

Structural completeness in basic hoops and BL-algebras

PAOLO AGLIANÒ, *DIISM, Università di Siena, 53100, Siena, Italy.*

E-mail: agliano@live.com

FRANCESCO MANFUCCI, *IIIA-CSIC, Campus UAB, 08193, Barcelona, Spain.*

E-mail: fmanfucci@gmail.com

Abstract

In the past twenty years many papers on basic logic (BL)-algebras have been published and many problems about them have been solved; thus, we may safely assume that our understanding of BL-algebras is quite mature. However there is at least one relevant topic which remains open i.e. describing the structurally complete and/or primitive varieties and quasivarieties of BL-algebras and basic hoops. This is precisely the argument of this paper. Though we do not reach a complete solution, we believe we have covered sufficient ground to make the investigation worthwhile. In detail, after recalling the state of the art for Wajsberg algebras and Wajsberg hoops, we add some further results about them. Next, we give a complete classification of locally finite varieties of BL-algebras and basic hoops that are structurally complete or primitive and we describe several other interesting classes of varieties with the same property. We then move our analysis to quasivarieties, where we find several non structural examples. Finally, we show how our techniques can be used to investigate beyond BL-algebras.

Keywords: Structural completeness, Algebraic logic, BL-algebras, Basic hoops, Free algebras

1 Introduction

A large class of substructural logics (i.e. logics that lack some structural rules) is given by the extensions of the *Full Lambek Calculus* $\mathcal{FL}$ (see [37] p. 76). All extensions of $\mathcal{FL}$ have a primitive connective $\mathbf{0}$ that denotes *falsum*; so if $\mathcal{L}$ is an extension of $\mathcal{FL}$, it makes sense to study its *positive* fragment $\mathcal{L}^+$. The language of $\mathcal{L}^+$ is obtained from the language of $\mathcal{L}$ by deleting $\mathbf{0}$ from the signature; the valid derivations in $\mathcal{L}^+$ are the valid derivations in $\mathcal{L}$ that contains only $\mathbf{0}$ -free formulas.

An important subfamily of substructural logics over $\mathcal{FL}$ consists of logics that satisfy both exchange and weakening but lack contraction; in this case the primitive connectives can be taken as $\vee, \wedge, \rightarrow, \mathbf{0}, \mathbf{1}$ where of course $\mathbf{1}$ represents *truth*. The minimal substructural logic of this kind is denoted by $\mathcal{FL}_{ew}$ and hence all the substructural logics lacking contraction are an extension of it. These logics are *algebraizable* with an *equivalent algebraic semantics* (in the sense of Blok–Pigozzi [22]) that is at least a quasivariety of algebras. Algebraizability in Blok–Pigozzi’s fashion entails that the deducibility relation of an algebraizable logic is characterized by means of the algebraic equational consequence of its equivalent algebraic semantics. Therefore, interesting properties of algebraizable logics can be studied via algebraic means, and structural completeness and hereditary structural completeness are among them.

This has been a well-travelled path both for logics in general and for substructural logics in particular, starting from the seminal paper by Bergman [18] that sparked interest in the subject. Among the many papers that have been devoted to the general topic in the past 20 years, we quote [26, 35] and the more recent [15]. For applications to substructural logics, we quote [45] and [29];

the characterization of the hereditarily structurally complete axiomatic extensions of Intuitionistic Logic and its positive fragment are both due to Citkin [30, 31].

Basic logic (BL)-algebras and their zero-free counterparts *basic hoops* (as equivalent algebraic semantics of Hajek's Basic Logic $\mathcal{BL}$ [43] and of its positive fragment $\mathcal{BL}^+$) are among the most investigated classes of algebras coming from substructural logics. One of the reasons is that there is a very powerful structure theorem that gives a decomposition of arbitrary totally ordered BL-algebras as ordinal sums of a Wajsberg algebra and a class of zero-free subreducts of Wajsberg-algebras, often called Wajsberg hoops [7]. Using this result, the lattice of subvarieties of BL-algebras (and basic hoops) has been investigated at length [1, 7, 8]; splitting algebras have been identified [2, 3] and strictly join irreducible subvarieties have been described [12]. Possibly the crowning achievement in the use of this technique is in [36], where all varieties of BL-algebras and basic hoops with the amalgamation property are identified.

Given all this, one may say that our understanding of BL-algebras is quite mature. However, there are still some stones that need to be turned; one of them is to describe the structurally complete and/or primitive varieties and quasivarieties of BL-algebras and basic hoops. The main technique used involves again ordinal sums; the idea is first to describe the varieties (and quasivarieties) of Wajsberg algebras and Wajsberg hoops that are structurally complete and/or primitive and then to use ordinal sums to 'climb the ladder', so to speak. The first part has been mostly carried out in [4, 6] and [38]; we will quote freely from them, but sometimes a little more context might be necessary. Another relevant part is played by the *generalized rotation of a basic hoop*; this is a construction foreshadowed in [25] and developed in [10, 11] and [13].

Obtaining a complete classification for varieties (and even worse, quasivarieties) seems out of reach. As a matter of fact, we believe that here we have pushed the known techniques to their limit: new techniques are needed and their discovery is the focus of an ongoing investigation.

Let's now describe the structure of the paper. In Section 2 we state some results and definitions that will be used in the sequel. In Section 3 we recall the state of the art for Wajsberg algebras and Wajsberg hoops and we add some further results. In Section 4 we give a complete classification of locally finite varieties of basic hoops that are structurally complete or primitive and we describe several other interesting classes of varieties with the same property. In Section 5 we do the same for BL-algebras. In Section 6 we tackle quasivarieties of basic hoops and BL-algebras. Our concluding remarks are in Section 7.

In order to get the results, some detours are needed; not to interrupt the flow of information, we created two Appendices. Appendix A deals with ordered sums of varieties, while Appendix B shows that some of our techniques can be applied beyond BL-algebras.

2 Preliminaries

2.1 Structural and primitive (quasi)varieties

We will use the language of universal algebra throughout most of this paper; for all unexplained notions the reader can consult [23] or [49]. Let $\mathbf{K}$ be a class of algebras; we denote by $\mathbb{I}, \mathbb{H}, \mathbb{P}, \mathbb{S}, \mathbb{P}_u$ the class operators sending $\mathbf{K}$ to the class of all isomorphic copies, homomorphic images, direct products, subalgebras and ultraproducts of members of $\mathbf{K}$. The operators can be composed in the obvious way; for instance $\mathbb{ISP}(\mathbf{K})$ denotes all algebras that are embeddable in a direct product of members of $\mathbf{K}$; moreover there are relations among the classes resulting from applying the operators in a specific orders, for instance $\mathbb{PS}(\mathbf{K}) \subseteq \mathbb{SP}(\mathbf{K})$ and $\mathbb{HSP}(\mathbf{K})$ is the largest class we can obtain composing the operators. We will use all the known relations without further notice following [52].

A class of algebras is a variety if it is closed under $\mathbb{H}, \mathbb{S}$ and $\mathbb{P}$, a quasivariety if it is closed under $\mathbb{I}, \mathbb{S}, \mathbb{P}$ and $\mathbb{P}_u$ and a universal class if it is closed under $\mathbb{I}, \mathbb{S}$ and $\mathbb{P}_u$. The following is well-known:

LEMMA 2.1.

Let $\mathbf{K}$ be any class of algebras. Then:

1. $\mathbf{K}$ is a quasivariety if and only if $\text{ISP}_u(\mathbf{K}) = \mathbf{K}$ if and only if it is the class of algebras in which a given set of quasiequations is valid;
2. $\mathbf{K}$ is a variety if and only if $\text{HSP}(\mathbf{K}) = \mathbf{K}$ if and only if it is the class of algebras in which a given set of equations is valid.

Finally, as it is traditional, we will often write $\mathbb{V}$ for HSP and $\mathbb{Q}$ for ISP_u .

If $\mathbf{Q}$ is a quasivariety and $\mathbf{A} \in \mathbf{Q}$, a **$\mathbf{Q}$ -congruence** of $\mathbf{A}$ is a $\theta \in \text{Con}(\mathbf{A})$ such that $\mathbf{A}/\theta \in \mathbf{Q}$; the $\mathbf{Q}$ -congruences of $\mathbf{A}$ form an algebraic lattice (that is not in general a sublattice of $\text{Con}(\mathbf{A})$) which we denote by $\text{Con}_{\mathbf{Q}}(\mathbf{A})$. An algebra $\mathbf{A} \in \mathbf{Q}$ is **$\mathbf{Q}$ -irreducible** if $\text{Con}_{\mathbf{Q}}(\mathbf{A})$ has a unique minimal element. There are two fundamental results that we will be using many times and deserve a spotlight.

Let $\mathbf{B}, (\mathbf{A}_i)_{i \in I}$ be algebras in the same signature; we say that $\mathbf{B}$ *embeds* in $\prod_{i \in I} \mathbf{A}_i$ if $\mathbf{B} \in \text{IS}(\prod_{i \in I} \mathbf{A}_i)$. Let p_i be the i th projection, or better, the composition of the embedding and the i th projection, from $\mathbf{B}$ to $\mathbf{A}_i$; the embedding is *subdirect* if for all $i \in I$, $p_i(\mathbf{B}) = \mathbf{A}_i$ and in this case we will write

$$\mathbf{B} \leq_{sd} \prod_{i \in I} \mathbf{A}_i.$$

If $\mathbf{V}$ is a variety, we denote by $\mathbf{V}_{\text{si}}$ the class of subdirectly irreducible algebras in $\mathbf{V}$.

THEOREM 2.2.

1. (Birkhoff [20]) Every algebra can be subdirectly embedded in a product of subdirectly irreducible algebras. So if $\mathbf{A} \in \mathbf{V}$, then $\mathbf{A}$ can be subdirectly embedded in a product of members of $\mathbf{V}_{\text{si}}$.
2. (Jónsson's Lemma [44]) Suppose that $\mathbf{K}$ is a class of algebras such that $\mathbf{V}(\mathbf{K})$ is congruence distributive; then $\mathbf{V}_{\text{si}} \subseteq \text{HSP}_u(\mathbf{K})$.

If $\mathbf{Q}$ is a quasivariety we denote by $\mathbf{Q}_{\text{rsi}}$ the class of $\mathbf{Q}$ -irreducible algebras in $\mathbf{Q}$.

THEOREM 2.3.

Let $\mathbf{Q}$ be any quasivariety.

1. (Mal'cev [47]) Every $\mathbf{A} \in \mathbf{Q}$ is subdirectly embeddable in a product of algebras in $\mathbf{Q}_{\text{rsi}}$.
2. (Czelakowski-Dziobiak [32]) If $\mathbf{Q} = \mathbb{Q}(\mathbf{K})$, then $\mathbf{Q}_{\text{rsi}} \subseteq \text{ISP}_u(\mathbf{K})$.

A quasivariety $\mathbf{Q}$ is **structurally complete** (**structural** for short) if for every quasivariety $\mathbf{Q}' \subseteq \mathbf{Q}$, if $\mathbb{H}(\mathbf{Q}) = \mathbb{H}(\mathbf{Q}')$ then $\mathbf{Q}' = \mathbf{Q}$. A quasivariety $\mathbf{Q}' \subseteq \mathbf{Q}$ is **equational** in $\mathbf{Q}$ if $\mathbf{Q}' = \mathbb{H}(\mathbf{Q}') \cap \mathbf{Q}$; a quasivariety is **primitive** if every subquasivariety $\mathbf{Q}' \subseteq \mathbf{Q}$ is equational in $\mathbf{Q}$.

LEMMA 2.4.

For a quasivariety $\mathbf{Q}$ the following are equivalent:

1. $\mathbf{Q}$ is structural;
2. every proper subquasivariety of $\mathbf{Q}$ is contained in a proper subvariety of $\mathbb{H}(\mathbf{Q})$ and
3. $\mathbf{Q} = \mathbb{Q}(\mathbf{F}_{\mathbf{Q}}(\omega))$.

COROLLARY 2.5.

A variety $\mathbf{V}$ is structural if and only if every proper subquasivariety of $\mathbf{V}$ is contained in a proper subvariety of $\mathbf{V}$; therefore if $\mathbf{A}$ is such that $\mathbb{V}(\mathbf{A})$ is structural, then $\mathbb{V}(\mathbf{A}) = \mathbb{Q}(\mathbf{A})$.

4 Structural completeness in basic hoops and *BL*-algebras

LEMMA 2.6.

For a quasivariety $\mathbf{Q}$ the following are equivalent:

1. $\mathbf{Q}$ is primitive;
2. every subquasivariety of $\mathbf{Q}$ is structural.

PROOF. If $\mathbf{Q}$ is primitive and $\mathbf{Q}' \subseteq \mathbf{Q}$, then $\mathbf{Q}' = \mathbb{H}(\mathbf{Q}') \cap \mathbf{Q}$; so (2) follows. If (2) holds, then let $\mathbf{Q}' \subseteq \mathbf{Q}$ and let $\mathbf{Q}'' = \mathbb{H}(\mathbf{Q}') \cap \mathbf{Q}$; then $\mathbb{H}(\mathbf{Q}'') = \mathbb{H}(\mathbf{Q}')$ and thus $\mathbf{Q}'' = \mathbf{Q}'$. Hence (1) holds. $\square$

COROLLARY 2.7.

For any variety $\mathbf{V}$ the following are equivalent:

1. $\mathbf{V}$ is primitive;
2. every subquasivariety of $\mathbf{V}$ is a variety and
3. $\mathbf{V}$ is structural and every proper subvariety of $\mathbf{V}$ is primitive.

We observe that condition (2) above is a sufficient condition for primitivity of quasivarieties.

LEMMA 2.8.

Let $\mathbf{Q}$ be a quasivariety such that every proper subquasivariety of $\mathbf{Q}$ is a variety; then $\mathbf{Q}$ is primitive.

PROOF. We must show that for all $\mathbf{Q}' \subseteq \mathbf{Q}$, $\mathbf{Q}' = \mathbb{H}(\mathbf{Q}') \cap \mathbf{Q}$. If $\mathbf{Q}' = \mathbf{Q}$ there is nothing to prove; otherwise $\mathbf{Q}'$ is a variety so $\mathbf{Q}' = \mathbf{Q}' \cap \mathbf{Q} = \mathbb{H}(\mathbf{Q}') \cap \mathbf{Q}$. $\square$

We have several sufficient conditions for structurality and a necessary and sufficient condition for primitivity for some varieties (see Corollary 2.10 and Theorem 2.12 below). However to get to them we need some definitions: let $\mathbf{A}$ be an algebra and $\mathbf{K}$ a class of algebras of the same type as $\mathbf{A}$. We say that:

- $\mathbf{A}$ is **projective** in $\mathbf{K}$ if for all $\mathbf{B} \in \mathbf{K}$ if $f : \mathbf{B} \rightarrow \mathbf{A}$ is a surjective epimorphism, then there is an embedding $g : \mathbf{A} \rightarrow \mathbf{B}$ with $gf = id_{\mathbf{A}}$;
- $\mathbf{A}$ is **weakly projective** in $\mathbf{K}$ if for all $\mathbf{B} \in \mathbf{K}$ if $\mathbf{A} \in \mathbb{H}(\mathbf{B})$, then $\mathbf{A} \in \mathbb{IS}(\mathbf{B})$.

It is clear that if $\mathbf{A}$ is projective in $\mathbf{K}$, then $\mathbf{A}$ is weakly projective in $\mathbf{K}$.

An algebra $\mathbf{A}$ is **finitely presented** in $\mathbf{Q}$ if it is nontrivial and can be defined by a finite set of generators and relations in $\mathbf{Q}$. This means that there is a finite set X and a compact congruence θ of $\mathbf{F}_{\mathbf{Q}}(X)$ such that $\mathbf{F}_{\mathbf{Q}}(X)/\theta \cong \mathbf{A}$. Now if $\mathbf{Q}$ is a quasivariety and $\mathbf{A} \in \mathbf{Q}$ let's define

$$[\mathbf{Q} : \mathbf{A}] = \{\mathbf{B} : \mathbf{A} \notin \mathbb{S}(\mathbf{B})\}.$$

If $\mathbf{A}$ is finitely presented and $\mathbf{Q}$ -irreducible, then $[\mathbf{Q} : \mathbf{A}]$ is a quasivariety (see Lemma 3.5 in [4]).

There is a deep connection between structural quasivarieties, weakly projective algebras and finitely presented algebras.

THEOREM 2.9.

Let $\mathbf{Q} = \mathbb{Q}(\mathbf{K})$ for some class $\mathbf{K}$ of algebras. Then:

1. if every $\mathbf{A} \in \mathbf{K}$ is weakly projective in $\mathbf{Q}$, then $\mathbf{Q}$ is structural;
2. if $\mathbf{Q}$ is structural, every algebra in $\mathbf{K}$ is $\mathbf{Q}$ -irreducible and for all $\mathbf{A}, \mathbf{A}' \in \mathbf{K}$, $\mathbf{A} \notin \mathbb{IS}(\mathbf{A}')$, then each finitely presented algebra in $\mathbf{K}$ is weakly projective in $\mathbf{Q}$.

PROOF. For (1), let $\mathbf{Q}' \subseteq \mathbf{Q}$ with $\mathbb{H}(\mathbf{Q}') = \mathbb{H}(\mathbf{Q})$; then for any $\mathbf{A} \in \mathbf{K}$ there exists a $\mathbf{B} \in \mathbf{Q}'$ with $\mathbf{A} \in \mathbb{H}(\mathbf{B})$. As $\mathbf{A}$ is weakly projective in $\mathbf{Q}$, $\mathbf{A} \in \mathbb{IS}(\mathbf{B})$ and so $\mathbf{A} \in \mathbf{Q}$. This implies $\mathbf{Q}' = \mathbf{Q}$ and so $\mathbf{Q}$ is structural.

For (2) we argue by counterpositive. Suppose that there is a finitely presented $\mathbf{A} \in \mathbf{K}$ which is not weakly projective in $\mathbf{Q}$. Then there is an algebra $\mathbf{B} \in \mathbf{Q}$ such that $\mathbf{A} \in \mathbb{H}(\mathbf{B})$ and $\mathbf{A} \notin \mathbb{IS}(\mathbf{B})$. Then $\mathbf{B} \in [\mathbf{Q} : \mathbf{A}]$ which is a quasivariety. Note also that if $\mathbf{A}' \in \mathbf{K}$ with $\mathbf{A} \neq \mathbf{A}'$, then $\mathbf{A}' \in [\mathbf{Q} : \mathbf{A}]$ by hypothesis. Since $\mathbf{A} \in \mathbb{V}([\mathbf{Q} : \mathbf{A}])$, as $\mathbf{A} \in \mathbb{H}(\mathbf{B})$, we get that $\mathbb{V}([\mathbf{Q} : \mathbf{A}]) = \mathbb{V}(\mathbf{Q})$; but this implies that $\mathbf{Q}$ is not structural. $\square$

We observe that given a class $\mathbf{K}$ of algebras it is always possible to find a class $\mathbf{K}'$ such that $\mathbb{Q}(\mathbf{K}) = \mathbb{Q}(\mathbf{K}')$ and $\mathbf{K}'$ satisfies the hypotheses of Theorem 2.9(2). As a consequence of Theorem 2.9(1) we have:

COROLLARY 2.10.

Let $\mathbf{Q}$ be a quasivariety; if either

1. every finitely generated algebra in $\mathbf{Q}$ is weakly projective, or
 2. every finitely presented algebra in $\mathbf{Q}$ is weakly projective, or
 3. every finitely generated $\mathbf{Q}$ -irreducible in $\mathbf{Q}$ is weakly projective,
- then $\mathbf{Q}$ is structural.

PROOF. These are consequences of Theorem 2.9, the well-known fact that every algebra is embeddable in an ultraproduct of its finitely generated subalgebras (for (1)) and that any quasivariety $\mathbf{Q}$ is generated by its finitely presented algebras ([42], Proposition 2.1.18). $\square$

Since the trivial algebra is a homomorphic image of any algebra, for varieties it is sufficient to ask that every nontrivial finitely presented algebra be weakly projective. We will use this observation in the paper without further mention.

For primitivity, Corollary 2.10(3) above becomes a necessary condition:

THEOREM 2.11.

Let $\mathbf{Q}$ be a quasivariety of finite type. If $\mathbf{Q}$ is primitive, then every finitely presented $\mathbf{Q}$ -irreducible algebra in $\mathbf{Q}$ is weakly projective in $\mathbf{Q}$.

Moreover:

THEOREM 2.12.

If $\mathbf{Q}$ is locally finite quasivariety, then the following are equivalent.

1. $\mathbf{Q}$ is primitive;
2. every $\mathbf{Q}$ -irreducible finitely presented algebra $\mathbf{A} \in \mathbf{Q}$ is weakly projective in $\mathbf{Q}$ and
3. every $\mathbf{Q}$ -irreducible finitely presented algebra $\mathbf{A} \in \mathbf{Q}$ is weakly projective in the class of finitely presented algebras in $\mathbf{Q}$.

