteq [\mathsf{Q} : \mathbf{A}]$ and $\mathbf{A}$ is splitting with conjugate quasivariety $[\mathsf{Q} : \mathbf{A}]$. $\quad\square$

It follows (from Lemma 3.5) that every finitely presented weakly projective algebra in Q has a conjugate variety.

5. Quasivarieties of Wajsberg hoops: chain generated subquasivarieties

A **left residuated semilattice** (short for *left residuated semilattice ordered monoid*) is an algebra $\mathbf{A} = \langle A, \wedge, \cdot, \rightarrow, 1 \rangle$ where

- $\langle A, \wedge \rangle$ is a semilattice;
- $\langle A, \cdot, 1 \rangle$ is a monoid;
- $(\cdot, \rightarrow)$ form a left residuated pair w.r.t. the semilattice ordering.

Left residuated semilattices form a variety; for an axiomatization the reader can consult [2] where they have been studied under the (rather unfortunate) name of em BI-monoids. A left residuated semilattice is **commutative** if so is the monoid operation; in that case the left residuation is a also a right residuation and the monoid is residuated. A left residuated semilattice is **integral** if 1 is the uppermost element of the ordering. We will denote by CIRS the variety of integral and commutative residuated semilattices.

Residuated lattices are defined in the obvious fashion (see [18] and we denote by CIRL the variety of commutative and integral residuated lattices. CIRSs share a good chunk of the theory with CIRLs since CIRS is exactly the class of $\vee$-less subreducts of CIRLs. They are congruence permutable with Mal'cev term

$$m(x, y, z) = ((x \rightarrow y) \rightarrow z) \wedge ((z \rightarrow y) \rightarrow x),$$

and moreover, since they have a semilattice term, they are also congruence distributive. The theory of congruences (and filters) is identical to the one of residuated lattices; as a matter of fact it can be easily shown that the congruences of a residuated lattice are exactly the congruences of its semilattice reduct. For a list of equations holding in residuated (semi)lattices the reader can consult [2] or [18]. A residuated (semi)lattice is **bounded** if it has a (necessarily unique) minimal element in the ordering.

A commutative and integral residuated semilattice $\mathbf{A}$ is **representable** if for all $a, b, c \in A$

$$(a \rightarrow b) \rightarrow c \leq ((b \rightarrow a) \rightarrow c) \rightarrow c;$$

it is **divisible** if for all $a, b, c \in A$

$$(a \rightarrow b)a = (b \rightarrow a)b.$$

We are mainly interested in quasivarieties of commutative and integral residuated lattices whose members are all representable and divisible. It is easy to check that any totally ordered CIRS is representable; moreover

Lemma 5.1. *Let* $\mathbf{A}$ *be a representable commutative and integral residuated semilattice; then*

1. $\mathbf{A}$ *is a subdirect product of totally ordered residuated semilattices;*
2. *the ordering of* $\mathbf{A}$ *is a lattice ordering where*

$$a \vee b := ((a \to b) \to b) \wedge ((b \to a) \to a).$$

For a proof the reader can consult [18] and/or [4]. It follows that any representable algebra in CIRS is really an algebra in CIRL; since both representability and divisibility are expressible by equations a (quasi)variety consists entirely of representable and/or divisible algebras if and only if it satisfies the corresponding equations.

A divisible CIRS is called a **hoop** [16]; a divisible and representable CIRS (that is also a CIRL by Lemma 5.1) is called a **basic hoop** [7]. We stress that a hoop is not in general a divisible CIRL even though many properties of hoops can be transferred easily to divisible CIRls. We will deal mainly with quasivarieties of basic hoops and the reason will be clear shortly.

A hoop $\mathbf{A}$ is **cancellative** if for all $a, b \in A$, $a \to ba = b$; it is an easy exercise to check that this corresponds to the underlying monoid being cancellative in the usual sense. A hoop $\mathbf{A}$ is a **Wajsberg hoop** if for all $a, b \in A$

$$(a \to b) \to b = (b \to a) \to a.$$

Lemma 5.2. *[16] Every Wajsberg hoop is a basic hoop and every cancellative hoop is a Wajsberg hoop.*

We will denote by BH, WH and CH the varieties of basic hoops, Wajsberg hoops and cancellative hoops respectively.

Wajsberg hoops have a *canonical representation*. Let $\mathbf{G}$ be a lattice ordered abelian group; by [42], if u is a strong unit of $\mathbf{G}$ we can construct a bounded Wajsberg hoop $\Gamma(\mathbf{G}, u) = \langle [0, u], \to, \cdot, 0, u \rangle$ where $ab = \max\{a + b - u, 0\}$ and $a \to b = \min\{u - a + b, u\}$. The main result of [42] is that any bounded Wajsberg hoop can be presented in this way (really there is a catecorical equivalence between the category abelian ℓ-groups with strong unit and the category of bounded Wajsberg hoops). Let now $\mathbb{Z} \times_l \mathbb{Z}$ denote the lexicographic product of two copies of $\mathbb{Z}$. In other words, the universe is the cartesian product, the group operations are defined componentwise and the ordering is the lexicographic ordering (w.r.t. the natural ordering of $\mathbb{Z}$); then $\mathbb{Z} \times_l \mathbb{Z}$ is a totally ordered abelian group and we can apply Γ to it. A **Wajsberg chain** is a totally ordered Wajsberg hoop. Let's define some useful Wajsberg chains:

- the finite Wajsberg chain with $n + 1$ elements $\mathbf{Ł}_n = \Gamma(\mathbb{Z}, n)$;
- the infinite finitely generated Wajsberg chain $\mathbf{Ł}_n^\infty = \Gamma(\mathbb{Z} \times_l \mathbb{Z}, (n, 0))$;
- the infinite finitely generated Wajsberg chain $\mathbf{Ł}_{n,k} = \Gamma(\mathbb{Z} \times_l \mathbb{Z}, (n, k))$;
- the infinite bounded Wajsberg chain $[0, 1]_{\mathbf{Ł}} = \Gamma(\mathbf{R}, 1)$, i.e. the real interval with operations induced by the *Wajsberg norm*. i.e. $xy = \max(x + y - 1, 0)$, $x \to y = \min(1 + x - y, 1)$;
- the infinite bounded Wajsberg chain $\mathbf{Q} = \Gamma(\mathbb{Q}, 1) = \mathbb{Q} \cap [0, 1]_{\mathbf{L}}$;
- the unbounded Wajsberg chain $\mathbf{C}_\omega$ that has as universe the free group on one generator, where the product is the group product and $a^l \to a^m = a^{\max(l-m, 0)}$;
- finally we fix once and for all an irrational number $\alpha \in [0, 1]$ and we let X be the totally ordered dense subgroup of $\mathbb{R}$ generated by α and 1; then $\mathbf{S}_n = \Gamma(X, n)$.

The proof of the following is a simple verification:

Lemma 5.3.

1. *For $n, m \in \mathbb{N}$, $\mathbf{Ł}_n \in \mathbf{S}(\mathbf{Ł}_m)$ if and only if $n \mid m$.*
2. *For $n, r, j \in \mathbb{N}$, $\mathbf{Ł}_n \in \mathbf{S}(\mathbf{Ł}_{r,j})$ if and only if $n \mid \gcd\{r, j\}$.*
3. *For $n, l \in \mathbb{N}$, $\mathbf{Ł}_n \in \mathbf{S}(\mathbf{S}_l)$ if and only if $n \mid l$.*

A **Wajsberg algebra** is a Wajsberg hoop with an additional constant 0 which is minimal in the ordering; it is straightforward to show that a bounded Wajsberg hoop is polynomially equivalent to a Wajsberg algebra so all the bounded chains above have their Wajsberg algebra counterpart; we will denote by $\mathbf{W}_n$ and $\mathbf{W}_n^\infty$ the Wajsberg algebras counterparts of $\mathbf{Ł}_n$ and $\mathbf{L}_n^\infty$. Similarly the variety of Wajsberg algebras will be denoted by WA. A complete description

of $\Lambda(\mathsf{WH})$ is in [9] while $\Lambda(\mathsf{WA})$ was first considered in [38] (for a complete description see and [44]); they are both countable and of course distributive.

On the other hand it is easy to see that CIRL has the congruence extension property and that each $\mathbf{Ł}_n$ is simple; so, by applying Lemma 2.5, to the set $\{\mathbf{Ł}_p : p \text{ prime}\}$ or $\{\mathbf{W}_p : p \text{ prime}\}$ we get at once that $\Lambda_q(\mathsf{WH})$ and $\Lambda_q(\mathsf{WA})$ are Q-universal, hence uncountable and extremely complex. Hence our only hope to understanding them, at least in part, is considering particular classes of subquasivarieties and then combining the information.

We first consider subquasivarieties generated by chains; we note in passing that those quasivarieties are exactly the subquasivarieties of WH that are **relative congruence distributive**, i.e. those quasivarieties Q for which $\mathrm{Con}_\mathsf{Q}(\mathbf{A})$ is a distributive lattice for any $\mathbf{A} \in \mathsf{Q}$. Let's explain briefly why:

- the finitely subdirectly irreducible Wajsberg hoops coincide with the totally ordered ones, hence the class of finitely subdirectly irreducible Wajsberg hoops is a universal class;
- if V is any congruence distributive variety then a subquasivariety Q of V is relative congruence distributive if and only if it is generated by a class of finitely subdirectly irreducible algebras in V ([24], Corollary 2.7);
- so the relative congruence distributive subquasivarieties of WH are exactly those generated by Wajsberg chains.

A lot of information about universal classes and subquasivarieties of Wajsberg algebras is contained in [31] and [32] respectively; however to justify our use of these results we need to explain the context better. First the results are stated in terms of *MV-algebras*; this is not a great problem since MV-algebras are easily proven to be term-equivalent to Wajsberg algebras and in fact they are two different avatars of the same concept. We recall that the operator $\mathbf{ISP}_u$ on Wajsberg hoops has been studied in [8] using the results about Wajsberg appeared in [31]; while we maintain that it should be clear why we can do this (and in [8] no explanation was given), maybe some clarification is useful. Wajsberg algebras are polynomially equivalent to bounded Wajsberg hoops; it is easy to see that if $\mathbf{O}$ is a class operator that is a composition of $\mathbf{I}, \mathbf{H}, \mathbf{S}, \mathbf{P}, \mathbf{P}_u, \mathbf{A}, \mathbf{B}$ are Wajsberg algebras and $\mathbf{A}_0, \mathbf{B}_0$ are their Wajsberg hoop reducts, then $\mathbf{O}(\mathbf{A}) \subseteq \mathbf{O}(\mathbf{B})$ if and only if $\mathbf{O}(\mathbf{A}_0) = \mathbf{O}(\mathbf{B}_0)$. This allows us to consider bounded Wajsberg hoops *as they were* Wajsberg algebras. Since a totally ordered Wajsberg hoop is either bounded or cancellative [16] we can use results about Wajsberg algebras and integrate them with the cancellative case.

