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Nonlinear Dynamics and Invariants of Motion in ReRAM Cells

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ABSTRACT

Memristor circuits are gaining increasing importance in modern electronics due to their ability to mimic neurons, synapses, and, more generally, the computation capabilities of biological neural systems. The ideal memristor model was introduced by L. Chua in a seminal paper back in 1971. Later on, several more complex memristor models, such as the popular Voltage ThrEshold Adaptive Memristor model, have been developed to better describe the state-dependent current-voltage behavior of physical memristor devices, in particular Resistive Random Access Memories, a nanotechnology of great appeal nowadays. While the study of circuits with ideal memristors is reaching a mature state, very few results are available on the analysis of the dynamics of circuits with real memristors. As demonstrated in this paper, peculiar dynamical properties of cells with ideal memristors are also displayed by some analogue electrical circuits with real memristors. In particular, simple circuits, including, as dynamical elements, one capacitor and a memristive one-port, composed of a pair of back-to-back Resistive Random Access Memory devices described through the Voltage ThrEshold Adaptive Memristor model, admit physical quantities, referred to as Invariants of Motion, which are conserved over time. As a consequence, their state space can be foliated into Invariant Manifolds, which induce the coexistence of infinitely many different monostable and bistable quiescent steady states. A second-order cell from the proposed class may be toggled between regimes of quiescent monostability and bistability by controlling the initial condition-dependent index of the manifold along which its dynamical variables are constrained to lie at all times. Moreover, at any equilibrium point, the voltage across its capacitor vanishes, which allows the memristive one-port to hold its state without energy consumption. These attractive properties make the proposed cells ideal candidates to serve as sensing-and-memcomputing units for innovative Cellular Neural Networks to be deployed for solving time-critical edge computing problems in resource-constrained environments.

1 | Introduction

Emerging devices such as memristors are expected to play an ever increasing role in modern electronics and futuristic neuro-morphic computers. One key reason is that memristors are able to electronically mimic biological synapses, neurons, and more generally the parallel and efficient computational capabilities of nervous systems [1–9]. In particular, memristors enable to

implement *in memory* computing schemes where computation and memorization of data are in the same physical location [10, 12–18, 73]. This permits, in principle, to break down the Von Neumann barrier of traditional computing machines where data are continuously exchanged between the central processing unit (CPU) and the random access memory (RAM), which are at different physical locations. Memristors have been used to implement high efficiency sensing systems in the visual, tactile,

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and auditory areas [19–23]. They have been also used to solve complex tasks in perception systems [24], pattern recognition [25], image processing [26] and decision-making [27].

The *ideal* memristor has been introduced by L. Chua in the seminal article [28] and is defined by a nonlinear relationship between flux (i.e., the integral of voltage) and charge (i.e., the integral of current). An ideal memristor is able to account for basic features such as non-volatility, nonlinearity, and memory, moreover, it displays a typical pinched hysteresis loop in the $v-i$ plane when subject to a sinusoidal excitation. Yet, it soon became apparent that an ideal memristor is in general unable to accurately represent the behavior of *real* memristor devices implemented in nanotechnology. Therefore, a number of real memristor models have been proposed in the literature that fall in the classes of generic or extended memristors [29–37]. In those models the state variable is not necessarily the flux or charge, moreover, more than one state variable can be used. There are models that, differently from an ideal memristor, are volatile, as in the case of a thermistor. Moreover, a key feature is that real memristor models usually display a voltage or current *threshold*, namely, the state undergoes negligible changes below threshold, while there are rapid changes above threshold. In this paper, we will mainly consider the Voltage ThrEshold Adaptive Memristor (VTEAM) memristor model proposed by Kvatinsky et al. in [34]. This is a relatively simple, general, robust and accurate model of non-volatile voltage-controlled Resistive Random Access Memories (ReRAMs). VTEAM turns out to be designer-friendly and it can be fit to numerous memristor models and experimental data. VTEAM is intrinsically characterized by a threshold voltage, which is important for the characterization of real memristor devices. VTEAM is a successor of TEAM, a current-control model introduced as an approximation of Simmons barrier model [33].

The analysis of circuits with ideal memristors is currently reaching a quite mature state. In particular, the flux-charge analysis method (FCAM) introduced in [38, 39] has enabled to thoroughly analyze the dynamics of basic circuits as memristor Chua's circuits, as Chua's [40] and Murali [41] ones, and classes of memristor neural networks [42–44]. Via FCAM, it has been shown that circuits with ideal memristors display a number of salient dynamical features. One main property is that, for structural reasons, they possess *invariants of motion* (or conserved quantities) and as a consequence their state space can be foliated in *invariant manifolds* where they display a reduced-order dynamics. In turn, there is coexistence of infinitely many different dynamics and attractors, a property termed *extreme multistability*. Moreover, memristor circuits can display bifurcations due to changing the initial conditions for fixed circuit parameters (*bifurcations without parameters*) [39, 45].

The dynamic analysis of circuits with real memristors is even more important than that of the ideal case in view of the practical implementation of memristor circuits in nanotechnology. However, only very few results are available in the literature in the real case. This is due to several reasons, the most important one being that the real memristors are described by much more complex models than ideal ones, which in turn leads to relevant additional difficulties. Among the available results, we can mention the stability analysis of neural networks for associative memories implemented via the Stanford memristor model [46]

or other real memristor models [47, 48], the investigation of the mem-computing capabilities of cellular neural networks (CNNs), hosting ReRAM devices inside the processing elements [49, 73], the analysis of bifurcations at the edge of chaos for circuits with locally-active volatile memristors [50–52] and the theoretic and experimental study of chaos in Chua's circuit with a real memristor [53, 54].

Goal of this paper is to study whether some of the peculiar dynamical properties of a circuit with ideal memristors are displayed also by circuits with real memristors. To this end, some simple memristor circuits containing capacitors and ReRAMs described by the VTEAM model are considered. Via an approach that generalizes FCAM, it is shown that for the considered circuits we can explicitly find invariants of motion and invariant manifolds. Moreover, the circuits display extreme multistability and bifurcations without parameters. In particular, a detailed dynamic analysis shows that there is co-existence of infinitely many different monostable and bistable behaviors. These unconventional dynamic properties of emerging components as memristors are not only of theoretic but also of practical interest. In particular, the coexistence of a monostable and a bistable behavior would permit to implement for instance basic cells of a CNN architecture along the lines discussed in [55].

2 | Memristor Models

2.1 | Ideal Memristor Model

The ideal memristor model has been introduced by L. Chua in the seminal paper [28] as the fourth basic passive circuit element in addition to the resistor, capacitor and inductor. In the flux-controlled case, an ideal memristor is defined by the constitutive relation $q = \hat{q}(\varphi)$, where $q(t) = \int_{-\infty}^t i(\tau) d\tau$ (resp., $\varphi(t) = \int_{-\infty}^t v(\tau) d\tau$) is the charge (resp., the flux) and $\hat{q}(\cdot)$ is a nonlinear function. By differentiation, the memristor satisfies

$$\begin{cases} i = \hat{q}'(\varphi)v \\ \dot{\varphi} = v \end{cases} \quad (1)$$

where $\hat{q}'(\cdot)$ has dimension Ohm^{-1} and is named memductance. The second equation, termed memristor state equation, defines how the state variable, i.e., the flux, depends upon the voltage. On the other hand, the first equation is the current-voltage characteristic, which is seen to amount to a state-dependent Ohm's law.

2.2 | VTEAM Memristor Model

In the literature on memristors' models two main classes may be identified. The first class includes models, which are meant to capture the main physical mechanisms [57], underlying the operating principles of a device, at the microscopic scale. Being based on physics' first principles, the models are ideally suited for developing strategies to optimize the layer stack in the fabrication process. A second class groups compact behavioral descriptions, which intend to reproduce the behavior of a device in terms of port voltage and port current, at the macroscopic scale. These

these models are extremely useful for circuit design and analysis, enabling hand calculations and facilitating the numerical exploration of the potential of the memristor for electronics applications. An example from this class is the VTEAM model. As all behavioral descriptions, the VTEAM model captures the impact of an un-specified scalar variable, encoding the memory state of a device, on the relationship between the voltage and current, which respectively fall across and flow through its nanoscale medium. Proposed in [34] as the voltage-controlled variant of the ThrEshold Adaptive Memristor (TEAM) model [33], introduced earlier on to model first-order current-controlled generic memristors [56] from the ReRAM class, the ordinary differential equation (ODE), denoting its state equation, admits the form

$$\dot{w} = \begin{cases} k_{\text{off}} \left(\frac{v}{v_{\text{off}}} - 1 \right)^{\alpha_{\text{off}}} f_{\text{off}}(w), & 0 < v_{\text{off}} < v \\ 0, & v_{\text{on}} < v < v_{\text{off}} \\ k_{\text{on}} \left(\frac{v}{v_{\text{on}}} - 1 \right)^{\alpha_{\text{on}}} f_{\text{on}}(w), & v < v_{\text{on}} < 0 \end{cases} \quad (2)$$

where w is an internal state variable¹, $k_{\text{off}} > 0, k_{\text{on}} < 0$ are given real constants, while $\alpha_{\text{off}}, \alpha_{\text{on}}$ are given natural constants. Moreover, $v_{\text{on}} < 0$ and $v_{\text{off}} > 0$ are given real constants playing the role of *threshold voltages*. Indeed, there is no change in the state variable w below threshold, i.e., when $v_{\text{on}} < v < v_{\text{off}}$, while w is subject to changes when the voltage is above the positive threshold v_{off} or below the negative threshold v_{on} . As a distinctive feature of the VTEAM, which is not present in ideal memristor models, the voltage across the device must decrease below (increase above) a negative (a positive) threshold for SET (RESET) switching phenomena to emerge across its medium. Importantly, supra-threshold activation kinetics are typically observed in memristive devices irrespective of the nanotechnologies employed for their fabrication [37, 63, 64]. Functions $f_{\text{off}}(\cdot)$ and $f_{\text{on}}(\cdot)$ are *window functions* that constrain the state w to evolve within the range $0 \leq w_{\text{on}} \leq w \leq w_{\text{off}}$, where $w_{\text{on}} < w_{\text{off}}$. As it was remarked in [34, Section III], several possible choices for the windows are possible. In this manuscript, mainly for simplicity of computation, we make use of the popular Joglekar window

$$F_{J,p}(w) = 1 - \left(2 \frac{w - w_{\text{on}}}{w_{\text{off}} - w_{\text{on}}} - 1 \right)^{2p}, \quad (3)$$

where $w \in [w_{\text{on}}, w_{\text{off}}]$ and $p = 1, 2, \dots$. With this choice, Equation (2) can be written in the form

$$\dot{w} = \gamma(v) F_{J,p}(w) \quad (4)$$

where

$$\gamma(v) \doteq \begin{cases} k_{\text{off}} \left(\frac{v}{v_{\text{off}}} - 1 \right)^{\alpha_{\text{off}}}, & 0 < v_{\text{off}} < v \\ 0, & v_{\text{on}} < v < v_{\text{off}} \\ k_{\text{on}} \left(\frac{v}{v_{\text{on}}} - 1 \right)^{\alpha_{\text{on}}}, & v < v_{\text{on}} < 0. \end{cases} \quad (5)$$

It is worth noting that the current-voltage relation is not defined explicitly in the VTEAM model. An exponential dependance of the resistance on the state is one popular choice [34, Section III]. Another typical choice, made also in our study, envisages a linear dependance of the resistance upon the state variable. The Ohm

law is thus defined as

$$i = \mathcal{G}(w)v \quad (6)$$

where the memductance function is defined as

$$\mathcal{G}(w) = \frac{1}{R_{\text{on}} + \frac{R_{\text{off}} - R_{\text{on}}}{w_{\text{off}} - w_{\text{on}}} (w - w_{\text{on}})} \quad (7)$$

with $0 < R_{\text{on}} < R_{\text{off}}$. Note that R_{on} is the resistance in the ON state, while R_{off} is that in the OFF state. $\mathcal{G}(\cdot)$ is independent of the voltage. Therefore, according to Chua's theory [56], the differential algebraic equation (DAE) set, consisting of the ODE in Equation (4), with $F_{J,p}(\cdot)$ and $\gamma(\cdot)$ respectively defined via Equations (3), and (5), and of the state-dependent Ohm's law in Equation (6), with $\mathcal{G}(\cdot)$ expressed through Equation (7), is a first-order voltage-controlled generic memristor model.

