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Forcing axioms via ground model interpretations

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We study principles of the form: if a name σ is forced to have a certain property φ, then there is a ground model filter g such that σ^g satisfies φ. We prove a general correspondence connecting these name principles to forcing axioms. Special cases of the main theorem are:

- Any forcing axiom can be expressed as a name principle. For instance, PFA is equivalent to:
  - A principle for rank 1 names (equivalently, nice names) for subsets of ω_1.
  - A principle for rank 2 names for sets of reals.
- λ-bounded forcing axioms are equivalent to name principles. Bagaria's characterisation of BFA via generic absoluteness is a corollary.

We further systematically study name principles where φ is a notion of largeness for subsets of ω_1 (such as being unbounded, stationary or in the club filter) and corresponding forcing axioms.

Contents

1. Introduction . . . 2
2. Some definitions . . . 5
   2.1. Forcing axioms . . . 6
   2.2. Name principles . . . 7
   2.3. Hybrid axioms . . . 11
3. Results for rank 1 . . . 12
   3.1. Basic implications . . . 12
   3.2. Extremely bounded name principles . . . 15
   3.3. Extremely bounded forcing axioms . . . 16
   3.4. Basic results on ub-FA . . . 17
   3.5. Characterisations of FA^+ and FA^{++} . . . 18
4. A correspondence for arbitrary ranks . . . 19
   4.1. The correspondence . . . 19
   4.2. The proofs . . . 20

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4.3. Generic absoluteness . . . 27
4.4. Boolean ultrapowers . . . 32
4.5. An application to ub-FA . . . 33
5. Specific classes of forcings . . . 34
5.1. Classes of forcings . . . 34
5.1.1. σ-distributive forcings . . . 34
5.1.2. σ-closed forcings . . . 35
5.1.3. c.c.c. forcings . . . 35
5.2. Specific forcings . . . 38
5.2.1. Cohen forcing . . . 38
5.2.2. Random forcing . . . 39
5.2.3. Hechler forcing . . . 42
5.2.4. Suslin trees . . . 43
5.2.5. Club shooting . . . 44
6. Conclusion . . . 44
Declaration of competing interest . . . 45
References . . . 45

1. Introduction

In this paper, we isolate and study *name principles*. They express that for names σ such that a certain property is forced for σ, there exists a filter g in the ground model V such that σ^g already has this property in V. In general, we fix a class Σ of names, for example nice names for sets of ordinals. Given a forcing $\mathbb{P}$ and a formula $\varphi(x)$, one can then study the principle:

"If $\sigma \in \Sigma$ and $\mathbb{P} \Vdash \varphi(\sigma)$ holds, then there exists a filter $g \in V$ on $\mathbb{P}$ such that $\varphi(\sigma^g)$ holds in V."

Such principles are closely related to Bagaria's work on generic absoluteness and forcing axioms [4]. Recall that the forcing axiom $\mathsf{FA}_{\mathbb{P},\kappa}$ associated to a forcing $\mathbb{P}$ and an uncountable cardinal κ states:

"For any sequence $\vec{D} = \langle D_\alpha : \alpha < \kappa\rangle$ of predense subsets of $\mathbb{P}$, there is a filter $g \in V$ on $\mathbb{P}$ such that $g \cap D_\alpha \neq \emptyset$ for all $\alpha < \kappa$."

Often, proofs from forcing axioms can be formulated by first proving a name principle and then obtaining the desired result as an application.

Example 1.1. $\mathsf{FA}_{\mathbb{P},\omega_1}$ implies that for any stationary subset S of ω_1, $\mathbb{P}$ does not force that S is nonstationary. We sketch an argument via a name principle. We shall show in Section 4 that $\mathsf{FA}_{\mathbb{P},\omega_1}$ implies the name principle for any nice name τ for a subset of ω_1 and any Σ_0-formula φ. So towards a contradiction, suppose there is a name τ for a club with $\Vdash_{\mathbb{P}} \tau \cap S = \emptyset$. Apply the name principle for the formula "τ is a club in ω_1 and $\tau \cap \check{S} = \emptyset$". Hence there is a filter $g \in V$ such that τ^g is a club and $\tau^g \cap S = \emptyset$. However, the existence of τ^g contradicts the assumption that S is stationary.