In [31] (Lemma 4.3) the author observed that a quasivariety generated by a class K of totally ordered MV-algebras is determined by the universal class generated by K. This is not a property of MV-algebras, so let us state and prove it for the most general case we are aware of.

Theorem 5.4. *Let* K, K' *be classes of commutative and integral residuated chains with the property that, for all* $\mathbf{A} \in \mathsf{K} \cup \mathsf{K}'$, 1 *is join irreducible in* $\mathbf{A}$. *Then* $\mathbf{Q}(\mathsf{K}) = \mathbf{Q}(\mathsf{K}')$ *if and only if* $\mathbf{ISP}_u(\mathsf{K}) = \mathbf{ISP}_u(\mathsf{K}')$.

Proof. Any universal class is axiomatized by a set of finite conjunctions of so-called universal basic sentences; a universal basic sentence is of the form

$$\Gamma \implies r_1 \approx s_1 \vee \cdots \vee r_n \approx s_n$$

where Γ is a finite disjunction of equations. But it is well-known (see for instance Lemma 3.1 in [29]) that in an integral and commutative residuated lattices the left hand side of the implication is equivalent to

$$\bigvee_{i=1}^{n} ((r_i \to s_i) \wedge (s_i \to r_i)) \approx 1.$$

But in any algebra in $\mathsf{K} \cup \mathsf{K}'$, 1 is join irreducible; so for any possible assignment of the variables the universal basic sentence in question is equivalent to a quasiequation. It follows that the quasivariety generated by K and the universal class generated by K satisfy the same quasiequations, from which the conclusion follows. $\square$

We remark that any chain satisfies the hypothesis of Theorem 5.4; but so does any finitely subdirectly irreducible n-potent (i.e. satisfying $x^n \approx x^{n+1}$) CIRL ([30], Lemma 3.60).

Now some definitions; the **radical** of a bounded Wajsberg chain $\mathbf{A}$, in symbols $\mathrm{Rad}(\mathbf{A})$, is the intersection of the maximal filters of $\mathbf{A}$; it is easy to see that $\mathrm{Rad}(\mathbf{A})$ is a cancellative basic subhoop of $\mathbf{A}$. We say that a bounded Wajsberg

hoop **A has rank** n, if $\mathbf{A}/\mathrm{Rad}(\mathbf{A}) \cong \mathbf{Ł}_n$. For any bounded Wajsberg hoop $\mathbf{A}$, $d_{\mathbf{A}}$, called the **divisibility index**, is the maximum k such that $\mathbf{Ł}_k$ is embeddable in $\mathbf{A}$ if any, otherwise $d_{\mathbf{A}} = \infty$.

Here is the summary of the main results about the rank and the divisibility index; the proofs are either trivial or can be found in [31] or [8].

Lemma 5.5. *For any* $n, k \geq 1$

1. $\mathbf{Ł}_n$ *is simple and* $\mathbf{Ł}_n \in \mathbf{S}(\mathbf{Ł}_k)$ *if and only if* $n \mid k$,
2. $\mathbf{Ł}_n$ *has rank* n *and divisibility index* n.
3. *For any* $k \geq 0$, $\mathbf{Ł}_{n,k}$ *has rank* n *and* $d_{\mathbf{Ł}_{n,k}} = \gcd(n, k)$; *in particular* $d_{\mathbf{Ł}_n^\infty} = n$.
4. $\mathbf{S}_n$ *has infinite rank and* $\mathbf{Ł}_k \in \mathbf{S}(\mathbf{S}_n)$ *if and only if* $k \mid n$; *hence* $d_{\mathbf{S}_n} = n$.
5. *If* $\mathbf{A}$ *is a nontrivial totally ordered cancellative hoop then* $\mathbf{ISP}_u(\mathbf{A}) = \mathbf{ISP}_u(\mathbf{C}_\omega)$.
6. *If* $\mathbf{A}$ *is a bounded Wajsberg chain of finite rank* k, *then* $d_{\mathbf{A}}$ *divides* k, *and* $\mathbf{ISP}_u(\mathbf{A}) = \mathbf{ISP}_u(\mathbf{Ł}_{k,d_{\mathbf{A}}})$.
7. *If* $\mathbf{A}$ *is a bounded Wajsberg chain of finite rank* n, *then* $\mathbf{ISP}_u(\mathbf{A}) = \mathbf{ISP}_u(\mathbf{Ł}_n^\infty)$ *if and only if* $d_{\mathbf{A}} = n$.
8. *If* $\mathbf{A}, \mathbf{B}$ *are a Wajsberg chains of infinite rank then* $\mathbf{ISP}_u(\mathbf{A}) \subseteq \mathbf{ISP}(\mathbf{B})$ *if and only if* $\{n : \mathbf{Ł}_n \in \mathbf{S}(\mathbf{A})\} \subseteq \{n : \mathbf{Ł}_n \in \mathbf{S}(\mathbf{B})\}$. *In particular* $\mathbf{A} \in \mathbf{ISP}_u(\mathbf{S}_n)$ *if and only if* $d_{\mathbf{A}} \mid n$.

A **v-presentation** is a triple $\{I, J, K\}$ where I, J are finite subsets of $\mathbb{N}$ and $K \subseteq \{0\}$; a v-presentation is **reduced** if:

- $I \cup K \cup J \neq \emptyset$;
- if $K = \{0\}$, then $J = \emptyset$;
- no $m \in I$ divides any $m' \in (I \setminus \{m\}) \cup J$;
- no $t \in J$ divides any $t' \in J \setminus \{t\}$.

Theorem 5.6. [9] *The proper subvarieties of* WH *are in one-to-one correspondence with the reduced triples via the mapping*

$$\{I, J, \emptyset\} \longmapsto \mathbf{V}(\{\mathbf{Ł}_m : m \in I\} \cup \{\mathbf{Ł}_t^\infty : t \in J\})$$

$$\{I, \emptyset, \{0\}\} \longmapsto \mathbf{V}(\{\mathbf{Ł}_m : m \in I\} \cup \{\mathbf{C}_\omega\}).$$

If $P = \{I, J, K\}$ is a v-presentation we denote by $\mathbf{V}(P)$ or by $\mathbf{V}(I, J, K)$ the variety associated with P. Now it is clear from the description of $\Lambda(\mathsf{WH})$ in [9] that if I, I' are reduced subsets of $\mathbb{N}$ then the relation

$$I \leq I' \quad \text{if and only if} \quad \text{for all } i \in I \text{ there is a } j \in I' \text{ with } i \mid j$$

is a partial ordering. The following lemma is quite obvious:

Lemma 5.7. *Let* $\mathsf{V} = \mathbf{V}(I, J, K)$, $\mathsf{V}' = \mathbf{V}(I', J, K')$ *be proper subvarieties of* WH*; then the following are equivalent:*

1. $\mathsf{V} \subseteq \mathsf{V}'$;
2. $I \leq I' \cup J'$, $J \leq J'$ *and* $K \subseteq K'$;
3. *if* $\mathbf{C}_\omega \in \mathsf{V}$, *then* $\mathbf{C}_\omega \in \mathsf{V}'$ *and* $I \leq I' \cup J'$, $J \leq J'$.

For quasivarieties things are slightly more complex. A **q-presentation** is a set $P = \{I, J, L, K\}$ such that $I, L, K \subseteq \mathbb{N}$, $J \subseteq \mathbf{N} \times \mathbf{N}$ such that:

- for any $(r, j) \in J$, $j \mid r$;
- $K \subseteq \{0\}$;
- if $K = \{0\}$, then $J = L = \emptyset$.

To any q-presentation P we associate sets of Wajsberg chains in the following way: where

$$Q_I = \{\mathbf{Ł}_i : i \in I\}$$

$$Q_J = \bigcup_{r \in J} \{\mathbf{Ł}_{r,j} : (r, j) \in J\}$$

$$Q_L = \{\mathbf{S}_l : l \in L\}$$

$$Q_K = \{\mathbf{C}_\omega\} \text{ if } K = \{0\} \text{ and } \emptyset \text{ if } K = \emptyset.$$

We will set

$$\mathbf{Q}(P) = \mathbf{Q}(I, J, L, K) = \mathbf{Q}(Q_I \cup Q_J \cup Q_L \cup Q_K)$$

so that $\mathbf{Q}(P)$ is the quasivariety defined by the q-presentation P.

Theorem 5.8. *Let* Q *be a quasivariety of Wajsberg hoops generated by chains; then* $\mathsf{Q} = \mathbf{Q}(P)$ *for some q-presentation* P.

Proof. Let C be a set of chains such that $\mathsf{Q} = \mathbf{Q}(\mathsf{C})$; since any algebra is embeddable in an ultraproduct of its finitely generated subalgebras we may consider only the finitely generated members of C. Let

$\mathsf{C}_1 = \{\mathbf{A} \in \mathsf{C} : \mathbf{A} \text{ is bounded and finite}\}$

$\mathsf{C}_2 = \{\mathbf{A} \in \mathsf{C} : \mathbf{A} \text{ is bounded, infinite and has finite rank}\}$

$\mathsf{C}_3 = \{\mathbf{A} \in \mathsf{C} : \mathbf{A} \text{ is bounded, infinite and has infinite rank}\}$

$\mathsf{C}_4 = \{\mathbf{A} \in \mathsf{C} : \mathbf{A} \text{ is cancellative}\}.$

We observe also that all algebras in $\mathsf{C} \setminus \mathsf{C}_4$ have finite divisibility index, since they are bounded and finitely generated. Now we define $P = \{I, J, L, K\}$ as:

$I = \{i : \mathbf{Ł}_i \in \mathsf{C}_1\}$

$J = \{(r, j) : \text{there is an } \mathbf{A} \in \mathsf{C}_2 \text{ with } \text{rank}(\mathbf{A}) = r \text{ and } d_{\mathbf{A}} = j\}$

$L = \{l : \text{there is an } \mathbf{A} \in \mathsf{C}_3 \text{ with } d_{\mathbf{A}} = l\}$

$K = \emptyset \text{ if } \mathsf{C}_2 \cup \mathsf{C}_3 \neq \emptyset.$

Observe that $\{I, J, L, K\}$ is a presentation, because of Lemma 5.5(6).