Theorems 2.11 and 2.12 were proved by V. Slavik for locally finite varieties of lattices [53] and extended to locally finite quasivarieties of algebras by Gorbunov ([42], Proposition 5.1.24); a self contained proof of Theorem 2.12 is in [15] (Theorem 4.11).

We close this section with a definition that will be useful in the sequel. A finitely presented algebra $\mathbf{A}$ is **unifiable** in a quasivariety $\mathbf{Q}$ if there is a homomorphism $f : \mathbf{A} \rightarrow \mathbf{P}$, where $\mathbf{P}$ is projective in $\mathbf{Q}$. A quasivariety $\mathbf{Q}$ has the **Baker–Beynon property** if every unifiable finitely presented algebra in $\mathbf{Q}$ is weakly projective in $\mathbf{Q}$. From Corollaries 2.10 and Theorem 2.11 we get:

THEOREM 2.13.

If a variety $\mathbf{V}$ has the Baker–Beynon property and every nontrivial finitely presented algebra in $\mathbf{V}$ is unifiable, then $\mathbf{V}$ is structural. If $\mathbf{V}$ is also locally finite, then it is primitive.

EXAMPLE 2.14.

Not surprisingly, having the Baker–Beynon property is way stronger than being primitive even in locally finite varieties. The variety of distributive lattices is locally finite and primitive (since it is an idempotent dual discriminator variety [15]), but it does not have the Baker–Beynon property. First notice that any lattice is unifiable, since the one-generated free lattice is trivial and every algebra has a homomorphism onto the trivial algebra; moreover, since lattices have finite type, any finite lattice is finitely presented. However, Balbes [17] showed that the finite distributive lattices that are (weakly) projective in the variety of distributive lattices must have the property that the join of two meet irreducible elements is again meet irreducible.

2.2 BL-algebras and basic hoops

A **hoop** is an algebra $\mathbf{A} = \langle A, \rightarrow, \cdot, 1 \rangle$ with two binary operations and a constant, where $\langle A, \cdot, 1 \rangle$ is a commutative monoid and moreover the following equations hold:

$$x \rightarrow (y \rightarrow z) \approx xy \rightarrow z$$

$$x \rightarrow x \approx 1$$

$$(x \rightarrow y)x \approx (y \rightarrow x)y.$$

For the basic (and not so basic) facts about hoops, the reader can consult [21]. Here we point out the following:

- in any hoop the operation $x \wedge y := (x \rightarrow y)x$ is a semilattice operation and 1 is the largest element in the ordering, that will be denoted by $\leq$;
- $\rightarrow$ and $\cdot$ form a residuated pair, i.e. for all $a, b, c \in A$

$$ab \leq c \quad \text{if and only if} \quad a \leq b \rightarrow c.$$

A hoop is **basic** if it satisfies

$$(x \rightarrow y) \rightarrow z \approx ((x \rightarrow y) \rightarrow z) \rightarrow z.$$

Basic hoops have been introduced and investigated in [5]; it turns out that every basic hoop is a subdirect product of totally ordered hoops and thus every subdirectly irreducible basic hoop is totally ordered. In a basic hoop one can define a join operation whose induced ordering agrees with the underlying semilattice ordering:

$$x \vee y := ((x \rightarrow y) \rightarrow y) \wedge ((y \rightarrow x) \rightarrow x).$$

Therefore any basic hoop is a commutative and integral residuated lattice and the entire theory of commutative residuated lattices can be applied to it; in particular there is a good interplay between congruences and certain subsets called filters. We will not enter into details here but we feel free to use the general theory as illustrated for instance in [14] and in many other papers on the subject.

A **BL-algebra** is an algebra $\langle A, \rightarrow, \cdot, 0, 1 \rangle$ where $\langle A, \rightarrow, \cdot, 1 \rangle$ is a basic hoop and 0 is a constant satisfying $0 \leq a$ for all $a \in A$. Of course there is a deep connection between basic hoops and BL-algebras; the theory of filters is identical (the congruences of a BL-algebra are the same as those of its hoop reduct) and a bounded basic hoop is termwise equivalent to a BL-algebra. This allows us to use theorems about BL-algebras when we deal with bounded basic hoops with almost no change. In

spite of this, there are also significant differences between the two theories, and we will see some of them in this note.

A hoop is a **Wajsberg hoop** if it satisfies

$$(x \rightarrow y) \rightarrow y \approx (y \rightarrow x) \rightarrow x;$$

a hoop is cancellative if its underlying monoid is cancellative in the usual sense. Every cancellative hoop is a Wajsberg hoop and every Wajsberg hoop is a basic hoop [21]. A **Wajsberg algebra** is a bounded hoop satisfying the above equation; again, a Wajsberg algebra is a BL-algebra, and moreover, Wajsberg algebras are termwise equivalent to the more famous MV-algebras (see [9]).

2.3 Ordinal sums

Let $\langle I, \leq \rangle$ be a totally ordered set and let $(\mathbf{A}_i)_{i \in I}$ be a family of basic hoops; the **ordinal sum** $\mathbf{A} = \bigoplus_{i \in I} \mathbf{A}_i$ of the family is an algebra such that

- the universe of $\mathbf{A}$ is $\bigcup_{i \in I} (A_i \setminus \{1\}) \cup \{1\}$;
- the ordering is defined by $a \leq b$ if and only if either $a, b \in A_i$ for some i and $a \leq b$ in $\mathbf{A}_i$, or else there are i, j such that $i < j$, $a \in A_i$ and $b \in A_j$;
- the operation $\rightarrow$ and $\cdot$ are defined as

$$a \rightarrow b = \begin{cases} a \rightarrow_{\mathbf{A}_i} b, & \text{if } a, b \in A_i \text{ for some } i; \\ 1, & \text{if } a \leq b; \\ b, & \text{if } a \in A_i, b \in A_j \text{ and } j < i. \end{cases}$$

$$ab = \begin{cases} a \cdot_{\mathbf{A}_i} b, & \text{if } a, b \in A_i \text{ for some } i; \\ b, & \text{if } a \in A_i, b \in A_j \text{ and } j < i; \\ a, & \text{if } a \in A_i, b \in A_j \text{ and } i < j. \end{cases}$$

It is obvious that the multiplication is commutative and that the ordinal sum of hoops is still a hoop; however, the ordinal sum of basic hoops is not in general a basic hoop, in that the resulting ordering might not be a lattice ordering. If $\mathbf{K}$ is a class of basic hoops such that the ordinal sum of any family of basic hoops from $\mathbf{K}$ is a basic hoop, we will say that **ordinal sums exist** in $\mathbf{K}$; it is easy to check that if $\mathbf{K}$ is the class of basic hoops in which 1 is strictly join irreducible, then ordinal sums exist in $\mathbf{K}$. In particular, any ordinal sum of totally ordered basic hoops is a basic hoop. If now $\langle I, \leq \rangle$ has a minimum, say 0, and $\mathbf{A}_0$ is BL-algebra then the ordinal sum construction, whenever it exists, produces a BL-algebra. Therefore, when we write that, for a basic hoop or BL-algebra $\mathbf{A}$, $\mathbf{A} = \bigoplus_{i \in I} \mathbf{A}_i$ we will tacitly assume that the ordinal sum exists. Next we will explore how the ordinal sum construction interacts with the class operators, slightly extending the results in [7].

LEMMA 2.15.

If $\mathbf{A} = \bigoplus_{i \in I} \mathbf{A}_i$ where $\mathbf{A}_i$ is a basic hoop, then

$$\mathbb{IS}(\mathbf{A}) = \left\{ \bigoplus_{i \in I} \mathbf{B}_i : \mathbf{B}_i \in \mathbb{IS}(\mathbf{A}_i) \right\}$$

($\mathbf{B}_i$ can be trivial). If $\mathbf{A} = \bigoplus_{i \in I} \mathbf{A}_i$ is a BL-algebra (hence I has 0 as a minimum), then

$$\mathbb{IS}(\mathbf{A}) = \left\{ \mathbf{B}_0 \oplus \bigoplus_{i > 0} \mathbf{B}_i : \mathbf{B}_0 \in \mathbb{IS}(\mathbf{A}_0), \mathbf{B}_i \in \mathbb{IS}(\mathbf{A}_i) \right\}$$

($\mathbf{B}_i$ can be trivial if $i > 0$).

LEMMA 2.16.

Let $\mathbf{A} = \bigoplus_{i \in I} \mathbf{A}_i$ be a basic hoop or a BL-algebra, let F be a filter of $\mathbf{A}$ and let

$$\widehat{I} = \{i : F \cap (A_i \setminus \{1\}) \neq \emptyset\}.$$

Then

1. if $\inf \widehat{I}$ does not exist then $A = F$;
2. if $\inf \widehat{I} = \widehat{i} \notin \widehat{I}$, then $F = \bigoplus_{i > \widehat{i}} \mathbf{A}_i$ and
3. if $\inf \widehat{I} = \widehat{i} \in \widehat{I}$ then

$$F = (F \cap A_{\widehat{i}}) \oplus \bigoplus_{i > \widehat{i}} \mathbf{A}_i.$$

PROOF. We observe that since a filter is upward closed, if $i \in \widehat{I}$ and $i \leq j$, then $j \in \widehat{I}$; for the same reason $F \cap A_j = A_j$ if $i < j$. The thesis follows. $\square$

COROLLARY 2.17.

Let $\mathbf{A} = \bigoplus_{i \in I} \mathbf{A}_i$ be a basic hoop or a BL-algebra; then $\mathbf{B} \in \mathbb{H}(\mathbf{A})$ if and only if one of the following hold

1. $\mathbf{B} \cong \mathbf{A}$;
2. there is a $\widehat{i} \in I$ such $\mathbf{B} \in \bigoplus_{i < \widehat{i}} \mathbf{A}_i \oplus \mathbb{H}(\mathbf{A}_{\widehat{i}})$.

Moreover if $I = \{0, \dots, n\}$ is finite

$$\mathbf{B} \in \mathbb{H}(\mathbf{A}_0) \cup \{\mathbf{A}_0 \oplus \mathbf{C} : \mathbf{C} \in \mathbb{H}(\mathbf{A}_1)\} \cup \dots \cup \{\mathbf{A}_0 \oplus \dots \oplus \mathbf{A}_{n-1} \oplus \mathbf{C} : \mathbf{C} \in \mathbb{H}(\mathbf{A}_n)\}.$$

LEMMA 2.18.

([7], Lemma 7.1) Let $\mathbf{A} \bigoplus_{i=0}^n \mathbf{A}_i$ be a totally ordered basic hoop or a BL-algebra. Then

$$\mathbb{ISP}_u(\mathbf{A}) = \left\{ \bigoplus_{i=0}^n \mathbf{B}_i : \mathbf{B}_i \in \mathbb{ISP}_u(\mathbf{A}_i) \right\}.$$

Therefore

$$\mathbb{HSP}_u(\mathbf{A}) = \mathbb{HSP}_u(\mathbf{A}_0) \cup \bigcup_{k=1}^n \left\{ \bigoplus_{i=0}^k \mathbf{B}_i : \mathbf{B}_i \in \mathbb{ISP}_u(\mathbf{A}_i) \text{ for } i < k, \mathbf{B}_k \in \mathbb{HSP}_u(\mathbf{A}_k) \right\}.$$

We have already mentioned that the theory of BL-algebras and that of basic hoops are deeply interconnected; this connection has been exploited first in [7].

THEOREM 2.19.

[7] Every totally ordered basic hoop is an ordinal sum of a family of Wajsberg hoops; every totally ordered BL-algebra is an ordinal sum of a family of Wajsberg hoops (the first of which is a Wajsberg algebra). Moreover, since the sum of two nontrivial Wajsberg hoops is never a Wajsberg hoop, the components of the sum are unique up to isomorphism.

For this reason, from now on, when we write a totally ordered hoop or BL-algebra as $\mathbf{A} = \bigoplus_{i \in I} \mathbf{A}_i$ we always mean that the ordinal sum is composed of Wajsberg hoops, which will be called the **Wajsberg components** of $\mathbf{A}$. To denote a totally ordered Wajsberg hoop or algebra, we will often use the term ‘Wajsberg chain’.

2.4 Wajsberg chains

Here we make a list of specific Wajsberg chains; this is possible because Wajsberg algebras (and hence bounded Wajsberg hoops) have a *standard representation*. Let $\mathbf{G}$ be an abelian ℓ -group; by [50], if u is a strong unit of $\mathbf{G}$ we can construct a bounded Wajsberg hoop $\Gamma(\mathbf{G}, u) = \langle [0, u], \rightarrow, \cdot, 0, u \rangle$ where $ab = \max\{a + b - u, 0\}$ and $a \rightarrow b = \max\{u - a + b, u\}$. Of course $\Gamma(\mathbf{G}, u)$ is a chain if and only if $\mathbf{G}$ is totally ordered; here we will look at $\mathbb{Z}, \mathbb{Q}$ and $\mathbb{R}$ as totally ordered ℓ -groups. Let now $\mathbb{Z} \times_l \mathbb{Z}$ denote the lexicographic product of two copies of $\mathbb{Z}$. In other words, the universe is the Cartesian product, the group operations are defined componentwise and the ordering is the lexicographic ordering (w.r.t. the natural ordering of $\mathbb{Z}$); then $\mathbb{Z} \times_l \mathbb{Z}$ is a totally ordered abelian ℓ -group and we can apply Γ to it. Now we introduce:

- the finite Wajsberg chain with $n + 1$ elements $\mathbb{L}_n = \Gamma(\mathbb{Z}, n)$;
- for $0 \leq k \leq n$, the infinite Wajsberg chain $\mathbb{L}_{n,k} = \Gamma(\mathbb{Z} \times_l \mathbb{Z}, (n, k))$;
- the infinite bounded Wajsberg chain $[0, 1]_{\mathbb{L}} = \Gamma(\mathbf{R}, 1)$, i.e. the real interval with operations induced by the *Wajsberg t -norm*, i.e. $xy = \max(x + y - 1, 0)$, $x \rightarrow y = \min(1 + x - y, 1)$ and
- the unbounded cancellative Wajsberg chain $\mathbf{C}_\omega$ that has as universe the free commutative monoid on one generator, where the product is the monoid product and $a^l \rightarrow a^m = a^{\max(m-l, 0)}$.

In many papers $\mathbb{L}_{n,0}$ is denoted by $\mathbb{L}_n^\infty$ and the same symbol is used for Wajsberg chains regardless of whether they are Wajsberg algebras or Wajsberg hoops. However in our case this distinction becomes rather significant so, we will denote the hoop reducts of $\mathbb{L}_n$ and $\mathbb{L}_{n,k}$ by $\mathbb{L}_n^+$ and $\mathbb{L}_{n,k}^+$ respectively. Similarly we will write $\mathbf{A}^+$ for the hoop reduct of the BL-algebra $\mathbf{A}$. The proofs of the following facts are either trivial or can be found in [7].

LEMMA 2.20.

For any $n, k \geq 1$

1. $\mathbb{L}_n$ is simple and $\mathbb{L}_n \in \mathbb{IS}(\mathbb{L}_k)$ if and only if $\mathbb{L}_n \in \mathbb{IS}(\mathbb{L}_{k,0})$ if and only if $n \mid k$.
2. For any $k \geq 0$, $\mathbb{L}_{n,k}$ is subdirectly irreducible, and $\mathbb{L}_m \in \mathbb{IS}(\mathbb{L}_{n,k})$ if and only if m divides $\gcd(n, k)$.
3. If $\mathbf{A}$ is a nontrivial totally ordered cancellative hoop then, $\mathbf{C}_\omega \in \mathbb{IS}(\mathbf{A})$.
4. If $\mathbf{A}$ is a nontrivial totally ordered cancellative hoop then $\mathbb{ISP}_u(\mathbf{A}) = \mathbb{ISP}_u(\mathbf{C}_\omega)$.

Moreover, (1) and (2) hold, mutatis mutandis, for the hoop reducts.

The **radical** of a bounded Wajsberg chain (hence of any totally ordered Wajsberg algebra) $\mathbf{A}$, in symbols $\text{Rad}(\mathbf{A})$, is the intersection of the maximal filters of $\mathbf{A}$; it is easy to see that $\text{Rad}(\mathbf{A})$ is a cancellative basic subhoop of $\mathbf{A}$. We say that a bounded Wajsberg hoop $\mathbf{A}$ has **rank** n , if $\mathbf{A}/\text{Rad}(\mathbf{A}) \cong \mathbb{L}_n$. For any bounded Wajsberg hoop $\mathbf{A}$, $d_{\mathbf{A}}$, called the **divisibility index**, is the maximum k such that $\mathbb{L}_k$ is embeddable in $\mathbf{A}$ if any; otherwise $d_{\mathbf{A}} = \infty$.

2.5 Connections with logic

Thanks mainly to the pioneering work of Blok and Pigozzi [22], there is a standard way to associate a (quasi)variety of algebras (the *equivalent quasivariety semantics*) to the so-called algebraizable

logics. As we said in the introduction, algebraizability in Blok–Pigozzi’s fashion entails that the deducibility relation of an algebraizable logic is characterized by means of the algebraic equational consequence of its equivalent algebraic semantics. Therefore, interesting properties of algebraizable logics can be studied via algebraic means and vice versa, algebraic properties reflect naturally into the structure of the logics. The translation matrix is well-known and explained many times in many papers (see for instance [15]) and we will not go over it. However:

PROPOSITION 2.21.

(Folklore) Let $\mathcal{L}$ be an algebraizable logic with equivalent quasivariety semantics $\mathbf{Q}$. Then there is a (dual) isomorphism between the lattice of finitary extensions of $\mathcal{L}$ and the lattice of subquasivarieties of $\mathbf{Q}$. Moreover, every axiomatic extension of $\mathcal{L}$ corresponds to an equational subquasivariety of $\mathbf{Q}$ and viceversa; if $\mathbf{Q}$ is a variety, then the lattice of axiomatic extensions of $\mathcal{L}$ is (dually) isomorphic with the lattice of subvarieties of $\mathbf{V}$.

It is well-known that:

- the variety of BL-algebras is the equivalent quasivariety semantics of *Hajek’s Basic Logic* $\mathcal{BL}$ and the variety of basic hoops is the equivalent quasivariety semantics of its positive fragment $\mathcal{BL}^+$;
- the variety of Wajsberg algebras is the equivalent quasivariety semantics of *Łukasiewicz Many Valued Logic* $\mathcal{MV}$ and the variety of Wajsberg hoops is the equivalent quasivariety semantics of its positive fragment $\mathcal{MV}^+$.

For us a **logic** $\mathcal{L}$ over an algebraic language ρ , is identified as a substitution invariant finitary consequence relation $\vdash \subseteq \mathbb{P}(\mathbf{T}_\rho(\omega)) \times \mathbf{T}_\rho(\omega)$. A **rule** of $\mathcal{L}$ is an ordered pair (Γ, δ) where $\Gamma \cup \delta$ is a finite set of formulas that we write suggestively as $\Gamma \Rightarrow \delta$. We say that a rule $\Gamma \Rightarrow \delta$ is **derivable** in $\mathcal{L}$ if $\Gamma \vdash_{\mathcal{L}} \delta$. If $\vdash_1$ and $\vdash_2$ are two logics over the same language, $\vdash_2$ is an **extension** of $\vdash_1$ if $\vdash_1 \subseteq \vdash_2$; $\vdash_2$ is an **axiomatic extension** of $\vdash_1$ if there is a set of formulas Γ such that $\vdash_2$ is the smallest logic that is an extension of $\vdash_1$ satisfying $\vdash_2 \gamma$ for all $\gamma \in \Gamma$. A rule $\Gamma \Rightarrow \delta$ is **admissible** in $\mathcal{L}$ if for every substitution σ :

$$\vdash \sigma(\alpha) \text{ for all } \alpha \in \Gamma \text{ implies } \vdash \sigma(\delta).$$

A logic $\mathcal{L}$ is **structurally complete** if every admissible rule of $\mathcal{L}$ is derivable; it is **hereditary structurally complete** if all its finitary extensions are structurally complete.

PROPOSITION 2.22.

(Folklore) Let $\mathcal{L}$ be an algebraizable logic with equivalent quasivariety semantics $\mathbf{Q}$. Then

1. $\mathcal{L}$ is structurally complete if and only if $\mathbf{Q}$ is structural;
2. $\mathcal{L}$ is hereditary structurally complete if and only if $\mathbf{Q}$ is primitive.

Therefore, structural completeness and hereditary structural completeness of extensions of those logics can be dealt with using purely algebraic arguments; for Łukasiewicz logic, the task has been carried out (in various stages) in [4, 38] and [6]. Here we take care of extensions of BL and its positive fragment.