Name principles for stationary sets have appeared implicitly in combination with forcing axioms.

Example 1.2. The forcing axiom PFA^+ states:

For any proper forcing $\mathbb{P}$, any sequence $\vec{D} = \langle D_\alpha : \alpha < \omega_1\rangle$ of predense subsets of $\mathbb{P}$ and any nice name σ for a stationary subset of ω_1, there is a filter g on $\mathbb{P}$ such that

- $g \cap D_\alpha \neq \emptyset$ for all $\alpha < \omega_1$ and
- σ^g is stationary.

Thus PFA^+ is a combination of the forcing axiom PFA with a name principle for stationary sets. Note that the formula "σ is stationary" is not Σ_0.

We aim for an analysis of name principles for their own sake. The main result of this paper is that name principles are more general than forcing axioms. In other words, all known forcing axioms can be reformulated as name principles. For instance, we have:

Theorem 1.3. *(see Theorem 4.1[1]) Suppose that $\mathbb{P}$ is a forcing and κ is a cardinal. Then the following statements are equivalent:*

(1) $\mathsf{FA}_{\mathbb{P},\kappa}$
(2) *The name principle $\mathsf{N}_{\mathbb{P},\kappa}$ for nice names σ and the formula $\sigma = \check{\kappa}$.*
(3) *The simultaneous name principle $\Sigma_0^{(\mathrm{sim})}$-$\mathsf{N}_{\mathbb{P},\kappa}$ for nice names σ and all first-order formulas over the structure $(\kappa, \in, \sigma)$.*

The main Theorems 4.1 and 4.2 are more general and cover: (i) arbitrary names instead of nice names and (ii) bounded forcing axioms.

Bagaria proved an equivalence between bounded forcing axioms and generic absoluteness principles [3,4]. The following corollary of 4.2 has Bagaria's result as a special case. Here $\mathsf{BFA}_{\mathbb{P},\kappa}$ denotes the usual bounded forcing axiom, i.e. for κ many predense sets of size at most κ, and the principle in (2) denotes the name principle for names of the form $\{(\check{\alpha}, p_\alpha) : \alpha \in \kappa\}$ and for all Σ_0-formulas simultaneously.

Theorem 1.4. *(see Theorems 4.17 and 4.22) Suppose that κ is an uncountable cardinal, $\mathbb{P}$ is a complete Boolean algebra and $\dot{G}$ is a $\mathbb{P}$-name for the generic filter. The following conditions are equivalent:*

(1) $\mathsf{BFA}_{\mathbb{P},\kappa}$
(2) $\Sigma_0^{(\mathrm{sim})}$-$\mathsf{BN}^1_{\mathbb{P},\kappa}$
(3) $\Vdash_{\mathbb{P}} V \prec_{\Sigma^1_1(\kappa)} V[\dot{G}]$

If $\mathrm{cof}(\kappa) > \omega$ or there is no inner model with a Woodin cardinal, then the next condition is equivalent to (1), (2) and (3):

(4) $\Vdash_{\mathbb{P}} H^V_{\kappa^+} \prec_{\Sigma_1} H^{V[\dot{G}]}_{\kappa^+}$

If $\mathrm{cof}(\kappa) = \omega$ and $2^{<\kappa} = \kappa$, then the next condition is equivalent to (1), (2) and (3):

(5) $1_{\mathbb{P}}$ *forces that no new bounded subset of κ are added.*

The second topic of this paper is the study of name principles for *specific* formulas $\varphi(x)$. In particular, we will consider these principles when $\varphi(x)$ denotes a notion of largeness for subsets of κ such as being unbounded, stationary, or in the club filter. For each of these notions, we also study the corresponding forcing axiom. For instance, the *unbounded forcing axiom* $\mathsf{ub}\text{-}\mathsf{FA}_{\mathbb{P},\kappa}$ states:

> *"For any sequence $\vec{D} = \langle D_\alpha : \alpha < \kappa \rangle$ of predense subsets of $\mathbb{P}$, there is a filter g on $\mathbb{P}$ such that $g \cap D_\alpha \neq \emptyset$ for unboundedly many $\alpha < \kappa$."*

[1] This follows from Theorem 4.1 (2) for $X = \kappa$ and $\alpha = 1$.