Let now $\mathbf{A} \in \mathsf{C}_2$ with $\text{rank}(\mathbf{A}) = r$; by Lemma 5.5(6) $\mathbf{ISP}_u(\mathbf{A}) = \mathbf{ISP}_u(\mathbf{Ł}_{r,d_{\mathbf{A}}})$, where $d_{\mathbf{A}} \mid r$. Then

$$C_2 \subseteq \mathbf{ISP}_u\left(\bigcup_{\mathbf{A} \in C_2} \mathbf{Ł}_{\text{rank}(\mathbf{A}),d_{\mathbf{A}}} \right)$$
$$= \mathbf{ISP}_u(\{\mathbf{Ł}_{r,d} : r = \text{rank}(\mathbf{A}), \mathbf{A} \in C_2, d \mid r\} \subseteq \mathbf{ISP}_u(Q_J).$$

If $\mathbf{A} \in \mathsf{C}_3$, then by Lemma 5.5(8), $\mathbf{A} \in \mathbf{ISP}_u(\mathbf{S}_{d_{\mathbf{A}}})$ whenever $d_{\mathbf{A}} \mid n$. Hence $C_3 \subseteq \mathbf{ISP}_u(\{\mathbf{S}_{\delta_{\mathbf{A}}} : \mathbf{A} \in C_3\}) \subseteq \mathbf{ISP}_u(Q_L)$. Since clearly $C_1 \subseteq \mathbf{ISP}_u(Q_I)$ and $C_4 \subseteq \mathbf{ISP}(Q_K)$ we get

$$C_1 \cup C_2 \cup C_3 \cup C_4 \subseteq \mathbf{ISP}_u(Q_I \cup Q_J \cup Q_L \cup Q_K).$$

Therefore

$$\mathsf{Q} = \mathbf{ISPP}_u(C_1 \cup C_2 \cup C_3 \cup C_4)$$
$$\subseteq \mathbf{Q}(Q_I \cup Q_J \cup Q_L \cup Q_K) \subseteq \mathsf{Q},$$

as wished. $\square$

The first question we want to answer is: which q-presentations P are **generic**, in the sense that $\mathbf{Q}(P) = \mathsf{WH}$? It is well-known that WH [16] has the FEP, hence by Lemma 5.13 they are both generated as a quasivarieties by their finite totally ordered algebras. This implies that any reduced presentation P in which $\mathbb{N} = I \cup \{j : (r, j)J\} \cup L$ is such that $\mathbf{Q}(P) = \mathsf{WH}$. Is this the only possibility? Everything boils down to characterizing the subsets $X \subseteq \mathbb{N}$ for which

$$\mathbf{Q}(\{\mathbf{Ł}_x : x \in X\}) = \mathsf{WH}.$$

A. Tarski proved long time ago [47] that $\mathbf{V}(\{\mathbf{Ł}_x : x \in X\}) = \mathsf{WH}$ if and only if X is infinite, but it this is not the case for $\mathbf{Q}(\{\mathbf{Ł}_x : x \in X\})$. To prove that we need a lemma which will be useful also in the sequel.

In any Wajsberg hoop we can define a derived operation

$$x \oplus y = (x \to xy) \to y$$

and then by induction

$$\mathbf{1}x = x$$
$$(\mathbf{n}+\mathbf{1})x = x \oplus \mathbf{n}x.$$

Lemma 5.9. *For every Wajsberg hoop* $\mathbf{A}$, $\mathbf{Ł}_n \in \mathbf{S}(\mathbf{A})$ *for* $n > 1$ *if and only if there is an* $a \in A$, $a \neq 1$ *such that*

$$(\mathbf{n}-\mathbf{1})a = a \to a^{n+1}.$$

For a proof the reader can look at [8] or (for an even more general case) [4]; the idea however traces back to [48].

Lemma 5.10. *Let* K *be any class of Wajsberg hoops. If* $\mathbf{A} \in \mathbf{ISP}_u(\mathsf{K})$ *is such that* $\mathbf{Ł}_n \in \mathbf{S}(\mathbf{A})$ *for some* n, *then* $\mathbf{A} \in \mathbf{ISP}_u(\{\mathbf{B} \in \mathsf{K} : \mathbf{Ł}_n \in \mathbf{S}(\mathbf{B})\})$.

Proof. Let $\mathbf{A}_i \in \mathsf{K}$ for all $i \in I$ and let U be an ultrafilter on I; suppose that $\mathbf{A} \in \mathbf{S}(\prod_{i \in I} \mathbf{A}_i / U)$. Then $\mathbf{Ł}_n \in \mathbf{S}(\prod_{i \in I} \mathbf{A}_i / U)$ so, by Lemma 5.9, there is an $\mathbf{a} \in \prod_{i \in I} \mathbf{A}_i / U$ such that $(\mathbf{n}-\mathbf{1})\mathbf{a} = \mathbf{a} \to \mathbf{a}^n$.

Let $K = \{i \in I : (\mathbf{n}-\mathbf{1})a_i = a_i \to a_i^n\} \in U$ and $J = \{i \in I : \mathbf{Ł}_n \in \mathbf{S}(\mathbf{A}_i)\}$; then $K \subseteq J$, so $J \in U$. It follows that $V = U \cap J$ is a ultrafilter on J and

$$\prod_{i \in I} \mathbf{A}_i / U = \prod_{j \in J} \mathbf{A}_j / V.$$

The conclusion follows. $\quad\square$

So let X be the set of prime numbers; if $\mathbf{Q}(\{\mathbf{Ł}_p : p \in X\}) = \mathsf{WH}$, then for any composite n, $\mathbf{Ł}_n \in \mathbf{Q}(\{\mathbf{Ł}_p : p \in X\})$. Then, by Lemma 5.10,

$$\mathbf{Ł}_n \in \mathbf{ISP}_u(\{\mathbf{Ł}_p : p \in X \text{ and } \mathbf{Ł}_n \in \mathbf{S}(\mathbf{Ł}_p)\});$$

but since n is composite the generating set is empty, a clear contradiction.

Theorem 5.11. *Let* P *be a presentation; then* $\mathbf{Q}(P) = \mathsf{WH}$ *if and only if for any* $n \in \mathbb{N}$ *there is an* $t \in I \cup \{j : (r, j) \in J\} \cup L$ *with* $n \mid t$.

Proof. Assume P has the desired property and let

$$\Psi = \bigwedge_{i=1}^n (p_i \approx q_i) \to r \approx s$$

a quasi equation such that $\mathsf{WH} \nvDash \Psi$. Then there exists a finitely generated totally ordered Wajsberg hoop $\mathbf{A}$ such that $\mathbf{A} \nvDash \Psi$. Since WH has the FEP we may argue as in Theorem 4.1 and find a finite totally ordered Wajsberg hoop $\mathbf{A}'$ with $\mathbf{A}' \nvDash \Psi$. Of course $\mathbf{A}' = \mathbf{Ł}_n$ for some n; by the property we can find a $t \in I \cup \{j : (r, j) \in J\} \cup L$ such that $n \mid t$, hence $\mathbf{Ł}_n \in \mathbf{S}(\mathbf{Ł}_t) \subseteq \mathbf{Q}(P)$ and $\mathbf{Q}(P) \nvDash \Psi$. This shows that $\mathbf{Q}(P) = \mathsf{WH}$.

Conversely suppose that $\mathbf{Q}(P) = \mathsf{WH} = \mathbf{Q}(\{\mathbf{Ł}_n : n \in \mathbb{N}\})$; this implies that $\mathbf{ISP}_u(P) = \mathbf{ISP}_u(\{\mathbf{Ł}_n : n \in \mathbb{N}\})$ and so $\mathbf{Ł}_n \in \mathbf{ISP}_u(P)$ for any n. By Lemma 5.10

$$\mathbf{Ł}_n \in \mathbf{ISP}_u(\{\mathbf{A} : \mathbf{A} \in Q_I \cup Q_J \cup Q_L \text{ and } \mathbf{Ł}_n \in \mathbf{S}(\mathbf{A})\});$$

since the generating set cannot be empty, there exists an $\mathbf{A}$ with $\mathbf{Ł}_n \in \mathbf{S}(\mathbf{A})$. But if $\mathbf{A} = \mathbf{Ł}_i$, then $n \mid i$; if $\mathbf{A} = \mathbf{Ł}_{r,j}$ then $n \mid j \in \{j : (r, j) \in J\}$ and if $\mathbf{A} \in \mathbf{S}_l$, then $n \mid l$. The conclusion follows. $\quad\square$

We have already observed that, due to the fact that a Wajsberg chain is either bounded of cancellative, we can extend many results about Wajsberg algebras (MV-algebras) to Wajsberg hoops, simply taking care of the cancellative case. Theorem 5.8 is an example of this and we can find many others. We proceed to illustrate some of them without going into details. If **A** is a bounded Wajsberg chain (say by 0) the **order** of $a \in A$ is

$$\begin{cases} \min\{n : a^n = 0\}, & \text{if such } n \text{ exists;} \\ \infty, & \text{otherwise;} \end{cases}$$

the **order** of **A** is

$$\operatorname{ord}(\mathbf{A}) = \sup\{n : \operatorname{ord}(a) = n \text{ for some } a \in A\}.$$

It is easily checked that $\operatorname{rank}(\mathbf{A}) = \operatorname{ord}(\mathbf{A}/\operatorname{Rad}(\mathbf{A}))$. From Lemma 5.5 and the analogous result in [31] (Theorem 4.4) about MV-algebras we get:

Theorem 5.12. *Let* **A**, **B** *be two Wajsberg chains; then* $\mathbf{Q}(\mathbf{A}) = \mathbf{Q}(\mathbf{B})$ *if and only if*

1. *either* **A** *and* **B** *are both cancellative;*
2. *or they are both bounded* $\operatorname{rank}(\mathbf{A}) = \operatorname{rank}(\mathbf{B})$, $\operatorname{ord}(\mathbf{A}) = \operatorname{ord}(\mathbf{B})$ *and* $\{\mathbf{Ł}_n : \mathbf{Ł}_n \in \mathbf{S}(\mathbf{A})\} = \{\mathbf{Ł}_n : \mathbf{Ł}_n \in \mathbf{S}(\mathbf{A})\}$.

Next we deal with inclusion properties between quasivarieties of Wajsberg hoops generated by chains. Let I, J, K, L be a q-presentation and for any $r \in J$ let $J_r = \{s : s \mid r\} = \{j \in \mathbb{N} : (j, r) \in J\}$. The q-presentation is **reduced** if

- for all $i \in I$, $i \nmid t$ for $t \in (I \setminus \{i\}) \cup \bigcup_{r \in J} J_r \cup L$;
- for all $l \in L$,
  (a) either there is an $i \in I$ with $l \nmid i$, or
  (b) there is an $r' \in J$ such that for all $j' \in J_{r'}$, $l \nmid j'$;
- for all $r \in J$ and $j \in J_r$
  (a) for all $l' \in L$, $j \nmid l'$;
  (b) for all $r' \in J \setminus \{r\}$, for all $j' \in J_{r'}$, either $r \nmid r'$ or $j \nmid j'$;
  (c) either there is an $i \in I$ with $j \nmid i$, or there is an $r' \in J$ such that for all $j' \in J_{r'}$, $j \nmid j'$.