3 Structural completeness in Wajsberg algebras and Wajsberg hoops

We will build our knowledge about (quasi) varieties of BL-algebras and basic hoops using Wajsberg algebras and Wajsberg hoops as building blocks of our knowledge. Here we recall some known facts (mostly from [6]) and also improve a little on them.

The proper subvarieties of both Wajsberg algebras and Wajsberg hoops have a nice description in terms of reduced pairs and triples of finite subsets of natural numbers. Let I, J be two finite subsets of positive integers. A pair (I, J) is **reduced** if

- no $i \in I$ divides an element $j \in I \cup J \setminus \{i\}$;
- no $j \in J$ divides an element $j' \in J \setminus \{j\}$ and
- $I \cup J \neq \emptyset$.

A triple (I, J, K) is **reduced** if (I, J) is reduced, $K \subseteq \{\omega\}$ and if $J \neq \emptyset$, then $K = \emptyset$.

3.1 Wajsberg hoops

If (I, J, K) is a reduced triple we define varieties of Wajsberg hoops

$$\begin{aligned}\mathbf{WH}(I, \emptyset, \{\omega\}) &= \mathbb{V}(\{\mathbf{L}_i^+ : i \in I\} \cup \{\mathbf{C}_\omega\}) \\ \mathbf{WH}(I, J, \emptyset) &= \mathbb{V}(\{\mathbf{L}_i^+ : i \in I\} \cup \{\mathbf{L}_{j,0}^+ : j \in J\}).\end{aligned}$$

THEOREM 3.1.

[9] The mapping $(I, J, K) \mapsto \mathbf{WH}(I, J, K)$ is a bijection between the reduced triples and the proper subvarieties of Wajsberg hoops; moreover, every proper subvariety of Wajsberg hoops is axiomatized by equations in just one variable.

Structural varieties of Wajsberg hoops have been completely described in [4] and [6].

THEOREM 3.2.

[4] [6] For a variety $\mathbf{V}$ of Wajsberg hoops the following are equivalent:

1. $\mathbf{V}$ is structural;
2. $\mathbf{V} = \mathbf{WH}(I, J, K)$ where $J \subseteq \{1\}$ and
3. $\mathbf{V}$ is primitive.

For structural quasivarieties of Wajsberg hoops we have:

THEOREM 3.3.

[6] Let $\mathbf{Q}$ be a quasivariety of Wajsberg hoops; then $\mathbf{Q}$ is structural if and only if it is either $\mathbf{Q} = \mathbb{Q}(\mathbf{F}_{\mathbf{WH}}(x))$ or there is a reduced triple (I, J, K) such that

$$\begin{aligned}\mathbf{Q} &= \mathbf{WH}(I, \emptyset, K) \quad \text{or} \\ \mathbf{Q} &= \mathbb{Q}(\{\mathbf{L}_i^+ : i \in I\} \cup \{\mathbf{L}_{j,1}^+ : j \in J\}) \quad \text{if } J \neq \emptyset.\end{aligned}$$

For primitivity we need a notation. Let $P = (I, J, K)$ be a triple (not necessarily reduced); then $\mathbf{Q}[I, J, K]$ is

$$\begin{aligned}\mathbf{Q}[I, \emptyset, \emptyset] &= \mathbb{Q}(\{\mathbf{L}_i^+ : i \in I\}) \\ \mathbf{Q}[I, \emptyset, \{\omega\}] &= \mathbb{Q}(\{\mathbf{L}_i^+ : i \in I\} \cup \{\mathbf{C}_\omega\}) \\ \mathbf{Q}[I, J, \emptyset] &= \mathbb{Q}(\{\mathbf{L}_i^+ : i \in I\} \cup \{\mathbf{L}_{j,1}^+ : j \in J\})\end{aligned}$$

If $I \subseteq \mathbb{N}$, we denote by $I \downarrow$ the set of all the divisors of elements of I .

THEOREM 3.4.

[6] Let $\mathbf{Q}$ be a quasivariety of Wajsberg hoops; then $\mathbf{Q}$ is primitive if and only if $\mathbf{Q} = \mathbf{Q}[I, J, K]$ where either $J = \emptyset$ or else $K = \emptyset$ and $I \downarrow \cap J \downarrow \subseteq \{1\}$.

3.2 Wajsberg algebras

If I, J is a reduced pair we define the variety of Wajsberg algebras

$$\mathbf{MV}(I, J) = \mathbb{V}(\{\mathbf{L}_i : i \in I\} \cup \{\mathbf{L}_{j,0} : j \in K\}).$$

THEOREM 3.5.

[46] The mapping $(I, J) \mapsto \mathbf{MV}(I, J)$ is a bijection between the reduced pairs and the proper subvarieties of Wajsberg algebras; moreover every proper subvariety of Wajsberg algebras is axiomatized by equations in just one variable.

For Wajsberg algebras the situation is different because of:

THEOREM 3.6.

(see [14] Proposition 5.10 and Corollary 5.16) Let $\mathbf{V}$ be a variety of *BL*-algebras; if $\mathbf{A} \in \mathbf{V}$ is weakly projective in $\mathbf{V}$ then $\mathbf{A}$ has $\mathbf{2}$ (the two-element boolean algebra) as a homomorphic image. If $\mathbf{V}$ is locally finite and $\mathbf{A}$ is finite, then the converse holds.

This clearly implies (via Theorem 2.11) that no variety of Wajsberg algebras containing a simple algebra different from $\mathbf{2} = \mathbf{L}_1$ can be primitive. Now all the $\mathbf{L}_n$ are simple and by the description of varieties of Wajsberg algebras in Theorem 3.5 if $\mathbf{MV}(I, J)$ is such that $I, J \not\subseteq \{1\}$, then $\mathbf{MV}(I, J)$ contains $\mathbf{L}_n$ for some $n > 1$; thus, it cannot be primitive. So the only candidates are $\mathbb{V}(\mathbf{L}_1)$ and $\mathbb{V}(\mathbf{L}_1^\infty)$. The former is the variety of Boolean algebras that is well-known to be primitive; for the latter, one can show that $\mathbb{V}(\mathbf{L}_1^\infty)$ covers $\mathbb{Q}(\mathbf{L}_1) = \mathbb{V}(\mathbf{L}_1)$ in the lattice of subquasivarieties (see the discussion after Lemma 8.3 in [4]). So to show that $\mathbb{V}(\mathbf{L}_1^\infty)$ is primitive, it is enough to show that it is structural.

Are there structural varieties of Wajsberg algebras that are not primitive? First we observe that $\mathbf{WA}$ is not structural; in fact, by Theorem 3.5, $\mathbb{H}(\mathbb{Q}(\mathbf{F}_{\mathbf{WA}}(x))) = \mathbf{WA}$. However $\mathbf{F}_{\mathbf{WA}}(x)$ satisfies the quasi equation

$$x^2 \approx x \text{ and } x \rightarrow 0 = 0 \Rightarrow x \approx 1.$$

This is well-known and follows from the description of free Wajsberg algebras in [51]. Thus $\mathbb{Q}(\mathbf{F}_{\mathbf{WA}}(x))$ is a proper subquasivariety of $\mathbf{WA}$ and $\mathbf{WA}$ is not structural. For proper locally finite subvarieties we have the following:

LEMMA 3.7.

(Lemma 8.3 in [4]) Let $\mathbf{V} = \mathbf{MV}(I, \emptyset)$ with $I \neq \{1\}$; then

$$\mathbb{Q}(\{\mathbf{L}_1 \times \mathbf{L}_i : i \in I\})$$

is a proper subquasivariety of $\mathbf{V}$.

As clearly $\mathbb{H}(\mathbb{Q}(\{\mathbf{L}_1 \times \mathbf{L}_i : i \in I\})) = \mathbf{V}$, $\mathbf{V}$ cannot be structural and hence the only structural locally finite variety of Wajsberg algebras is $\mathbb{V}(\mathbf{L}_1)$, the variety of Boolean algebras. For proper non locally finite subvarieties we have:

LEMMA 3.8.

Let $\mathbf{V} = \mathbf{MV}(I, J)$ with $J \not\subseteq \{1\}$; then $\mathbf{V}$ is not structural.

PROOF. Just copy the proof of the forward direction of Theorem 4.3 in [6], making the obvious modifications to the case of Wajsberg algebras. $\square$

So again the only possible candidate is $\mathbb{V}(\mathbb{L}_{1,0})$. To prove that it is structural, we can argue as in the other direction of Theorem 4.3 in [6]; so we need an analogue of Theorem 3.10 in [6] for Wajsberg algebras. But the analogue has been proved in [38] (Theorem 3.3 and Lemma 3.4) so we conclude that

THEOREM 3.9.

The only structural varieties of Wajsberg algebras are the variety of Boolean algebras and the variety generated by $\mathbb{L}_{1,0}$; they are also the only primitive subvarieties.

Quasivarieties are harder to deal with for the same reason as Wajsberg hoops. The classification of structural quasivarieties of Wajsberg algebras is essentially due to J. Gispert [38] (but see also [4], Section 7). Again, let us denote, for I, J finite subsets of $\mathbb{N}$

$$\mathbf{Q}[I, J] = \mathbf{Q}(\{\mathbb{L}_1 \times \mathbb{L}_i : i \in I\} \cup \{\mathbb{L}_1 \times \mathbb{L}_{j,1} : j \in J\}).$$

THEOREM 3.10.

[38] A quasivariety of Wajsberg algebras is structural if and only if either $\mathbf{Q} = \mathbf{Q}(\mathbf{F}_{\mathbf{WA}}(x))$ or $\mathbf{Q} = \mathbf{Q}[I, J]$ for some reduced pair (I, J) .

Now obviously $\mathbf{Q}(\mathbb{L}_1 \times \mathbb{L}_1) = \mathbf{Q}(\mathbb{L}_1)$ and $\mathbf{Q}(\mathbb{L}_1 \times \mathbb{L}_{1,1}) = \mathbf{Q}(\mathbb{L}_{1,1})$ (see [38]), so $\mathbf{Q}(\mathbb{L}_1)$ and $\mathbf{Q}(\mathbb{L}_{1,1})$ are the only structural varieties of Wajsberg algebras generated by chains.

Primitivity is easy for locally finite quasivarieties; by the results in [40] a quasivariety $\mathbf{Q}$ of Wajsberg algebras is locally finite if and only if it is finitely generated. Now we have:

LEMMA 3.11.

Let $\mathbf{Q} = \mathbf{Q}[I, \emptyset]$; if $\mathbf{Q}'$ is a subquasivariety of $\mathbf{Q}$, then $\mathbf{Q}' = \mathbf{Q}[I', \emptyset]$ for some $I' \subseteq I \downarrow$.

PROOF. $\mathbf{Q}$ is locally finite and so is $\mathbf{Q}'$, hence, by Theorem 2.9 in [40], $\mathbf{Q}'$ has to be generated by its critical algebras. Moreover, by Theorem 2.11 in [40], every critical algebra is isomorphic to $\mathbb{L}_{n_1} \times \cdots \times \mathbb{L}_{n_m}$, where

1. $n_s \neq n_t$ for every $1 \leq s \neq t \leq m$;
2. there exists at most one s such that n_s is a multiple of an n_t with $s \neq t$.

Let $\mathbf{A}$ be a critical algebra in $\mathbf{Q}'$; in particular $\mathbf{A} \in \mathbf{Q}$, so $\mathbf{A} \in \mathbf{ISP}_u(\{\mathbb{L}_1 \times \mathbb{L}_i : i \in I\})$, and, since $\{\mathbb{L}_1 \times \mathbb{L}_i : i \in I\}$ is a finite set of finite Wajsberg algebras, we have that $\mathbf{A} \in \mathbf{ISP}(\{\mathbb{L}_1 \times \mathbb{L}_i : i \in I\})$; in other words $\mathbf{A}$ is embeddable in $\mathbf{B} = \prod_{k \in K} \mathbb{L}_1 \times \mathbb{L}_{i_k}$, for some K and $i_k \in I$ for every $k \in K$. This means that $\mathbf{B}$ is of the form $\prod_{j \in J} \mathbb{L}_{n_j}$, where $\{n_j : j \in J\} \subseteq I \cup \{1\}$; thus, the family $\{n_j : j \in J\}$ is finite and we are in the hypothesis of Lemma 2.10 in [40].

Now, $\mathbf{A} \cong \mathbb{L}_{n_1} \times \cdots \times \mathbb{L}_{n_m}$ is embeddable in $\mathbf{B} = \prod_{k \in K} \mathbb{L}_1 \times \mathbb{L}_{i_k}$, and $\mathbf{B}$ always has at least one factor equal to $\mathbb{L}_1$, so by Theorem 2.10 (2) in [40], there exists at least one factor of $\mathbf{A}$ equal to $\mathbb{L}_1$. Recalling the second condition of the characterization of critical algebras at the beginning of this proof, if a factor of $\mathbf{A}$ is $\mathbb{L}_1$ then, either $\mathbf{A} \cong \mathbb{L}_1$, or $\mathbf{A}$ has only two factors, because every natural number is a multiple of 1. Moreover, if $\mathbf{A}$ has a second factor $\mathbb{L}_n$ then n has to divide an $i \in I$.

Therefore, $\mathbf{Q}'$ is generated by algebras of the form $\mathbb{L}_1 \times \mathbb{L}_n$, where $n \in I \downarrow$, so $\mathbf{Q}' = \mathbf{Q}[I', \emptyset]$ for some $I' \subseteq I \downarrow$. $\square$

As a corollary we get at once.

COROLLARY 3.12.

For a locally finite quasivariety $\mathbf{Q}$ of Wajsberg algebras the following are equivalent:

1. $\mathbf{Q}$ is structural;
2. $\mathbf{Q} = \mathbf{Q}[I, \emptyset]$ for some finite set I and
3. $\mathbf{Q}$ is primitive.

PROOF. Assume that $\mathbf{Q}$ is structural; since it is locally finite, by Theorem 3.10, it is equal to $\mathbf{Q}[I, \emptyset]$ for some I and (2) holds. But if it such, then by Lemma 3.11 all of its subvarieties have the same form and thus they are all structural. Therefore $\mathbf{Q}$ is primitive and (3) holds. Finally (3) trivially implies (1). $\square$

The general case is still open. However we remark that, as observed in [6], the condition in Theorem 3.4 is still necessary for a quasivariety of Wajsberg algebras to be primitive. It follows that there are quasivarieties that are structural but not primitive.

4 Structural varieties of basic hoops

4.1 Two constructions

4.1.1 The ordered sum of varieties

Every variety of BL-algebras or basic hoops is totally determined by its chains and each chain is the ordinal sum of totally ordered Wajsberg hoops that are either bounded or cancellative (and in the case of BL-algebras then the first component is necessarily bounded and it is a Wajsberg algebra). This suggests a way to describe varieties of BL-algebras in a fashion in which the varieties generated by the Wajsberg components are in evidence. Of course not every variety of BL-algebras or basic hoops can be described in this way, but the definition captures enough varieties to allow some meaningful results. First if $\mathbf{V}$ is a variety of BL-algebras or basic hoops we denote by $\mathbf{V}^t$ the class of chains in $\mathbf{V}$. Next, if $\mathbf{V}$ is a variety of basic algebras or basic hoops and $\mathbf{W}$ is the variety of basic hoops we define the **ordered sum** of $\mathbf{V}$ and $\mathbf{W}$ as

$$\mathbf{V} \oplus^t \mathbf{W} = \mathbb{V}(\{\mathbf{A} \oplus \mathbf{B} : \mathbf{A} \in \mathbf{V}^t, \mathbf{B} \in \mathbf{W}^t\}),$$

The operation is clearly associative, so we may dispose of the parentheses in case of three or more summands. It is also obvious that $\mathbf{V} \oplus^t \mathbf{W}$ is a variety of BL-algebras if and only if $\mathbf{V}$ is such. Moreover, if for a variety $\mathbf{V}$ of BL-algebras we denote by $\mathbf{V}^+$ the variety of its hoop subreducts (it is always a variety by Proposition 1.10 in [5]) then it follows immediately from congruence distributivity and Lemma 2.18 that

$$(\mathbf{V} \oplus^t \mathbf{W})^+ = \mathbf{V}^+ \oplus^t \mathbf{W}.$$

In general there is a way to axiomatize $\mathbf{V} \oplus^t \mathbf{BH}$ where $\mathbf{V}$ is a variety of Wajsberg algebras. First we remind that every variety proper variety $\mathbf{V}$ of Wajsberg algebras or Wajsberg hoops is axiomatized (relative to Wajsberg algebras or Wajsberg hoops) by a single equation in one variable $t_V(x) \approx 1$ [9, 34].

Recall that in every variety with a constant 0 in the signature, one can define a new unary operation $\neg x$ in the usual way as $x \rightarrow 0$.

LEMMA 4.1.

If $\mathbf{V}$ is a proper variety of Wajsberg algebras, then $\mathbf{V} \oplus^t \mathbf{BH}$ is axiomatized modulo $\mathbf{BH}$ by the single equation $t_V(\neg\neg x) \approx 1$.

PROOF. Let $\mathbf{A} = \mathbf{A}_0 \oplus \mathbf{A}_1$ be a totally ordered algebra in $\mathbf{V} \oplus^t \mathbf{BH}$; then for any $a \in A$, $\neg\neg a \in A_0$, which proves that $t_V(\neg\neg x) \approx 1$ holds in $\mathbf{A}$ and hence $\mathbf{V} \oplus^t \mathbf{BH}$. Conversely, suppose that $\mathbf{A} \notin \mathbf{V} \oplus^t \mathbf{BH}$; by replacing $\mathbf{A}$ with one of its subdirectly irreducible factors we may assume that $\mathbf{A}$ is totally ordered and thus the first component $\mathbf{A}_0 \notin \mathbf{W}$. Hence there is an $a \in \mathbf{A}_0$ with $t_V(a) \neq 1$; as $\neg\neg a = a$ we obtain that $t_V(\neg\neg x) \approx 1$ does not hold in $\mathbf{A}$.

As $\mathbf{V} \oplus^t \mathbf{BH} = \mathbb{V}(\{\mathbf{A} \oplus \mathbf{B} : \mathbf{A} \in \mathbf{V}^t, \mathbf{B} \in \mathbf{BH}^t\})$, if $\mathbf{C}$ is a totally ordered algebra in $\mathbf{V} \oplus^t \mathbf{BH}$ by Jónsson Lemma

$$\mathbf{C} \in \mathbf{HSP}_u(\{\mathbf{A} \oplus \mathbf{B} : \mathbf{A} \in \mathbf{V}^t, \mathbf{B} \in \mathbf{BH}^t\}).$$

Now a straightforward application of Corollaries 2.17 and 2.18 yields that $\mathbf{C} = \mathbf{C}_0 \oplus \mathbf{C}_1$ where $\mathbf{C}_0 \in \mathbf{HSP}_u(\mathbf{A}_0)$. So $\mathbf{C} \in \mathbf{V}^t$ as wished. $\square$

For example, as an easy consequence, we get that the variety $\mathbb{V}(\mathbf{L}_{1,0}) \oplus^t \mathbf{BH}$, often referred to as **dIBL**, is axiomatized by

$$(\neg\neg x)^2 \oplus (\neg\neg x)^2 \approx (\neg\neg x \oplus \neg\neg x)^2,$$

where $x \oplus y := \neg(\neg x \cdot \neg y)$.

The case in which we have the ordered sum of proper subvarieties (and/or more than two summands) is more complex; not to interrupt the flow of information, we have moved all the proofs to Appendix A.

4.1.2 Generalized rotations

Generalized rotations of BL-algebras and basic hoops have been studied in several papers [10, 11, 13, 41]. Here we will deal mostly with a special kind of generalized rotations: given any basic hoop $\mathbf{A}$ the n -**lifting** of $\mathbf{A}$ is the algebra $\mathbf{L}_n \oplus \mathbf{A}$. Given a variety $\mathbf{V}$ of basic hoops its n -**lifting** is the variety $\mathbb{V}(\mathbf{L}_n) \oplus^t \mathbf{V}$; by Lemma 4.1 it follows that

$$\mathbb{V}(\mathbf{L}_n) \oplus^t \mathbf{V} = \mathbb{V}(\{\mathbf{L}_k \oplus \mathbf{A} : k \mid n \text{ and } \mathbf{A} \text{ is a totally ordered algebra in } \mathbf{V}\}).$$

Note that in our definition the n -lifting of a variety of basic hoops is always a variety of Wajsberg algebras. We will also consider its **BH**- n -lifting as the variety

$$\mathbb{V}(\mathbf{L}_n^+) \oplus^t \mathbf{V} = \mathbb{V}(\{\mathbf{L}_k^+ \oplus \mathbf{A} : k \mid n \text{ and } \mathbf{A} \text{ is a totally ordered algebra in } \mathbf{V}\}).$$

However, we will often drop the decoration **BH** when the situation is clear from the context. We will deal with a different kind of generalized rotations in Appendix B.