Thanks to the adjustability of the window functions, to the tuneability of the coefficients, governing the impact of positive and negative voltages, dropping across the device, on its off and on switching kinetics, respectively, and to the adaptable form of the state-dependent Ohm's law, the VTEAM model has been fitted to physical memristor realizations of various kind ([58, 59]) over the years, which explains its extensive use in memristor research ([60–62]).

2.3 | VTEAM Memristors in Anti-Parallel

In the case of a linear dependance of the resistance on the state variable w , as in Equation (7), the memductance $\mathcal{G}(w)$ is a monotonically decreasing function of w (Figure 1a). In the applications discussed in the paper, we find it useful to consider memductances which do not depend monotonically upon w . This can be simply obtained by a combination of VTEAM memristors. In particular, consider two VTEAM memristors in anti-parallel, as shown in Figure 2b). To make the computations easier, we suppose that the two memristors are identical and that γ is odd, i.e., $\gamma(-v) = -\gamma(v)$. This is ensured provided $k = k_{\text{off}} = -k_{\text{on}}$, $v_{\sigma} = v_{\text{off}} = -v_{\text{on}}$, and $\alpha = \alpha_{\text{off}} = \alpha_{\text{on}}$. In this case, we can write Equation (5) as follows

$$\gamma(v) \doteq \begin{cases} k \left(\frac{v}{v_{\sigma}} - 1 \right)^{\alpha}, & 0 < v_{\sigma} < v \\ 0, & -v_{\sigma} < v < v_{\sigma} \\ -k \left(\frac{v}{-v_{\sigma}} - 1 \right)^{\alpha}, & v < -v_{\sigma} < 0. \end{cases} \quad (8)$$

The stated conditions may not be exactly verified in practice. Yet, imposing them permits to simplify the analysis and arrive at a basic understanding of the dynamic behavior. Moreover, as in [33], we let $w_{\text{on}} = 0$ and introduce the normalized state variable $x = w/w_{\text{off}} \in [0, 1]$. Hence, Equation (3) becomes

$$f_{J,p}(x) = 1 - (2x - 1)^{2p}. \quad (9)$$

A plot of $f_{J,p}(\cdot)$ for various values of p is in Figure 3. Moreover, Equations (6) and (7) become, respectively

$$i = \mathcal{G}(x)v \quad (10)$$

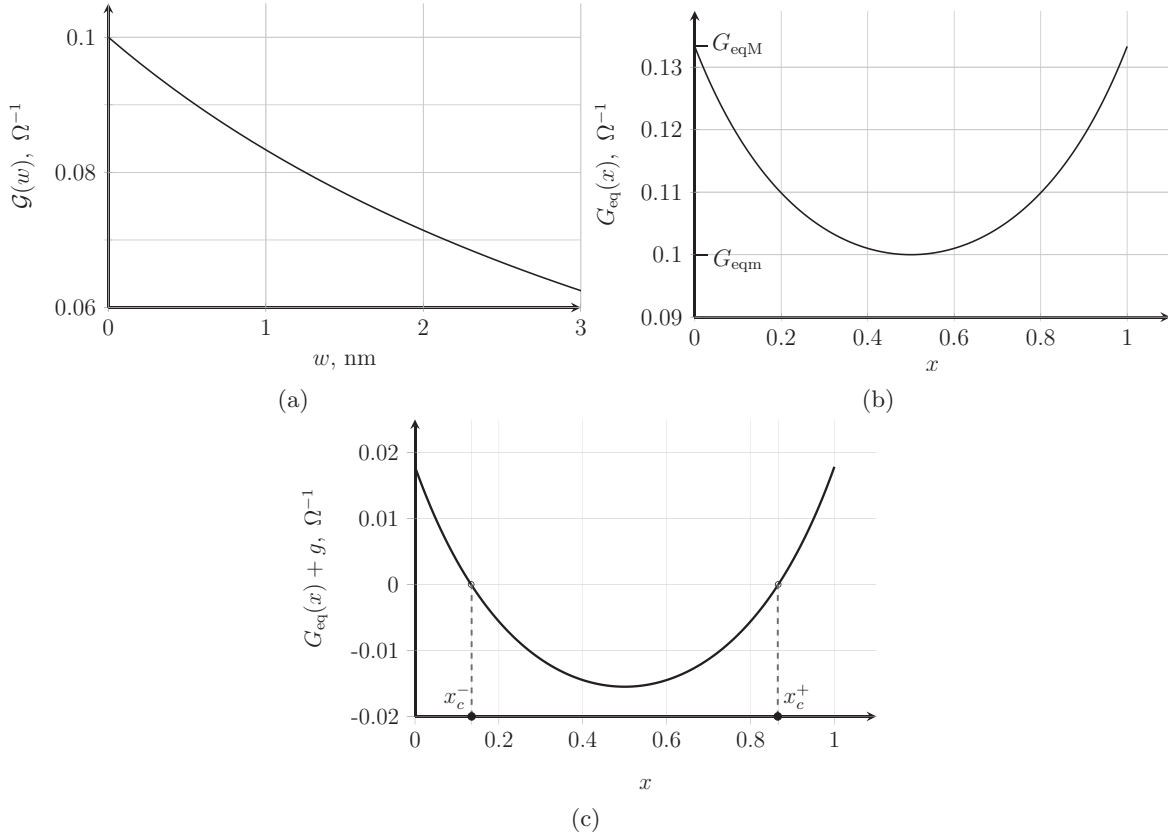


FIGURE 1 | (a) Monotonically decreasing memductance $G(w)$ of a single real VTEAM memristor as a function of the width w . The maximum value of G is $1/R_{on}$ and is reached at $w = 0$, while the minimum value $1/R_{off}$ is reached at $w = w_{off}$. (b) Non-monotonic memductance $G_{eq}(x)$ of two real VTEAM memristors in anti-parallel as a function of the normalized gap width $x = w/w_{off}$ for $x_1(-\infty) = x_2(-\infty) = 1/2$ and for $p = 1$. The maximum value $1/R_{on} + 1/R_{off}$ of G_{eq} is reached at $x = 0$ and at $x = 1$, while the minimum value $4/(R_{on} + R_{off})$ is reached at $x = 1/2$. (c) Same as in (b) where a negative conductance g satisfying Equation (39) is added in parallel to the two memristors. In case (c) the memductance is active, i.e., negative, in the range $x \in (x_c^-, x_c^+)$. The following values are used for the VTEAM memristor (cf. Example 1): $w_{off} = 3$ nm, $w_{on} = 0$ nm, $R_{on} = 10 \Omega$, $R_{off} = 30 \Omega$, $g = -0.114 \Omega^{-1}$. For function γ in Equation (8) we have $k = 3 \times 10^{-3}$ m/s and $\alpha = 3$. The threshold voltage is $v_\sigma = 0.5$ V. Joglekar window with $p = 1$ is used to model each of the two identical VTEAM memristors. With these values, we have $x_c^- = 0.14$ and $x_c^+ = 0.86$.

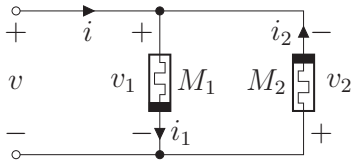


FIGURE 2 | Two real VTEAM memristors M_1 and M_2 in anti-parallel.

and

$$G(x) = \frac{1}{R_{on} + (R_{off} - R_{on})x}. \quad (11)$$

Since $i = i_1 - i_2$ and $v = v_1 = -v_2$, we obtain

$$i = (G(x_1) + G(x_2))v$$

$$\dot{x}_1 = \frac{1}{w_{off}} \gamma(v) f_{J,p}(x_1)$$

$$\dot{x}_2 = -\frac{1}{w_{off}} \gamma(v) f_{J,p}(x_2).$$

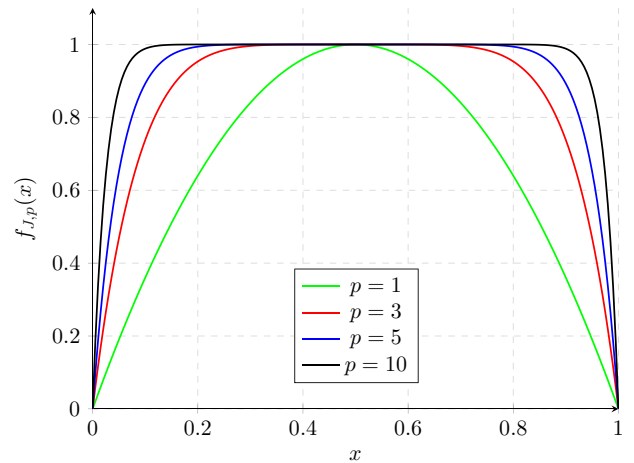


FIGURE 3 | Joglekar window for various values of p .

Therefore

$$\frac{\dot{x}_1}{f_{J,p}(x_1)} = -\frac{\dot{x}_2}{f_{J,p}(x_2)}. \quad (12)$$

Suppose that $t = -\infty$ denotes the instant where the two memristors are connected in parallel and assume they stay connected for $t > -\infty$. Let $x_1(-\infty)$ and $x_2(-\infty)$ be the state variables when they are first connected. Integrating Equation (12), we obtain

$$\int_{-\infty}^t \frac{\dot{x}_1(\tau)}{f_{J,p}(x_1(\tau))} d\tau = - \int_{-\infty}^t \frac{\dot{x}_2(\tau)}{f_{J,p}(x_2(\tau))} d\tau$$

and

$$\int_{x_1(-\infty)}^{x_1(t)} \frac{1}{f_{J,p}(\xi)} d\xi = - \int_{x_2(-\infty)}^{x_2(t)} \frac{1}{f_{J,p}(\xi)} d\xi.$$

Due to the window functions, we have $x_1(t), x_2(t) \in (0, 1)$ for any t . Consider function

$$H_p(y) \doteq \int_{\frac{1}{2}}^y \frac{1}{f_{J,p}(\xi)} d\xi.$$

Since $f_{J,p}(\cdot)$ assumes positive values, $H_p(\cdot)$ is strictly increasing and, hence, it is invertible for $y \in (0, 1)$. Then, we obtain

$$H_p(x_1(t)) - H_p(x_1(-\infty)) = -H_p(x_2(t)) + H_p(x_2(-\infty)) \quad (13)$$

and

$$x_2(t) = H_p^{-1}(H_p(x_1(-\infty)) + H_p(x_2(-\infty)) - H_p(x_1(t))). \quad (14)$$

It is easy to compute function $H_p(\cdot)$ for any p . Yet, for the sake of simplicity and to avoid the use of cumbersome formula, in the following we consider the simplest case $p = 1$. We have $f_{J,1}(x) = 1 - (2x - 1)^2$ and

$$H_1(y) = \int_{\frac{1}{2}}^y \frac{1}{1 - (2\xi - 1)^2} d\xi = \frac{1}{4} \ln \left(\frac{y}{1 - y} \right).$$

From Equation (13), we obtain

$$\ln \left(\frac{x_2(t)}{1 - x_2(t)} \right) = \ln \left(\Gamma \frac{1 - x_1(t)}{x_1(t)} \right)$$

where

$$\Gamma = \frac{x_1(-\infty)x_2(-\infty)}{(1 - x_1(-\infty))(1 - x_2(-\infty))}. \quad (15)$$

Therefore, we have

$$x_2(t) = \frac{\Gamma(1 - x_1(t))}{x_1(t) + \Gamma(1 - x_1(t))}. \quad (16)$$

This leads to a useful conclusion. Namely, given the initial conditions $x_1(-\infty), x_2(-\infty) \in (0, 1)$ and for $p = 1$, the two VTEAM memristors in anti-parallel are *equivalent to a VTEAM memristor* obeying

$$\begin{cases} i = G_{eq}(x_1)v \\ \dot{x}_1 = \frac{1}{w_{off}} \gamma(v) f_{J,1}(x_1) \end{cases} \quad (17)$$

where the equivalent memductance is given by

$$G_{eq}(x_1) = G(x_1) + G \left(\frac{\Gamma(1 - x_1)}{x_1 + \Gamma(1 - x_1)} \right). \quad (18)$$

The equivalent memristor can be described by using only one state variable x_1 . Given x_1 , we can uniquely find x_2 via Equation (16).