Finally we remark that we can find an axiomatization of any quasivariety generated by chain; here the key point is that, as for Wajsberg algebras, every proper subvariety of WH can be axiomatized (modulo WH) by a single equation in one variable [9]. Using this (and the fact that cancellative hoops are axiomatized modulo WH by the single equation $x \to x^2 \approx x$), one can find an analogue to Theorem 4.5 in [31].

A further question that we want to answer is for which q-presentation P, $\mathbf{Q}(P)$ is a variety; let's start with a lemma of general interest.

Lemma 5.13. *Let* K *be any class of basic hoops; then* $\mathsf{Q} = \mathbf{Q}(\mathsf{K})$ *is a variety if and only if all finitely generated totally ordered members of* $\mathbf{V}(\mathsf{K})$ *are in* $\mathbf{ISP}_u(\mathsf{K})$. *If* Q *has the FEP, then* $\mathsf{Q} = \mathbf{Q}(\mathsf{K}_{fin})$.

Proof. If $\mathbf{Q}(\mathsf{K})$ is a variety, then $\mathbf{Q}(\mathsf{K}) = \mathbf{V}(\mathsf{K})$ and by Theorem 2.3 every relatively subdirectly irreducible in $\mathbf{Q}(\mathsf{K})$ is in $\mathbf{ISP}_u(\mathsf{K})$. But since $\mathbf{Q}(\mathsf{K})$ is a variety a relatively subdirectly irreducible is subdirectly irreducible and the conclusion follows.

Conversely let F be the class of all finitely generated totally ordered members of $\mathbf{V}(\mathsf{K})$. Since any algebra is embeddable in an ultraproduct of its finitely generated subalgebras and any basic hoop is a subdirect product of totally ordered basic hoops we get

$$\mathbf{V}(\mathsf{K}) \subseteq \mathbf{ISPSP}_u(\mathsf{F}) \subseteq \mathbf{ISPSP}_u\mathbf{SP}_u(\mathsf{k}) \subseteq \mathbf{ISPP}_u(\mathsf{K}) = \mathbf{Q}(\mathsf{K}).$$

Hence $\mathbf{Q}(\mathsf{K}) = \mathbf{V}(\mathsf{K})$ and $\mathbf{Q}(\mathsf{K})$ is a variety. The second claim is a straightforward consequence of Theorem 4.1. $\square$

Theorem 5.14. *Let* **A** *be a Wajsberg chain; then* $\mathbf{Q}(\mathbf{A})$ *is a variety if and only if*

- $Ł_n$ *is embeddable in* $\mathbf{A}$ *for all n, or*
- $\mathbf{A}$ *is finite, or*
- $\mathbf{A}$ *is cancellative, or*
- $\mathbf{A}$ *is infinite, bounded and the rank of* $\mathbf{A}$ *is equal to* $d_\mathbf{A}$.

Proof. Since WH is generated as a quasivariety by its finite algebras [16] if every $Ł_n$ is embeddable in $\mathbf{A}$, then $\mathbf{Q}(\mathbf{A}) = \mathsf{WH}$. If $\mathbf{A}$ is finite, $\mathbf{Q}(\mathbf{A})$ is locally finite so it is primitive; if $\mathbf{A}$ is cancellative, then $\mathbf{ISP}_u(\mathbf{A}) = \mathbf{ISP}_u(\mathbf{C}_\omega)$ and so

$$\mathbf{Q}(\mathbf{A}) = \mathbf{ISPP}_u(\mathbf{A}) = \mathbf{ISPP}_u(\mathbf{C}_\omega) = \mathsf{C}.$$

Finally assume that $\mathbf{A}$ is bounded and infinite (so it is not simple) and it's rank is $d = d_\mathbf{A}$. Then $\mathbf{ISP}_u(\mathbf{A}) = \mathbf{ISP}_u(Ł_{d,d})$; now any totally ordered member of $\mathbf{V}(Ł_{d,d})$ is either cancellative or else it is bounded Wajsberg hoop $\mathbf{B}$ such $d_\mathbf{B}$ divides d and all these chains are in $\mathbf{ISP}_u(Ł_{d,d})$ (see for instance Lemmas 6.1 and 6.3 in [8]). Therefore by Lemma 5.13, $\mathbf{Q}(\mathbf{A})$ is a variety.

Conversely suppose that $\mathbf{A}$ is either of infinite rank and $Ł_k$ is not embeddable in $\mathbf{A}$ for some k, or else $\mathbf{A}$ has finite rank and $d_\mathbf{A}$ is strictly less than the rank of $\mathbf{A}$. In the first case $\mathbf{V}(\mathbf{A}) = \mathsf{WH}$ ([9], Theorem 2.4); now $\mathbf{Q} = [0,1]_{Ł} \cap \mathbb{Q}$ is a simple member of WH that does not belong to $\mathbf{ISP}_u(\mathbf{A})$ (since $Ł_k$ is embeddable in $\mathbf{Q}$ for all k). Hence by Lemma 5.13 $\mathbf{Q}(\mathbf{A})$ is not a variety. In the second case consider $Ł_n^\infty$; then $Ł_n^\infty \in \mathbf{V}(\mathbf{A})$ (since it has rank n), but $Ł_n^\infty \notin \mathbf{ISP}_u(\mathbf{A})$ (by Lemma 5.5. Again by Lemma 5.13 $\mathbf{Q}(\mathbf{A})$ is not a variety. $\quad\square$

We can use Theorem 5.14 for proper subquasivaries of WH generated by chains applying Jónnson's Lemma to varieties of basic hoops; for a variety V of basic hoops let V_t be the class of totally ordered members of V.

Lemma 5.15. *Let* $\mathbf{A}_1, \dots, \mathbf{A}_n$ *be basic hoops, let* $\mathsf{V}^i = \mathbf{V}(\mathbf{A}_i)$ *and let* $\mathsf{V} = \mathbf{V}(\mathbf{A}_1, \dots, \mathbf{A}_n)$; *then*

$$\mathsf{V}_t = \mathsf{V}_t^1 \cup \dots \cup \mathsf{V}_t^n.$$

Let us remark that, due to the existence of 1 which is idempotent w.r.t. any operation, if $\mathbf{A}, \mathbf{B} \in \mathsf{CIRL}$, then $\mathbf{A}$ and $\mathbf{B}$ are both embeddable in $\mathbf{A} \times \mathbf{B}$. This means that any subvariety of CIRL has the *joint embedding property* and thus every subquasivariety of CIRL (and so every subquasivariety of WH) is generated by a single algebra ([34], Proposition 2.1.19). In particular if $\mathbf{A}_1, \dots, \mathbf{A}_n \in \mathsf{WH}$, then

$$\mathbf{Q}(\mathbf{A}_1, \dots, \mathbf{A}_n) = \mathbf{Q}(\mathbf{A}_1 \times \dots \times \mathbf{A}_n).$$

Theorem 5.16. *Let* $\mathbf{A}_1, \dots, \mathbf{A}_n$ *be totally ordered Wajsberg hoops; if for* $i = 1, \dots, n$

- $Ł_n$ *is embeddable in* $\mathbf{A}_i$ *for all n, or*
- $\mathbf{A}_i$ *is finite, or*
- $\mathbf{A}_i$ *is cancellative, or*
- $\mathbf{A}_i$ *is infinite, bounded and the rank of* $\mathbf{A}$ *is equal to* $d_\mathbf{A}$,

then $\mathbf{Q}(\mathbf{A}_1, \dots, \mathbf{A}_n)$ *is a variety.*

Proof. By Lemma 5.13, to show that $\mathbf{Q}(\mathbf{A}_1, \dots, \mathbf{A}_n)$ is a variety, it is enough to prove that every Wajsberg chain in $\mathbf{V}(\mathbf{A}_1, \dots, \mathbf{A}_n)$ is in $\mathbf{ISP}_u(\mathbf{A}_1, \dots, \mathbf{A}_n)$. But each of such chains is a totally ordered member of $\mathbf{V}(\mathbf{A}_i)$ for some i (by Lemma 5.15) and by Theorem 5.14 it is in $\mathbf{ISP}_u(\mathbf{A}_i)$. $\quad\square$

Corollary 5.17. *Let P be a reduced q-presentation; if*

- *P is finite (i.e. all the sets involved are finite),*
- $L = \emptyset$,
- $(r, j) \in J$ *implies* $r = j$,

then $Q(P)$ is a variety.

Actually a little more is true; by looking at the description of the proper subvarieties of WH and observing that $\mathbf{ISP}_u(\mathbf{Ł}_n^\infty) = \mathbf{ISP}_u(\mathbf{Ł}_{n,n})$ (Lemma 5.5(7)) it is clear that any proper subvariety of Wajsberg hoop is $\mathbf{Q}(P)$ for some finite q-presentation P of the type described above. This means that the class of quasivarieties of WH generated by chains contains all the subvarieties of WH and also that $\Lambda_q(\mathsf{WH})$ contains $\Lambda(\mathsf{WH})$ as a distributive sublattice.

6. Primitivity and structural completeness in Wajsberg hoops

The question of primitivity in varieties of commutative and integral residuated lattices has been tackled in several papers [23] [37] in connection with the corresponding logics. Here need only a result that appears in [11]:

Theorem 6.1. *[11] Every finite hoop is finitely projective, i.e. it is projective in the class of finite hoops.*

By Theorem 3.9 we get:

Corollary 6.2. *Every locally finite variety of basic hoops is primitive.*

In particular, since every locally finite quasivariety is contained in a locally finite variety, any locally finite quasi-variety of Wajsberg hoops is a primitive variety. On the other hand the variety of cancellative hoops $\mathsf{CH} = \mathbf{Q}(\mathbf{C}_\omega)$ is not locally finite but by Lemma 5.5(5) if **A** and **B** are totally ordered cancellative hoops, then [8]

$$\mathbf{ISP}_u(\mathbf{A}) = \mathbf{ISP}_u(\mathbf{B}).$$

It follows that CH has no proper nontrivial subquasivarieties and thus is primitive. So there are (quasi)varieties of Wajsberg hoops that are primitive without being locally finite. Now let V be any proper variety of Wajsberg hoops; then $\mathsf{V} = \mathbf{V}(I, J, K)$ for some reduced v-presentation $\{I, J, K\}$.