4.2 The Baker–Beynon property in varieties of basic hoops

For any variety $\mathbf{V}$ of hoops the $\mathbf{V}$ -free algebra on the empty set is trivial. This implies at once that every hoop is unifiable and thus if a variety of hoops has the Baker–Beynon property, then it is structural and if it is also locally finite, then it is primitive (Theorem 2.13).

So, which varieties of basic hoops have the Baker–Beynon property? Let's start with locally finite varieties. The locally finite varieties of hoops have been described a long time ago in [21], Theorem 3.6. The proof therein uses a particular construction from which it is clear that the result holds for BL-algebras and basic hoops as well. However there is an easy direct proof, that we display here for the reader's sake.

THEOREM 4.2.

For a variety $\mathbf{V}$ of basic hoops the following are equivalent:

1. $\mathbf{V}$ is locally finite;
2. $\mathbf{V}$ satisfies the equation $x^k \approx x^{k+1}$ for some $k \geq 1$ and
3. for all totally ordered $\mathbf{A} = \bigoplus_{i \in I} \mathbf{A}_i \in \mathbf{V}$ (where the decomposition is in its Wajsberg components), for all i there is an $m_i \leq k + 1$ such that $\mathbf{A}_i \cong \mathbf{L}_{m_i}^+$.

PROOF. (1) implies (2) trivially; if $\mathbf{V}$ is locally finite, then its free algebra on the single generator x is finite and hence it must satisfy $x^k \approx x^{k+1}$ for some k .

Assume (2) and let $\mathbf{A} = \bigoplus_{i \in I} \mathbf{A}_i \in \mathbf{V}$; then each component must satisfy $x^k \approx x^{k+1}$. But it is well-known (and rather obvious) that the only totally ordered Wajsberg hoops satisfying such an equations are the $\mathbf{L}_n$ with $n \leq k + 1$.

Finally assume (3) and let $\mathbf{A}$ be a totally ordered m -generated algebra in $\mathbf{V}$. It is clear that $\mathbf{A}$ has at most m Wajsberg components and each component has at most $k + 1$ elements and this implies that $|A| \leq (k + 1)m$. So there is a bound for the size of the m -generated totally ordered algebras (hence for all the subdirectly irreducible algebras) which depends only on m . By the well-known Mal'cev criterion ([48], Theorem 14.1.3) $\mathbf{V}$ is locally finite. $\square$

In [14] it was shown that any locally finite quasivariety of (basic) hoops has the Baker–Beynon property, thus:

PROPOSITION 4.3.

Every locally finite variety of basic hoops is primitive.

For the non locally finite varieties in [13] the authors, building on the early work by Beynon [19] and Baker [16], showed that the variety $\mathbf{C}$ of cancellative hoops (that are basic hoops) has the Baker–Beynon property. Another important contribution is [41]; there the author showed that both the variety $\mathbb{V}(\mathbf{L}_{1,0}^+)$, usually called **dIWH**, and the variety **PH** of product hoops have the Baker–Beynon property and thus are structural. Since $\mathbf{PH} = \mathbb{V}(\mathbf{L}_2^+) \oplus^t \mathbf{C} = \mathbb{V}(\mathbf{L}_2^+ \oplus \mathbf{C}_\omega)$ (by Theorem A.6) it is only natural to ask whether the result can be generalized to $\mathbb{V}(\mathbf{L}_n^+) \oplus^t \mathbf{C}$ for $n \geq 2$. This will turn out to be the case, but proving it will take a surprising amount of work which will occupy most of this section. The difficult part, like in the **PH** case, will turn out to be characterizing free algebras in these varieties.

We start from the following well known fact (see [54], Theorem 3.2).

THEOREM 4.4.

Let $\mathbf{V}$ be a variety of bounded commutative residuated lattices, and $\mathbf{V}^+$ the variety of commutative residuated lattices given by its 0-free subreducts. Then, for any set X , $\mathbf{F}_{\mathbf{V}^+}(X)$ is isomorphic to the subalgebra of the 0-free reduct of the free algebra $\mathbf{F}_{\mathbf{V}}(X)$ generated by X .

This allows one to give a representation for $\mathbf{F}_{\mathbf{V}^+}(X)$ starting from a representation of $\mathbf{F}_{\mathbf{V}}(X)$, by properly characterizing the 0-free subreduct of $\mathbf{F}_{\mathbf{V}}(X)$ generated by X . Recall that we can see the elements of a free algebra $\mathbf{F}_{\mathbf{V}}(X)$ as equivalence classes of terms in the language of $\mathbf{V}$ written over variables in X ; thus characterizing $\mathbf{F}_{\mathbf{V}^+}(X)$ corresponds to characterizing the equivalence classes of the 0-free terms. We will call these elements of $\mathbf{F}_{\mathbf{V}}(X)$ **positive**.

Therefore, in order to get a description of the free algebras of the **BH**- n -liftings of $\mathbf{C}$, which is the thing we are interested in, we need to start from the characterization of free **BL**-algebras in $\mathbf{PL}_n := \mathbb{V}(\mathbf{L}_n \oplus \mathbf{C}_\omega)$. This description can be found in [24] or [13] and we will follow the latter more closely. The representation is given as the weak Boolean product of directly indecomposable algebras over the Stone space of the Boolean skeleton of the free algebras $\mathbf{F}_{\mathbf{PL}_n}(X)$; for a proper definition of the weak Boolean product of directly indecomposable algebras over the Stone space, we direct

the reader to [13], Section 3. For any BL-algebra $\mathbf{A}$, we define its **MV**-skeleton as $\mathbf{MV}(\mathbf{A}) = \{a \in A \mid \neg\neg a = a\}$ and its Boolean skeleton as $\mathbf{Bool}(\mathbf{A}) = \{a \in A \mid a \vee \neg a = 1 \text{ and } a \wedge \neg a = 0\}$. The **MV**-skeleton of $\mathbf{A}$ is a Wajsberg algebra and it is the greatest Wajsberg algebra contained in $\mathbf{A}$; similarly the Boolean skeleton is the greatest Boolean algebra contained in $\mathbf{A}$. An **MV**-retraction term on $\mathbf{A}$ is any unary term which induces a retraction from $\mathbf{A}$ into $\mathbf{MV}(\mathbf{A})$. It can be proved that the term $\neg\neg x$ is always an **MV**-retraction term [28] and thus respects all other operations.

PROPOSITION 4.5.

[24] Given any set X , the **MV**-skeleton of the free algebra in $\mathbf{PL}_n$ over X is the free algebra in $\mathbb{V}(\mathbb{L}_n)$ over the set $\{\neg\neg x \mid x \in X\}$.

It is not hard to see that, in the case of algebras in $\mathbf{PH} = \mathbf{PL}_1^+$, the **MV**-skeleton actually coincides with the Boolean skeleton; this is what made the analysis in [41] much easier. In order to understand the Boolean skeleton of free $\mathbf{PL}_n$ -algebras we shall use a characterization of the Boolean skeleton of free algebras in $\mathbb{V}(\mathbb{L}_n)$ in [24], to which we refer the reader. This makes use of the so-called **Moisil operators**. Let $\mathbf{A} \in \mathbb{V}(\mathbb{L}_n)$; Moisil operators on $\mathbf{A}$ are one-variable terms $\sigma_1^n(x), \dots, \sigma_n^n(x)$ in the language $(\neg, \rightarrow, 1)$ that map $\mathbf{A}$ to $\mathbf{Bool}(\mathbf{A})$, such that:

- $x \in \mathbf{Bool}(\mathbf{A})$ if and only if $\sigma_n^n(x) = x$;
- $\sigma_n^n(x) = \min\{b \in \mathbf{Bool}(\mathbf{A}) \mid x \leq b\}$ and
- $\sigma_1^n(x) = \max\{b \in \mathbf{Bool}(\mathbf{A}) \mid b \leq x\}$.

In particular, if they are evaluated in the Wajsberg chain $\mathbb{L}_n = \{\frac{0}{n}, \dots, \frac{1}{n}\}$ they are such that

$$\sigma_i^n(\frac{j}{n}) = 1 \text{ if } i + j > n, 0 \text{ otherwise.}$$

REMARK 4.6.

Moisil operators applied on a set X of generators of a free algebra in $\mathbb{V}(\mathbb{L}_n)$ result in a poset $(X_\sigma, \leq)$, where for each $x \in X$, $\sigma_1^n(x) < \dots < \sigma_n^n(x)$, and elements from different such chains are incomparable.

Recall that an algebra $\mathbf{F}$ generated by a set X in a variety $\mathbf{V}$ is free over a poset $(X, \leq)$ if for each $\mathbf{A} \in \mathbf{V}$ and for each non-decreasing function $f : X \rightarrow \mathbf{A}$, f can be uniquely extended to a homomorphism from $\mathbf{F}$ to $\mathbf{A}$. From now on, given a set X , we define $X^{\gamma\sigma}$ as $\{\sigma_i^n(\neg\neg x) \mid x \in X, i = 1, \dots, n\}$.

THEOREM 4.7.

[24]

1. (Theorem 2.4 and Corollary 2.13) $\mathbf{Bool}(\mathbf{F}_{\mathbf{PL}_n}(X)) = \mathbf{Bool}(\mathbf{MV}(\mathbf{F}_{\mathbf{PL}_n}(X)))$ is the free Boolean algebra over the poset $(X^{\gamma\sigma}, \leq)$.
2. (Lemma 1.6) The correspondence that assigns each upwards closed subset $S \subseteq X^{\gamma\sigma}$ to the Boolean ultrafilter U_S generated by the set $S \cup \bar{S}$, where $\bar{S} = \{\neg y \mid y \in X^{\gamma\sigma} - S\}$, defines a bijection from the set of upwards closed subsets of $X^{\gamma\sigma}$ onto the ultrafilters of $\mathbf{Bool}(\mathbf{F}_{\mathbf{PL}_n})$.

Thus, the free algebra $\mathbf{F}_{\mathbf{PL}_n}$ can be represented as a weak Boolean product indexed by the upwards closed subsets $S \subseteq X^{\gamma\sigma}$. Each factor is then going to be a quotient of $\mathbf{F}_{\mathbf{PL}_n}(X)$ with respect to the filter generated in $\mathbf{F}_{\mathbf{PL}_n}(X)$ by U_S ; let us call this filter F_S . These factors have

been completely characterized in [24], starting from free algebras in the variety of cancellative hoops $\mathbf{F}_C$.

THEOREM 4.8.

[24] Given any set X , $\mathbf{F}_{\mathbf{PL}_n}(X)$ is representable as the weak Boolean product of the family

$$\{\mathbf{F}_{\mathbf{PL}_n}(X)/F_S : S \text{ upward closed subset of } X^{\gamma^\sigma}\}.$$

In particular, for each upward closed subset $S \subseteq X^{\gamma^\sigma}$:

$$\mathbf{F}_{\mathbf{PL}_n}(X)/F_S \cong \mathbf{L}_{k_S} \oplus (\mathbf{F}_C(X_S)),$$

for some k_S that divide n and with $X_S = \{x/F_S : \neg\neg(x/F_S) = 1\} \cup \{\neg x/F_S : \neg\neg(x/F_S) = 0\}$.

Now, all we need to do is characterize the positive elements of this weak Boolean product. This will rely heavily on the fact that the MV-retraction term $\neg\neg x$ is positive, i.e. it can be written without using the constant 0.

LEMMA 4.9.

In $\mathbf{PL}_n$, the MV-retraction term $\neg\neg x$ is equivalent to the term $\gamma_n(x) = (x^n \rightarrow x^{n+1}) \rightarrow x$ and therefore it is positive.

PROOF. Since $\mathbf{PL}_n$ is generated by $\mathbf{L}_n \oplus \mathbf{C}_\omega$, it suffices to verify that $\neg\neg x = \gamma_n(x)$ in $\mathbf{L}_n \oplus \mathbf{C}_\omega$.

Take $x \in \mathbf{L}_n \oplus \mathbf{C}_\omega$.

- If $x \in \mathbf{L}_n - \{1\}$, then $\neg\neg x = x$, while

$$\gamma_n(x) = (x^n \rightarrow x^{n+1}) \rightarrow x = (0 \rightarrow 0) \rightarrow x = 1 \rightarrow x = x.$$

- If $x \in \mathbf{C}_\omega$, then $\neg\neg x = \neg 0 = 1$, while

$$\gamma_n(x) = (x^n \rightarrow x^{n+1}) \rightarrow x = x \rightarrow x = 1.$$

□

This implies that, given any set X , the set of the generators of the MV-skeleton of $\mathbf{F}_{\mathbf{PL}_n}(X)$ can be taken also as $\{\gamma_n(x) \mid x \in X\}$; furthermore, we can also prove that all Moisil operators are positive as well.

LEMMA 4.10.

In $\mathbb{V}(\mathbf{L}_n)$, the terms $\sigma_1^n(x), \dots, \sigma_n^n(x)$ are positive.

PROOF. Since, for what we want, we will only need to know that those terms are positive, we will use a trick to show their positivity without showing their equivalent positive terms (i.e. those written without the 0). As shown in [9], an MV-term $t(x)$ in $\mathbb{V}(\mathbf{L}_n)$ is positive if and only if $t(1) = 1$ in $\mathbf{L}_n$. In this case, for every $i = 1, \dots, n$ we get $\sigma_i^n(1) = \sigma_i^n(\frac{n}{n})$, that is 1 because $i + n > n$ for every $i = 1, \dots, n$. □

Combining Lemmas 4.9 and 4.10 we have that all the elements in X^{γ^σ} are positive. We are now ready to ‘rewrite’ elements of the free algebras in these varieties, in order to be able to describe the positive ones; the strategy here is similar to the one in [41] for the variety $\mathbf{PH}$, but in a more complex and general scenario.

Let X be a set of variables, and let $p \in \mathbf{F}_{\mathbf{PL}_n}(X)$; notice that there are $x_1, \dots, x_m \in X$ such that $p = u(x_1, \dots, x_m)$ for some term u in the language of $\mathbf{PL}_n$. Let $X_m = \{x_1, \dots, x_m\}$, then p actually

belongs to the subalgebra $\mathbf{FPL}_n(X_m)$ of $\mathbf{FPL}_n(X)$. Let us now consider the atoms of the Boolean skeleton of $\mathbf{FPL}_n(X_m)$. Since (X_m) is finite, so is $(X_m^{\gamma\sigma}, \leq)$. By Theorem 4.7, $\mathbf{Bool}(\mathbf{FPL}_n(X_m))$ is the free Boolean algebra over the poset $(X_m^{\gamma\sigma}, \leq)$, hence it is finite too. It is well known that the atoms of a finite Boolean algebra are in one to one correspondence to the ultrafilters of the same algebra; indeed, the ultrafilters are all principal and generated by the atoms. By Theorem 4.7 once again, we have that given any upwards closed subset $S \subseteq X_m^{\gamma\sigma}$, we can associate an atom

$$a_S = \bigwedge_{y \in S} y \wedge \bigwedge_{z \in X_m^{\gamma\sigma} - S} \neg z, \quad (1)$$

and all atoms can be written in this way. To simplify the notation we shall write a_+ for $a_{X_m^{\gamma\sigma}}$ and U_+ for its associated ultrafilter, since these two will play an important role in what follows. Notice that $U_+ = U_{X_m^{\gamma\sigma}}$ is an ultrafilter of $\mathbf{Bool}(\mathbf{FPL}_n(X_m))$ that is a finite subalgebra of $\mathbf{Bool}(\mathbf{FPL}_n(X))$, hence in general it is different from $U_{X^{\gamma\sigma}}$. Nevertheless U_+ is always contained in $U_{X^{\gamma\sigma}}$ for all $X_m \subseteq X$, because it is its restriction to $\mathbf{Bool}(\mathbf{FPL}_n(X_m))$.

a_+ is the meet of all elements in $X_m^{\gamma\sigma}$, but by Remark 4.6, it is equivalent to saying that it is the meet of all $\{\sigma_1^n(\neg\neg x_1), \dots, \sigma_1^n(\neg\neg x_m)\}$, because for every i, j we have $\sigma_1^n(\neg\neg x_j) < \sigma_i^n(\neg\neg x_j)$. This means that U_+ is generated by $\sigma_1^n(\neg\neg x_1) \wedge \dots \wedge \sigma_1^n(\neg\neg x_m)$. Moreover we also have the following.

LEMMA 4.11.

The equation $\sigma_1^n(x) \leq x$ holds in $\mathbb{V}(\mathbb{L}_n)$.

PROOF. Once again it suffices to verify the inequality in $\mathbb{L}_n$ because it generates the variety. By definition $\sigma_1^n(\frac{j}{n})$ is 1 if $j + 1 > n$, so if $j = n$, otherwise it is 0, hence it is less or equal than the identity. $\square$

Lemma 4.11 gives that, for every $x \in X$, we have $\sigma_1^n(\neg\neg x) \leq \neg\neg x$. Since every $\sigma_1^n(\neg\neg x)$ is in $U_{X^{\gamma\sigma}} \subseteq F_{X^{\gamma\sigma}}$ and $F_{X^{\gamma\sigma}}$ is closed upwards, this implies that:

COROLLARY 4.12.

For every $x \in X$ we have $\neg\neg x \in F_{X^{\gamma\sigma}}$.

As a consequence we also have that $\{x/F_{X^{\gamma\sigma}} : \neg\neg(x/F_{X^{\gamma\sigma}}) = 1\} = X$ hence, by Theorem 4.8, $\mathbf{FPL}_n(X)/F_{X^{\gamma\sigma}} \cong \mathbb{L}_k \oplus (\mathbf{FC}(X))$ for some k that divides n . Now let us show that one can use the atoms of finitely generated free algebras to rewrite the elements of possibly infinitely generated algebras. From now on, given a poset $(P, \leq)$, we will denote with $\mathbf{Ups}(P)$ the set of all upward closed subsets S of P .

Let X be any set and $p = u(x_1, \dots, x_m) \in \mathbf{FPL}_n(X)$. Since $\bigwedge_{S \in \mathbf{Ups}(X_m^{\gamma\sigma})} \neg a_S = 0$,

$$p = p \vee \bigwedge_{S \in \mathbf{Ups}(X_m^{\gamma\sigma})} \neg a_S = \bigwedge_{S \in \mathbf{Ups}(X_m^{\gamma\sigma})} p \vee \neg a_S = \bigwedge_{S \in \mathbf{Ups}(X_m^{\gamma\sigma})} a_S \rightarrow p, \quad (2)$$

where the second equality is just an application of distributivity, and the last equality holds because a_S is a Boolean element, so $\neg a_S \vee p = a_S \rightarrow p$ as in Boolean algebras.

REMARK 4.13.

With respect to the weak Boolean product representation

$$\mathbf{FPL}_n(X) \leq \prod_{S \in \mathbf{Ups}(X^{\gamma\sigma})} \mathbf{FPL}_n(X)/F_S,$$

we observe that an element $p \in \mathbf{FPL}_n(X)$ embeds into the tuple

$$(p/F_S)_{S \in \mathbf{Ups}(X^{\gamma\sigma})} = ((p \vee \neg a_S)/F_S)_{S \in \mathbf{Ups}(X^{\gamma\sigma})}.$$

Since $a_S \in U_S \subseteq F_S$ and then $a_S/F_S = 1/F_S$. We will often use this representation in proofs.

Now we will show that one can reduce the analysis of the positivity of an element p in free algebras to its ‘components’ $p \vee \neg a_S$. The first result that we will prove in this direction is the following.

LEMMA 4.14.

$b \in \mathbf{Bool}(\mathbf{FPL}_n(X))$ is positive if and only if $b \in U_{X^{\gamma\sigma}}$.

PROOF. We start by showing the left-to-right direction by contrapositive; assume that $b \notin U_{X^{\gamma\sigma}}$. Then, since $U_{X^{\gamma\sigma}}$ is an ultrafilter, $\neg b \in U_{X^{\gamma\sigma}}$, i.e. there are $y_1, \dots, y_m \in X^{\gamma\sigma}$ such that $y_1 \wedge \dots \wedge y_m \leq \neg b$. Then we get that $b \wedge y_1 \wedge \dots \wedge y_m \leq b \wedge \neg b = 0$, and thus $b \wedge y_1 \wedge \dots \wedge y_m = 0$. Since by Lemmas 4.9 and 4.10 all $y \in X^{\gamma\sigma}$ are positive, if b were to be positive then 0 would be positive as well; the latter yields a contradiction. Thus if $b \notin U_{X^{\gamma\sigma}}$, b is not positive.