In the analysis in Section 3.3, we will let $x_1(-\infty) = x_2(-\infty) = 1/2$. Then, $\Gamma = 1$, which yields

$$x_2(t) = 1 - x_1(t) \quad (19)$$

and

$$G_{eq}(x_1) = G(x_1) + G(1 - x_1). \quad (20)$$

The corresponding behavior of $G_{eq}(x_1)$ as a function of x_1 is depicted in Figure 1b. The memductance $G_{eq}(\cdot)$ is monotone decreasing in the interval $[0, 1/2)$ and monotone increasing in the interval $(1/2, 1]$. Moreover, it has a maximum

$$\begin{aligned} G_{eqM} &= G_{eq}(0) = G_{eq}(1) = G(0) + G(1) \\ &= 1/R_{on} + 1/R_{off} \end{aligned} \quad (21)$$

and a minimum

$$G_{eqm} = G_{eq}(1/2) = 4/(R_{on} + R_{off}) < G_{eqM}. \quad (22)$$

3 | Simple Circuits with Real Memristors

In this section, we analyze the dynamic behavior of some elementary circuits with memristors. We start with the simplest case where we have an ideal memristor in parallel to a capacitor. Then, we consider the case of a real VTEAM memristor in parallel to a capacitor and, finally, the case case where there are two VTEAM memristors in anti-parallel, combined once again in parallel with a capacitor and a negative conductance.

3.1 | Circuit with Ideal Memristor and Capacitor

Consider the circuit with an ideal flux-controlled memristor satisfying (1) and an ideal capacitor C (Figure 5a). The circuit obeys the second-order state equations (SEs)

$$\begin{cases} C\dot{v} = -\hat{q}'(\varphi)v \\ \dot{\varphi} = v. \end{cases} \quad (23)$$

Since

$$\hat{q}'(\varphi)v = \frac{d}{dt} \hat{q}(\varphi)$$

we have

$$\frac{d}{dt} (Cv + \hat{q}(\varphi)) = 0.$$

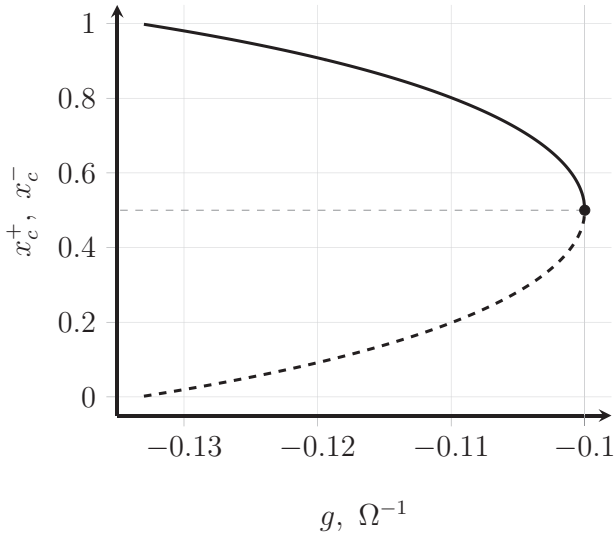


FIGURE 4 | Behavior of x_c^- (dashed line) and x_c^+ (solid line) as a function of the negative conductance g . Memristor parameters as in Figure 1.

Therefore, function

$$Q(v, \varphi) = Cv + \hat{q}(\varphi) \quad (24)$$

is constant along the trajectories of the circuit and represents an *invariant of motion* (or conserved quantity). As a consequence, the state-space (v, φ) of the circuit can be *foliated* in one-dimensional invariant manifolds

$$\mathcal{M}(Q_0) = \{(v, \varphi) \in \mathbb{R}^2 : Q(v, \varphi) = Cv + \hat{q}(\varphi) = Q_0\}$$

where

$$Q_0 = Cv(0) + \hat{q}(\varphi(0))$$

is a quantity depending upon the initial conditions for the state variables which is termed *manifold index*. Due to the foliation property, on each manifold $\mathcal{M}(Q_0)$ the circuit can be described by a *reduced-order* dynamic model, i.e., a first-order differential equation in the state variable φ

$$C\dot{\varphi} = -\hat{q}(\varphi) + Q_0. \quad (25)$$

These results are in agreement with those obtainable via FCAM developed in [38, 39] for the analysis of circuits with ideal memristors.

3.2 | Circuit with a Real VTEAM Memristor and a Capacitor

Consider now the circuit in Figure 5(b) where the memristor is a ReRAM device, satisfying the VTEAM DAE set, consisting of the ODE in Equation (4), with $F_{J,p}(\cdot)$ and $\gamma(\cdot)$ respectively defined via Equations (3), and (5), and of the state-dependent Ohm's law in Equation (6), with $\mathcal{G}(\cdot)$ expressed through Equation (7). The

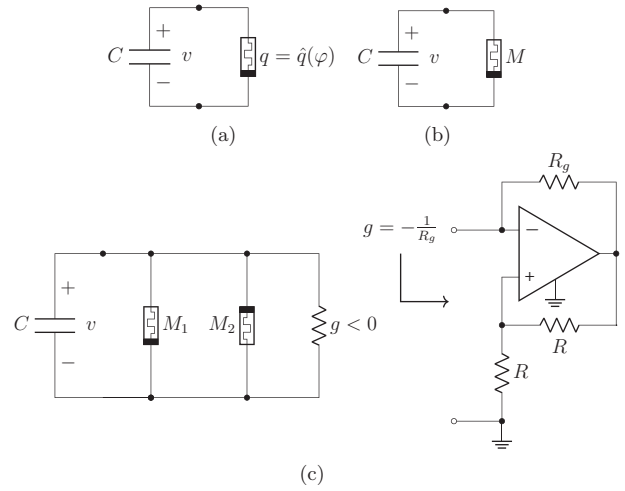


FIGURE 5 | (a) Simple circuit with a capacitor C and an ideal memristor with flux-charge characteristic $q = \hat{q}(\varphi)$; (b) circuit with a capacitor and a real VTEAM memristor M ; (c) circuit with a capacitor, two real VTEAM memristors M_1 and M_2 in anti-parallel and a negative (active) conductance g . This conductance, whose value is given by $g = -1/R_g$, is implemented via a typical negative impedance converter circuit (NIC) using an operational amplifier, operating in the linear region, and three positive (passive) resistances, two of value R and one of value R_g .

dynamics is described by the second-order SEs

$$\begin{cases} C\dot{v} = -\mathcal{G}(w)v \\ \dot{w} = \gamma(v)F_{J,p}(w) \end{cases} \quad (26)$$

We are interested to investigate if these SEs admit an invariant of motion. To proceed, we consider the *ansatz*

$$I(v, w) = a(v) + b(w), \quad (27)$$

where $a(\cdot)$ and $b(\cdot)$ are unknown functions, and we impose that the time derivative of $I(\cdot)$ along the trajectories of Equation (26) is zero. We obtain

$$\begin{aligned} \dot{I}(v, w) &= a'(v)\dot{v} + b'(w)\dot{w} \\ &= -\frac{1}{C}a'(v)\mathcal{G}(w)v + b'(w)\gamma(v)F_{J,p}(w) = 0 \end{aligned}$$

and separating variables

$$\frac{a'(v)v}{C\gamma(v)} = b'(w)\frac{F_{J,p}(w)}{\mathcal{G}(w)} = \lambda$$

where λ is a real constant. Therefore, we have

$$a(v) = \lambda C \int^v \frac{\gamma(\xi)}{\xi} d\xi$$

and

$$b(w) = \lambda \int^w \frac{\mathcal{G}(\xi)}{F_{J,p}(\xi)} d\xi.$$

Due to Equation (5), the integral in $a(\cdot)$ is well defined. The same holds for the integral defining $b(\cdot)$, since, due to the window

function $F_{J,p}(\cdot)$, the state variable w is constrained to evolve in the interval $(w_{\text{on}}, w_{\text{off}})$.

An invariant of motion is therefore given by

$$I(v, w) = \lambda C \int_0^v \frac{\gamma(\xi)}{\xi} d\xi + \lambda \int_{\frac{w_{\text{on}}+w_{\text{off}}}{2}}^w \frac{G(\xi)}{F_{J,p}(\xi)} d\xi. \quad (28)$$

Clearly, we have $I(v(t), w(t)) = I(v(0), w(0)) = I_0$, where

$$I_0 = \lambda C \int_0^{v(0)} \frac{\gamma(\xi)}{\xi} d\xi + \lambda \int_{\frac{w_{\text{on}}+w_{\text{off}}}{2}}^{w(0)} \frac{G(\xi)}{F_{J,p}(\xi)} d\xi \quad (29)$$

is a quantity depending upon the initial conditions $v(0)$ and $w(0)$ for the state variables.

Summing up, the considered circuit admits an invariant of motion and its dynamics evolves on the invariant manifolds

$$\mathcal{M}(I_0) = \{(v, w) \in \mathbb{R} \times (w_{\text{on}}, w_{\text{off}}) : I(v, w) = I_0\}$$

where the manifold index I_0 is given in Equation (29).

3.3 | Circuit with Two Real VTEAM Memristors in Anti-Parallel, a Capacitor and a Negative Conductance

Finally, we consider a circuit containing two VTEAM memristors in anti-parallel as in Section 2.3, a capacitor and a negative (active) conductance $g < 0$ (Figure 5c). The conductance can be obtained via a negative impedance converter (NIC) operating in the linear region [65] and its value amounts to $g = -1/R_g$, where $R_g > 0$ is a passive resistor.

The circuit dynamics is described by the second-order SEs

$$\begin{cases} C\dot{v} = -(G_{\text{eq}}(x) + g)v \\ \dot{x} = \frac{1}{w_{\text{off}}} \gamma(v) f_{J,1}(x) \end{cases} \quad (30)$$

where we let $x_1 = x$ in (17) and $G_{\text{eq}}(\cdot)$ is defined in Equation (18). Moreover, we considered Joglekar window with $p = 1$.

3.3.1 | Invariants of Motion

The analysis of invariants can be conducted as before provided we consider $G_{\text{eq}}(x) + g$ in place of $G(x)$. The circuit admits an invariant of motion and this can be expressed as

$$\begin{aligned} I(v, x) &= a(v) + b(x) \\ &= \frac{C}{w_{\text{off}}} \int_0^v \frac{\gamma(\xi)}{\xi} d\xi + \int_{\frac{1}{2}}^x \frac{G_{\text{eq}}(\xi) + g}{f_{J,1}(\xi)} d\xi. \end{aligned} \quad (31)$$

Note that, with some abuse of notation, we used for $I(\cdot, \cdot)$ and $b(\cdot)$ the same symbols as for the memristor circuit in Figure 5b (cf. Section 3.2). Moreover, we have chosen $\lambda = 1/w_{\text{off}}$ (see Equation (28)). We have $\dot{I}(v, x) = 0$ along the trajectories of

Equation (30), hence $I(v(t), x(t)) = I(v(0), x(0)) = I_0$, where

$$I_0 = \frac{C}{w_{\text{off}}} \int_0^{v(0)} \frac{\gamma(\xi)}{\xi} d\xi + \int_{\frac{1}{2}}^{x(0)} \frac{G_{\text{eq}}(\xi) + g}{f_{J,1}(\xi)} d\xi \quad (32)$$

is a quantity depending upon the initial conditions $v(0)$ and $x(0)$ for the state variables.

Due to these properties, the circuit dynamics evolves on the invariant manifolds

$$\mathcal{M}(I_0) = \{(v, x) \in \mathbb{R} \times (0, 1) : I(v, x) = I_0\} \quad (33)$$

where the manifold index I_0 is given in Equation (32).