All these principles are defined formally in Section 2. The next diagram displays some results about them. Solid arrows denote non-reversible implications, dotted arrows stand for implications whose converse remains open, and dashed lines indicate that no implication is provable. The numbers indicate where to find the proofs.

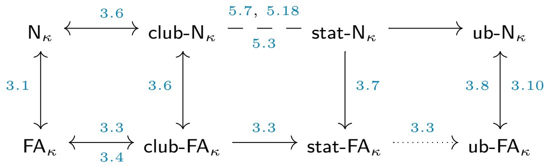

Fig. 1. Forcing axioms and name principles for regular κ.

We also investigate whether similar implications hold for λ-bounded name principles and forcing axioms, where λ is any cardinal. The results about the cases $\kappa \le \lambda$, $\omega \le \lambda < \kappa$ and $1 \le \lambda < \kappa$ are displayed in the next diagrams. Here a CBA is a complete Boolean algebra.

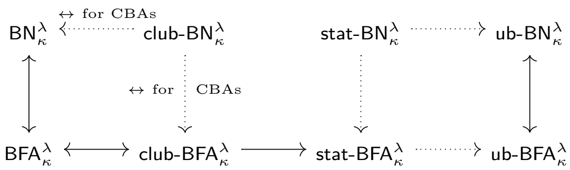

Fig. 2. λ-bounded forcing axioms and name principles for regular κ and $\lambda \ge \kappa$.

It is open whether club-$\mathsf{BN}^\lambda_{\mathbb{P},\kappa}$ implies stat-$\mathsf{BN}^\lambda_{\mathbb{P},\kappa}$. Conversely, there are forcings $\mathbb{P}$ where stat-$\mathsf{BN}^\lambda_{\mathbb{P},\kappa}$ holds for all λ, but club-$\mathsf{BN}^\lambda_{\mathbb{P},\kappa}$ fails for all $\lambda \ge \omega$ (see Section 5.1.3, Lemma 5.7 and Remark 5.18).

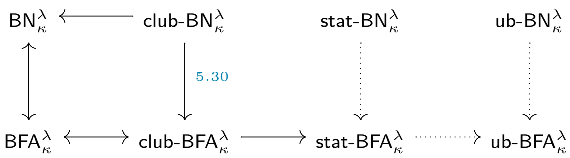

Fig. 3. λ-bounded forcing axioms and name principles for regular κ and $\omega \le \lambda < \kappa$.

Again, it is open whether club-$\mathsf{BN}^\lambda_{\mathbb{P},\kappa}$ implies stat-$\mathsf{BN}^\lambda_{\mathbb{P},\kappa}$, but the converse implication does not hold by the previous remarks.

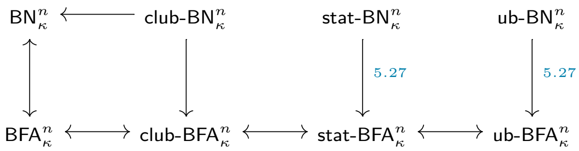

Fig. 4. n-bounded forcing axioms and name principles for regular κ and $1 \le n < \omega$.

The principles in the bottom row and BN^n_κ are all provable.

The implications and separations in the previous diagrams are proved using specific forcings such as Cohen forcing, random forcing and Suslin trees. For instance, we have the following results:

Proposition 1.5. *(see Lemma 5.20) Let $\mathbb{P}$ denote random forcing. The following are equivalent:*

(1) $\mathsf{FA}_{\mathbb{P},\omega_1}$
(2) $\mathsf{ub\text{-}FA}_{\mathbb{P},\omega_1}$
(3) 2^ω *is not the union of* ω_1 *many null sets*

Proposition 1.6. *(see Corollary 5.27) Suppose that a Suslin tree exists. Then there exists a Suslin tree T such that* $\mathsf{stat\text{-}BN}^1_{T,\omega_1}$ *fails.*

For some forcings, most of Fig. 1 collapses. In particular, if $\mathsf{ub\text{-}FA}_{\mathbb{P},\kappa}$ implies $\mathsf{FA}_{\mathbb{P},\kappa}$, then all entries other than $\mathsf{stat\text{-}N}_{\mathbb{P},\kappa}$ are equivalent. We investigate when this implication holds. For instance:

Proposition 1.7. *(see Lemma 5.1) For any $<\kappa$-distributive forcing $\mathbb{P}$, we have* $\mathsf{ub\text{-}FA}_{\mathbb{P},\kappa} \implies \mathsf{FA}_{\mathbb{P},\kappa}$.