Assume now that b is not positive, we show that $b \notin U_{X^{\gamma\sigma}}$. Notice that, since $\mathbf{Bool}(\mathbf{FPL}_n(X))$ is generated by $X^{\gamma\sigma}$, there exists a Boolean term p such that $b = p(y_1, \dots, y_m)$ for some $y_1, \dots, y_m \in X^{\gamma\sigma}$. It is well known that Boolean terms are either positive or negation of a positive term. Since every y_i is positive and b is not, we get that necessarily $p = \neg p'$ for some p' written in the 0-free language. Now, each $y_i \in U_X^{\gamma\sigma}$ by definition, and (ultra)filters are closed under the operations $\{\cdot, \rightarrow, \wedge, \vee\}$, thus $p'(y_1, \dots, y_m) \in U_X$, which implies that $b = \neg p'(y_1, \dots, y_m) \notin U_X$. This completes the proof. $\square$

It is now immediate to see that given a finite set $X_m = \{x_1, \dots, x_m\}$, the only positive atom of $\mathbf{Bool}(\mathbf{FPL}_n(X_m))$ is a_+ (since it is the only one in $U_{X_m^{\gamma\sigma}}$), while all other atoms a_S for $S \neq X_m^{\gamma\sigma}$ are not positive. Conversely, all $\neg a_S$ for $S \neq X_m^{\gamma\sigma}$ are positive, while $\neg a_+$ is not.

LEMMA 4.15.

Let X be any set of variables and consider $p(x_1, \dots, x_m) \in \mathbf{FPL}_n(X)$; let $X_m = \{x_1, \dots, x_m\}$ and $S \in \mathbf{Ups}(X_m^{\gamma\sigma})$ with $S \neq X_m^{\gamma\sigma}$. Then $p \vee \neg a_S$ is positive.

PROOF. As observed right above, $\neg a_S$ is positive as a consequence of Lemma 4.14. Consider the term p' obtained starting from p and substituting each occurrence of 0 with $\neg a_S$ (if p has no occurrences of 0, it means that it is positive, and we get $p' = p$); p' is 0-free and thus positive by construction.

We show that $p \vee \neg a_S = p' \vee \neg a_S$. Let us consider their representative in the weak Boolean product representation, i.e. respectively the tuples

$$(p \vee \neg a_S/F_{S'})_{S' \in \mathbf{Ups}(X^{\gamma\sigma})} \text{ and } (p' \vee \neg a_S/F_{S'})_{S' \in \mathbf{Ups}(X^{\gamma\sigma})}.$$

We distinguish two cases: $\neg a_S \in U_{S'}$ and $\neg a_S \notin U_{S'}$. In the former case, we get that $p \vee \neg a_S, p' \vee \neg a_S \in F_{S'}$ and then $(p \vee \neg a_S)/F_{S'} = (p' \vee \neg a_S)/F_{S'} = 1/F_{S'}$. In the latter case, $\neg a_S \notin U_{S'}$ implies that $\neg a_S/F_{S'} = 0/F_{S'}$ and then by the definition of p' we get that $(p \vee \neg a_S)/F_{S'} = (p' \vee \neg a_S)/F_{S'}$. Thus $p \vee \neg a_S = p' \vee \neg a_S$, so it is positive. $\square$

The previous result is extremely useful for our purpose, because essentially it yields that the positivity of a term p is completely determined by its ‘component’ $p \vee \neg a_+$. We will now show that such an element is positive if and only if its component $(p \vee \neg a_+)/F_{X^{\gamma\sigma}}$ in the weak Boolean product

representation is such that

$$\neg\neg((p \vee \neg a_+)/F_{X^{\gamma\sigma}}) = 1/F_{X^{\gamma\sigma}}.$$

Note that $\mathbf{F}_{\mathbf{PL}_n}(X)/F_{X^{\gamma\sigma}} \cong \mathbb{L}_k \oplus (\mathbf{F}_{\mathbf{C}}(X))$ for some k that divides n , hence saying $\neg\neg((p \vee \neg a_+)/F_{X^{\gamma\sigma}}) = 1/F_{X^{\gamma\sigma}}$ is equivalent to saying that $(p \vee \neg a_+)/F_{X^{\gamma\sigma}}$ is in the summand $\mathbf{F}_{\mathbf{C}}(X)$ of the above ordinal sum.

THEOREM 4.16.

Let X be any set of variables and consider $p(x_1, \dots, x_m) \in \mathbf{F}_{\mathbf{PL}_n}(X)$; let $a_+ = \sigma_1^n(\neg x_1) \wedge \dots \wedge \sigma_1^n(\neg x_m)$. Then p is positive if and only if $p \vee \neg a_+$ is positive if and only if $\neg\neg(p \vee \neg a_+) \in U_{X^{\gamma\sigma}}$.

PROOF. The first equivalence follows directly from Lemma 4.15. Indeed, if p is positive, then $p \vee \neg a_+ = a_+ \rightarrow p$ by Equation (2), and the latter is positive because p and a_+ are. Conversely, suppose $p \vee \neg a_+$ is positive; $p = \bigwedge_{S \in \text{Ups}(X_m^{\gamma\sigma})} p \vee \neg a_S$ so, when $S \neq X_m^{\gamma\sigma}$ we have that $p \vee \neg a_S$ is positive by Lemma 4.15, and $p \vee \neg a_+$ is positive by hypothesis; hence p is a meet of positive terms, therefore it is positive.

Let us now show the second equivalence, starting from the direction $p \vee \neg a_+$ positive implies $\neg\neg(p \vee \neg a_+) \in U_{X^{\gamma\sigma}}$. Let us consider the representation of the element $p \vee \neg a_+$ in the weak Boolean product, i.e. the tuple $((p \vee \neg a_+)/F_S)_{S \in \text{Ups}(X^{\gamma\sigma})}$. Since we assume that $p \vee \neg a_+$ is positive, there is a 0-free term over variables $x_1, \dots, x_m$ in X such that $p \vee \neg a_+ = q(x_1, \dots, x_m)$. Since $\neg\neg$ is an MV-retraction term, by definition it induces a homomorphism, thus $\neg\neg(q(x_1, \dots, x_m)) = q(\neg\neg(x_1), \dots, \neg\neg(x_m))$; since each $\neg\neg(x_i) \in U_{X^{\gamma\sigma}}$ by Corollary 4.12, and q is written with operations in $\{\cdot, \rightarrow, \wedge, \vee\}$ under which $U_{X^{\gamma\sigma}}$ is closed, $\neg\neg(p \vee \neg a_+) = q(\neg\neg(x_1), \dots, \neg\neg(x_m)) \in U_{X^{\gamma\sigma}}$.

Conversely, suppose that $\neg\neg(p \vee \neg a_+) \in U_{X^{\gamma\sigma}}$, we will show that $p \vee \neg a_+$ is positive by finding a 0-free term q such that $p \vee \neg a_+ = q \vee \neg a_+$; first we assume without loss of generality that $p \vee \neg a_+ = p'(x_1, \dots, x_m)$ for some term p' in the language of $\mathbf{PL}_n$. Notice that $\neg\neg(p \vee \neg a_+)$ is an element of the Boolean algebra $\mathbf{Bool}(\mathbf{F}_{\mathbf{PL}_n}(X))$ hence it can be written as a Boolean term over a finite number of variables in $X^{\gamma\sigma}$; it follows that if we consider the subalgebra of $\mathbf{F}_{\mathbf{PL}_n}(X)$ given by the finitely generated free Boolean algebra $\mathbf{Bool}(\mathbf{F}_{\mathbf{PL}_n}(X_m))$ (where we take a suitably $X_m = \{x_1, \dots, x_m\}$), then

$$\neg\neg(p \vee \neg a_+) \in U_{X^{\gamma\sigma}} \subseteq \mathbf{F}_{\mathbf{PL}_n}(X) \text{ if and only if } \neg\neg(p \vee \neg a_+) \in U_{X_m^{\gamma\sigma}} \subseteq \mathbf{F}_{\mathbf{PL}_n}(X_m).$$

Let us now consider the representation of $\mathbf{F}_{\mathbf{PL}_n}(X_m)$ as a weak Boolean product, i.e.

$$\mathbf{F}_{\mathbf{PL}_n}(X_m) \leq \prod_{S \in \text{Ups}(X_m^{\gamma\sigma})} \mathbf{F}_{\mathbf{PL}_n}(X_m)/F_S.$$

Let us have a look at the factor $\mathbf{F}_{\mathbf{PL}_n}(X_m)/F_{X_m^{\gamma\sigma}}$; this is a directly indecomposable algebra that is isomorphic to $\mathbb{L}_k \oplus (\mathbf{F}_{\mathbf{C}}(X_m))$ for some k that divides n . Since we are assuming that $\neg\neg(p \vee \neg a_+) \in U_{X_m^{\gamma\sigma}} \subseteq F_{X_m^{\gamma\sigma}}$, it follows that $\neg\neg(p \vee \neg a_+)/F_{X_m^{\gamma\sigma}} = 1/F_{X_m^{\gamma\sigma}}$, which implies that $(p \vee \neg a_+)/F_{X_m^{\gamma\sigma}}$ is in the summand $\mathbf{F}_{\mathbf{C}}(X_m)$ of the above ordinal sum, that is generated as a cancellative hoop by X_m . In the language of cancellative hoops there is no constant 0, therefore there is a 0-free term q such that $(p \vee \neg a_+)/F_{X_m^{\gamma\sigma}} = q(x_1, \dots, x_m)/F_{X_m^{\gamma\sigma}}$. We will finally show that $p \vee \neg a_+ = q \vee \neg a_+$; since the latter is positive, this will conclude the proof.

Consider $p \vee \neg a_+$ and $q \vee \neg a_+$ as their representatives given by the tuples $((p \vee \neg a_+)/F_S)_{S \in \text{Ups}(X_m^{\gamma\sigma})}$ and $((q \vee \neg a_+)/F_S)_{S \in \text{Ups}(X_m^{\gamma\sigma})}$. Now if $S = X_m^{\gamma\sigma}$, then

$$(p \vee \neg a_+)/F_{X_m^{\gamma\sigma}} = q(x_1, \dots, x_m)/F_{X_m^{\gamma\sigma}} = (q(x_1, \dots, x_m) \vee \neg a_+)/F_{X_m^{\gamma\sigma}}$$

since $\neg a_+/F_{X_m^{\gamma\sigma}} = 0/F_{X_m^{\gamma\sigma}}$. If $S \neq X_m^{\gamma\sigma}$, then $a_+ \notin U_S$ and also $a_+ \notin F_S$, thus $\neg a_+/F_S = 1/F_S$ and then

$$(p \vee \neg a_+)/F_S = (q(x_1, \dots, x_m) \vee \neg a_+)/F_S = 1/F_S.$$

We have shown that $p \vee \neg a_+ = q \vee \neg a_+$, and thus $p \vee \neg a_+$ is positive, which concludes the proof. $\square$

Now, combining Theorem 4.4 with the previous result we are finally ready to characterize the free algebra in the positive reducts of $\mathbf{PL}_n$. According to what we said before we denote the variety of basic hoops generated by $\mathbf{L}_n^+ \oplus \mathbf{C}_\omega$ by $\mathbf{PL}_n^+$.

THEOREM 4.17.

Given any set X , $\mathbf{F}_{\mathbf{PL}_n^+}(X)$ is representable as the weak Boolean product of the family

$$\{\mathbf{F}_{\mathbf{PL}_n^+}(X)/G_S : S \text{ upward closed subset of } X^{\gamma\sigma}\},$$

where G_S is the restriction of the filter F_S of $\mathbf{F}_{\mathbf{PL}_n}(X)$ to $\mathbf{F}_{\mathbf{PL}_n^+}(X)$. In particular the factors can be read as follows: if $S \neq X^{\gamma\sigma}$ then

$$\mathbf{F}_{\mathbf{PL}_n^+}(X)/G_S \cong \mathbf{L}_{k_S}^+ \oplus \mathbf{F}_{\mathbf{C}}(X_S),$$

for some k_S that divide n and with $X_S = \{x/G_S : \neg\neg(x/G_S) = 1\} \cup \{(x \rightarrow \neg\neg x)/G_S : \neg\neg(x/G_S) = 0\}$, and

$$\mathbf{F}_{\mathbf{PL}_n^+}(X)/G_{X^{\gamma\sigma}} \cong \mathbf{F}_{\mathbf{C}}(X).$$

PROOF. The proof follows immediately by applying to the representation of free $\mathbf{PL}_n$ algebras in 4.8, Theorems 4.4 and 4.16. The only thing left to prove is the definition of X_S , that follows from the easy observation that $(x \rightarrow \neg\neg x)/F_S = (\neg x)/F_S$ for all $x \in X$ such that $\neg\neg(x/F_S) = 0$. $\square$

Moreover, when considering a finite set of variables, the weak Boolean product representation yields an isomorphism, so we get the following

COROLLARY 4.18.

For every finite set X it holds that

$$\mathbf{F}_{\mathbf{PL}_n^+}(X) \cong \left(\prod_{\substack{S \in \text{Ups}(X^{\gamma\sigma}) \\ S \neq X^{\gamma\sigma}}} \mathbf{L}_{k_S}^+ \oplus \mathbf{F}_{\mathbf{C}}(X_S) \right) \times \mathbf{F}_{\mathbf{C}}(X),$$

for some k_S that divide n and with

$$X_S = \{x/G_S : \neg\neg(x/G_S) = 1\} \cup \{(x \rightarrow \neg\neg x)/G_S : \neg\neg(x/G_S) = 0\}.$$

Now that we have a representation for free algebras in these varieties of hoops, it is easy to study finitely presented algebras. Indeed, in the same way as in Theorem 5.4 in [41], one can prove the following theorem.

THEOREM 4.19.

In $\mathbf{PL}_n^+$, every finitely presented algebra is projective; in particular, for every n , $\mathbf{PL}_n^+$ has the Baker–Beynon property.

PROOF. The proof is exactly the same as in the case of $\mathbf{PH}$, shown in [41]. $\square$

COROLLARY 4.20.

The variety $\mathbf{PL}_n^+$ is structural for every $n \geq 1$.

We stress that we have already reached the limit of this technique. The problem is that it is more difficult to describe free algebras in varieties of basic hoops than in varieties of BL-algebras. Here we started from a description of the free BL-algebras in a variety $\mathbf{V}$ and from that we described the free basic hoops in $\mathbf{V}^+$. In doing so, we heavily relied on the fact that in $\mathbf{V}$ the double negation is an MV-retraction term and, most of all, $\neg\neg$ is positive. For every variety of BL-algebras, the double negation is an MV-retraction term [28], but for the majority of these varieties $\neg\neg$ is not positive. In particular, we have the following:

PROPOSITION 4.21.

Let $\mathbf{V}$ be a variety of BL-algebras and let $\mathbf{A} \oplus \bigoplus_{i \in I} \mathbf{A}_i$ be a totally ordered algebra in $\mathbf{V}$; if there exists $i \in I$ such that $\mathbf{A}_i$ is bounded, then $\neg\neg$ is not positive in $\mathbf{V}$.

PROOF. Suppose by contradiction that there exists $i \in I$ such that $\mathbf{A}_i$ is bounded and $\neg\neg$ is positive, so that there exists a 0-free term $p(x_1, \dots, x_m)$ such that $\neg\neg$ is equivalent to p . Let us call a the lower bound of $\mathbf{A}_i$ and 0 the lower bound of $\mathbf{A} \oplus \bigoplus_{i \in I} \mathbf{A}_i$. Since p is an MV-retraction term, $p(0) = \neg\neg 0 = 0$, while $p(a) = \neg\neg a = 1$. But note that, $0 \cdot 0 = 0$ and in the same way $a \cdot a = a$; similarly, $0 \rightarrow 0 = 1$ and $a \rightarrow a = 1$; in other words they behave in the same way under all positive operations. Hence, if p positive and $p(0) = 0$, we must have $p(a) = a$, and this contradicts the hypothesis that p is an MV-retraction term, because $\neg\neg a = 1$. $\square$

4.3 Other primitive varieties

The variety $\mathbf{PH}$ of product hoops is primitive. In fact it is structural since it is equal to $\mathbf{PL}_1^+$ and moreover in [5] there is a complete description of the lattice of its subvarieties. There are exactly three nontrivial subvarieties and they all consists of Wajsberg hoops, namely $\mathbf{WH}(1, \emptyset, \emptyset)$, $\mathbf{WH}(1, \emptyset, \{\omega\})$ and $\mathbf{WH}(\emptyset, \emptyset, \{\omega\})$. By Theorem 3.2 they are all primitive and so, by Corollary 2.7, $\mathbf{PH}$ is primitive.

What about $\mathbf{dIWH} = \mathbb{V}(\mathbf{L}_{1,0}^+)$? Again the reader can easily check (or look at the diagram in [9], p.360) that there are only three nontrivial subvarieties that are the same as those of $\mathbf{PH}$. Thus $\mathbf{dIWH}$ is primitive as well.

For $n > 1$ $\mathbf{PL}_n^+$ is structural, as we saw, but we cannot conclude that it is primitive. First we observe the following simple fact whose proof is straightforward and it is left to the reader.

LEMMA 4.22.

There is a bijection between the reduced triples (I, J, K) , where $I, J \subseteq \{n\} \downarrow$ and the subvarieties of $\mathbf{PL}_n^+ = \mathbb{V}(\mathbf{L}_n^+ \oplus \mathbf{C}_\omega)$. Every subvariety is of the form

$$\mathbb{V}(\{\mathbf{L}_i^+ : i \in I\} \cup \{\mathbf{L}_j^+ \oplus \mathbf{C}_\omega : j \in J\}).$$

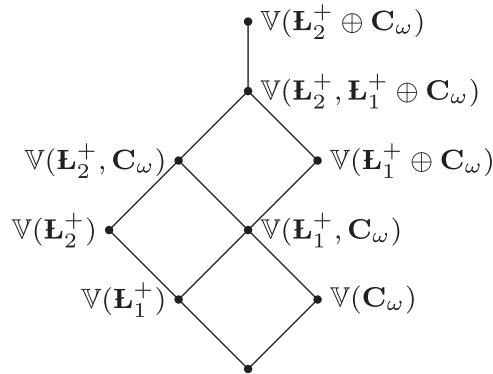


FIGURE 1

By Lemma 4.22 the lattice of subvarieties of $\mathbf{PL}_2^+$ is the one in Figure 1.

The problem is that we do not know if $\mathbb{V}(\mathbf{L}_2^+, \mathbf{L}_1^+ \oplus \mathbf{C}_\omega)$ is primitive or even structural. On the other hand we cannot exclude it either. If $\mathbf{V}$ is a variety of basic hoops and $\mathbf{A} = \bigoplus_{i \in I} \mathbf{A}_i$ is a totally ordered algebra in $\mathbf{V}$ written as the sum of its irreducible components, then $\mathbb{V}(\mathbf{A}_i) \subseteq \mathbb{V}(\mathbf{A})$. So if $\mathbf{V}$ is primitive then $\mathbb{V}(\mathbf{A}_i)$ must be primitive for all i ; it follows that the Wajsberg components of $\mathbf{A}$ must be chosen among the finite totally ordered Wajsberg hoops, the totally ordered cancellative hoops and the algebras in $\mathbf{HSP}_u(\mathbf{L}_{1,0})$ (i.e. those of rank 1 and divisibility index 1). Note that $\mathbb{V}(\mathbf{L}_2^+)$ and $\mathbb{V}(\mathbf{L}_1^+ \oplus \mathbf{C}_\omega)$ are both structural, so in particular they are equal to $\mathbb{Q}(\mathbf{L}_2^+)$ and $\mathbb{Q}(\mathbf{L}_1^+ \oplus \mathbf{C}_\omega)$ respectively, and this implies that also $\mathbb{V}(\mathbf{L}_2^+, \mathbf{L}_1^+ \oplus \mathbf{C}_\omega) = \mathbb{Q}(\mathbf{L}_2^+, \mathbf{L}_1^+ \oplus \mathbf{C}_\omega)$. Since $\mathbb{Q}(\mathbf{L}_2^+, \mathbf{L}_1^+ \oplus \mathbf{C}_\omega)$ is the smallest quasivariety generated by chains that generates $\mathbb{V}(\mathbf{L}_2^+, \mathbf{L}_1^+ \oplus \mathbf{C}_\omega)$, either $\mathbb{V}(\mathbf{L}_2^+, \mathbf{L}_1^+ \oplus \mathbf{C}_\omega)$ is structural or its structural core is not generated by chains.

5 Structural completeness in varieties of BL-algebras

A necessary condition for structurality for varieties of BL-algebras has been given in [15]. Let **dIBL** be the variety of BL-algebras axiomatized by

$$(x^2 \oplus x^2) \approx (x \oplus x)^2,$$

where $x \oplus y := ((x \rightarrow 0)(y \rightarrow 0)) \rightarrow 0$. Then Theorem 5.24 in [15] implies:

THEOREM 5.1.

If a variety $\mathbf{V}$ of BL-algebras is structural, then $\mathbf{V} \subseteq \mathbf{dIBL}$.