It is possible to give an explicit expression of the invariants. First, let us normalize the conductance $G(\cdot)$ as follows

$$G(x) = \frac{1}{R_{\text{on}}} \frac{1}{1 + (\rho - 1)x} \quad (34)$$

where we let $\rho = R_{\text{off}}/R_{\text{on}}$. Also, as it was done in Section 2.3, choose $x_1(-\infty) = x_2(-\infty) = 1/2$, so that

$$G_{\text{eq}}(x) = G(x) + G(1 - x). \quad (35)$$

The two functions in the expression of $I(\cdot)$ can now be computed explicitly as

$$I(v, x) = a(v) + b(x) \quad (36)$$

where, considering Equation (8), we have

$$a(v) = \begin{cases} \frac{kC}{w_{\text{off}}} \bar{a}(v), & 0 < v_{\sigma} < v \\ 0, & -v_{\sigma} < v < v_{\sigma} \\ -\frac{kC}{w_{\text{off}}} \bar{a}(-v), & v < -v_{\sigma} < 0 \end{cases} \quad (37)$$

with

$$\bar{a}(v) = (-1)^{\alpha} \ln \left(\frac{v}{v_{\sigma}} \right) + \sum_{j=1}^{\alpha} \frac{(-1)^{\alpha-j}}{j} \left(\frac{v}{v_{\sigma}} - 1 \right)^j.$$

Moreover, considering Equation (9) with $p = 1$ and Equation (35), we obtain

$$\begin{aligned} b(x) &= \frac{1}{4R_{\text{off}}} \left[(\rho + 1) \ln \left(\frac{x}{1-x} \right) \right. \\ &\quad \left. - (\rho - 1) \ln \left(\frac{(\rho - 1)x + 1}{\rho - (\rho - 1)x} \right) \right] \\ &\quad + g \frac{1}{4} \ln \left(\frac{x}{1-x} \right). \end{aligned} \quad (38)$$

3.3.2 | Nonlinear Dynamics

Let us now analyze in detail the memristor circuit dynamics. Consider again the conductance $G_{\text{eq}}(\cdot)$ of the two memristors

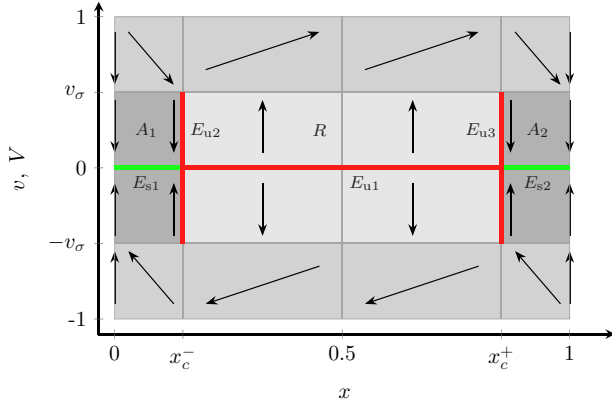


FIGURE 6 | The arrows indicate the qualitative behavior of the vector field defining the SEs in Equation (30). Trajectories move to the right in the supra-threshold region $v > v_\sigma$, while they move to the left in the region $v < -v_\sigma$. Trajectories move vertically in the subthreshold region $|v| \leq v_\sigma$. The sets of stable EPs E_{s1} and E_{s2} are indicated in green, while the sets E_{u1} , E_{u2} and E_{u3} of unstable EPs are in red. Also indicated are the subthreshold regions A_1 and A_2 attracted to E_{s1} and E_{s2} , respectively, and the repelling region R .

in antiparallel. Suppose as a design choice that the active conductance g satisfies (cf. Equations (21) and (22))

$$-G_{\text{eqM}} < g < -G_{\text{eqm}} < 0 \quad (39)$$

which can be equivalently rewritten using Equation (34) as

$$-\frac{1}{R_{\text{on}}} \left(1 + \frac{1}{\rho} \right) < g < -\frac{4}{R_{\text{on}}(1 + \rho)} < 0. \quad (40)$$

Under this condition, there exist $x_c^- \in (0, 1/2)$ and $x_c^+ \in (1/2, 1)$ such that $G_{\text{eq}}(x_c^-) + g = G_{\text{eq}}(x_c^+) + g = 0$. These are expressed as

$$x_c^- = \frac{1}{2} - \Delta, \quad x_c^+ = \frac{1}{2} + \Delta \quad (41)$$

where

$$\Delta = \sqrt{\frac{1}{4} + \frac{\rho}{(\rho - 1)^2} + \frac{\rho + 1}{gR_{\text{on}}(\rho - 1)^2}}.$$

We have a passive conductance $g + G_{\text{eq}}(x) \geq 0$ if $x \in [0, x_c^-] \cup [x_c^+, 1]$, where equality applies at x_c^- and x_c^+ , while there is an active conductance $g + G_{\text{eq}}(x) < 0$ if $x \in (x_c^-, x_c^+)$ (Figure 1c). Figure 4 shows the behavior of x_c^- and x_c^+ , which are symmetric about $1/2$, as a function of g .

Some basic dynamical properties of the SEs in Equation (30) can be highlighted as follows. Denote by $S = \{(v, x) : v \in \mathbb{R}, x \in [0, 1]\}$ the state space where the dynamics evolves. The equilibrium points (EPs) are given by $E = E_{s1} \cup E_{s2} \cup E_{u1} \cup E_{u2} \cup E_{u3}$ where $E_{s1} = \{(x, v) \in S : x \in [0, x_c^-], v = 0\}$, $E_{s2} = \{(x, v) \in S : x \in (x_c^+, 1], v = 0\}$, $E_{u1} = \{(x, v) \in S : x \in (x_c^-, x_c^+), v = 0\}$, $E_{u2} = \{(x, v) \in S : x = x_c^-, v \in [-v_\sigma, v_\sigma]\}$ and $E_{u3} = \{(x, v) \in S : x = x_c^+, v \in [-v_\sigma, v_\sigma]\}$ (Figure 6). It can be easily checked that any EP in $E_{s1} \cup E_{s2}$ is stable, while any EP in $E_{u1} \cup E_{u2} \cup E_{u3}$ is unstable. It is worth to notice that at any stable EP the capacitor voltage vanishes.

If the initial conditions are such that $x(0) = 0, v(0) \neq 0$, then there is convergence to the stable EP $(\bar{v}, \bar{x}) = (0, 0)$ along the straight line $x = 0$. Similarly, if $x(0) = 1, v(0) \neq 0$, we have convergence to the stable EP $(\bar{v}, \bar{x}) = (0, 1)$ along the straight line $x = 1$. Due to the uniqueness of solutions, this implies that if $x(0) \in (0, 1)$, then we have $x(t) \in (0, 1)$ for any $t \geq 0$. Therefore, it suffices to study the dynamics in the open region $S^0 = \{(v, x) : v \in \mathbb{R}, x \in (0, 1)\}$.

A simple but useful observation is as follows. Since the x axis is made of EPs, due to the uniqueness of solutions, it cannot be crossed by trajectories. Therefore, $v(0) \geq 0$ implies $v(t) \geq 0$ for any $t \geq 0$. Since $\dot{x} = (1/w_{\text{off}})\gamma(v)f_{J,1}(x) \geq 0$ for $v \geq 0$, it follows that $x(\cdot)$ is a monotone non-decreasing function. Therefore, $x(\cdot)$ is convergent, i.e., $x(t) \rightarrow x(\infty) \in [0, x_c^-] \cup (x_c^+, 1]$ as $t \rightarrow +\infty$. An analogous argument shows that $v(0) \leq 0$ implies $v(t) \leq 0$ for any $t \geq 0$, moreover $x(\cdot)$ is a monotone non-increasing function. Therefore, $x(\cdot)$ is convergent, i.e., $x(t) \rightarrow x(\infty) \in [0, x_c^-] \cup (x_c^+, 1]$ as $t \rightarrow +\infty$. Due to these properties, trajectories move to the right (or vertically) in the upper set $v \geq 0$, while they move to the left (or vertically) in the lower set $v \leq 0$. Figure 6 shows the behavior of the vector field defining the SEs in Equation (30) according to these properties.

The dynamic behavior in the subthreshold region $\{(v, x) \in S^0 : |v| \leq v_\sigma, x \in (0, 1)\}$ is easy to analyze. Indeed, in this region we have $\gamma(v) = 0$, hence $\dot{x} = 0$ and so x is constant. On the other hand, since $C\dot{v} = -(G_{\text{eq}}(x) + g)v$ with x constant, then v satisfies a simple first-order differential equation. It is seen that there are two regions, $A_1 = \{(v, x) : |v| \leq v_\sigma, 0 < x < x_c^-\}$ and $A_2 = \{(v, x) : |v| \leq v_\sigma, x_c^+ < x < 1\}$ that are attracted by the stable EPs and a repelling region $R = \{(v, x) : |v| \leq v_\sigma, x_c^- < x < x_c^+\}$ (Figure 6). If $(v(0), x(0)) \in A_1, v(0) \neq 0$, the trajectory moves vertically and it converges exponentially to the stable EP $(0, x(0)) \in E_{s1}$. An analogous situation occurs in A_2 . If $(v(0), x(0)) \in R, v(0) \neq 0$, the trajectory moves vertically and it escapes in finite time from R . The behavior of the vector field in the subthreshold region is shown in Figure 6.

Further useful information on the phase portrait can be obtained by analyzing the geometry of the invariant manifolds in Equation (30), which actually represent parameterizations of the trajectories. It is easier to study the geometry of the invariant manifolds if we let in a first phase $v_\sigma = 0$, i.e., we assume the memristor model does not possess a subthreshold region. In a second phase, we will re-introduce also the dynamics in the sub-threshold region. In the first phase, consider in system in Equation (8), in place of $\gamma(\cdot)$, the following function

$$\tilde{\gamma}(v) \doteq \begin{cases} \gamma(v + v_\sigma) = k\left(\frac{v}{v_\sigma}\right)^\alpha, & 0 < v \\ 0, & v = 0 \\ \gamma(v - v_\sigma) = -k\left(\frac{v}{-v_\sigma}\right)^\alpha, & v < 0. \end{cases} \quad (42)$$

Accordingly, we replace $a(v)$ with $A(v) = (C/w_{\text{off}}) \int_0^v \tilde{\gamma}(\xi)/\xi d\xi$, which is given by (cf. Equation (37))

$$A(v) = \begin{cases} \frac{kC}{w_{\text{off}}} \tilde{A}(v), & 0 < v \\ 0, & v = 0 \\ -\frac{kC}{w_{\text{off}}} \tilde{A}(-v), & v < 0 \end{cases} \quad (43)$$

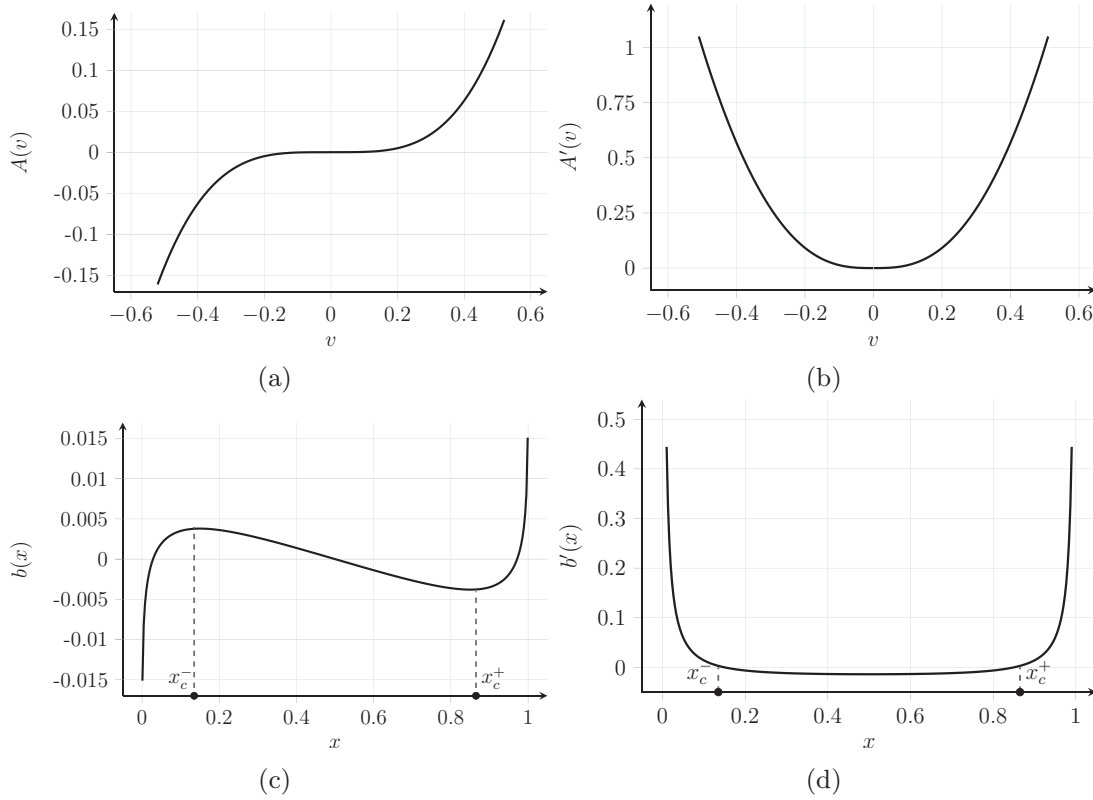


FIGURE 7 | Behavior of relevant functions involved in the dynamical analysis of the SEs in Equation (30). (a) Function $A(\cdot)$ and (b) its derivative $A'(\cdot)$. Note that $A(\cdot)$ is a strictly increasing and globally invertible function. (c) Function $b(\cdot)$ and (d) its derivative $b'(\cdot)$. Function $b(\cdot)$ has a local maximum at x_c^- and a local minimum at x_c^+ . Memristor parameters are as in Figure 1.

where we let

$$\tilde{A}(v) = (-1)^\alpha \ln \left(\frac{v + v_\sigma}{v_\sigma} \right) + \sum_{j=1}^{\alpha} \frac{(-1)^{\alpha-j}}{j} \left(\frac{v}{v_\sigma} \right)^j.$$

The relevant functions $A(\cdot)$ and $b(\cdot)$ involved in the dynamics have the following main properties. We have $A'(v) = C\tilde{\gamma}(v)/(w_{\text{off}}v) > 0$ if $v \neq 0$ and $A'(0) = kC/(v_\sigma w_{\text{off}}) > 0$ if $\alpha = 1$, while $A'(0) = 0$ if $\alpha > 1$. Then, $A(\cdot)$ is a strictly increasing, globally invertible function (Figure 7). Moreover, $b'(x) = (G_{\text{eq}}(x) + g)/f_{J,1}(x)$ is positive in the intervals $(0, x_c^-)$ and $(x_c^+, 1)$, while it assumes negative values in (x_c^-, x_c^+) . Hence, $b(\cdot)$ is increasing in the first two intervals, while it decreases in the third interval (Figure 7).