Lemma 6.3. *For any $n > 1$, $\mathbf{V}(\mathbf{Ł}_n^\infty)$ is not structurally complete; hence if $\mathsf{V} = \mathbf{V}(I, J, K)$ is primitive, then $J \subseteq \{1\}$.*

Proof. Since WH is not primitive, V must be proper, so it has a reduced v-presentation $\{I, J, K\}$. Suppose then that there is an $n > 1$ such that $\mathbf{Ł}_n^\infty \in \mathsf{V}$. Then by Lemma 5.5(7), $\mathbf{Ł}_{n,n} \in \mathsf{V}$ and it is easily checked that $\mathbf{Ł}_{n,1} \in \mathbf{S}(\mathbf{Ł}_{n,n})$; but by Theorem 5.14 $\mathbf{Ł}_{n,n} \notin \mathbf{ISP}_u(\mathbf{Ł}_{n,1})$, so $\mathbf{ISP}_u(\mathbf{Ł}_{n,n}) \neq \mathbf{ISP}_u(\mathbf{Ł}_{n,1})$. By Theorem 3.5, $\mathbf{Q}(\mathbf{Ł}_{n,1}) \subsetneq \mathbf{Q}(\mathbf{Ł}_{n,n}) = \mathbf{V}(\mathbf{Ł}_{n,n})$; however by Lemma 5.5(6,7)

$$\mathbf{V}(\mathbf{Ł}_{n,1}) = \mathbf{V}(\mathbf{Ł}_{n,n}) = \mathbf{V}(\mathbf{Ł}_n^\infty).$$

Therefore $\mathbf{V}(\mathbf{Ł}_n^\omega)$ is not its own least Q-quasivariety and so it is not structurally complete. $\square$

Now if $J = K = \emptyset$, then V is locally finite, hence primitive. It follows that a primitive variety of Wajsberg hoops must be either $\mathbf{V}(I, \emptyset, \emptyset)$ or $\mathbf{V}(I, \emptyset, \{0\})$ or $\mathbf{V}(I, \{1\}, \emptyset)$. To proceed further we need to observe that every variety of Wajsberg hoops can be axiomatized by a single equation in one variable [9]. It follows that for any quasivariety Q of Wajsberg hoops, its least Q-quasivariety is $\mathbf{Q}(\mathbf{F}_\mathsf{Q}(x))$.

In all the following proofs we will write $\mathbf{Ł}_m$, $m \geq 1$, in *multiplicative* notation (as opposed to the additive notation suggested by the Mundici's representation). In other words $\mathbf{Ł}_m = \{1, a, a^2, \ldots, a^m\}$ where

$$a^k \rightarrow a^n = a^{\max(0, n-k)} \qquad a^k a^n = a^{\min(k+n, m)}.$$

We have the following easy lemma, whose proof is left to the reader.

Lemma 6.4.

1. a^h generates $\mathbf{Ł}_k$ if and only if $h < k$ and k, h are relatively prime;
2. there is an element $c \in \mathbf{C}_\omega$ such that c generates $\mathbf{C}_\omega$ and if c' generates $\mathbf{C}_\omega$, then $c' = c$.

3. *if* **C** *is cancellative and* $c' \in C$ *with* $c' \neq 1$, *then* c' *generates a subalgebra of* **C** *isomorphic with* $\mathbf{C}_\omega$.
4. *there is an element* $d \in \mathbf{Ł}_1^\infty$ *such that* d^2 *is the minimum,* d *generates* $\mathbf{Ł}_1^\infty$ *and if* d' *generates* $\mathbf{Ł}_1^\infty$, *then* $d' = d$;
5. *if* $d' \in \mathbf{Ł}_1^\infty$ *and* $d' \leq d' \to d^2$, *then* d' *generates a subalgebra of* $\mathbf{Ł}_1^\infty$ *isomorphic with* $\mathbf{Ł}_1^\infty$.

We simply observe that if we represent $\mathbf{Ł}_1^\infty$ a $\Gamma(\mathbb{Z} \times_l \mathbb{Z}, (1,0))$ then $d = (0,1)$.

Lemma 6.5. *Let* $\mathsf{V} = \mathbf{V}(I, \emptyset, \{0\})$ *and let*

$$\Delta_I = \{(k,h) : k \mid m \text{ for some } m \in I \text{ and } h, k \text{ are relatively prime}\}.$$

Let $J = \Delta_i \cup \{0\}$ *and let's define for* $j \in J$

$$\mathbf{A}_j = \begin{cases} \mathbf{Ł}_k, & \text{if } j = (k,h); \\ \mathbf{C}_\omega, & \text{if } j = 0; \end{cases}$$

then define $g \in \prod_{j \in J} \mathbf{A}_j$ *by*

$$g_j = \begin{cases} a^h, & \text{if } j = (k,h); \\ c, & \text{if } j = 0, \end{cases}$$

where c *is the generator of* $\mathbf{C}_\omega$. *If* **B** *is the subalgebra of* $\prod_{j \in J} \mathbf{A}_j$ *generated by* g, *then* $\mathbf{B} \cong \mathbf{F}_\mathsf{V}(x)$.

Proof. First note that, since a^h generates $\mathbf{Ł}_k$ whenever k, h are relatively prime the embedding of **B** into $\prod_{j \in J} \mathbf{A}_j$ is subdirect. Next suppose that there is an equation in one variable $t(x) \approx s(x)$ that fails in V; then it must fail in some one-generated Wajsberg chain in V. This chain is either bounded or cancellative; if it is cancellative than it must be isomorphic with $\mathbf{C}_\omega$, since it is one-generated. We claim that if it is bounded then it must be finite. In fact let **C** be an infinite bounded chain in V; since V is a proper subvariety **C** cannot have infinite rank, as any chain of infinite rank generates WH ([9], Theorem 2.4). Hence **C** must have rank n an thus, by Lemma 5.5, $\mathbf{Ł}_n^\infty \in \mathsf{V}$; but this contradicts Theorem 5.6, hence **C** must be finite. Therefore $\mathbf{C} \cong \mathbf{Ł}_k$ for $k \mid m$ and $m \in I$.

Now if the equation fails in $\mathbf{C}_\omega$, then it fails for some $d \in \mathbf{C}_\omega$; clearly $d \neq 1$, so the subalgebra generated by d in $\mathbf{C}_\omega$ is isomorphic with $\mathbf{C}_\omega$. Therefore $t(c) \neq s(c)$ in $\mathbf{C}_\omega$; since $p_0(g) = c$ it follows that $t(g) \neq s(g)$ in **B**. If the equation fails in some $\mathbf{Ł}_k$ with $k \mid m$ and $m \in I$, then there is a generator b of $\mathbf{Ł}_k$ such that $t(b) \neq t(c)$. Such generator must a^h for some h which is relatively prime with k; since $g_{(k,h)} = a^h$, we have that $t(g) \neq s(g)$ in **B**.

The conclusion follows. □

Theorem 6.6. *For each* I, $\mathsf{V} = \mathbf{V}(I, \emptyset, \{0\})$ *is structurally complete.*

Proof. Let **B**, $\prod_{j \in J} \mathbf{A}_j$ and Δ_I as in Lemma 6.5; then

$$\mathbf{B} \leq_{sd} \prod_{j \in J} \mathbf{A}_j.$$

Let $I = \{n_1, \ldots, n_k\}$ and let g be the generator of **B** i.e.

$$g = ((a^h)_{(n_k,h) \in \Delta_I}, c);$$

if and $m = n_1 n_2 \cdots n_k$, then $g^m \to g^{2m} \in B$; since $(a^h)^m = (a^h)^{2m}$ for all $(n_k, h) \in \Delta_I$, we get that

$$g' = g^m \to g^{2m} = ((1)_{(n_k,h) \in \Delta_I}, c^m).$$

By Lemma 6.4 g' generates a subalgebra of **B** isomorphic with $\mathbf{C}_\omega$, thus $\mathbf{C}_\omega \in \mathbf{IS}(\mathbf{B})$.

Now consider $g'' = g' \to g$; of course $g'' = ((a^h)_{(n_k,h) \in \Delta_I}, 1)$. Then g'' generates a subalgebra **C** of **B** which is isomorphic with a subalgebra of $\prod_{(k,h) \in \Delta_I} \mathbf{Ł}_{n_k}$. Hence **C** is finite and, since a^h generates $\mathbf{Ł}_{n_k}$ for all $(k,h) \in \Delta_I$, $p_{(n_k,h)}(\mathbf{C}) = \mathbf{Ł}_{n_k}$. So $\mathbf{Ł}_{n_k} \in \mathbf{H}(\mathbf{C})$ and, since any finite hoop is finitely projective (Theorem 6.1),

$$\mathbf{Ł}_{n_k} \in \mathbf{S}(\mathbf{C}) \subseteq \mathbf{S}(\mathbf{B})$$

for all n_k.

Now we have

$$\begin{aligned}
\mathsf{V} &= \mathbf{V}(\{\mathbf{Ł}_{n_k} : n_k \in I\}, \mathbf{C}_\omega) \\
&= \mathbf{Q}(\{\mathbf{Ł}_{n_k} : n_k \in I\}, \mathbf{C}_\omega) \\
&\subseteq \mathbf{Q}(\mathbf{B}) = \mathbf{Q}(\mathbf{F}_\mathsf{V}(x)) \subseteq \mathsf{V}.
\end{aligned}$$

Thus $\mathsf{V} = \mathbf{Q}(\mathbf{F}_\mathsf{V}(x))$ and hence V is structurally complete. $\square$

Lemma 6.7. *Let* $\mathsf{V} = \mathbf{V}(I, \{1\}, \emptyset)$*; let*

$$\Delta_I = \{(k, h) : k \,|\, m \text{ for some } m \in I \text{ and } h, k \text{ are relatively prime}\}.$$

Let $J = \Delta_i \cup \{0, 1\}$ *and let's define for* $j \in J$

$$\mathbf{A}_j = \begin{cases} \mathbf{Ł}_k, & \text{if } j = (k, h); \\ \mathbf{C}_\omega, & \text{if } j = 0; \\ \mathbf{Ł}_1^\infty, & \text{if } j = 1; \end{cases}$$

then define $g \in \prod_{j \in J} \mathbf{A}_j$ *by*

$$g_j = \begin{cases} a^h, & \text{if } j = (k, h); \\ c, & \text{if } j = 0; \\ d \to d^2, & \text{if } j = 1; \end{cases}$$

where c *is the generator of* $\mathbf{C}_\omega$ *and* d *is the generator of* $\mathbf{Ł}_1^\infty$*. If* $\mathbf{B}$ *is the subalgebra of* $\prod_{j \in J} \mathbf{A}_j$ *generated by* g*, then* $\mathbf{B} \cong \mathbf{F}_\mathsf{V}(x)$*.*

Proof. The proof is almost identical to the one of Lemma 6.5. We have only to observe that any chain in V is either finite or cancellative or else has rank equal to 1; since we can consider only one-generated chains, the cancellative ones are isomorphic with $\mathbf{C}_\omega$ and bounded ones are isomorphic with $\mathbf{Ł}_1^\infty$. $\square$

Theorem 6.8. *If* $\mathsf{V} = \mathbf{V}(I, \{1\}, \emptyset)$*, then* V *is structurally complete.*

Proof. Let $I = \{n_1, \ldots, n_k\}$ and let Δ_I, $\mathbf{B}$ and g as in Lemma 6.7. Let

$$m = \begin{cases} n_1 n_2 \cdots n_k, & \text{if } I \neq \emptyset; \\ 2, & \text{otherwise.} \end{cases}$$

Then d^m is the bottom of $\mathbf{Ł}_1^\omega$ because the presentation is reduced and therefore if $I \neq \emptyset$, then $n_k \geq 2$ for all k. As in the proof of Theorem 6.6

$$g^m \to g^{2m} = ((1)_{(n_k,h) \in \Delta_I}, c^m, 1)$$

and therefore $\mathbf{C}_\omega \in \mathbf{S}(\mathbf{B})$.