Let $\mathbf{A} = \bigoplus_{i \geq i_0} \mathbf{A}_i$ be a totally ordered algebra in $\mathbf{V}$ where $\mathbf{V} \subseteq \mathbf{dIBL}$; then $\mathbf{A}_{i_0}$ is a Wajsberg algebra satisfying the above equation hence it is a **perfect** Wajsberg algebra in the sense of [33]. Moreover, if $a \in A_i$, with $i > i_0$ then $a \oplus a = a^2 \oplus a^2 = 1$ (since $a^2 \in A_i$ as well). It follows that any totally ordered algebra in $\mathbf{V}$ has a perfect Wajsberg algebra in the first component. Since the variety of perfect Wajsberg algebras is $\mathbb{V}(\mathbf{L}_{1,0})$ [33], any structurally complete variety of BL-algebras must be generated by totally ordered algebras whose first component is in $\mathbf{HSP}_u(\mathbf{L}_{1,0})$. This of course implies that **dIBL** is equal to $\mathbb{V}(\mathbf{L}_{1,0}) \oplus' \mathbf{BH}$.

In BL-algebras not every algebra is unifiable, so the Baker–Beynon property is not enough to guarantee structurality. However:

LEMMA 5.2.

A finitely presented nontrivial BL-algebra is unifiable in any nontrivial quasivariety to which it belongs if and only if it has $\mathbb{L}_1$ as a homomorphic image.

PROOF. It is not difficult to see that $\mathbb{L}_1$ is projective in any nontrivial variety of BL-algebras (since it belongs to them all). Therefore if $\mathbf{A}$ has $\mathbb{L}_1$ as a homomorphic image, then it is unifiable. Conversely if $\mathbf{A}$ is finitely presented and unifiable then there is an $f : \mathbf{A} \rightarrow \mathbf{P}$, where $\mathbf{P}$ is projective in $\mathbf{Q}$. But by Proposition 5.10 in [14], $\mathbf{P}$ must have $\mathbb{L}_1$ as homomorphic image and therefore so does $\mathbf{A}$. $\square$

Moreover, unlike in basic hoops, the Baker–Beynon property is preserved by generalized rotations of BL-algebras under certain (reasonable) assumptions; in particular it is preserved by n -liftings. The reader can easily check that the following is a consequence of Theorem 4.10 in [13].

THEOREM 5.3.

Let $\mathbf{V}$ be a variety of basic hoops with the Baker–Beynon property; then $\mathbb{V}(\mathbb{L}_n) \oplus^t \mathbf{V}$ has the Baker–Beynon property.

Given these two results, we can investigate structurality in BL-algebras with some success.

5.1 Locally finite varieties

The locally finite varieties of BL-algebras can be described in the same way as those of basic hoops, namely:

THEOREM 5.4.

For a variety $\mathbf{V}$ of BL-algebras the following are equivalent:

1. $\mathbf{V}$ is locally finite;
2. $\mathbf{V}$ satisfies the equation $x^k \approx x^{k+1}$ for some $k \geq 1$ and
3. for all totally ordered $\mathbf{A} = \bigoplus_{i \in I} \mathbf{A}_i \in \mathbf{V}$ (where the decomposition is in its Wajsberg components), for all i there is an $m_i \leq k + 1$ such that $\mathbf{A}_i \cong \mathbb{L}_{m_i}$.

Next we observe:

LEMMA 5.5.

Let $\mathbf{W}$ be a variety of hoops. Then every nontrivial algebra in $\mathbb{V}(\mathbb{L}_1) \oplus^t \mathbf{W}$ is unifiable.

PROOF. Let $\mathbf{A}$ be a nontrivial totally ordered algebra in $\mathbb{V}(\mathbb{L}_1) \oplus^t \mathbf{W}$; then by Lemma 4.1 $\mathbf{A} \cong \mathbb{L}_1 \oplus \mathbf{B}$. Thus $\mathbf{A}$ has $\mathbb{L}_1$ as homomorphic image, and the thesis follows from Lemma 5.2 $\square$

Therefore, via Theorems 2.13 and 5.3:

COROLLARY 5.6.

Let $\mathbf{V}$ be a variety of basic hoops with the Baker–Beynon property; then $\mathbb{V}(\mathbb{L}_1) \oplus^t \mathbf{V}$ is structural.

The 1-lifting of the entire variety of basic hoops, i.e. $\mathbb{V}(\mathbb{L}_1) \oplus^t \mathbf{BH}$, is usually called the variety of *Stonean BL-algebras* (denoted by **SBL**); by Lemma 4.1 is axiomatized by

$$\neg\neg x \vee \neg x \approx 1.$$

So a locally finite variety contained in **SBL** is just a variety satisfying Theorem 5.4 and such that the first Wajsberg component of any totally ordered algebra therein is equal to $\mathbb{L}_1$. The main example of this kind is the variety of **Gödel algebras** (introduced in [43], but see also [5]); it is the equivalent algebraic semantics of *Dummett's Logic* and it is generated by the chain $\bigoplus_{n \in \mathbb{N}} \mathbb{L}_1$.

Another important characteristic of **SBL** is that any of its subvarieties comes in a very precise way from a variety of basic hoops. Unfortunately the result describing this connection (Corollary 2.8 in [11]) is not stated correctly. However it is easily fixable; any variety $\mathbf{V}$ of **BL**-algebras has a unique atom, $\mathbb{V}(\mathbb{L}_1)$, so the nontrivial subvarieties of $\mathbf{V}$ form an algebraic lattice. Then:

LEMMA 5.7.

The mapping

$$\mathbf{V} \mapsto \mathbb{V}(\mathbb{L}_1) \oplus^t \mathbf{V}$$

is a lattice isomorphism between the lattice of subvarieties of basic hoops and the lattice of nontrivial subvarieties of **SBL** which sends locally finite subvarieties into locally finite subvarieties.

PROOF. The proof of the first part of the statement is the proof of Corollary 2.8 in [11], once we have corrected the statement. For the second, suppose that $\mathbf{V}$ is a locally finite variety of basic hoops and let $\mathbf{W} = \mathbb{V}(\mathbb{L}_1) \oplus^t \mathbf{V}$. Let $\mathbf{A} \in \mathbf{W}$ be subdirectly irreducible and finitely generated. Then $\mathbf{A} = \mathbb{L}_1 \oplus \mathbf{B}$ where $\mathbf{B}$ is a totally ordered member of $\mathbf{V}$ which must be finitely generated. Hence $\mathbf{B}$ is finite and so is $\mathbf{A}$, which implies that $\mathbf{W}$ is locally finite. The converse is obtained by simply reversing the argument. $\square$

We stress that we can obtain an axiomatization of those varieties (relative to **BL**); it is well known that every variety of basic hoops can be axiomatized by a set of equations all of the form $p(x_1, \dots, x_n) \approx 1$ and, if it is finitely axiomatizable, then a single equation is enough.

LEMMA 5.8.

Let $\mathbf{V}$ be a nontrivial variety of basic hoops axiomatized by

$$\{p_\gamma(x_1, \dots, x_{n_\gamma}) \approx 1 : \gamma \in \Gamma\}.$$

Then $\mathbb{V}(\mathbb{L}_1) \oplus^t \mathbf{V}$ is axiomatized relative to **SBL** by

$$\left\{ \bigvee_{i=1}^{n_\gamma} \neg x_i \vee p(x_1, \dots, x_{n_\gamma}) \approx 1 : \gamma \in \Gamma \right\}.$$

PROOF. Take a totally ordered algebra $\mathbf{B} \in \mathbb{V}(\mathbb{L}_1) \oplus^t \mathbf{V}$. Then $\mathbf{B} \cong \mathbb{L}_1 \oplus \mathbf{A}$, where $\mathbf{A}$ is a totally ordered algebra in $\mathbf{V}$. Let $\varepsilon_\gamma := \bigvee_{i=1}^{n_\gamma} \neg x_i \vee p(x_1, \dots, x_{n_\gamma}) \approx 1$ an equation of the type above; if $a_1, \dots, a_{n_\gamma} \in B$ and $a_i = 0$ for some i , then $\mathbf{B}$ satisfies ε_γ . Otherwise $a_1, \dots, a_{n_\gamma} \in A$, thus $p(a_1, \dots, a_{n_\gamma}) = 1$ and again $\mathbf{B}$ satisfies ε_γ .

Conversely suppose that $\mathbf{B} = \mathbb{L}_1 \oplus \mathbf{A}$ is a totally ordered algebra satisfying ε_γ ; and let $a_1, \dots, a_{n_\gamma} \in A$. Then $\neg a_i = 0$ for $i = 1, \dots, n_\gamma$; so ε_γ reduces to $p(a_1, \dots, a_{n_\gamma}) = 1$. Since this happens for every γ , $\mathbf{A} \in \mathbf{V}$, as wished. $\square$

Now we can produce our classification theorem for locally finite primitive varieties of **BL**-algebras.

THEOREM 5.9.

For a locally finite variety $\mathbf{V}$ of **BL**-algebras the following are equivalent:

1. $\mathbf{V}$ is primitive;
2. $\mathbf{V}$ is structural;
3. $\mathbf{V} \subseteq \mathbf{SBL}$;
4. $\mathbf{V} = \mathbb{V}(\mathbb{L}_1) \oplus^t \mathbf{W}$, where $\mathbf{W}$ is a locally finite variety of basic hoops and
5. every finite subdirectly irreducible algebra in $\mathbf{V}$ has $\mathbb{L}_1$ as a homomorphic image.

PROOF. Clearly (1) implies (2). Let $\mathbf{V}$ be a structurally complete variety of BL-algebras, then $\mathbf{V} \subseteq \mathbf{dIBL} = \mathbb{V}(\mathbb{L}_1) \oplus^t \mathbf{BH}$. Let $\mathbf{A} = \bigoplus_{i \geq i_0} \mathbf{A}_i$ be a totally ordered algebra in $\mathbf{V}$; as $\mathbb{V}(\mathbf{A}_{i_0}) \subseteq \mathbf{V}$ and $\mathbf{V}$ is locally finite $\mathbb{V}(\mathbf{A}_{i_0})$ is a locally finite variety of Wajsberg algebras contained in $\mathbf{dIBL}$. It is clear that the only possibility is $\mathbb{V}(\mathbf{A}_0) = \mathbb{V}(\mathbb{L}_1)$ so $\mathbf{A}_0 = \mathbb{L}_1$ and thus $\mathbf{V} \subseteq \mathbf{SBL}$.

Next assume that $\mathbf{V} \subseteq \mathbf{SBL}$ and let $\mathbf{A}$ be a subdirectly irreducible algebra in $\mathbf{V}$; then $\mathbf{A}$ is totally ordered and $\mathbf{A}_0 = \mathbb{L}_1$. Since $\mathbf{A}_0$ is a homomorphic image of $\mathbf{A}$ and $\mathbf{V}$ is locally finite, $\mathbf{A}$ is weakly projective in $\mathbf{V}$ (Theorem 3.6). By Theorem 2.11 $\mathbf{V}$ is primitive. So we have shown that (1),(2) and (3) are equivalent.

Next, (3) and (4) are equivalent by Lemma 5.7.

(4) implies (5) is the same proof as the one of Lemma 5.5, while (5) implies (4) immediately follows from the characterization of locally finite varieties of BL-algebras, and from the well-known fact (see [7]) that if a BL-chain is of the form $\mathbf{A} = \mathbf{A}_1 \bigoplus_{i \in I} \mathbf{A}_i$ (where the decomposition is in its Wajsberg components), then $\mathbf{A}$ has $\mathbb{L}_1$ as homomorphic image if and only if $\mathbf{A}_1 \cong \mathbb{L}_1$. $\square$

5.2 Non locally finite varieties

For non locally finite varieties contained in $\mathbf{SBL}$ we have some information. By Corollary 5.6 the variety $\mathbb{V}(\mathbb{L}_1) \oplus^t \mathbf{C}$ is a structural variety contained in $\mathbf{SBL}$; it is not locally finite though (really, it has no finite models except for $\mathbb{L}_1$). This variety is also known as the variety **PA** of **product algebras** and was introduced in [27] as the equivalent algebraic semantics of **product logic**; it can be axiomatized (relative to **BL**) by

$$\neg\neg z \rightarrow ((xz \rightarrow yz) \rightarrow (x \rightarrow y)) = 1.$$

Moreover, by Theorem A.6 $\mathbf{PA} = \mathbb{V}(\mathbb{L}_1 \oplus \mathbf{C}_\omega)$.

PROPOSITION 5.10.

The variety **PA** is primitive.

PROOF. Since $\mathbf{PA} = \mathbb{V}(\mathbb{L}_1 \oplus \mathbf{C}_\omega)$, there is only one proper nontrivial subvariety of **PA**, i.e. the variety of Boolean algebras (generated by $\mathbb{L}_1$) which is primitive. Therefore, by Corollary 2.7 **PA** is primitive. $\square$

We can go one step more; since $\mathbf{PL}_n^+$ has the Baker–Beynon property for all n (via Theorem 4.19), applying Corollary 5.6, we can derive the following theorem.

THEOREM 5.11.

The varieties $\mathbb{V}(\mathbb{L}_1) \oplus^t \mathbf{PL}_n^+$ are structural for every $n \in \mathbb{N}$.

Notice that, by Theorem 5.3, the varieties $\mathbb{V}(\mathbb{L}_m) \oplus^t \mathbf{PL}_n^+ = \mathbb{V}(\mathbb{L}_m \oplus \mathbb{L}_n^+ \oplus \mathbf{C}_\omega)$ all have the Baker–Beynon property for every $m, n \in \mathbb{N}$. However as we will see in the next section, they are not structural whenever $m > 1$ (Example 6.4).

The remaining question is: are the varieties $\mathbb{V}(\mathbb{L}_1) \oplus^t \mathbf{PL}_n^+$ also primitive? Let's look closely at the case $n = 1$ by considering $\mathbb{V}(\mathbb{L}_1) \oplus^t \mathbf{PL}_1^+$; by Lemma 5.7 its lattice of nontrivial subvarieties is the one in Figure 2.

The problem here is the variety $\mathbb{V}(\mathbb{L}_1 \oplus \mathbb{L}_1^+, \mathbb{L}_1 \oplus \mathbf{C}_\omega)$; it is easily checked that it is $\mathbb{V}(\mathbb{L}_1) \oplus^t \mathbf{WH}(\{1\}, \emptyset, \{\omega\})$. Now $\mathbf{WH}(\{1\}, \emptyset, \{\omega\})$ is primitive by Theorem 3.2 but we do not know if it has the Baker–Beynon property. If it had the Baker–Beynon property, then it would be structural and hence

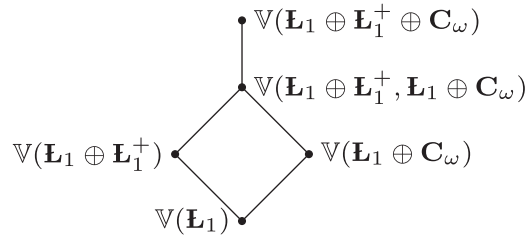


FIGURE 2

primitive, since we know that all the varieties below it are primitive; and that of course would imply that $\mathbb{V}(\mathbf{L}_1 \oplus \mathbf{L}_1^+ \oplus \mathbf{C}_\omega)$ is primitive.

For primitive varieties not contained in **SBL** we know next to nothing. We know that $\mathbb{V}(\mathbf{L}_{1,0})$ is structural; it is primitive as well since, as for **PA**, its only subvariety is the variety of Boolean algebras. And this is the only variety contained in **dIBL** of which we know something. Our impression is that the theory is not yet ripe enough to answer this type of questions. Some new techniques must be discovered.

6 Quasivarieties

Quasivarieties are way more complex to deal with for several reasons, the main one is that a quasivariety $\mathbf{Q}$ in general is no longer generated by chains but rather by its $\mathbf{Q}$ -irreducible algebras that do not need to be chains. For this reason we will restrict the majority of our analysis to quasivarieties generated by chains, where we will be able to start from some results about **WA** and **WH**. Let us start by recalling these results.

Remember that, if an algebra $\mathbf{A}$ is subdirectly irreducible in a variety $\mathbf{V}$ and we consider a quasivariety $\mathbf{Q} \subseteq \mathbf{V}$ such that $\mathbf{A} \in \mathbf{Q}$, then $\mathbf{A}$ is also $\mathbf{Q}$ -irreducible; moreover, if $\mathbf{Q}$ is generated by a class of algebras $\mathbf{K}$, by Theorem 2.3 (2) we have that $\mathbf{A} \in \mathbb{ISP}_u(\mathbf{K})$. Combining this fact with the characterization of structural quasivarieties in **WA** and in **WH** in Section 3 we have the following corollary:

COROLLARY 6.1.

In **WA**:

1. if $n \neq 1$, then $\mathbb{V}(\mathbf{L}_n) = \mathbb{V}(\mathbf{L}_1 \times \mathbf{L}_n)$ but $\mathbf{L}_n \notin \mathbb{Q}(\mathbf{L}_1 \times \mathbf{L}_n)$, that is equivalent to say that $\mathbf{L}_n \notin \mathbb{ISP}_u(\mathbf{L}_1 \times \mathbf{L}_n)$;
2. if $n \neq 1$, then $\mathbb{V}(\mathbf{L}_{n,k}) = \mathbb{V}(\mathbf{L}_1 \times \mathbf{L}_{n,1})$ but $\mathbf{L}_{n,k} \notin \mathbb{Q}(\mathbf{L}_1 \times \mathbf{L}_{n,1})$, that is equivalent to say that $\mathbf{L}_{n,k} \notin \mathbb{ISP}_u(\mathbf{L}_1 \times \mathbf{L}_{n,1})$ and

In **WH**:

3. if $n, k \neq 1$, then $\mathbb{V}(\mathbf{L}_{n,k}^+) = \mathbb{V}(\mathbf{L}_{n,1}^+)$ but $\mathbf{L}_{n,k}^+ \notin \mathbb{Q}(\mathbf{L}_{n,1}^+)$, that is equivalent to say that $\mathbf{L}_{n,k}^+ \notin \mathbb{ISP}_u(\mathbf{L}_{n,1}^+)$;

Let us show how we can lift some of these results in order to say something about structural quasivarieties in **BL** and **BH**.

6.1 Structural completeness

For **BL**-algebras, the main idea is encoded in this rather technical lemma.

LEMMA 6.2.

Let $\mathbf{A} = \mathbf{B} \oplus \bigoplus_{i>0} \mathbf{B}_i$ be a BL-chain (seen as an ordinal sum of its Wajsberg components); if $\mathbf{A} \in \mathbf{ISP}_u(\mathbb{L}_1 \times \mathbf{A})$ then $\mathbf{B} \in \mathbf{ISP}_u(\mathbb{L}_1 \times \mathbf{B})$.

PROOF. Let $\mathbf{D}_U = (\prod_{j \in J} \mathbb{L}_1 \times \mathbf{A})/U$ be an ultraproduct of $\mathbb{L}_1 \times \mathbf{A}$ that has a subalgebra $\mathbf{C}$ isomorphic to $\mathbf{A}$. To prove the statement we first need two claims.

CLAIM. $D_U = (\prod_{j \in J} \mathbb{L}_1 \times (B \setminus \{1\}))/U \cup (\prod_{j \in J} \mathbb{L}_1 \times \bigoplus_{i>0} B_i)/U$ and this union is disjoint. $\square$

Proof of the claim. For every $a \in \prod_{j \in J} \mathbb{L}_1 \times A$, we have that a is of the form $((s, x)_j)_{j \in J}$ for some $(s, x)_j \in \mathbb{L}_1 \times A$; now, define the two sets of indexes

$$J_B = \{j \in J \mid (s, x)_j \in \mathbb{L}_1 \times (B \setminus \{1\})\}, \quad J_{\bigoplus B_i} = \left\{j \in J \mid (s, x)_j \in \mathbb{L}_1 \times \bigoplus_{i>0} B_i\right\}.$$

Notice that $J_{\bigoplus B_i}$ is the complement of J_B , so at least one of them has to be in the ultrafilter U . Suppose $J_B \in U$, then we can take the element

$$b = \begin{cases} (s, x)_j & \text{if } j \in J_B, \\ (0, 0) & \text{otherwise.} \end{cases}$$

b is in $\prod_{j \in J} \mathbb{L}_1 \times (B \setminus \{1\})$ by construction, moreover $a_U = b_U$ because they coincide over J_B that is in U , hence $a_U \in (\prod_{j \in J} \mathbb{L}_1 \times (B \setminus \{1\}))/U$; analogously we can prove that if $J_{\bigoplus B_i} \in U$ then $a_U \in (\prod_{j \in J} \mathbb{L}_1 \times \bigoplus_{i>0} B_i)/U$. The fact that the union is disjoint is obvious by construction. $\square$

Recall that we defined $\mathbf{C}$ as the subalgebra of $\mathbf{D}_U$ isomorphic to $\mathbf{A}$ that exists by hypothesis; let us divide $\mathbf{C}$ into two parts: $\mathbf{C}_0$, that is the union of the top element 1_U of D_U with the restriction of $\mathbf{C}$ to $(\prod_{j \in J} \mathbb{L}_1 \times (B \setminus \{1\}))/U$, and $\mathbf{C}_1$, that is the restriction of $\mathbf{C}$ to $(\prod_{j \in J} \mathbb{L}_1 \times \bigoplus_{i>0} B_i)/U$. Notice that $C_0 \cap C_1 = \{1_U\}$ by construction.