Consider now the set

$$\hat{\mathcal{M}}(I_0) = \{(v, x) \in S^0 : A(v) + b(x) = I_0\}. \quad (44)$$

It is easy to see that, given any $I_0 \in \mathbb{R}$, then $\hat{\mathcal{M}}(I_0) \neq \emptyset$. Since $A(\cdot)$ is globally invertible, we can write v as a function of x , i.e.,

$$v = \hat{v}(x) = A^{-1}[I_0 - b(x)].$$

It can be verified that $v \rightarrow +\infty$ as $x \rightarrow 0^+$, while $v \rightarrow -\infty$ as $x \rightarrow 1^-$. We have

$$\hat{v}'(x) = \frac{1}{A'(A(v))}(-b'(x)) = \frac{1}{A'(I_0 - b(x))}(-b'(x))$$

for any $A(v) \neq 0$, i.e., $v \neq 0$. Suppose first that $x \in (0, x_c^-)$. Since $A'(v) > 0$ if $v \neq 0$ and $b'(x) > 0$, we obtain $\hat{v}'(x) < 0$, i.e., v is a strictly decreasing function of x . The same holds if $x \in (x_c^+, 1)$. If, instead, $x_c^- < x < x_c^+$, then $A'(v) > 0$ if $v \neq 0$ and $b'(x) < 0$, hence $\hat{v}'(x) > 0$, i.e., v is a strictly increasing function of x . Function $\hat{v}(\cdot)$ has a local minimum at x_c^- , depending upon I_0 , which is given by

$$\hat{v}_m(I_0) = \hat{v}(x_c^-) = A^{-1} \left[I_0 - \int_{\frac{1}{2}}^{x_c^-} \frac{G_{\text{eq}}(\xi) + g}{f_{J,1}(\xi)} d\xi \right]$$

and it has a local maximum at x_c^+ , also depending upon I_0 , given by

$$\hat{v}_M(I_0) = \hat{v}(x_c^+) = A^{-1} \left[I_0 - \int_{\frac{1}{2}}^{x_c^+} \frac{G_{\text{eq}}(\xi) + g}{f_{J,1}(\xi)} d\xi \right]$$

and $\hat{v}_M(I_0) > \hat{v}_m(I_0)$.

Now, we discuss the main cases we can observe for set $\hat{\mathcal{M}}(I_0)$ when we vary I_0 . Define the critical manifold index

$$I_{0c} = b(x_c^-) = \int_{\frac{1}{2}}^{x_c^-} \frac{G_{\text{eq}}(\xi) + g}{f_{J,1}(\xi)} d\xi \quad (45)$$

where $b(x_c^-)$ can be explicitly found using Equation (38) for $x_1(-\infty) = x_2(-\infty) = 1/2$ (recall that we let $x = x_1$).

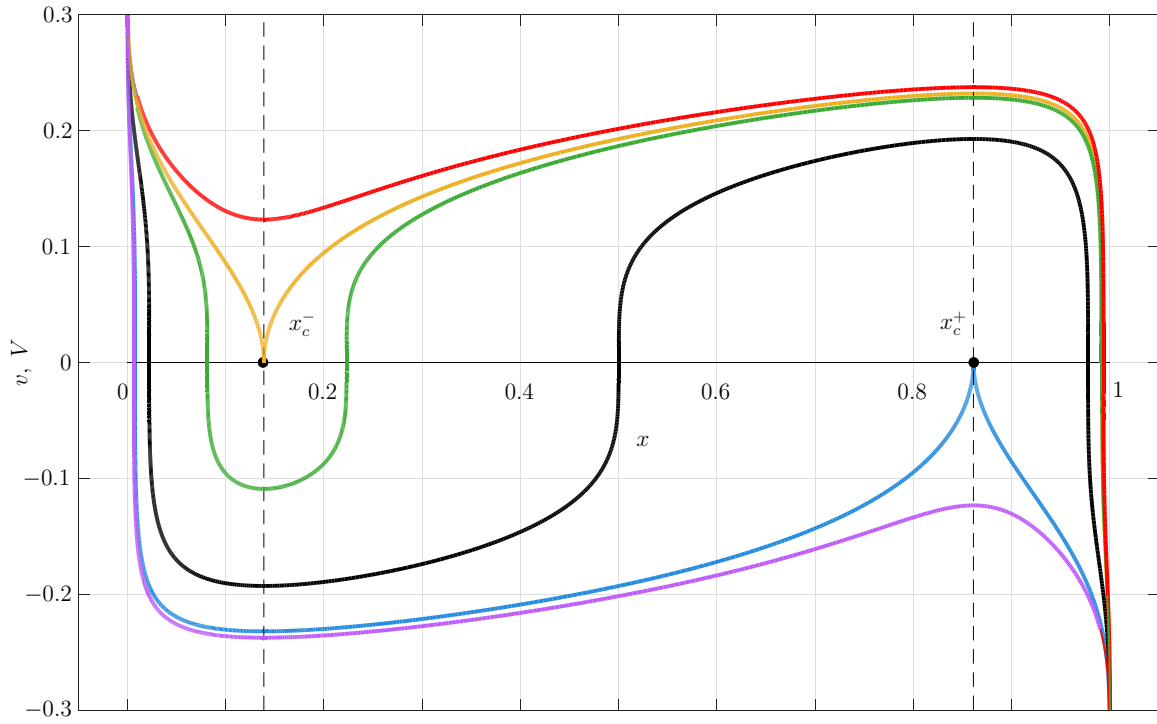


FIGURE 8 | Sets $\hat{\mathcal{M}}(I_0)$ as in Equation (44) obtained by varying I_0 in Equation (30) with $\tilde{\gamma}(\cdot)$ in place of $\gamma(\cdot)$. When $I_0 > I_{0c}$, $\hat{\mathcal{M}}(I_0)$ intersects the x -axis only once at a point such that the x -variable belongs to the interval $(x_c^+, 1)$ (red curve), whereas, when $I_0 < -I_{0c}$, $\hat{\mathcal{M}}(I_0)$ intersects the x -axis only once at a point in $(0, x_c^-)$ (violet curve). When $|I_0| < I_{0c}$, $\hat{\mathcal{M}}(I_0)$ intersects the x -axis at a point in $(0, x_c^-)$, at a point in $(x_c^+, 1)$ and at a point in (x_c^-, x_c^+) (green or black curve). In particular, the black curve corresponds to $\hat{\mathcal{M}}(0)$. The orange and blue curves correspond to the special cases $I_0 = I_{0c}$ and $I_0 = -I_{0c}$. Parameter values are as in Figure 1. In this figure we assume the VTEAM memristor does not have a sub-threshold region, i.e., we have $v_\sigma = 0$. In the next Figure 9 we assume $v_\sigma > 0$ and consider the behavior of the memristor model both in the sub-threshold region $|v| \leq v_\sigma$ and in the supra-threshold region $|v| > v_\sigma$.

- Case 1. If $I_0 > I_{0c}$, then we have $\hat{v}_m(I_0) > 0$ and the set $\hat{\mathcal{M}}(I_0)$ intersects the x -axis only once at a point such that the variable x is in $(x_c^+, 1)$ (red curve in Figure 8). Note that $\hat{v}_m(I_0) > 0$ is made up of two parts, one containing points (v, x) with $v \geq 0$ and one with points (v, x) with $v < 0$.
- Case 1'. For symmetry reasons, if $I_0 < -I_{0c}$, then we have $\hat{v}_m(I_0) < 0$ and the set $\hat{\mathcal{M}}(I_0)$ intersects the x -axis only once at a point in $(0, x_c^-)$ (violet curve in Figure 8).
- Case 2. If $|I_0| < I_{0c}$, then have $\hat{v}_m(I_0) < 0 < \hat{v}_m(I_{0c})$. Now, $\hat{\mathcal{M}}(I_0)$ intersects the x -axis at a point in $(0, x_c^-)$, at a point in $(x_c^+, 1)$ and at a point in (x_c^-, x_c^+) (green curve in Figure 8).
- Case 3. If $I_0 = I_{0c}$, then $\hat{v}_m(I_{0c}) = 0$ and the set $\hat{\mathcal{M}}(I_{0c})$ intersects the x -axis at two points, namely, x_c^- and a point in $(x_c^+, 1)$. Actually, $\hat{\mathcal{M}}(I_{0c})$ is a *separatrix* between Case 1 and Case 2 (orange curve in Figure 8).
- Case 3'. For symmetry reasons, if $I_0 = -I_{0c}$, then $\hat{v}_m(-I_{0c}) = 0$ and the set $\hat{\mathcal{M}}(-I_{0c})$ is a *separatrix* between Case 1' and Case 2 (blue curve in Figure 8).

Now, reconsider the original SEs in Equation (30). The global behavior of the manifolds $\mathcal{M}(I_0)$, when also the subthreshold region is considered, i.e., when we let $v_\sigma > 0$, is as shown in

Figure 8. Each manifold $\mathcal{M}(I_0)$ can be easily obtained by means of the set $\hat{\mathcal{M}}(I_0)$ previously defined. Recall the definition in Equation (33) of $\mathcal{M}(I_0)$ and note from Equation (37) that $a(v) = 0$ when $|v| \leq v_\sigma$. Then, we can proceed as follows. Suppose for instance that $I_0 > I_{0c}$ and that $(0, \bar{x}(I_0)) \in E_{s2}$ is the only intersection of set $\hat{\mathcal{M}}(I_0)$ with the x -axis. Then, $\mathcal{M}(I_0)$ is given by $\hat{\mathcal{M}}(I_0)$ and the vertical segment $\{(v, x) : |v| \leq v_\sigma, x = \bar{x}(I_0)\}$. If $|I_0| < I_{0c}$, then $\hat{\mathcal{M}}(I_0)$ intersects the x -axis at three points, one in E_{s1} , one in E_{u1} and one in E_{s2} . Then, $\mathcal{M}(I_0)$ is obtained by adding to $\hat{\mathcal{M}}(I_0)$ three vertical segments with $|v| \leq v_\sigma$, in correspondence with these intersections, in a way similar to the previous case. This rule holds for any manifold. It is worth to note that, in the case of the separatrix $I_0 = I_{0c}$ the vertical segment for $x = x_c^-$ coincides with the set of unstable EPs E_{u2} .

Now, recall that the trajectories in the subthreshold region move vertically. On this basis, Figure 9 also shows via arrows the direction of motion along the manifolds as obtained in the preceding analysis.