In a broader range of cases, $\mathsf{ub\text{-}FA}_{\mathbb{P},\kappa}$ implies most of the entries in Fig. 2:

Proposition 1.8. *(see Lemma 4.25) If κ an uncountable cardinal and $\mathbb{P}$ is a complete Boolean algebra that does not add bounded subsets of κ, then*

$$(\forall q \in \mathbb{P}\ \mathsf{ub\text{-}FA}_{\mathbb{P}_q,\kappa}) \Longrightarrow \mathsf{BFA}^\kappa_{\mathbb{P},\kappa}.$$

The previous result is a corollary to the proof of Theorem 1.4.

We collect some definitions in Section 2. In Section 3, we prove the positive implications in Fig. 1. In Section 4, we prove a general correspondence between forcing axioms and name principles. Theorem 1.3 is a special case. We further derive results about generic absoluteness and other consequences of the correspondence. In Section 5, we study the principles in Figs. 1-4 for specific classes of forcings such as σ-distributive and c.c.c. and for specific forcings such as Cohen and random forcing. We use these results to separate some of the principles in the figures.

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2. Some definitions

In this section, we introduce the axioms we will be finding equivalences between. We will also define a few concepts that we will want to use repeatedly.

Definition 2.1. Let X be a set and α an ordinal. We recursively define $\mathcal{P}^\alpha(X)$ and $\mathcal{P}^{<\alpha}(X)$:

$\mathcal{P}^0(X) = X$
$\mathcal{P}^{<\alpha}(X) = \bigcup_{\beta<\alpha} \mathcal{P}^\beta(X)$
$\mathcal{P}^\alpha(X) = \mathcal{P}(\mathcal{P}^{<\alpha}(X))$ for $\alpha > 0$.

The axioms we are working with come under two headings: *forcing axioms* and *name principles*. Within these headings there are a variety of different axioms we will be working with.

A *forcing* is a partial order with a largest element 1. Throughout this section, assume that $\mathbb{P}$ is a forcing and $\mathcal{C}$ is a class of forcings. G will be a generic filter (on $\mathbb{P}$); g will be a filter on $\mathbb{P}$ which is contained in the ground model V (and therefore certainly not generic, if $\mathbb{P}$ is atomless).

2.1. Forcing axioms

Notation 2.2. In the following, $\vec{D} = \langle D_\gamma : \gamma < \kappa \rangle$ always denotes a sequence of dense (or predense) subsets of a forcing $\mathbb{P}$. If g is a subset of $\mathbb{P}$, then its *trace* with respect to $\vec{D}$ is defined as the set

$$\mathrm{Tr}_{g,\vec{D}} = \{\alpha < \kappa : g \cap D_\alpha \neq \emptyset\}.$$

Definition 2.3. Let κ be a cardinal. The forcing axiom $\mathsf{FA}_{\mathbb{P},\kappa}$ says:

"For any $\vec{D}$, there exists a filter $g \in V$ with $\mathrm{Tr}_{g,\vec{D}} = \kappa$."

The forcing axiom $\mathsf{FA}_{\mathcal{C},\kappa}$ asserts that $\mathsf{FA}_{\mathbb{P},\kappa}$ holds for all $\mathbb{P} \in \mathcal{C}$.

Of course, we could just as well have written "predense" instead of "dense" in the above definition.

We will suppress the $\mathbb{P}$ or $\mathcal{C}$ in the above notation when it is clear which forcing we are referring to. If $\kappa = \omega_1$ we will suppress it too, just writing $\mathsf{FA}_{\mathbb{P}}$ (or just FA if $\mathbb{P}$ is clear as well).

We can weaken this axiom: instead of insisting that g must meet every D_γ, we could insist only that it meets "many" of them in some sense. The following forcing axioms do exactly that, for various senses of "many".