CLAIM. $\mathbf{C} = \mathbf{C}_0 \oplus \mathbf{C}_1$.

Proof of the claim. We start from the following observation: take an element $a \in \prod_{j \in J} \mathbb{L}_1 \times (B \setminus \{1\})$ and an element $b \in \prod_{j \in J} \mathbb{L}_1 \times \bigoplus_{i>0} B_i$ and let us say that $a = ((s, x)_j)_{j \in J}$ and $b = ((t, y)_j)_{j \in J}$. Let us fix a $j \in J$; notice that $x < y$ in the order of $\mathbf{A}$ by definition of ordinal sum, hence we can have the following cases:

1. if $s = 0$, then $(s, x)_j < (t, y)_j$ in $\mathbb{L}_1 \times \mathbf{A}$ because $s \leq t$ in $\mathbb{L}_1$ and $x < y$ in $\mathbf{A}$;
2. if $s = 1$ and $t = 0$, then $(s, x)_j$ and $(t, y)_j$ are incomparable in $\mathbb{L}_1 \times \mathbf{A}$ because $s > t$ in $\mathbb{L}_1$ but $x < y$ in $\mathbf{A}$ and
3. if $s = 1$ and $t = 1$, then $(s, x)_j < (t, y)_j$ in $\mathbb{L}_1 \times \mathbf{A}$ because $x < y$ in $\mathbf{A}$.

Since this property is true for every coordinate, this means that also for every $a_U \in (\prod_{j \in J} \mathbb{L}_1 \times (B \setminus \{1\}))/U$ and $b_U \in (\prod_{j \in J} \mathbb{L}_1 \times \bigoplus_{i>0} B_i)/U$ we must have either $a_U < b_U$ or they are incomparable. Given that $\mathbf{C}$ is a chain, this means that all elements of $\mathbf{C}_0 \setminus \{1_U\}$ are strictly less than all elements of $\mathbf{C}_1$.

Now, we need to show that $\mathbf{C}_0$ is closed under all operation of BL-algebras and $\mathbf{C}_1$ is closed under all operations of basic hoops. First, notice that the latter is a direct consequence of the fact that $\mathbf{C}$ and $(\prod_{j \in J} \mathbb{L}_1 \times \bigoplus_{i>0} \mathbf{B}_i)/U$ are closed under all BH operations. For the former, notice

that $(0, 1)_{j \in J} \notin \mathbf{C}$, because otherwise, also its negation $(1, 0)_{j \in J}$ would be in $\mathbf{C}$, but this cannot be the case because $(0, 1)_{j \in J}$ and $(1, 0)_{j \in J}$ are incomparable and $\mathbf{C}$ is a chain. This means that $\mathbf{C}_0$ is also the restriction of $\mathbf{C}$ to $(\prod_{j \in J} \mathbb{L}_1 \times \mathbf{B})/U$. Since $\mathbf{C}$ and $(\prod_{j \in J} \mathbb{L}_1 \times \mathbf{B})/U$ are closed under all BL-operations, so is $\mathbf{C}_0$.

For the last step, if $a_U \in \mathbf{C}_0$ and $b_U \in \mathbf{C}_1$, then $a_U \cdot b_U = a_U \wedge b_U$, because in the ultraproduct the operations are defined coordinate-wise, the product in $\mathbb{L}_1$ coincides with the meet and the product between an element of $\mathbf{B}$ and an element of $\bigoplus_{i>0} \mathbf{B}_i$ is also equal to the meet by definition of ordinal sum.

Combining all the previous observations, we finally have that $\mathbf{C} = \mathbf{C}_0 \oplus \mathbf{C}_1$, where in principle $\mathbf{C}_0$ can still be an ordinal sum of more than one algebra. $\square$

Now, we also know that $\mathbf{C} \cong \mathbf{A} = \mathbf{B} \oplus \bigoplus_{i>0} \mathbf{B}_i$, where $\mathbf{B}$ cannot be written as an ordinal sum of any other algebras, so $\mathbf{C}_0$ must have the first component of its ordinal sum isomorphic to $\mathbf{B}$, hence in particular it has a subalgebra isomorphic to $\mathbf{B}$. $\mathbf{C}_0$ is closed under all BL operations and it is contained in $(\prod_{j \in J} \mathbb{L}_1 \times (\mathbf{B} \setminus \{1\})) / U \cup \{1_U\} \subset (\prod_{j \in J} \mathbb{L}_1 \times \mathbf{B}) / U$ by construction, so $\mathbf{C}_0$ is a subalgebra of $(\prod_{j \in J} \mathbb{L}_1 \times \mathbf{B}) / U$. Hence we have $\mathbf{B} \in \mathbb{S}(\mathbf{C}_0)$ and $\mathbf{C}_0 \in \mathbb{ISP}_u(\mathbb{L}_1 \times \mathbf{B})$, so $\mathbf{B} \in \mathbb{ISP}_u(\mathbb{L}_1 \times \mathbf{B})$.

From this we immediately get the following theorem.

THEOREM 6.3.

Let $\mathbf{A}$ be a BL-chain isomorphic to $\mathbf{B} \oplus \bigoplus_{i>0} \mathbf{B}_i$, where $\mathbf{B} \cong \mathbb{L}_n$ or $\mathbf{B} \cong \mathbb{L}_{n,k}$ with $n \neq 1$; then $\mathbb{Q}(\mathbf{A})$ is not structural.

PROOF. We prove the statement for $\mathbf{B} \cong \mathbb{L}_n$; the other case follows analogously using Corollary 6.1 (2).

By Corollary 6.1 (1), we know that $\mathbb{L}_n \notin \mathbb{ISP}_u(\mathbb{L}_1 \times \mathbb{L}_n)$ and, by Lemma 6.2, this gives us that also $\mathbf{A} \notin \mathbb{ISP}_u(\mathbb{L}_1 \times \mathbf{A})$. Since $\mathbb{L}_1 \times \mathbf{A} \in \mathbb{Q}(\mathbf{A})$, we must have $\mathbb{Q}(\mathbb{L}_1 \times \mathbf{A}) \subset \mathbb{Q}(\mathbf{A})$; moreover $\mathbb{V}(\mathbf{A}) = \mathbb{V}(\mathbb{L}_1 \times \mathbf{A})$, hence $\mathbb{Q}(\mathbf{A})$ is not structural. $\square$

EXAMPLE 6.4.

Let $\mathbf{V}$ be a variety of basic hoops generated by a single chain and with the Baker–Beynon property. Theorem 6.3 tells us that, for every $n > 1$, no n -lifting of $\mathbf{V}$ is structural, despite having the Baker–Beynon property by Theorem 5.3. The same holds for any n -lifting of a quasivariety generated by chains.

Moreover, Corollary 6.6 will extend this result also to varieties and quasivarieties generated by a finite number of chains.

Similarly, we can prove the following theorem starting from the results about Wajsberg hoops.

THEOREM 6.5.

Let $\mathbf{A}$ be a BL-chain (or a BH-chain) isomorphic to $\bigoplus_{i=0}^n \mathbf{B}_i \oplus \mathbf{B}$, where $\mathbf{B} \cong \mathbb{L}_{n,k}^+$ with $n, k \neq 1$; then $\mathbb{Q}(\mathbf{A})$ is not structural;

PROOF. Consider the algebra $\mathbf{C} \cong \bigoplus_{i=0}^n \mathbf{B}_i \oplus \mathbb{L}_{n,1}^+$; now, $\mathbb{V}(\mathbf{C}) = \mathbb{V}(\mathbf{A})$ because $\mathbb{V}(\mathbb{L}_{n,k}^+) = \mathbb{V}(\mathbb{L}_{n,1}^+)$, but $\mathbf{A} \notin \mathbb{ISP}_u(\mathbf{C})$, because $\mathbb{L}_{n,k}^+ \notin \mathbb{ISP}_u(\mathbb{L}_{n,1}^+)$ by Corollary 6.1 (3) and

$$\mathbb{ISP}_u(\mathbf{C}) = \left\{ \bigoplus_{i=0}^n \mathbf{D}_i \oplus \mathbf{D} : \mathbf{D}_i \in \mathbb{ISP}_u(\mathbf{B}_i) \text{ and } \mathbf{D} \in \mathbb{ISP}_u(\mathbb{L}_{n,1}^+) \right\}.$$

So we have that $\mathbf{A}$ and $\mathbf{C}$ generates the same variety but $\mathbb{Q}(\mathbf{C}) \subset \mathbb{Q}(\mathbf{A})$; hence $\mathbb{Q}(\mathbf{A})$ is not structural. $\square$

It is then natural to extend these results also to quasivarieties generated by finitely many algebras.

COROLLARY 6.6.

Let $\mathbf{Q} = \mathbb{Q}(\mathbf{A}_1, \dots, \mathbf{A}_m)$ be a quasivariety of BL-algebras (or basic hoops) such that for every $r \neq s$ it holds $\mathbf{A}_r \notin \text{ISP}_u(\mathbf{A}_s)$. If there exists j such that at least one of the following holds

1. $\mathbf{A}_j = \mathbb{L}_n \oplus \bigoplus_{i>0} \mathbf{B}_i$ is a BL-chain for some $n \neq 1$,
2. $\mathbf{A}_j = \mathbb{L}_{n,k} \oplus \bigoplus_{i>0} \mathbf{B}_i$ is a BL-chain for some $n \neq 1$ and
3. $\mathbf{A}_j = \bigoplus_{i=0}^n \mathbf{B}_i \oplus \mathbb{L}_{n,k}^+$ is a BL-chain (or BH-chain) for some $n, k \neq 1$,

then $\mathbf{Q}$ is not structural.

PROOF. The proof follows immediately from the two previous theorems and the fact that $\text{ISP}_u(\mathbf{A}_1, \dots, \mathbf{A}_m) = \text{ISP}_u(\mathbf{A}_1) \cup \dots \cup \text{ISP}_u(\mathbf{A}_m)$, because it is an ultraproduct of finitely many algebras. $\square$

Notice that the condition in the above theorem that $\mathbf{A}_r \notin \text{ISP}_u(\mathbf{A}_s)$ for every r, s is only saying that no generator of the quasivariety $\mathbf{Q}$ is redundant.

The previous results show that, when considering if $\mathbb{Q}(\mathbf{A})$ is structural, the first (in the case of BL-algebras) and the last Wajsberg component of $\mathbf{A}$ play a special role. The idea is that for all other components of $\mathbf{A}$, we are almost tied up by how homomorphic images of an ordinal sums work. For example, suppose we take a chain $\mathbf{A} = \mathbf{B}_0 \oplus \mathbb{L}_{n,k}^+ \oplus \mathbf{B}_2$ with $n, k \neq 1$ and we try to prove that $\mathbb{Q}(\mathbf{A})$ is not structural. If we want to rely on the results about Wajsberg hoops, the idea could be to do it like in the proof of Theorem 6.5, so we would like to consider $\mathbf{C} = \mathbf{B}_0 \oplus \mathbb{L}_{n,1}^+ \oplus \mathbf{B}_2$ and $\mathbb{Q}(\mathbf{C})$. While it is always true that $\mathbf{A} \notin \text{ISP}_u(\mathbf{C})$ (like in the proof of Theorem 6.5) and so $\mathbb{Q}(\mathbf{C}) \subset \mathbb{Q}(\mathbf{A})$, this time $\mathbf{A}$ and $\mathbf{C}$ do not generate the same variety, indeed, $\text{HSP}_u(\mathbf{C}) = \mathbb{H}(\text{SP}_u(\mathbf{B}_0) \oplus \text{SP}_u(\mathbb{L}_{n,1}^+) \oplus \text{SP}_u(\mathbf{B}_2))$, and this is equal to

$$\text{HSP}_u(\mathbf{B}_0) \cup (\text{ISP}_u(\mathbf{B}_0) \oplus \text{HSP}_u(\mathbb{L}_{n,1}^+)) \cup (\text{ISP}_u(\mathbf{B}_0) \oplus \text{ISP}_u(\mathbb{L}_{n,1}^+) \oplus \text{HSP}_u(\mathbf{B}_2));$$

since $\mathbf{A}$ has three Wajsberg components, this means that $\mathbf{A} \in \mathbb{V}(\mathbf{C})$ if and only if $\mathbf{A} \in \text{HSP}_u(\mathbf{C})$ if and only if $\mathbf{A} \in \text{ISP}_u(\mathbf{B}_0) \oplus \text{ISP}_u(\mathbb{L}_{n,1}^+) \oplus \text{HSP}_u(\mathbf{B}_2)$, but this is impossible because $\mathbb{L}_{n,k}^+ \notin \text{ISP}_u(\mathbb{L}_{n,1}^+)$.

We close this subsection remarking that, finding quasivarieties of BL or BH (not locally finite) that are structural appears to be way more difficult. Indeed, one should check not only those quasivarieties that are generated by a product of chains (like in the WA and WH case), but also quasivarieties generated by ordinal sums in which every component can be a product of chains.

6.2 Primitivity

As a consequence of Corollary 6.6 we get the following:

COROLLARY 6.7.

Every structural quasivariety of BL-algebras generated by a finite number of chains is contained in dBL, hence also every primitive quasivariety generated by a finite number of chains.

If $\mathbf{Q}$ is a locally finite quasivariety we can say something more, because it is easily seen that $\mathbb{V}(\mathbf{Q})$ is locally finite as well, therefore:

PROPOSITION 6.8.

Let $\mathbf{Q}$ be a locally finite quasivariety; if $\mathbf{Q} \subseteq \text{SBL}$ then $\mathbf{Q}$ is primitive. Conversely if $\mathbf{Q}$ is generated by chains and primitive then it is contained in SBL.

PROOF. Let $\mathbf{Q}$ be locally finite; if $\mathbf{Q} \subseteq \mathbf{SBL}$ then $\mathbb{V}(\mathbf{Q}) \subseteq \mathbf{SBL}$ and hence, by Theorem 5.9, is primitive. So $\mathbf{Q}$ is primitive as well.

Conversely suppose that $\mathbf{Q}$ is generated by chains and $\mathbf{Q} \not\subseteq \mathbf{SBL}$; then there must be a chain $\mathbf{A} \in \mathbf{Q}$ which is not in $\mathbf{SBL}$. Now it is not possible that all finitely generated subchains of $\mathbf{A}$ are in $\mathbf{SBL}$, otherwise $\mathbf{A} \in \mathbf{SBL}$, a contradiction. So there must be a finitely generated, hence finite, subchain $\mathbf{B}$ of $\mathbf{A}$ that is not in $\mathbf{SBL}$. Since $\mathbf{B}$ is finite, the first component of $\mathbf{B}$ must be equal to $\mathbb{L}_n$ for some $n > 1$, so $\mathbb{L}_1 \times \mathbb{L}_n \in \mathbb{Q}(\mathbf{B})$. By Lemma 6.2 $\mathbb{L}_n \notin \mathbb{S}(\mathbb{L}_1 \times \mathbb{L}_n)$ so $\mathbb{L}_n$ is not weakly projective in $\mathbb{Q}(\mathbf{B})$. So, by Theorem 2.11, $\mathbb{Q}(\mathbf{B})$ is not primitive and neither is $\mathbf{Q}$. $\square$

REMARK 6.9.

The second part of Proposition 6.8 fails for quasivarieties not generated by chains already in Wajsberg algebras. By Corollary 3.12 the quasivarieties $\mathbb{Q}(\mathbb{L}_1 \times \mathbb{L}_n)$ are all primitive and also locally finite. However they are not contained in $\mathbf{SBL}$; in fact the only quasivariety of Wajsberg algebras contained in $\mathbf{SBL}$ is the variety of Boolean algebras.

Regarding basic hoops, every locally finite quasivariety is contained in a locally finite variety of basic hoops which is primitive. Hence every locally finite quasivariety of basic hoops is a locally finite variety.

Contrary to Wajsberg algebras every quasivariety of Wajsberg hoops that is structural is generated by chains; therefore we have

PROPOSITION 6.10.

Let $\mathbf{Q} = \mathbb{Q}(\mathbf{A})$ be a primitive quasivariety of basic hoops. If $\mathbf{A} = \bigoplus_{i \in I} \mathbf{A}_i$ is a chain then every Wajsberg component $\mathbf{A}_i$ is either finite, cancellative or else it has rank equal to 1 and divisibility index $d_{\mathbf{A}_i} = 1$.

PROOF. Since $\mathbf{Q}$ is primitive, $\mathbb{Q}(\mathbf{A}_i)$ must be primitive; hence by Theorem 2.6 $\mathbb{Q}(\mathbf{A}_i)$ is either equal to $\mathbb{Q}(\mathbb{L}_n)$, or else it is equal to $\mathbf{C}$ or else it is equal to $\mathbb{Q}(\mathbb{L}_{1,1})$. The rest follows. $\square$

7 Further investigations and remarks

Although we have obtained some important results in this article, the problem of structural completeness in $\mathbf{BL}$ and $\mathbf{BH}$ remains far from solved. There are two main reasons that make a complete characterization of all structural quasivarieties incredibly challenging:

- the first reason is that we lack a clear representation for free algebras in subvarieties of $\mathbf{BL}$ and $\mathbf{BH}$. This is crucial because, as shown during the article, structural quasivarieties are precisely the ones generated by free algebras over countably many generators. In Section 4.2, we found a characterization for free algebras in $\mathbf{PL}_n^+$, and the reader will have noticed how that was incredibly convoluted! The problem is that we still do not have a general method for representing $\mathbf{FV}(\mathbf{A} \oplus \mathbf{B})$ starting from $\mathbf{FV}(\mathbf{A})$ and $\mathbf{FV}(\mathbf{B})$. Understanding this will probably allow one to solve the problem of structural completeness in $\mathbf{BL}$ and $\mathbf{BH}$ building on the known solution about $\mathbf{WA}$ and $\mathbf{WH}$.

Another, perhaps simpler, problem concerning free algebras is that we would like to describe $\mathbf{FV}(\mathbf{A}, \mathbf{B})$ starting from $\mathbf{FV}(\mathbf{A})$ and $\mathbf{FV}(\mathbf{B})$. Such a description would, for instance, allow one to answer the question about primitivity posed at the end of Section 4.2.

- The second major obstacle on our path were quasivarieties that are not generated by chains. It is known that the lattices of subquasivarieties of $\mathbf{BL}$ and $\mathbf{BH}$ are not numerable (neither are the

ones of **WA** and **WH**) and that there exist quasivarieties which cannot be generated by chains. This is a problem because very little is known about these quasivarieties.

For what concerns structural completeness, what matters are structural cores, and when these structural cores are quasivarieties not generated by chains it is very difficult to describe them. For example, in **WH** the authors showed that every structural core (except for that of **WH** itself) is generated by chains, and this made possible a complete characterization of all structural and primitive quasivarieties of **WH**. In contrast, Gispert [38] showed that in **WA** almost every structural core is not generated by chains (except for those of $\mathbb{V}(\mathbf{L}_1)$ and of $\mathbb{V}(\mathbf{L}_1^\infty)$). This is the main reason why the problem of characterizing all primitive quasivarieties of **WA** is way more difficult and still open. Notice that this implies that in the lattice of subquasivarieties of **BL** and **BH**, quasivarieties not generated by chains appear quite early.

These problems are difficult to overcome but not insurmountable; for this reason we propose some concrete directions that we think could be the easiest way to tackle the problem:

1. characterize all primitive quasivarieties of Wajsberg algebras;
2. characterize free algebras in varieties of **BL** and **BH** of the form $\mathbb{V}(\mathbf{A} \oplus \mathbf{B})$ and $\mathbb{V}(\mathbf{A}, \mathbf{B})$;
3. understand if the ordered sum preserve the Baker–Beynon property. Even easier, if $\mathbb{V}(\mathbf{A})$ and $\mathbb{V}(\mathbf{B})$ have the Baker–Beynon property, does $\mathbb{V}(\mathbf{A} \oplus \mathbf{B})$ have it? And does $\mathbb{V}(\mathbf{A}, \mathbf{B})$ have it?
4. Try to understand if the converse implication of Corollary 6.6 is true. Otherwise, there exist more conditions on the generators of **Q** that make **Q** not structural?
5. Understand better the lattice of subquasivarieties of **BL** and **BH**; in particular, try to understand if there are portions of the lattice where quasivarieties not generated by chains do not arise.