3.3.3 | Coexistence of Monostable and Bistable Behaviors

On the basis of the previous dynamic analysis, we conclude the following. For any manifold index $I_0 > I_{0c}$, we have a *monostable*

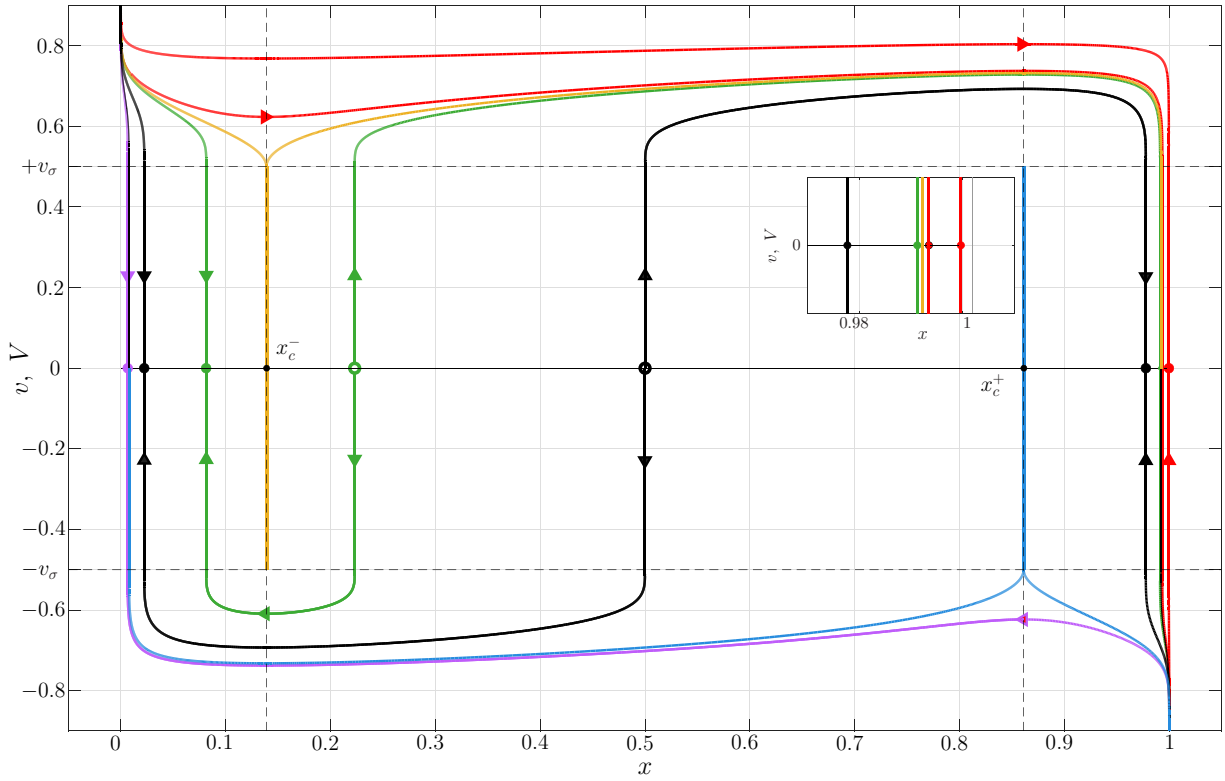


FIGURE 9 | Invariant manifolds $\mathcal{M}(I_0)$ of the SEs in Equation (30) obtained by varying the manifold index I_0 for the memristor circuit in Example 1. The arrows highlight the direction of motion along the manifolds. When $I_0 > I_{0c} = 0.0042$, we have a monostable behavior on $\mathcal{M}(I_0)$, i.e., any trajectory starting on $\mathcal{M}(I_0)$ converges to the same stable EP in E_{s2} , see for instance the lower red curve with index $I_0 = 0.005$ or the upper red curve with index $I_0 = 0.019$. Similarly, when $I_0 < -I_{0c} = -0.0042$, we have a monostable behavior on $\mathcal{M}(I_0)$, i.e., any trajectory starting on $\mathcal{M}(I_0)$ converges to the same stable EP in E_{s1} , see for instance the violet curve with index $I_0 = -0.005$. When $|I_0| < I_{0c}$, we have a bistable behavior on $\mathcal{M}(I_0)$, i.e., depending upon the initial conditions, a trajectory starting on $\mathcal{M}(I_0)$ converges either to a stable EP in E_{s2} or to a stable EP in E_{s1} . See, for instance, the green curve with index $I_0 = 0.0038$ or the black curve with index $I_0 = 0$. The upper orange curve with index $I_0 = 0.0042$ and the lower blue curve with index $I_0 = -0.0042$ are separatrices between a monostable and a bistable behavior. The inset shows a magnification of the invariant manifolds close to $v = 0$ and $x = 1$.

behavior on $\mathcal{M}(I_0)$, namely, each trajectory starting on $\mathcal{M}(I_0)$ converges as $t \rightarrow +\infty$ to the same stable EP in the set E_{s2} (e.g., the lower red curve in Figure 9). On the other hand, if we choose a different manifold, i.e., $I'_{01} > I_{0c}$, $I'_{01} \neq I_0$, then we have convergence to a different stable EP in E_{s2} (upper red curve in Figure 9). Similarly, if $I_0 < -I_{0c}$, we have once more a monostable behavior on $\mathcal{M}(I_0)$, namely, each trajectory starting on $\mathcal{M}(I_0)$ converges as $t \rightarrow +\infty$ to the same stable EP in the set E_{s1} (violet curve in Figure 9). In any case the capacitor voltage tends to 0 at a stable EP.

For any manifold index I_0 such that $|I_0| < I_{0c}$, we have a *bistable* behavior on $\mathcal{M}(I_0)$, namely, *depending on the initial condition* on $\mathcal{M}(I_0)$, the trajectory converges either to a stable EP in E_{s1} or to a stable EP in E_{s2} (green or black curve in Figure 9). Different manifolds correspond to different pairs of stable EPs. Once more, the capacitor voltage tends to 0 at the two locally stable EPs.

In Figure 10 we have synthesized the global dynamic behavior of the memristor circuit in the phase-space (v, x) . In the red region $\mathfrak{M}_1$ the circuit displays a monostable behavior with convergence of solutions to stable EPs in E_{s2} . For symmetry reasons, in the violet region $\mathfrak{M}_2$ there is a monostable behavior with convergence of solutions to stable EPs in E_{s1} . In the green

region $\mathfrak{B}_1 \cup \mathfrak{B}_2$ we have a bistable behavior. It is seen that that the set $\mathfrak{S}_{MB1} \doteq \mathcal{M}(I_{0c}) \setminus E_{u2}$ (orange curve in Figure 10) and $\mathfrak{S}_{MB2} \doteq \mathcal{M}(-I_{0c}) \setminus E_{u3}$ (blue curve in Figure 10) are separatrices between a monostable and a bistable behavior. Let us examine in more detail the region of bistable behavior. Consider the set of unstable EPs $E_u \doteq E_{u1} \cup E_{u2}^+ \cup E_{u3}^-$, where $E_{u2}^+ = \{(x, v) \in S : x = x_c^-, v \in [0, v_\sigma]\}$ and $E_{u3}^- = \{(x, v) \in S : x = x_c^+, v \in [-v_\sigma, 0]\}$ (three-segment black curve in Figure 10). This set is a separatrix between solutions within the bistability region that converge to stable EPs in E_{s2} (dark green region $\mathfrak{B}_1$), and those that converge to stable EPs in E_{s1} (light green region $\mathfrak{B}_2$). Note that we neglected the two sets of unstable EPs $E_{u2} \setminus E_{u2}^+$ and $E_{u3} \setminus E_{u3}^-$ whose measure is zero.

Remark 1. The memristor circuit in Figure 5c is in principle a third-order circuit, since there are two state variables x_1 and x_2 for the memristors in anti-parallel and one state variable v for the capacitor. As we discussed in Section 2.3, once we fixed the state variables for memristors when they were first connected, then we obtain a second-order circuit described by the state variables $x = x_1$ and v (cf. (30)). The main result that follows from the existence of an invariant of motion is that there is a further reduction of order. Indeed, on any manifold the circuit dynamics can be analyzed as that of a first-order circuit. To better

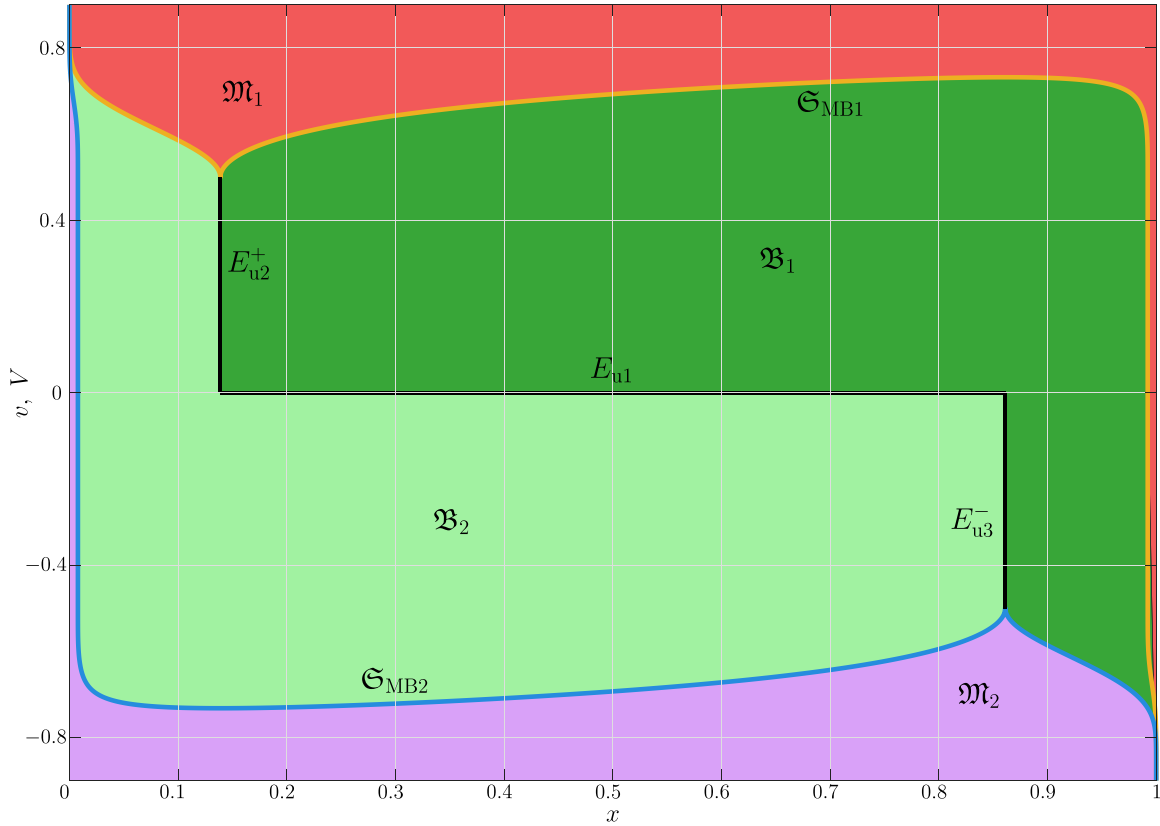


FIGURE 10 | Global dynamic behavior in the phase space of the memristor circuit. $\mathfrak{M}_1$ (respective, $\mathfrak{M}_2$) is the region where there is a monostable behavior with solutions that converge to a stable EP in E_{s2} (respective, E_{s1}). In the region $\mathfrak{B}_1 \cup \mathfrak{B}_2$ we have a bistable behavior. The orange curve $\mathfrak{S}_{MB1} \doteq \mathcal{M}(I_{0c}) \setminus E_{u2}$ and the blue curve $\mathfrak{S}_{MB2} \doteq \mathcal{M}(-I_{0c}) \setminus E_{u3}$ are separatrices between a monostable and a bistable behavior. The set $E_u \doteq E_{u1} \cup E_{u2}^+ \cup E_{u3}^-$, i.e., the three-segment black curve, is a separatrix, within the bistability region, between solutions that converge to stable EPs in E_{s2} and solutions that converge to stable EPs in E_{s1} . We neglected the two sets of unstable EPs $E_{u2} \setminus E_{u2}^+$ and $E_3 \setminus E_{u3}^-$ whose measure is zero.

see this, note that for $|v| > v_\sigma$ the function $a(\cdot)$ is invertible and we have $v = a^{-1}(I_0 - b(x))$. Hence, from (30), the circuit satisfies the first-order SE

$$\dot{x} = \frac{1}{w_{\text{off}}} \gamma(a^{-1}(I_0 - b(x))) f_{J,1}(x).$$

On the other hand, when $|v| < v_\sigma$ we have a simple linear first-order dynamics, i.e.,

$$\dot{v} = -\frac{(G_{\text{eq}}(\bar{x}) + g)v}{C}$$

where $\bar{x}$ is fixed. Overall, the behavior is analogous to that of a circuit with an ideal memristor (cf. Equation (25)).