Definition 2.4. Suppose that κ is a cardinal and $\varphi(x)$ is a formula. The axiom φ-$\mathsf{FA}_{\mathbb{P},\kappa}$ states:

"For any $\vec{D}$, there is a filter g on $\mathbb{P}$ such that $\varphi(\mathrm{Tr}_{g,\vec{D}})$ holds."

In particular, we will consider the following formulas:

(1) $\mathsf{club}(x)$ states that x contains a club in κ. club-$\mathsf{FA}_{\mathbb{P},\kappa}$ is called the *club forcing axiom*.
(2) $\mathsf{stat}(x)$ states that x is stationary in κ. stat-$\mathsf{FA}_{\mathbb{P},\kappa}$ is called the *stationary forcing axiom*.
(3) $\mathsf{ub}(x)$ states that x contains an unbounded subset of κ. ub-$\mathsf{FA}_{\mathbb{P},\kappa}$ is called the *unbounded forcing axiom*.
(4) ω-$\mathsf{ub}(x)$ states that x contains ω as a subset and is also unbounded in κ. ω-ub-$\mathsf{FA}_{\mathbb{P},\kappa}$ is called the *ω-unbounded forcing axiom*.

We define club-$\mathsf{FA}_{\mathcal{C},\kappa}$, stat-$\mathsf{FA}_{\mathcal{C},\kappa}$, ub-$\mathsf{FA}_{\mathcal{C},\kappa}$ and ω-ub-$\mathsf{FA}_{\mathcal{C},\kappa}$ in the same way as we defined $\mathsf{FA}_{\mathcal{C},\kappa}$ in Definition 2.3.

ω-ub-FA can also be expressed as a combined version of two forcing axioms: that given a κ long sequence $\vec{D}$ and a separate ω long sequence $\vec{E}$ of (pre)dense sets, we can find a filter g such that $\mathrm{Tr}_{g,\vec{D}}$ is unbounded and $\mathrm{Tr}_{g,\vec{E}} = \omega$.

Again, we will suppress $\mathbb{P}$ or $\mathcal{C}$ where they are obvious, and will suppress κ when $\kappa = \omega_1$.

We can also weaken the axiom by insisting that every dense set D_γ be *bounded* in cardinality, by some small cardinal.

Definition 2.5. Let κ and λ be cardinals. The bounded forcing axiom $\mathsf{BFA}^{\lambda}_{\mathbb{P},\kappa}$ says

> "Whenever $\langle D_\gamma : \gamma < \kappa\rangle$ is a sequence of *predense* subsets of $\mathbb{P}$, *and for all* γ *we have* $|D_\gamma| \leq \lambda$, then there is a filter $g \in V$ such that for all $\gamma < \kappa$, $g \cap D_\gamma \neq \emptyset$."

We define $\mathsf{BFA}^{\lambda}_{\mathcal{C},\kappa}$, $\text{club-}\mathsf{BFA}^{\lambda}_{\mathbb{P},\kappa}$ and so forth in the natural way, using definitions analogous to those in 2.3 and 2.4.

Again, we will suppress notation as described above. We will suppress the λ if $\lambda = \kappa$.

Note that we are definitely looking at predense sets here, since actual dense sets are likely to be rather large and the axiom would be likely to be trivial if we had to use dense sets. These bounded forcing axioms are only really of interest when $\mathbb{P}$ is a Boolean algebra, since they always contain (nontrivial) predense sets with as few as two elements so the axiom will not be vacuous.

There is one more forcing axiom we want to introduce, but it requires some additional notation so we will postpone it until later in this section.

2.2. Name principles

We ought to define name principles at this point, but we need to cover some other terminology first in order to express the definitions.

As one might expect, name principles are about different $\mathbb{P}$ names, and it will be useful to have some measure of how complex a name is. The following three definitions are all different ways of doing this; we will be using all of them.

Definition 2.6. Let X be a set (in V). We recursively define a name's rank as follows.

σ is an α rank X name (or a rank α name for short) if either:

- $\alpha = 0$ and $\sigma = \check{x}$ for some $x \in X$; or
- σ is not rank 0 and $\alpha = \sup\{\mathrm{rank}(\tau) + 1 : \exists p \in \mathbb{P}(\tau, p) \in \sigma\}$

We also call a 1 (or 0) rank X name a *good* name. Of course, we will also talk about rank $\leq \alpha$ names, meaning names which are either rank $<\alpha$ or rank α.