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Appendix A: The ordered sum of varieties

In this appendix we will consider varieties of the form $V_1 \oplus^t \dots \oplus^t V_n$ where $V_1, \dots, V_n$ are varieties of Wajsberg hoops or V_1 is possibly a variety of Wajsberg algebras. The first case we consider is when each $V_i = WH$ (and possibly $V_1 = WA$). Clearly the chains in $\bigoplus_{i \leq n}^t WH$ are exactly the chains in BH or BL having at most n Wajsberg components. If A is a totally ordered BL -algebra or a basic hoop, the **index** of A , denoted by $\#A$ is the number of Wajsberg components of A if there are finitely many of them; otherwise $\#A$ is infinite. It is an easy exercise to check that any finitely generated BL -algebra or basic hoop has finite index. We have

THEOREM A.1.

([7], Lemma 4.2) Let $\mathbf{A}$ be a totally ordered BL-algebra or Wajsberg hoop. Then $\#\mathbf{A} = n$ if and only if the following equation holds in $\mathbf{A}$

$$\bigwedge_{i=1}^n ((x_{i+1} \rightarrow x_i) \rightarrow x_i) \leq \bigvee_{i=1}^{n+1} x_i. \quad (\lambda_n)$$

COROLLARY A.2.

The variety of BL-algebras (basic hoops) axiomatized by λ_n is $\mathbf{WA} \oplus \bigoplus_{2 \leq i \leq n}^t \mathbf{WH} (\bigoplus_{i \leq n}^t \mathbf{WH})$. A totally ordered BL-algebra (basic hoop) has index n if and only if it belongs to $\mathbf{WA} \oplus \bigoplus_{2 \leq i \leq n}^t \mathbf{WH} (\bigoplus_{i \leq n}^t \mathbf{WH})$.

Combining this fact with Lemma 4.1, if $\mathbf{V}$ is a proper variety of Wajsberg algebras, an axiomatization for $\mathbf{V} \oplus^t \bigoplus_{i \leq n}^t \mathbf{WH}$ is λ_{n+1} and $t_V(\neg \neg x) \approx 1$. What happens if we consider $\bigoplus_{i \leq n}^t \mathbf{V}_i$ where the $\mathbf{V}_i$'s are arbitrary varieties of Wajsberg hoops (the first of which is possibly a variety of Wajsberg algebras)? The solution by Montagna and the first author, published in [7] Theorem 5.1, is not correct for at least two reasons. First it does not take good care of the case in which some of the varieties of Wajsberg hoops involved are not proper; secondly the proposed equations are too strong and it may happen (as the reader might want to check) that some algebras belonging to $\bigoplus_{i \leq n}^t \mathbf{V}_i$ do not satisfy those equations¹. However the general ideas behind the proof are still sound and we are glad to have the opportunity to fix it.

Let's consider first $\bigoplus_{i \leq n}^t \mathbf{V}_i$ where each $\mathbf{V}_i$ is a variety of Wajsberg hoops; we observe that the case in which $\mathbf{V}_i = \mathbf{WH}$ for $i \geq 1$ has already been covered by Theorem A.1. So we may assume that there is an $i \geq 1$ such that $\mathbf{V}_i \neq \mathbf{WH}$ or equivalently that the set $[n] = \{i \geq 1 : \mathbf{V}_i \neq \mathbf{WH}\}$ is nonempty. To deal with ordered sum we use a notation introduced in [7]; if $\mathbf{A} = \bigoplus_{i \in I} \mathbf{A}_i$ is the decomposition of a totally ordered Wajsberg hoop into its Wajsberg components and $a, b \in A$ we say that $a \ll b$ if either $b = 1$ or else a lies in a Wajsberg component strictly below the Wajsberg component in which b lies. The reader can easily check that

- if $a \ll b$, then $(b \rightarrow a) \rightarrow a = 1$;
- if $a \not\ll b$, then $(b \rightarrow a) \rightarrow a = a \vee b$.

Now for all $k \leq n$ we consider the set SEQ_k of all strictly increasing sequences of length k with values in $\{1, \dots, n\}$; in other words $s \in SEQ_k$ if $s : \{1, \dots, k\} \rightarrow \{1, \dots, n\}$ and $s_i < s_{i+1}$ for $i = 1, \dots, k-1$. For each $k \leq n$ we define two terms

$$u_k(x_1, \dots, x_k) := \bigwedge_{i=1}^{k-1} (x_{i+1} \rightarrow x_i) \rightarrow x_i$$

$$v_k(x_1, \dots, x_k) := \bigvee_{i=1}^k x_i \vee \bigvee_{s \in SEQ_k} \left(\bigwedge_{j \leq k, s_j \in [n]} t_{s_j}(x_j) \right),$$

where $t_{s_j}(x) \approx 1$ axiomatizes the proper subvariety $\mathbf{V}_{s_j}$.

¹ This mistake was found by Ugolini who communicated it to the first author. Later she and Galatos found a correct proof; the proof in this note is different in parts and slightly improves theirs.

THEOREM A.3.

Let $\mathbf{V}_1, \dots, \mathbf{V}_n$ be varieties of Wajsberg hoops; then $\bigoplus_{i \leq n}^t \mathbf{V}_i$ is axiomatized by λ_n and $u_k \leq v_k$ for $k \leq n$.

PROOF. First we prove that every generator of $\bigoplus_{i \leq n}^t \mathbf{V}_i$ satisfies the equations. Such a generator is of the form $\mathbf{B} = \bigoplus_{i=1}^n \mathbf{B}_i$ where $\mathbf{B}_i \in \mathbf{V}_i^t$ and satisfies λ_n by Theorem A.1.

Let $b_1, \dots, b_k \in B$; if for some i , $b_i \not\ll b_{i+1}$, then one of the meetands in u_k is $b_i \vee b_{i+1}$. So $u_k(b_1, \dots, b_k) \leq b_i \vee b_{i+1} \leq v_k(b_1, \dots, b_k)$, where the second inequality is obvious. A similar conclusion is easy to reach if $b_i = 1$ for some i , hence the only case missing is when

$$b_1 \ll b_1 \ll \dots \ll b_k \ll 1.$$

In this case we consider $s \in SEQ_k$ defined by $s_j = l$ where $b_j \in B_l$ for some $j \leq k$; clearly $t_{s_j}(b_j) = 1$ whenever $s_j \in [n]$ so $\left(\bigwedge_{j \leq k, s_j \in [n]} t_{s_j}(b_j)\right) = 1$. This shows that $v_k(b_1, \dots, b_k) = 1$ and hence $u_k \leq v_k$ holds in $\mathbf{B}$.

Conversely suppose that $\mathbf{B}$ is a finitely generated totally ordered hoop; then $\mathbf{B}$ has finite index. If $\#\mathbf{B} > n$, then $\mathbf{B} \not\models \lambda_n$ by Theorem A.1, so we may assume that $\mathbf{B} = \bigoplus_{j=1}^k \mathbf{B}_j$ for $k \leq n$.

Let $I_j = \{i : \mathbf{B}_j \notin \mathbf{V}_i\}$ and let $\mathbf{W}_j = \mathbb{V}(\{\mathbf{V}_i : i \in I_j\})$; we observe that $I_j \subseteq [n]$ for all i . First let's assume that I_j is nonempty; we are in a congruence distributive variety, so Jónsson's Lemma applies. One of its many consequences is that the finitely subdirectly irreducible algebras (i.e. the totally ordered algebras in our case) of a finite join of subvarieties are exactly the totally ordered algebras that belong to at least one of joinands. This implies immediately that $\mathbf{W}_i^t$ consists of all the totally ordered hoops satisfying the universal sentence

$$\forall x \bigwedge_{i \in I_j} (t_i(x) \approx 1).$$

But for a totally ordered hoop, since 1 is join irreducible, this is clearly equivalent to satisfying the equation $\bigvee_{i \in I_j} t_i(x) \approx 1$ which therefore axiomatizes $\mathbf{W}_j$. Now $\mathbf{B}_j$ is a totally ordered hoop that does not belong to $\mathbf{V}_i$ for all $i \in I_j$; so $\mathbf{B}_j \notin \mathbf{W}_i$. Therefore there is a $b_j \in B_j$ which falsifies $\bigvee_{i \in I_j} t_i(x) \approx 1$, which implies that $t_i(b_j) \neq 1$ for all $i \in I_j$.

If, on the other hand, I_j is empty, then we let b_j be any element of $\mathbf{B}_j$ different from 1.

Suppose now that $\mathbf{B} \notin \bigoplus_{i \leq n}^t \mathbf{V}_i$; we claim that $b_1, \dots, b_k$ falsify $u_k \leq v_k$. Note that by construction $b_1 \ll b_2 \ll \dots \ll b_k$ so $u_k(b_1, \dots, b_k) = 1$. On the other hand the first join in v_k evaluates to $b_k < 1$. Since $\mathbf{B} \notin \bigoplus_{i \leq n}^t \mathbf{V}_i$ for every sequence $s \in SEQ_k$ there is a $j \leq k$ such that if $s_j \in [n]$ then $\mathbf{B}_j \notin \mathbf{V}_{s_j}$. It follows that $s_j \in I_j$ and so $t_{s_j}(b_j) \neq 1$; hence $\bigwedge_{j \leq k, s_j \in [n]} t_{s_j}(b_j) < 1$. Since this happens for every $s \in SEQ_k$, the second join in v_k is less than 1. In conclusion, $v_k(b_1, \dots, b_k) < 1$ and thus $u_k \leq v_k$ fails in $\mathbf{B}$. So every finitely generated totally ordered algebra not in $\mathbf{V}$ fails either λ_n or $u_k \leq v_k$ for some k . This concludes the proof. $\square$

If now in the sum $\bigoplus_{i \leq n}^t \mathbf{V}_i$ the first component is a Wajsberg algebra, then we have to modify the argument a little. The main change is that we must consider only the sequences in $s \in SEQ_k$ such that $s_1 = 1$. If we call SEQ_k^1 the set of those sequences then we can state and prove an analogue of Theorem A.3. whose proof is left to the reader, without any problem.

THEOREM A.4.

Let $\mathbf{V}_1, \dots, \mathbf{V}_n$ be varieties of Wajsberg hoops; then $\bigoplus_{i \leq n}^t \mathbf{V}_i$ is axiomatized by λ_n , $\neg\neg(t_1(x)) \approx 1$

and for all $k \leq n$

$$\bigwedge_{i=1}^{k-1} (x_{i+1} \rightarrow x_i) \rightarrow x_i \leq \bigvee_{i=1}^k x_i \vee \bigvee_{s \in SEQ_k^1} \left(\bigwedge_{j \leq k, s_j \in [n]} t_{s_j}(x_j) \right).$$

The next thing we want to deal with is the case in which every summand in $\bigoplus_{i \leq n}^t \mathbf{V}_i$ is generated by a single Wajsberg chain $\mathbf{A}_i$; we are interested in the case in which $\mathbb{V}(\bigoplus_{i=1}^n \mathbf{A}_i) = \bigoplus_{i \leq n}^t \mathbb{V}(\mathbf{A}_i)$. The following combines results from [7], [1] and a lemma that has many versions; the one we use is from [4] (Lemma 5.13).

LEMMA A.5.

Let $\mathbf{K}$ be any class of basic hoops or BL-algebras; then $\mathbf{Q} = \mathbf{Q}(\mathbf{K})$ is a variety if and only if all the totally ordered members of $\mathbb{V}(\mathbf{K})$ are in $\mathbb{ISP}_u(\mathbf{K})$.

Next, according to [1], we say that a totally ordered BL-algebra $\mathbf{A} = \bigoplus_{i=1}^n \mathbf{A}_i$ is **uniform** if for all $i \leq n$

- $\mathbf{L}_n$ is embeddable in $\mathbf{A}_i$ for all n , or
- $\mathbf{A}_i$ is finite, or
- $i > 0$ and $\mathbf{A}_i$ is cancellative, or else
- $\mathbf{A}_i$ is not simple (i.e. $\text{Rad}(\mathbf{A}_i)$ is proper) and $\text{rank}(\mathbf{A}_i) = d(\mathbf{A}_i)$.

The definition of a uniform totally ordered basic hoop is the same with the obvious modifications.

THEOREM A.6.

Let $\mathbf{A} = \bigoplus_{i=1}^n \mathbf{A}_i$ be a totally ordered BL-algebra (basic hoop) of finite index. Then the following are equivalent:

1. $\mathbf{A}$ is uniform;
2. $\mathbf{Q}(\mathbf{A}) = \bigoplus_{i \leq n}^t \mathbb{V}(\mathbf{A}_i)$;
3. $\mathbb{V}(\mathbf{A}) = \bigoplus_{i \leq n}^t \mathbb{V}(\mathbf{A}_i)$;
4. $\mathbf{Q}(\mathbf{A})$ is a variety.

PROOF. The equivalence of (1), (2) and (3) comes from Theorems 7.4 and 7.6 in [7] and (2) clearly implies (4).

Assume then that (1) fails; then there is $j \leq n$ such that either $\mathbf{A}_j$ has infinite rank and $\mathbf{L}_k$ is not embeddable in $\mathbf{A}_j$ for some k , or else $\mathbf{A}_j$ has finite rank and $d_{\mathbf{A}_j}$ is strictly less than the rank of $\mathbf{A}_j$. In the first case $\mathbb{V}(\mathbf{A}_j) = \mathbf{WH}$ ([9], Theorem 2.4); now $[0, 1]_{\mathbf{L}}$ is a simple member of $\mathbf{WH}$ that does not belong to $\mathbb{ISP}_u(\mathbf{A}_j)$ (since $\mathbf{L}_k$ is embeddable in $[0, 1]_{\mathbf{L}}$ for all k). In the second suppose that the rank of $\mathbf{A}_j$ is l ; then $\mathbf{L}_{l,0}$ is subdirectly irreducible, $\mathbf{L}_{l,0} \in \mathbb{V}(\mathbf{A}_j)$ (since it has rank l), but $\mathbf{L}_{l,0} \notin \mathbb{ISP}_u(\mathbf{A}_j)$ (by Theorem 6.3 in [7]). So in both cases we can find a subdirectly irreducible algebra $\mathbf{B}$ that is in $\mathbb{V}(\mathbf{A}_j)$ but not in $\mathbb{ISP}_u(\mathbf{A}_j)$. Let $\mathbf{C} = \mathbf{A}_1 \oplus \cdots \oplus \mathbf{A}_{j-1} \oplus \mathbf{B}$; then $\mathbf{C}$ is subdirectly irreducible, $\mathbf{C} \in \mathbb{V}(\mathbf{A})$ but $\mathbf{C} \notin \mathbb{ISP}_u(\mathbf{A})$. By Lemma A.5 $\mathbf{Q}(\mathbf{A})$ is not a variety. So (4) implies (1) and (1)-(4) are equivalent. $\square$

The combination of Theorems A.3 and A.6 yields that if $\mathbf{A}$ is uniform then $\mathbb{V}(\mathbf{A})$ is finitely based. The converse, however, does not hold; there are non-uniform chains of finite index that generate a finitely based variety (see [7], p.127). The problem of characterizing the chains of finite index that generate a finitely based variety is still open; also, there are other relevant problems about varieties

of finite chains and/or ordered sums of varieties still awaiting a solution. The interested reader can consult the final part of [1] for more information.

Appendix B: Disconnected rotations: beyond BL-algebras

We have introduced generalized rotations in Section 4.1 concentrating on n -liftings. Another form of generalized rotation is the **disconnected rotation**. Given a basic hoop $\mathbf{A}$ one can define its disconnected rotation $\mathbf{A}^\delta$ as the stacking of two copies of $\mathbf{A}$, the lower one with the dual order, with the operations defined in such a way that it is a (bounded) commutative integral residuated lattice (see [11] for a proper definition). If $\mathbf{V}$ is a variety of hoops then $\mathbf{V}^\delta$ is the variety generated by the disconnected rotations of (totally ordered) members of $\mathbf{V}$.

An MTL-algebra is a commutative, prelinear, integral and bounded residuated lattice; thus BL-algebras are simply MTL-algebras satisfying the equation $x(x \rightarrow y) \approx y(y \rightarrow x)$. The zero-free subreducts of MTL algebras are called GMTL-algebras and so basic hoops are GMTL-algebras satisfying the equation above. Now while it is obvious that any n -lifting of a variety of basic hoops is a variety of BL-algebras (basic hoops), this does not happen for disconnected rotations. The following is easy to verify (see Proposition 3.4 in [11]).

PROPOSITION B.1.

If $\mathbf{A}$ is a basic hoop then $\mathbf{A}^\delta$ is an MTL-algebra. It is a BL-algebra if and only if the $\mathbf{A}$ is a cancellative hoop.

Given this it is easily checked that $\mathbb{L}_{1,0} = \mathbf{C}_\omega^\delta$ (as a BL-algebra or a basic hoop), $\mathbb{V}(\mathbb{L}_{1,0}) = \mathbf{C}^\delta$ is the only variety of BL-algebras that can be obtained as a disconnected rotation of a variety of basic hoops. Therefore, disconnected rotations are more suitable to treat varieties that fall outside the realm of BL-algebras. Disconnected rotations have been studied at length in [11]: the following lemma, which is obvious for n -liftings, requires a proof for disconnected rotations.

LEMMA B.2.

[11] Let $\mathbf{V}$ be a variety of hoops; then the totally ordered members of $\mathbf{V}^\delta$ are exactly the rotations of totally ordered members of $\mathbf{V}$.

Next we observe that again as a consequence of Theorem 4.10 in [13] we get:

THEOREM B.3.

Let $\mathbf{V}$ be a variety of basic hoops with the Baker–Beynon property; then $\mathbf{V}^\delta$ has the Baker–Beynon property.

We also have a stronger analogue of Lemma 5.5 and Corollary 5.6

LEMMA B.4.

Let $\mathbf{W}$ be a variety of hoops. Then every nontrivial algebra in $\mathbf{W}^\delta$ is unifiable.

PROOF. Let $\mathbf{A}$ be a nontrivial totally ordered algebra in $\mathbf{W}^\delta$; then by Lemma B.2 $\mathbf{A} \cong \mathbf{B}^\delta$ for some $\mathbf{B} \in \mathbf{W}$. It is easy to check that B is a filter in $\mathbf{B}^\delta$ and that $\mathbf{B}^\delta/B \cong \mathbb{L}_1$. So $\mathbf{A}$ has $\mathbb{L}_1$ as homomorphic image. The conclusion follows from an analogous reasoning as the one of the proof of Lemma 5.2. $\square$

COROLLARY B.5.

Let $\mathbf{W}$ be a variety of basic hoops with the Baker–Beynon property; then $\mathbf{W}^\delta$ is a structural variety of MTL-algebras.

This is a good point to recap the variety of hoops that we know that have the Baker–Beynon property:

- the variety $\mathbf{C}$ of cancellative hoops;

- the variety $\mathbf{dIWH} = \mathbb{V}(\mathbb{L}_{1,0}^+)$;
- the varieties $\mathbf{PL}_n^+$ for $n \geq 1$ and
- every locally finite variety.

We can apply the disconnected rotations to all of them to get structural varieties of **MTL**-algebras. For instance, let's consider the variety **GH** of Gödel hoops, which is locally finite. Then it is easy to check that $\mathbf{GH}^\delta$ is the variety $\mathbf{NM}^-$ of **nilpotent minimum algebras without negative fixpoint**; so $\mathbf{NM}^-$ has the Baker–Beynon property and since it is also locally finite, it is primitive. So we have obtained the same result as in [39] with a different approach. Actually, this is an instance of a more general result.

LEMMA B.6.

Let $\mathbf{V}$ be a locally finite variety of basic hoops; then $\mathbf{V}^\delta$ is locally finite.

PROOF. Let $\mathbf{B}$ be an m -generated totally ordered algebra in $\mathbf{V}^\delta$; then by Lemma B.2 there is a totally ordered algebra $\mathbf{A} \in \mathbf{V}$ such that $\mathbf{B} \cong \mathbf{A}^\delta$. Now $\mathbf{A}$ is at most m generated, so there exists a function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that $|A| \leq f(m)$ (because of Theorem 14.1.3 in [48]). Clearly $|B| \leq 2f(n)$ so again by the quoted result by Mal'cev, $\mathbf{V}^\delta$ is locally finite. $\square$

COROLLARY B.7.

If $\mathbf{V}$ is a locally finite variety of basic hoops, then $\mathbf{V}^\delta$ is primitive.

In **MTL**-algebras we can also use n -liftings; Theorem 4.10 in [13] when relativized to **MTL** algebra is:

THEOREM B.8.

Let $\mathbf{V}$ be any variety of **GMTL**-algebras with the Baker–Beynon property. Then, for every $n \geq 1$, $\mathbb{V}(\mathbb{L}_n \oplus^f \mathbf{V})$ is a variety of **MTL**-algebras with the Baker–Beynon property.

Here the problem is that the Baker–Beynon property has not been investigated in **GMTL**-algebras (we have only results about basic hoops). If one wants to develop further the circle of ideas considered in this appendix, such an investigation is of paramount importance.

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