Remark 2. Given any manifold index I_0 , the EPs on $\mathcal{M}(I_0)$ that belong to the x -axis can be simply obtained by noting that $v = 0$ implies $a(v) = 0$ (cf. Equation (37)), hence at any EP $(0, \bar{x})$ we have $b(\bar{x}) = I_0$. From Figure 7c we can therefore derive the x -component of the EPs as a function of I_0 (Figure 11). In accordance with the previous analysis, we have only one stable EP in $\mathcal{M}(I_0)$ when $|I_0| > I_{0c}$, while there are three EPs on $\mathcal{M}(I_0)$, two of which are stable and one is unstable, when $|I_0| < I_{0c}$. We observe that the manifold index I_0 can be changed by varying the initial conditions $v(0)$ and $x(0)$ for the state variables

of the memristor circuit in Figure 5c while keeping the circuit parameters constant. This implies that the SEs describing the circuit undergoes a *saddle-node bifurcation without parameters* with the appearance (respective, disappearance) of two EPs when the manifold index increases and reaches the critical value $-I_{0c}$ (respective, I_{0c}) [66].

Remark 3. The analysis in Section 3.3.2 has shown that, for the circuit in Figure 5c there is coexistence of infinitely many monostable behaviors. This is because trajectories that start on different manifolds in Case 1 or Case 1' tend to different stable EPs. Each manifold in fact intersects the x axis at a different stable EP (cf. Figure 11). Similarly, in Case 2 there is coexistence of infinitely many different bistable behaviors. Indeed, each manifold intersects the x axis at a different pair of stable EPs. Overall, there are infinitely many different attractors for the considered dynamics, a peculiar dynamical property termed *extreme multistability*. It is worth to stress that this is true for a fixed set of circuit parameters including the VTEAM memristor model coefficients. These considerations also highlight the importance of the choice of the initial conditions $(v(0), x(0))$ in memristor circuits. In fact, the considered circuits display a different dynamical behavior and attractor set, depending on the choice of the invariant manifold index, i.e., on the choice of the initial conditions.

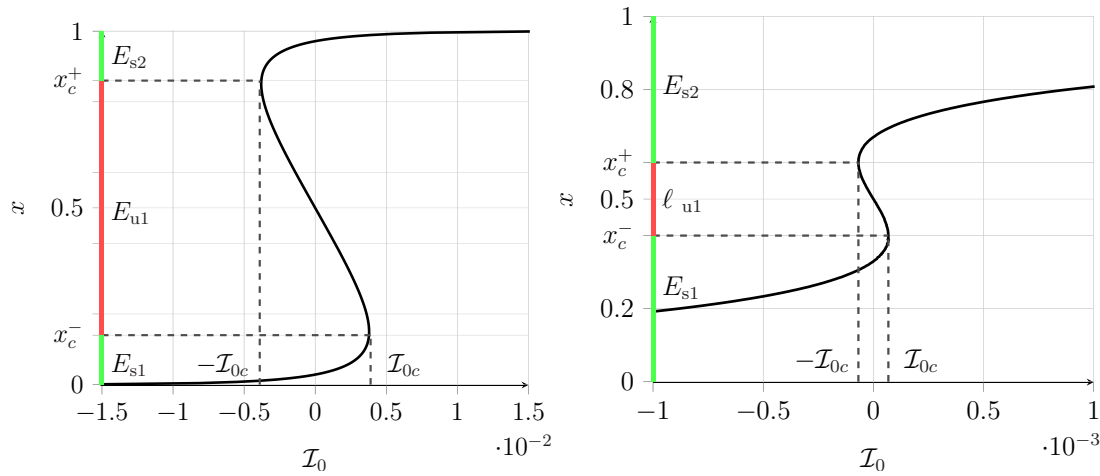


FIGURE 11 | Horizontal axis: manifold index I_0 and critical index I_{0c} . Vertical axis: number and location (x -component) of EPs on manifold $\mathcal{M}(I_0)$ as a function of I_0 for the memristor circuits in Figure 5c. Points x_c^- and x_c^+ are such that $[0, x_c^-)$ (respectively, $(x_c^+, 1]$) are the x -components of the stable EPs in E_{s1} (respectively, E_{s2}), while (x_c^-, x_c^+) are the x -components of the unstable EPs in E_{u1} . There is only one stable EP in $\mathcal{M}(I_0)$ when $|I_0| > I_{0c}$. On the other hand, when $|I_0| < I_{0c}$, there are three EPs on $\mathcal{M}(I_0)$, more precisely, two stable EPs and one unstable EP. The SEs describing the memristor circuit undergoes two saddle-node bifurcations without parameters when the manifold index reaches the critical values $\pm I_{0c}$. (a) Parameter values as in Figure 1. Active conductance $g = -0.114 \Omega^{-1}$ (cf. Example 1). We have $I_{0c} = 0.0042$, $x_c^- = 0.14$ and $x_c^+ = 0.86$. (b) Parameters as in (a), with the only difference that $g = -0.101 \Omega^{-1}$ (cf. Example 2). We have $I_{0c} = 6.8 \cdot 10^{-5}$, $x_c^- = 0.4$ and $x_c^+ = 0.6$.

Remark 4. We have seen that the x -component of the stable EPs of the circuit belongs either to the interval $[0, x_c^-)$ or the interval $(x_c^+, 1]$. Recall that x_c^- and x_c^+ are symmetric about $1/2$ and their value can be adjusted by suitably selecting the active conductance g (cf. Equation (41) and Figure 4). If we choose x_c^- close to 0, hence x_c^+ is close to 1, then any trajectory converges to an x -value close either to 0 or to 1. One such circuit is useful to process signal by providing, both in the monostable and bistable mode, binary, i.e., black or white x -values. This will be further illustrated in Example 1 in Section 3.3.4. On the other hand, if x_c^- is close to $1/2$, hence x_c^+ is close to $1/2$, then the stable EPs include also non-binary (i.e., grey scale) x -values. See Example 2 in Section 3.3.4.

Remark 5. One of the hottest topics in the research on memristors is to exploit their unconventional dynamical properties to develop novel architectures with respect to the state-of-the-art in electronic circuits. One basic property, simple memristor circuits exhibit, at steady state, is quiescent or oscillatory *bistability* [67].

The availability of a novel memristive circuit, as proposed in this manuscript (see Figure 5c), which may be toggled between quiescent monostability and bistability by exploiting the invariants of motion it admits, is of great interest for the design of innovative sensing-and-mem-computing building blocks for bio-inspired CNNs [68]–[69] targeting time-critical resource-constrained edge computing applications. In these applications these arrays of locally-coupled continuous-time dynamical systems are employed to process data, acquired from close-by sensors, in real time. In state-of-the-art hardware realisations [70], the results of the data processing operation of a CNN are read from the EPs its cells converge to. Also, each cell typically admits a single dynamical variable – the voltage across a capacitor – which evolves continuously over time while obeying a task-dependent set of rules involving also the neighbors. Importantly, each of the rule specifies the value of the so-called *offset current*,

which flows into the cell capacitor, and crucially determines the monostable or bistable nature of the cell quiescent steady-state [71]. Employing the second-order circuit of Figure 5c as core processing unit for a novel CNN design, leveraging disruptive ReRAM nanotechnologies, *the role of the offset current for the traditional cell would be assumed by the index of the invariant manifold for the memristive counterpart* [42]. Remarkably, unlike what is the case in state-of-the-art CNN cells, the capacitor voltage vanishes at any of the admissible EPs for the circuit of Figure 5c, which is beneficial from an energy consumption perspective. The output of each cell of the novel CNN could be defined simply as the asymptotic value of the corresponding memristive one-port state, which allows, if desired, to store the outcome of an equilibria-based computation in memory. Moreover, choosing the design parameters for the circuit of Figure 5c in such a way to push x_c^- and x_c^+ as farther apart from (as close to) each other as possible, as discussed in Remark 4, the resulting nonlinear dynamic array would deliver results of data processing operations in binary (analogue) format, along the lines discussed in ([72, 73]).

3.3.4 | Illustrative Examples

Example 1 To illustrate the dynamics of the circuit composed of the parallel combination of two VTEAM memristors in anti-parallel, a capacitor and a negative conductance (Figure 5c), suppose to choose the following parameters. For the VTEAM model we let $R_{on} = 10 \Omega$, $R_{off} = 30 \Omega$, $w_{on} = 0$ and $w_{off} = 3 \text{ nm}$. For the odd function $\gamma(\cdot)$ in Equation (8) we choose $k = 3 \times 10^{-3} \text{ m/s}$, $\alpha = 3$ and the threshold voltage v_σ is set to 0.5 V . Finally, we set $C = 1 \mu\text{F}$, while $g = -0.114 \Omega^{-1}$ is such that condition in Equation (39) is satisfied. Figure 1a shows the behavior of $\mathcal{G}(w)$ in Equation (7), while Figure 1b,c, respectively, show $G_{eq}(x)$ in Equation (35) and $G_{eq}(x) + g$ as a function of $x = w/w_{off}$, for $p = 1$, $x_1(-\infty) = x_2(-\infty) = 1/2$ and $x = x_1$. The

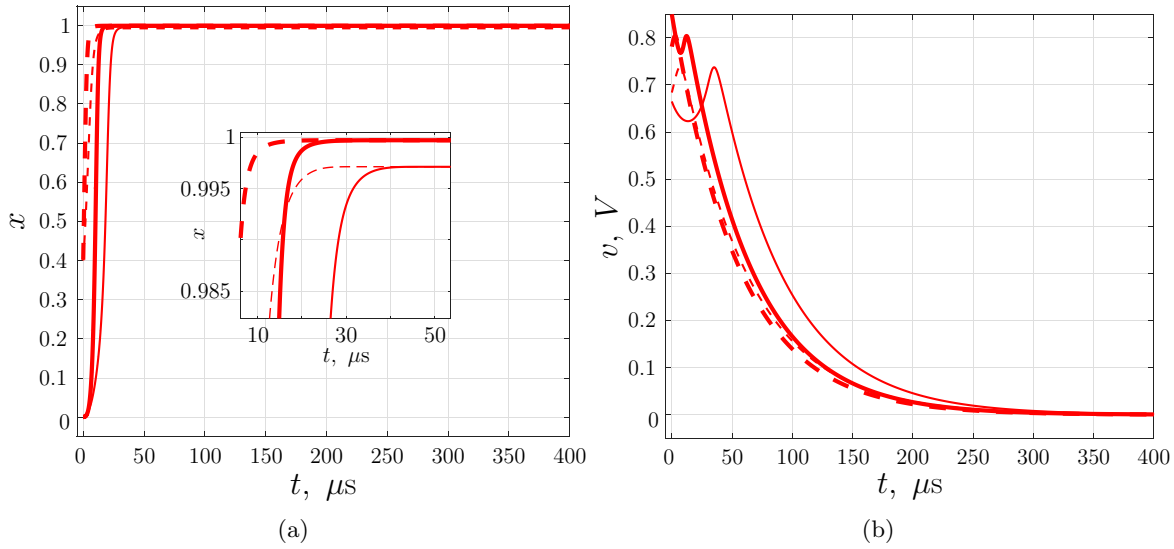


FIGURE 12 | Simulation results for the memristor circuit in Example 1. (a) Time-domain behavior of the state variable x for the SEs in Equation (30). Solid thick red curve (respective, dashed thick red curve): trajectory starting at the initial conditions $(v_a(0), x_a(0)) = (0.665, 0.001)$ and $(v_b(0), x_b(0)) = (0.684, 0.4)$ on manifold $\mathcal{M}(I_{01})$ with $I_{01} = 0.042$. Solid thin red curve (respective, dashed thin red curve): trajectory starting at the initial conditions $(v_c(0), x_c(0)) = (0.85, 0.001)$ and $(v_d(0), x_d(0)) = (0.781, 0.4)$ on manifold $\mathcal{M}(I_{02})$ with $I_{02} = 0.005$. Inset: magnification of the time-domain behavior in the interval $[10, 50]$ μs . (b) Corresponding time-domain behavior of the state variable v for the SEs in Equation (30).

conductance $G_{\text{eq}}(x) + g$ vanishes at points $x_c^- = 0.14$ and $x_c^+ = 0.86$ and assumes negative (active) values in the interval (x_c^-, x_c^+) .