This definition is a name analogue to saying that $\sigma \in \mathcal{P}^\alpha(X)$, where X is transitive. Most of the time, we will be interested in the case where X is some cardinal, most often either 0 or ω_1. Note that every $\mathbb{P}$ name is an α rank X name for some α.

Definition 2.7. Let σ be a $\mathbb{P}$ name and κ be a cardinal. We say σ is *locally* κ *small* if there are at most κ many names τ such that for some $p \in \mathbb{P}$, we have $(\tau, p) \in \sigma$. A name σ is κ *small* if it is locally κ small, and every name τ in the above definition is κ small.

If being rank α is analogous to being in $\mathcal{P}^\alpha$ (or $\mathcal{P}^\alpha(X)$) then the analogue of being κ small would be being in H_{κ^+}. We could also easily define a version of this for $H_{\kappa^+}(X)$ if we wanted. However, we don't actually need to: in all the cases we're going to be interested in, $\bar{X}$ will have cardinality $\leq \kappa$ and the definition would be equivalent to the above one.

The following proposition says that we only really need to worry about κ smallness when we go above rank 1 names.

Proposition 2.8. *Let X be transitive, and of size at most κ. Let σ be a 0 rank or 1 rank X name. Then σ is κ small.*

On the other hand if X has size greater than κ then no interesting rank 1 name will be κ small.

The next definition does not have an easy analogue, but is a kind of complement to the previous one and is critical when we work with bounded forcing axioms.

Definition 2.9. Let σ be a $\mathbb{P}$ name and λ be a cardinal. We say σ is *locally λ bounded* if it can be written as

$$\sigma = \{(\tau, p) : \tau \in T, p \in S_\tau\}$$

where T is some set of names, and for $\tau \in T$ the set S_τ is a subset of $\mathbb{P}$ of size at most λ. A name σ is *λ bounded* if it is locally λ bounded, and every name $\tau \in T$ in the above definition is λ bounded.

A good name which is 1 bounded is known as a *very good* name. A check name $\check{x}$ has the form $\{(\check{y}, 1) : y \in x\}$ and is therefore guaranteed to be λ bounded for any $\lambda > 0$.

We will be talking about interpreting names with respect to a filter. Unfortunately, the literature uses two different meanings of the word "interpretation", which only coincide if the filter is generic. For clarity:

Definition 2.10. Let σ be a name, and g a filter. (Here, g may be inside V or in some larger model.) When we refer to the *interpretation σ^g* of σ, we mean the recursive interpretation:

$$\sigma^g := \{\tau^g : \exists p \in g \ (\tau, p) \in \sigma\}$$

When we refer to the *quasi-interpretation $\sigma^{(g)}$*, we mean the following set:

$$\sigma^{(g)} := \{x \in V : \exists p \in g \ p \Vdash \check{x} \in \sigma\}$$

Proposition 2.11. *$\sigma^g = \sigma^{(g)}$ if σ is a 1 rank X name (for some X) and either*

(1) *g is generic; or*
(2) *σ is 1 bounded.*

Proposition 2.12. *Suppose $\mathbb{P}$ is a complete Boolean algebra, and σ is a 1 rank X name. Then we can find a name τ such that for every filter g, $\tau^g = \tau^{(g)} = \sigma^{(g)}$.*

Proof. For $x \in X$ let $p_x = \sup\{p \in \mathbb{P} : (\check{x}, p) \in \sigma\}$ (so $p_x \in \mathbb{P} \cup \{0\}$). Let $\tau = \{(\check{x}, p_x) : x \in X, p_x \neq 0\}$. $\square$

We can now define our name principles. Here, we take $\mathbb{P}$ to be a forcing, $\mathcal{C}$ a class of forcings, and X an arbitrary set.