On the other hand

$$(g^m \to g^{2m}) \to g = ((a^h)_{(n_k,h) \in \Delta_I}, 1, d)$$

generates a subalgebra $\mathbf{C}$ of $\mathbf{B}$ which is isomorphic with a subalgebra of

$$\prod_{(n_k,h) \in \Delta_I} \mathbf{Ł}_{n_k} \times \mathbf{Ł}_1^\infty.$$

Now, identifying $\mathbf{C}$ with its isomorphic copy, we may assume that $\mathbf{C}$ is generated by $((a^h)_{(n_k,h) \in \Delta_I}, d)$ and by setting $\mathbf{a} = (a^h)_{(n_k,h) \in \Delta_I}$ we may assume it is generated by $(\mathbf{a}, d)$. Observe that $(\mathbf{m}\mathbf{a}, \mathbf{m}d) = (\mathbf{1}, \mathbf{m}d) \in C$; now the reader can easily check that $(0, 1) \longmapsto (\mathbf{1}, \mathbf{m}d)$ defines an embedding of $\mathbf{Ł}_1^\infty$ in $\mathbf{C}$. Therefore $\mathbf{Ł}_1^\infty \in \mathbf{IS}(\mathbf{C}) \subseteq \mathbf{IS}(\mathbf{B})$.

Next if we denote d^2 by 0 we get

$$(\mathbf{1}, \mathbf{m}d)^2 = (\mathbf{1}, 0) \in C.$$

On the other hand if we denote by 0 the bottom of each $\mathbb{Ł}_{n_k}$ we get that $(\mathbf{a}^m, d^m) = (\mathbf{0}, 0) \in C$ (it is also the minimum of $\mathbf{C}$) and thus

$$(\mathbf{1}, 0) \to (\mathbf{0}, 0) = (\mathbf{0}, 1) \in C.$$

Let $\mathbf{C}'$ be the filter generated by $(\mathbf{0}, 1)$ in $\mathbf{C}$; this is a bounded Wajsberg subalgebra of $\mathbf{C}$ which is isomorphic with a subalgebra of $\prod_{(n_k, h) \in \Delta_I} \mathbb{Ł}_{n_k}$. Hence $\mathbf{C}'$ is finite and since clearly $(\mathbf{a}, 1) \in C'$ we get that $p_{n_k}(\mathbf{C}') = \mathbb{Ł}_{n_k}$. As in the proof of Theorem 6.6 we may deduce that $\mathbb{Ł}_{n_k} \in \mathbf{IS}(\mathbf{B})$ for all k.

In conclusion

$$\begin{aligned} \mathsf{V} &= \mathbf{V}(\{\mathbb{Ł}_{n_k} : n_k \in I\}, \mathbb{Ł}_1^\infty) \\ &= \mathbf{Q}(\{\mathbb{Ł}_{n_k} : n_k \in I\}, \mathbb{Ł}_1^\infty) \\ &\subseteq \mathbf{Q}(\mathbf{B}) \\ &= \mathbf{Q}(\mathbf{F}_\mathsf{V}(x)) \subseteq \mathsf{V}. \end{aligned}$$

Thus $\mathbf{Q}(\mathbf{F}_\mathsf{V}(x)) = \mathsf{V}$; therefore by Theorem 3.1 V is structurally complete. □

Remark 6.9. We already know that $\mathbf{V}(\mathbb{Ł}_n^\infty)$ is not structurally complete for $n > 1$, so the proof of Theorem 6.8 must fail. The failure of structural completeness can be witnessed by

$$\mathbf{Q}(\{\mathbb{Ł}_i : i \in I\}, \mathbb{Ł}_{n,1}) \subsetneq \mathbf{V}(\{\mathbb{Ł}_i : i \in I\}, \mathbb{Ł}_{n,1}) = \mathbf{V}(\{\mathbb{Ł}_i : i \in I\}, \mathbb{Ł}_n^\infty). \quad □$$

We define the **coradical** of a Wajsberg chain $\mathbf{A}$ in the following way:

- if $\mathbf{A}$ is cancellative, then $\mathrm{Corad}(\mathbf{A}) = \emptyset$;
- if $\mathbf{A}$ is bounded, say by 0, then

$$\mathrm{Corad}(\mathbf{A}) = \{a \to 0 : a \in \mathrm{Rad}(\mathbf{A})\}.$$

A Wajsberg chain $\mathbf{A}$ is **perfect** if $\mathbf{A} = \mathrm{Rad}(\mathbf{A}) \cup \mathrm{Corad}(\mathbf{A})$ (hence every cancellative chain is perfect and $\mathbb{Ł}_1^\infty$ is perfect). For an example of a bounded perfect chain different from $\mathbb{Ł}_1^\infty$ we may take a totally ordered group $\mathbf{G}$ and consider $\Gamma(\mathbf{G} \times_l \mathbb{Z}, (1, 0))$. Really it can be shown that every bounded perfect Wajsberg chain can be obtained in this way and this in turn implies that $\mathbf{A} \in \mathbf{ISP}_u(\mathbb{Ł}_1^\infty)$ whenever $\mathbf{A}$ is a perfect chain. Hence the variety generated by all the perfect Wajsberg chains is exactly $\mathbf{V}(\mathbb{Ł}_1^\infty)$ and it is axiomatized by $2x^2 \approx (2x)^2$. The reader can easily verify that all these facts can be deduced from the corresponding results about perfect MV-algebras (see for instance [25]). Now we can characterize all primitive and structurally complete varieties of Wajsberg hoops.

Theorem 6.10. *A proper variety V of Wajsberg hoops is primitive if and only if every chain in V is either finite or perfect.*

Proof. Let $\mathsf{V} = \mathbf{V}(I, J, K)$ be such variety; suppose that in V there is a chain that is neither finite nor perfect. Then such chain cannot belong to $\mathbf{V}(\mathbb{Ł}_1^\infty)$, otherwise it would be perfect; this implies that there is at least a $j \in J$ with $j > 1$. It follows that $\mathbb{Ł}_j^\infty \in \mathsf{V}$ and since $\mathbf{V}(\mathbb{Ł}_j^\infty)$ is not primitive, neither is V.

For the converse, if every chain in V is either finite or perfect and V is proper, then either $\mathsf{V} = \mathbf{V}(I, \{1\}, \emptyset)$ or else $\mathsf{V} = \mathbf{V}(I, \emptyset, K)$ for some finite set $I = \{n_1, \ldots, n_k\}$. If $m = n_1 n_2 \cdots n_k$ then $\mathsf{V} \subseteq \mathbf{V}(\mathbb{Ł}_m, \mathbb{Ł}_1^\infty)$, which is structurally complete by Theorem 6.8.

If D is the set of divisors of m, then every subvariety of $\mathbf{V}(\mathbb{Ł}_m, \mathbb{Ł}_1^\infty)$ is of the form $\mathbf{V}(I', \emptyset, K)$ or $\mathbf{V}(I', \{1\}, \emptyset)$ for some reduced set $I' \subseteq D$. But those varieties are all structurally complete because they are either locally finite o else they satisfy the hypotheses of Theorem 6.6 or Theorem 6.8. By Lemma 3.10 $\mathbf{V}(\mathbb{Ł}_m, \mathbb{Ł}_1^\infty)$ is primitive and so is V. □

Corollary 6.11. *A variety of Wajsberg hoops is structurally complete if and only if it is primitive.*

What about primitive quasivarieties (that are not varieties) of Wajsberg hoops? For those generated by chains there seems to be a promising path to their description (and we will talk about it in Section 9). We will consider some quasivarieties not generated by chains in the next section.

7. Quasivarieties of Wajsberg hoops: highlighting the complexity

Observe that $\mathbf{Q}(\mathbf{Ł}_1)$ and $\mathbf{Q}(\mathbf{C}_\omega)$ are varieties that are the only two atoms in $\Lambda(\mathsf{WH})$ [12]. Since they are both primitive, they are also atoms in $\Lambda_q(\mathsf{WH})$. Are there any other atoms in $\Lambda_q(\mathsf{WH})$? This question was asked first a long time ago [27]; we still do not have a solution, but we can clarify the matter a little bit. First we observe that for all $n, k \geq 1$, $\mathbf{Ł}_n$, $\mathbf{Ł}_n^\infty$ and $\mathbf{Ł}_{n,k}$ are finitely generated and subdirectly irreducible, so they are splitting in $\Lambda_q(\mathsf{WH})$ (Lemma 4.3). We also observe that none of them can be weakly projective in WH; really no bounded finitely generated Wajsberg hoop can be weakly projective in WH. In fact if $\mathbf{A}$ is n-generated and weakly projective, then it must be a subalgebra of $\mathbf{F}_{\mathsf{WH}}(n)$; but it is clear from the description of $\mathbf{F}_{\mathsf{WH}(n)}$ in [9] that the only idempotent element therein is 1. This prevents any bounded Wajsberg hoop from being embeddable in $\mathbf{F}_{\mathsf{WH}}(n)$. Also $\mathbf{C}_\omega$ is finitely generated, subdirectly irreducible (hence splitting) and not weakly projective in WH (this has been proved by S. Ugolini in [49] using geometrical methods). It follows from Lemma 3.5 that the conjugate quasivarieties $[\mathsf{WH} : \mathbf{Ł}_n]$, $[\mathsf{WH} : \mathbf{Ł}_n^\infty]$ and $[\mathsf{WH} : \mathbf{Ł}_{n,k}]$ for $n, k \geq 1$ and $[\mathsf{WH} : \mathbf{C}_\omega]$ are all proper quasivarieties. Now a possible third atom must be $\mathbf{Q}(\mathbf{A})$ for some $\mathbf{A}$ (since WH has the joint embedding property) and moreover

$$\mathbf{Q}(\mathbf{A}) \subseteq [\mathsf{WH} : \mathbf{Ł}_1] \cap [\mathsf{WH} : \mathbf{C}_\omega].$$

Of course $\mathbf{A}$ can be taken to be 1-generated and moreover no relative subdirectly irreducible in $\mathbf{Q}(\mathbf{A})$ can be subdirectly irreducible otherwise it would be totally ordered, hence bounded or cancellative, hence containing either $\mathbf{Ł}_1$ or $\mathbf{C}_\omega$. In conclusion $\mathbf{Q}(\mathbf{A})$ would be a very strange object indeed, even though we know nothing in the theory that prevents its existence.