The critical manifold index separating a monostable behavior from a bistable one is given by $I_{0c} = b(x_c^-) = 0.0042$ (cf. Equation (45)). First, we let $I_{01} = 0.019 > I_{0c}$. Manifold $\mathcal{M}(I_{01})$ corresponds to the upper red curve in Figure 9, where a monostable behavior is expected. We chose two different initial conditions $(v_a(0), x_a(0)) = (0.665, 0.001)$ and $(v_b(0), x_b(0)) = (0.684, 0.4)$ on $\mathcal{M}(I_{01})$. The corresponding trajectories for the SEs Equation (30), as obtained with MATLAB simulations, are shown in Figure 12. It is seen that both trajectories converge to the same stable EP $(0, 0.9971)$ in the set E_{s2} . Then, we have chosen a different manifold index $I_{02} = 0.005 > I_{0c}$ where once more we expect a monostable behavior. Manifold $\mathcal{M}(I_{02})$ corresponds to the lower red curve in Figure 9. Figure 12 shows the trajectories with two different initial conditions $(v_c(0), x_c(0)) = (0.85, 0.001)$ and $(v_d(0), x_d(0)) = (0.781, 0.4)$ on $\mathcal{M}(I_{02})$. Both trajectories converge to the stable EP $(0, 0.9997)$ in the set E_{s2} . As expected, the EP on $\mathcal{M}(I_{01})$ is (slightly) different from that on $\mathcal{M}(I_{02})$. This confirms that the circuit is convergent and displays a monostable behavior on each of the two considered manifolds. Moreover, there is coexistence of different monostable behaviors, since different manifolds correspond to different stable EPs. Also, as predicted, for any trajectory the capacitor voltage tends to 0.

In a subsequent simulation, we let $I_{03} = 0.0038$, which satisfies $|I_{03}| < I_{0c}$. On $\mathcal{M}(I_{03})$, which corresponds to the green curve in Figure 9, we expect a bistable behavior. This is confirmed by the two trajectories shown in Figure 13 which start at the two different initial conditions on $\mathcal{M}(I_{03})$ given by $(v_e(0), x_e(0)) = (0.6, 0.068)$ and $(v_f(0), x_f(0)) = (0.6, 0.256)$. The first trajectory converges to the stable EP $(0, 0.081)$ in the set E_{s1} , while the second converges to the stable EP $(0, 0.991)$ in the set E_{s2} . Finally, we let $I_{04} = 0$, which corresponds to the black curve in Figure 9. On $\mathcal{M}(0)$ we expect once more a bistable behavior.

Figure 13 shows two trajectories which start at the two different initial conditions on $\mathcal{M}(0)$ given by $(v_g(0), x_g(0)) = (0.6, 0.02)$ and $(v_h(0), x_h(0)) = (0.6, 0.523)$. The first trajectory converges to the stable EP $(0, 0.022)$ in the set E_{s1} , while the second converges to the stable EP $(0, 0.978)$ in the set E_{s2} , confirming a bistable behavior on $\mathcal{M}(0)$. It is seen that the pair of stable EPs on $\mathcal{M}(I_{03})$ differs from that on $\mathcal{M}(0)$, i.e., there is co-existence of different bistable behaviors. Since in this example $x_c^- = 0.14$ is close to 0 and $x_c^+ = 0.86$ is close to 1, in any case we have convergence to almost binary values for the x -variable (cf. Remark 4). This is confirmed by Figure 11a, which gives the x -component of the EPs as a function of the manifold index I_0 for the chosen parameter set.

Example 2 In a second example, we let $g = -0.101 \Omega^{-1}$, while all the other parameters are as in Example 1. According to Equation (41), for this choice of g we have $x_c^- = 0.4$ and $x_c^+ = 0.6$. Now, the stable set E_{s1} (respective, E_{s2}) contains all x -values in the interval $[0, 0.4)$ (respective, $(0.6, 1]$). This is confirmed by Figure 11b, that shows the EPs of the circuit of Figure 5c (x -values) as a function of the manifold index I_0 . This should be compared with Figure 11a of Example 1. Here, the critical manifold index is given by $I_{0c} = 6.8 \cdot 10^{-5}$. We conducted MATLAB simulations analogous to those in Example 1 by varying the manifold index. As an example, we have chosen $I_0 = 4.5 \cdot 10^{-5} < I_{0c} = 6.8 \cdot 10^{-5}$. Now, depending on the initial conditions on $\mathcal{M}(I_0)$, the solution converges either to the stable EP $(0, 0.641)$ in the set E_{s2} or the stable EP $(0.313, 0)$ in the set E_{s1} (Figure 14). In any case, the output, i.e., the x -variable of the stable EP, is not binary.

Remark 6. These simulations have confirmed that, although the memristor circuit of Figure 5c is not complex, it is however capable to display interesting dynamical behaviors. In particular, for the same set of circuit parameters and memristor nonlinearities, due to the existence of invariants and the foliation property of the state space, there is coexistence of infinitely many different

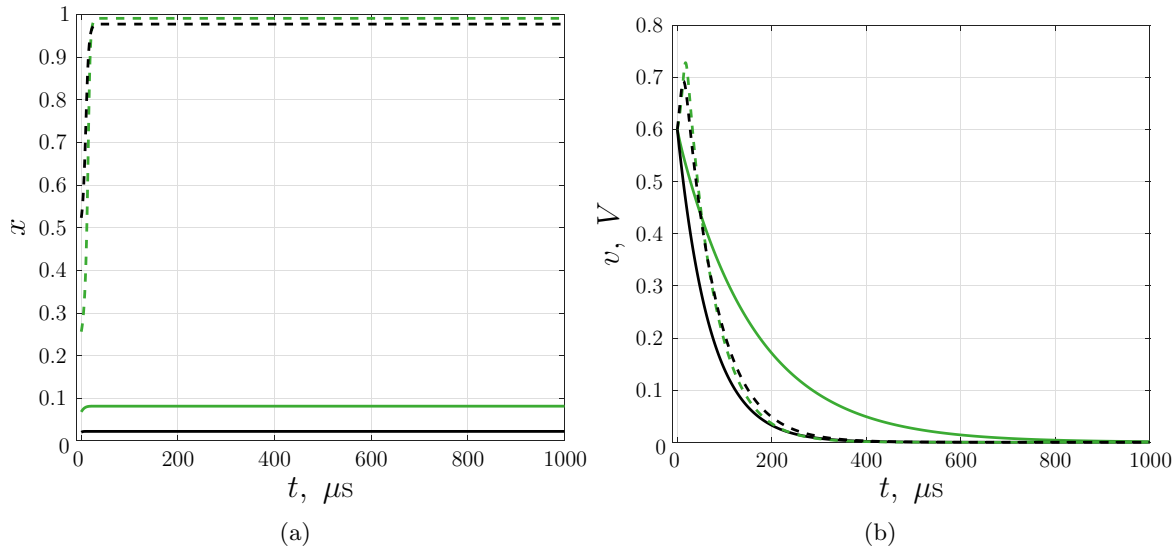


FIGURE 13 | Simulation results for the memristor circuit in Example 1. Time-domain behavior of the state variable x for the SEs in Equation (30). Solid green curve (respectively, dashed green curve): trajectory starting at the initial conditions $(v_e(0), x_e(0)) = (0.6, 0.068)$ (respectively, $(v_f(0), x_f(0)) = (0.6, 0.256)$) on manifold $\mathcal{M}(I_{03})$ with $I_{03} = 0.0038$. Solid black curve (respectively, dashed black curve): trajectory starting at the initial condition $(v_g(0), x_g(0)) = (0.6, 0.02)$ (respectively, $(v_h(0), x_h(0)) = (0.6, 0.523)$) on manifold $\mathcal{M}(0)$. (b) Corresponding time-domain behavior of the state variable v for the SEs in Equation (30).

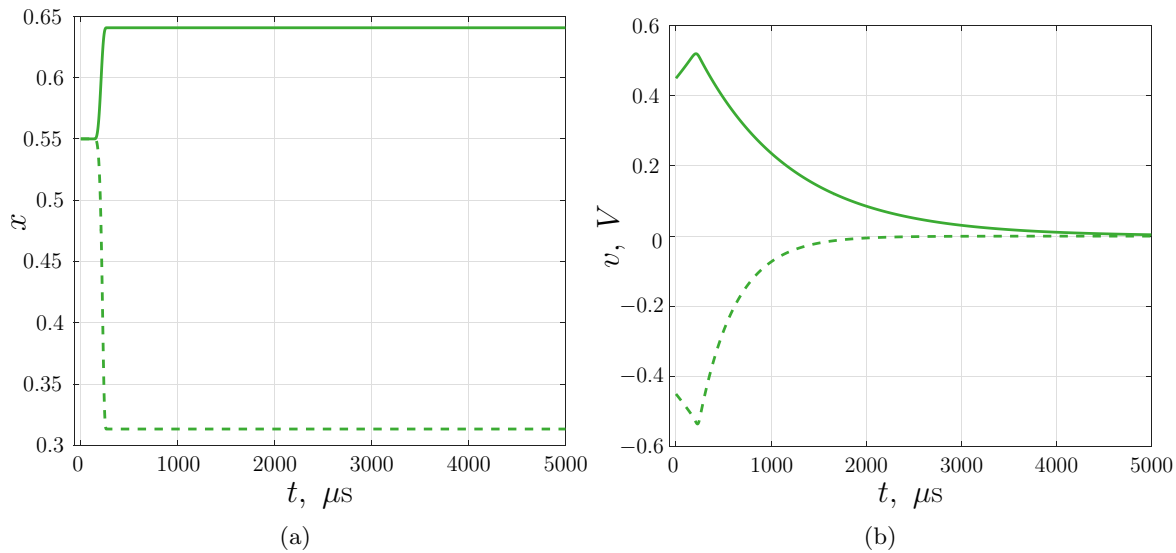


FIGURE 14 | Simulation results for the memristor circuit in Example 2. (a) Time-domain behavior of the state variable x for the SEs in Equation (30). Solid green curve (respectively, dashed green curve), trajectory starting at the initial conditions $(v_a(0), x_a(0)) = (0.45, 0.55)$ (respectively, $(v_a(0), x_a(0)) = (-0.45, 0.55)$) on manifold $\mathcal{M}(I_0)$ with $I_0 = 4.5 \cdot 10^{-5} < I_{0c} = 6.8 \cdot 10^{-5}$. Both initial conditions have been chosen on the vertical segment within the sub-threshold region. (b) Corresponding time-domain behavior of the state variable v for the SEs in Equation (30).

monostable and bistable behaviors. Moreover, by a suitable choice of the active conductance g , the state variable x of the cell, which defines its output, converges either to binary or analogue values.

4 | Conclusion

The manuscript has provided a thorough dynamical analysis of a class of simple circuits, hosting Resistive Random

Access Memory devices, under the assumption they obey a simple, yet general first-order Differential Algebraic Equation set, known as Voltage ThrEshold Adaptive Memristor, which may be flexibly adapted to voltage-controlled devices implemented in various nanotechnologies, resulting in an analytically-treatable and computationally-efficient mathematical model for circuit designers. One main result is that circuits with Resistive Random Access Memory devices admit conserved quantities, known as *Invariants of Motion*, which were so far demonstrated in cells with ideal memristors, only. The analysis and its results have a broad

range of potential applications, since the Voltage Threshold Adaptive Memristor model can be used to model accurately several real voltage-controlled Resistive Random Access Memory devices. The paper has also shown that, in the considered circuits, infinitely many different quiescent monostable and bistable steady states coexist. Since the proposed circuits may be easily toggled between monostability and bistability by manipulating the *Index of the Invariant Manifold*, along which their dynamical variables are bound to lie at all times, they may be employed as cells for Cellular Neural Networks, a non von Neumann computing architecture characterized by a massively-parallel computing power, and thus ideally suited for solving time-critical edge computing problems. Moreover, unlike previous Memristor Cellular Neural Network designs ([72, 73]), the proposed cell offers an attractive energy advantage, given that at any equilibrium point, it may admit, the voltage across its capacitor vanishes. Additionally, defining the output of our second-order circuit as the memristive one-port state, the locally-coupled arrays, based on the novel cell design, may store the results of an equilibrium-based computation in memory. Another interesting research path, our scientific study opens up, regards the determination of conditions for the existence of *Invariants of Motion* for higher-order circuits with Resistive Random Access Memory devices. Given that any given circuit is governed by a specific set of state equations, the results of analyses of this kind may differ from case to case. Finally, it would be important to verify that the emergence of *Invariants of Motion*, inducing the coexistence of infinitely many quiescent monostable and bistable steady-state solutions, predicted for memristor-based cells through our theoretical analysis based on realistic mathematical models, are indeed experimentally observable in laboratory prototypes.

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Conflicts of Interest

The author declare no conflicts of interest.

Data Availability Statement

The data that support the findings of this study are available from the corresponding author upon reasonable request.

Endnotes

¹As an example, w may represent the width of the gap between an electrode and the tip of the conductive filament.

²These values have been massaged from those available in the literature (cf. Equation [34]) to better display some dynamic features of the circuit in Figure 5c. Yet, we remark that the overall qualitative behavior remains the same also for different choices of the parameters.

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