To proceed further it is clear that the finitely generated covers of $\mathbf{Q}(\mathbf{Ł}_1)$ are the $\mathbf{Q}(\mathbf{Ł}_p)$ with p prime; a further cover is $\mathbf{Q}(\mathbf{Ł}_1, \mathbf{C}_\omega)$ and the reason is that it is a primitive variety, so it has no other subquasivarieties except for $\mathbf{Q}(\mathbf{Ł}_1)$ and $\mathbf{Q}(\mathbf{C}_\omega)$. The existence of other covers depends on the existence of other atoms; if there are no more these are the only covers of $\mathbf{Q}(\mathbf{Ł}_1)$. Clearly $\mathbf{Q}(\mathbf{Ł}_1, \mathbf{C}_\omega)$ is also a cover of $\mathsf{CH} = \mathbf{Q}(\mathbf{C}_\omega)$ and it is the only cover above $\mathbf{Q}(\mathbf{Ł}_1)$; since $\mathbf{Ł}_1$ is splitting with conjugate quasivariety $[\mathsf{WH} : \mathbf{Ł}_1]$ any other cover of CH must lie in $[\mathsf{WH} : \mathbf{Ł}_1]$.

Let's look now at the quasivarieties $[\mathsf{WH} : \mathbf{Ł}_n]$ for $n \geq 1$; it is an easy exercise to check that $\mathbf{A} \in [\mathsf{WH} : \mathbf{Ł}_1]$ if and only if $\mathbf{A}$ has no idempotent different from 1; this means that $[\mathsf{WH} : \mathbf{Ł}_1]$ is axiomatized by the single quasiequation

$$x \approx x^2 \quad \Rightarrow \quad x \approx 1. \tag{q1}$$

If $n \geq 2$ then we can use Lemma 5.9; from that it follows that $[\mathsf{WH} : \mathbf{Ł}_n]$ is axiomatized by the single quasiequation

$$(\mathbf{n-1})x \approx x \to x^n \quad \Rightarrow \quad x \approx 1,$$

which is of course the splitting quasiequation for $\mathbf{Ł}_n$. In Fig. 1 we see the slices relative kp for a prime p and $k \in \mathbb{N}$; both chains of subvarieties in the figure have WH as their join, because of Theorem 5.11.

Next we will show that none of the $[\mathsf{WH} : \mathbf{Ł}_n]$ is structurally complete; it is obvious from Tarski's result that $\mathbf{H}([\mathsf{WH} : \mathbf{Ł}_n]) = \mathsf{WH}$ for $n \geq 2$. The following lemma is a consequence of some very general facts (see [20], Chapter II, §10):

Lemma 7.1. *For any variety* V *the smallest quasivariety* Q *such that* $\mathbf{H}(\mathsf{Q}) = \mathsf{V}$ *is exactly* $\mathbf{Q}(\mathbf{F}_\mathsf{Q}(\omega))$.

Now every subvariety of WH is axiomatized by a single equation [9], so

$$\mathbf{H}(\mathbf{Q}(\mathbf{F}_\mathsf{WH}(x))) = \mathbf{H}(\mathbf{Q}(\mathbf{F}_\mathsf{WH}(\omega))).$$

It is clear from the description of $\mathbf{F}_\mathsf{WH}(x)$ in [9] that $\mathbf{F}_\mathsf{WH}(x)$ satisfies (q1), so (using Lemma 7.1),

$$\mathsf{WH} = \mathbf{H}(\mathbf{Q}(\mathbf{F}_\mathsf{WH}(x))) \subseteq \mathbf{H}([\mathsf{WH} : \mathbf{Ł}_1]) \subseteq \mathsf{WH}.$$

The final step is to show that $\mathbf{Q}(\mathbf{F}_\mathsf{WH}(x)) \subsetneq \mathbf{Q}([\mathsf{WH} : \mathbf{Ł}_1])$; but this is obvious since there are clearly subquasivarieties Q such that $\mathbf{H}(\mathsf{Q}) = \mathsf{WH}$ but $[\mathsf{WH} : \mathbf{Ł}_1] \nsubseteq \mathsf{Q}$ (for instance the quasivariety generated by all the $\mathbf{Ł}_p$ with p prime).

Now $[\mathsf{WH} : \mathbf{Ł}_1] \subsetneq [\mathsf{WH} : \mathbf{Ł}_n]$ for all n and

$$\mathbf{H}([\mathsf{WH} : \mathbf{Ł}_1]) = \mathbf{H}([\mathsf{WH} : \mathbf{Ł}_n]);$$
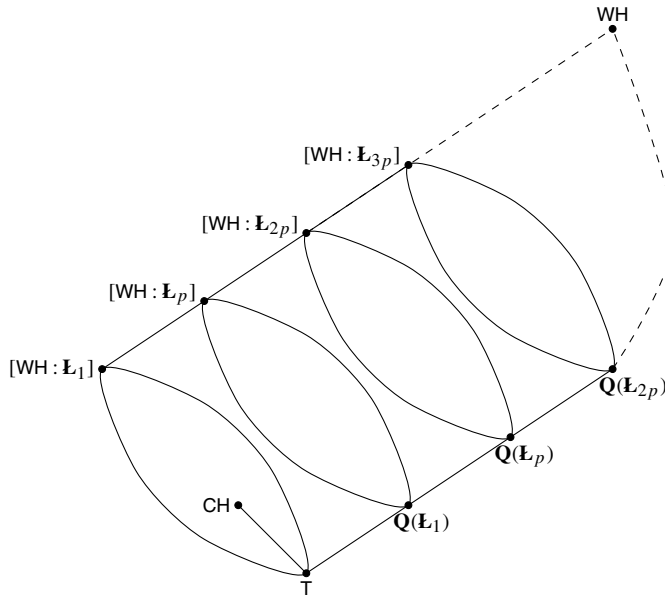
Fig. 1. The slices for $Ł_{kp}$.

so the conclusion holds by Theorem 3.1.

We close this section with an example of an uncountable set of proper quasivarieties; for a set Σ of Wajsberg hoops we define

$$[\mathsf{WH} : \Sigma] = \{\mathbf{B} : \text{for all } \mathbf{A} \in \Sigma, \mathbf{A} \notin \mathbf{IS}(\mathbf{B})\}.$$

We have the following lemma whose proof is similar to the "if" direction of Lemma 3.5(3).

Lemma 7.2. *If Σ consists entirely of finitely presented Wajsberg hoops, then $[\mathsf{WH} : \Sigma]$ is a quasivariety.*

Let now P be the set of primes; for any subset $Q \subseteq P$ we let

$$\Sigma_Q = \{Ł_q : q \in \mathsf{Q}\};$$

then for any $Q \subseteq P$, $[\mathsf{WH} : \Sigma_Q]$ is a quasivariety; moreover if $Q, Q' \subseteq P$, then $[\mathsf{WH} : Q] = [\mathsf{WH} : Q']$ if and only if $Q = Q'$. In fact let $q \in Q \setminus Q'$; then $Ł_q \in [\mathsf{WH} : Q'] \setminus [\mathsf{WH} : Q]$. It follows that the set of such quasivarieties is in 1-1 correspondence with the subsets of a countable set, i.e. it is uncountable. We cannot prove that they are all quasivarieties that are not varieties since the proof of the "only if" direction of Lemma 3.5(3) does not work; however since there are only countably many subvarieties of Wajsberg hoops, at least uncountably many of them are not varieties.

8. A glance at quasivarieties of Wajsberg algebras

Though quasivarieties of Wajsberg algebras have been studied thoroughly ([28], [33], [31], [32]), there are some observations that we can make. We recall that $\mathbf{W}_n, \mathbf{W}_n^\infty, \mathbf{W}_{n,k}$ denote the bounded versions of $Ł_n, Ł_n^\infty, Ł_{n,k}$ respectively and that BL-algebras are the bounded version of basic hoops. Now it is obvious that a version of Lemma 5.13 holds for classes of BL-algebras and from that we get at once:

Theorem 8.1. *Let $\mathbf{A}$ be a totally ordered Wajsberg algebra; then $\mathbf{Q}(\mathbf{A})$ is a variety if and only if*

- *$\mathbf{W}_n$ is embeddable in $\mathbf{A}$ for all n, or*

- **A** *is finite, or*
- **A** *is infinite, bounded and the rank of* **A** *is equal to* $d_\mathbf{A}$.

Theorem 8.2. *Let* $\mathbf{A}_1, \ldots, \mathbf{A}_n$ *be totally ordered Wajsberg algebras; if for* $i = 1, \ldots, n$

- $\mathbf{W}_n$ *is embeddable in* $\mathbf{A}_i$ *for all* n, *or*
- $\mathbf{A}_i$ *is finite, or*
- $\mathbf{A}_i$ *is infinite, bounded and the rank of* **A** *is equal to* $d_\mathbf{A}$,

then $\mathbf{Q}(\mathbf{A}_1, \ldots, \mathbf{A}_n)$ *is a variety.*

Another similarity is that any proper subvariety of Wajsberg algebras admits a v-presentation. In other words every proper subvariety V is generated by finitely many chains that are either finite or $\mathbf{W}_n^\infty$ for some n. If $\{\mathbf{W}_i : i \in I\}$ and $\{\mathbf{W}_j^\infty : j \in J\}$ are the generators of V we can denote V by $\mathsf{V}(I, J)$. As for Wajsberg hoops we can define the concept of reduced pair in the obvious way and it turns out that the set of proper subvarieties of WA is in 1-1 correspondence with $\{\mathsf{V}(I, J) : (I, J) \text{ a reduced pair}\}$ [44].

Now lattice $\Lambda(\mathsf{WH})$ is indubitably more complex that $\Lambda(\mathsf{WA})$, the reason being that there is an entire variety of algebras, the variety of cancellative hoops, that simply was not there before. On the other hand adding the constant makes it harder for an algebra to be a subalgebra of something else; in particular it is no longer true that $\mathbf{A}, \mathbf{B} \leq \mathbf{A} \times \mathbf{B}$ a fact that conceivably should make $\Lambda_q(\mathsf{WA})$ more complex than the interval $[\mathsf{V}(\mathbf{Ł}_1), \mathsf{WH}]$ in $\Lambda_q(\mathsf{WH})$. Let's confirm this intuition: in $\Lambda_q(\mathsf{WH})$, $\mathbf{Q}(\mathbf{Ł}_p)$ covers $\mathbf{Q}(\mathbf{Ł}_1)$ for every prime p. This is not the case for $\Lambda_q(\mathsf{WA})$ because of the following:

Lemma 8.3. *Let* $\mathsf{V} = \mathbf{V}(\mathbf{W}_{n_1}, \ldots, \mathbf{W}_{n_k})$; *then*

$$\mathbf{Q}(\mathbf{W}_1 \times \mathbf{W}_{n_1}, \ldots, \mathbf{W}_1 \times \mathbf{W}_{n_k})$$

is a proper subquasivariety of V.

Proof. The proof is based on the fact that $\mathbf{W}_n$ is embeddable in a chain in WA if and only if there is an $a \in A$ such that $(\mathbf{n} - \mathbf{1})a = \neg a$ (where $\neg a = a \to 0$) ([8], Lemma 4.4).

We will prove the case $k = 1$ and $n_1 = 2$; the procedure is general and the interested reader can fill the details for any other case. $\mathbf{W}_2$ is embeddable in **A** if and only if there is an $a \in A$ with $a = \neg a$. If **A** is any subalgebra of a power of $\mathbf{W}_1 \times \mathbf{W}_2$, then no such $a \in A$ can exist since the negation switches 0 and 1 in all the "coordinates" coming from copies of $\mathbf{W}_1$. This shows that $\mathbf{W}_2 \notin \mathbf{Q}(\mathbf{W}_1 \times \mathbf{W}_2)$ and hence the conclusion. $\square$

Now in [32] it is shown that $\mathbf{Q}(\mathbf{W}_1 \times \mathbf{W}_p)$ covers $\mathbf{Q}(\mathbf{W}_1)$ for every prime p and that

$$\mathbf{Q}(\mathbf{W}_1 \times \mathbf{W}_{1,1}) = \mathbf{Q}(\mathbf{W}_1^\infty) = \mathbf{V}(\mathbf{W}_1^\infty).$$

This implies that $\{\mathbf{Q}(\mathbf{W}_1 \times \mathbf{W}_p) : p \text{ prime}\} \cup \mathbf{Q}(\mathbf{W}_1^\infty)$ is a complete set of covers of $\mathbf{Q}(\mathbf{W}_1)$, which is of course the only atom in $\Lambda_q(\mathsf{WA})$.

The problem of describing all the structurally complete quasivarieties of Wajsberg algebras has been almost solved by J. Gispert in [32]; in fact therein he was able to describe all the least V-quasivarieties where V is a proper subvariety of V.

Theorem 8.4. *If* $\mathsf{V} = \mathsf{V}(I, J)$ *is a proper variety of Wajsberg algebras and* (I, J) *is a reduced pair, then the least* V-*quasivariety of* $\mathsf{V}(I, J)$ *is*

$$\mathsf{Q}(I, J) = \mathbf{Q}(\{\mathbf{W}_1 \times \mathbf{W}_i : i \in I\} \cup \{\mathbf{W}_1 \times \mathbf{W}^{j,1} : j \in J\}).$$

So the only structurally complete quasivarieties of Wajsberg algebras, besides $\mathbf{Q}(\mathbf{F}_{\mathsf{WA}}(x))$ for which no description is available, are those of type $\mathbf{Q}(I, J)$ where (I, J) is a reduced pair. It follows for instance that $\mathbf{Q}(\mathbf{W}_1^\infty) = \mathsf{Q}(\emptyset, \{1\})$

is structurally complete and together with $\mathbf{Q}(\mathbf{W}_1) = \mathsf{Q}(\{1\}, \emptyset)$ is the only quasivariety generated by a chain that it is structurally complete.

Which of those structurally complete varieties are also primitive? It is evident that all the covers of the unique atom in $\Lambda_q(\mathsf{WA})$ are such; in fact they are all structurally complete and their unique subvariety is $\mathbf{Q}(\mathbf{W}_1)$ which is also structurally complete. More generally a quasivariety Q of Wajsberg algebras is primitive if and only if all its subquasivarieties are the least V-quasivariety for some $\mathsf{V} \subseteq \mathbf{H}(\mathsf{Q})$. Let's see that this is not always the case by generalizing an argument in [32]. First we observe:

Lemma 8.5. *[32] If $(I, J), (I', J')$ are two residuated pairs then $\mathsf{Q}(I, J) \subseteq \mathsf{Q}(I', J')$ if and only if for every $n \in I$, $n > 1$, there is an $n' \in I'$ with $n \mid n'$ and for any $m \in J$ there is an $m' \in J'$ such that $m \mid m'$.*

If (I, J) is a reduced pair and $I = \{n\}$, $J = \{m\}$ we will write $\mathsf{V}(n, m)$ for $\mathsf{V}(I, J)$. Let p, q, r be three distinct primes; then

$$\mathsf{V}(pq, pr) \cap \mathsf{V}(pr, pq) = \mathsf{V}(\emptyset, p).$$

It follows that every least V-quasivariety contained in $\mathsf{Q}(pq, pr)$ and $\mathsf{Q}(pr, pq)$ is a least $\mathsf{V}(\emptyset, p)$-quasivariety. Using Lemma 8.5 it is easy to verify that those are exactly $\{\mathsf{Q}(1, \emptyset), \mathsf{Q}(\emptyset, 1), \mathsf{Q}(p, \emptyset), \mathsf{Q}(\emptyset, p)\}$; clearly $\mathsf{Q}(p, \emptyset)$ and $\mathsf{Q}(\emptyset, p)$ are maximal in that set and they are incomparable by Lemma 8.5. So $\mathsf{Q}(pr, pq) \cap \mathsf{Q}(pq, pr)$ is a quasivariety that is not a least V-quasivariety. Therefore for instance $\mathsf{Q}(2pqr, 3pqr)$ of which both $\mathsf{Q}(pr, pq)$ and $\mathsf{Q}(pq, pr)$ are subquasivarieties, cannot be primitive. Of course this argument can be further generalized in many ways to exclude primitivity for many structurally complete quasivarieties of Wajsberg algebras.

9. Conclusion and further investigations

The main problem that has not been solved in this paper is determining the structurally complete subquasivarieties of Wajsberg hoops. A possible path (as observed also by Reviewer # 1 of this paper) is to use the classification of free Wajsberg hoops in [9] to generalize the arguments in Lemmas 6.5 and 6.7. The conjecture formulated by Reviewer #1 is that $\mathsf{Q}(\mathbf{Ł}_{n,1})$ is structurally complete and it is indeed the least $\mathbf{V}(\mathbf{Ł}_n^\infty)$-quasivariety. We do agree with that conjecture and we plan to work on it in the next future.

Another possible path of investigation comes from the fact that Wajsberg hoops constitute the building blocks of basic hoops in a precise sense; let $\mathbf{A}_0, \mathbf{A}_1 \in \mathsf{CIRL}$ such that $\mathbf{A}_0 \cap \mathbf{A}_1 = \{1\}$, and consider $A_0 \cup A_1$. We define operations in the following way: the ordering intuitively stacks A_1 on top of $A_0 \setminus \{1\}$ and more precisely it is given by

$$a \le b \quad \text{if and only if} \quad \begin{cases} b = 1, \text{ or} \\ a \in A_0 \setminus \{1\} \text{ and } b \in A_1 \setminus \{1\} \text{ or} \\ a, b \in A_i \setminus \{1\} \text{ and } a \le_{A_i} b, i = 0, 1. \end{cases}$$

Moreover we define the product inside of the two components to be the original one, and between the two different components to be the meet:

$$a \cdot b = \begin{cases} a, & \text{if } a \in A_0 \setminus \{1\} \text{ and } b \in A_1; \\ b, & \text{if } a \in A_1 \text{ and } b \in A_0 \setminus \{1\}; \\ a \cdot_{A_i} b, & \text{if } a, b \in A_i, i = 0, 1. \end{cases}$$

$$a \to b = \begin{cases} b, & \text{if } a \in A_1 \text{ and } b \in A_0 \setminus \{1\}; \\ 1, & \text{if } a \in A_0 \setminus \{1\} \text{ and } b \in A_1; \\ a \to_{A_i} b, & \text{if } a, b \in A_i, i = 0, 1. \end{cases}$$

The resulting structure is called the **ordinal sum** of $\mathbf{A}_0$ and $\mathbf{A}_1$ and we denote it by $\mathbf{A}_0 \oplus \mathbf{A}_1$. It is easily checked that $\mathbf{A}_0 \oplus \mathbf{A}_1$ is a commutative and integral and residuated *semilattice*. However, it might not be a residuated lattice: for instance if 1_{A_0} is not join irreducible, $\mathbf{A}_1$ is not bounded and $a, b \in A_0 \setminus \{1\}$ are such that $a \vee_{A_0} b = 1_{A_0}$, then all the upper bounds of $\{a, b\}$ lie in A_1. Since A_1 is not bounded there can be no least upper bound of $\{a, b\}$ in $\mathbf{A}_0 \oplus \mathbf{A}_1$, thus the ordering cannot be a lattice ordering. However it is clear that the ordinal sum of two totally ordered CIRLs is again a totally ordered CIRL. This allows us to define the ordinal sum of a set of totally ordered CIRLs in the

obvious way; for more details about this construction the reader can consult [4] or [8]. A totally ordered CIRL is **sum irreducible** if it cannot be expressed as the ordinal sum of two nontrivial CIRLs; moreover any totally ordered CIRL can be decomposed into a ordinal sum of sum irreducible CIRLSs in an essentially unique way ([4], Theorem 3.2). The key result is:

Theorem 9.1. *[8]*

1. *A totally ordered hoop is sum irreducible if and only if it is a Wajsberg hoop. Hence every totally ordered hoop is the ordinal sum of Wajsberg hoops.*
2. *A totally ordered* BL*-algebra is sum irreducible if and only if it is a Wajsberg algebra. Hence every totally ordered* BL *algebra is a bounded ordinal sum of Wajsberg hoops, the first of which is a Wajsberg algebra.*

So it is not unreasonable to think that some properties of quasivarieties of Wajsberg hoops (and Wajsberg algebras) can be lifted to basic hoops (BL-algebras) via ordinal sums. There are some results that are quite straightforward to generalize, while others require a great deal of attention; we plan to investigate this topic in the future.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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