Definition 2.13. Let α be an ordinal, κ a cardinal and X a transitive set of size at most κ. The name principle $\mathsf{N}_{\mathbb{P},X,\kappa}(\alpha)$ says the following:

"Whenever σ is a κ small $\leq\alpha$ rank X name, and $A \in H_{\kappa^+} \cap \mathcal{P}^\alpha(X)$ is a set such that $\mathbb{P} \Vdash \sigma = \check{A}$, there is a filter $g \in V$ such that $\sigma^g = A$."

$\mathsf{N}_{\mathcal{C},X,\kappa}(\alpha)$ is the statement that $\mathsf{N}_{\mathbb{Q},X,\kappa}(\alpha)$ holds for all $\mathbb{Q} \in \mathcal{C}$. $\mathsf{N}_{\mathbb{P},\kappa}(\infty)$ (resp. $\mathsf{N}_{\mathcal{C},\kappa}(\infty)$) is the statement that $\mathsf{N}_{\mathbb{P},X,\kappa}(\alpha)$ (resp. $\mathsf{N}_{\mathcal{C},X,\kappa}(\alpha)$) holds for all $\alpha \in \mathrm{Ord}$ and all $X \in H_{\kappa^+}$. (Equivalently, we could just require that it holds for $\alpha \leq \kappa^+$ and all $X \in H_{\kappa^+}$.)

Some comments on this definition: It is easy to see that if σ is a κ small X name, and $g \in V$, then $\sigma^g \in H_{\kappa^+}$. If σ is rank $\leq\alpha$, then it is also easy to see that $\sigma^g \in \mathcal{P}^\alpha(X)$. So if we didn't require that $A \in H_{\kappa^+} \cap \mathcal{P}^\alpha(X)$, then the principle would fail trivially for most forcings. The only forcings on which it could hold would be those which don't force any names to be equal to such large A anyway.

This argument also shows that the name principle fails trivially if, for some $\lambda < \kappa$, there is a λ small σ which is forced to be equal to some $A \notin H_{\lambda^+}$. So we might think we should exclude such names from the principle as well. But in fact, we shall see in Section 4 that it makes little difference: the proof of Theorem 4.1 shows that if a name principle fails because of such a name, then it also fails for non-trivial reasons.

We can easily see that if σ is a κ-small 1 rank X name, and is forced to be equal to A, then $A \subseteq X$ and $|A| \leq \kappa$. Hence, when we're dealing with $\mathsf{N}(1)$, we don't need to worry about checking if the names we're working with are in $H_{\kappa^+} \cap \mathcal{P}(X)$, as this is automatically true. On the other hand, once we go above rank 1, these names can exist, even for small values of α and κ. For example, [13, Lemma 7.1] has an ω bounded rank 2 name which is forced to be equal to $(2^\omega)^V$.

One might ask why we allowed X-names for all $X \in H_{\kappa^+}$ in the definition of $\mathsf{N}_{\mathbb{P},\kappa}(\infty)$. This is because any such name can be understood as an $\emptyset$-name of some high rank, so these principles already follow from the conjunction of $\mathsf{N}_{\mathbb{P},\emptyset,\kappa}(\alpha)$ for all $\alpha \in \mathrm{Ord}$.

As with the forcing axioms, we will sometimes omit part of this notation. We will drop $\mathbb{P}$ and $\mathcal{C}$ when they are clear from context. We will omit α when $\alpha = 1$. While X is formally just some arbitrary set, most of the time it can be thought of as a cardinal; we will omit it in the case that $X = \kappa$, and will then omit κ as well if $\kappa = \omega_1$.

Most often, these omissions will come up when we're assuming $\alpha = 1$ and taking X to be some cardinal. In that situation, κ smallness is essentially trivial: if $\kappa < X$ then our class of names is too restrictive to do anything interesting, and if $\kappa \geq X$ then every 1 rank X name will be κ small, automatically. So when $\alpha = 1$ and X is a cardinal we can find out everything we need to know just by looking at the case $X = \kappa$.

We can also define variations analogous to club-FA, stat-FA, etc. However, this only really makes sense when we know σ a subset of some cardinal. For this reason, we only define these variations for the case where $\alpha = 1$ (also dropping the requirement of κ-smallness) and where X is a cardinal.

Definition 2.14. Let κ be a cardinal and $\varphi(x)$ a formula. The axiom φ-$\mathsf{N}_{\mathbb{P},\kappa}$ states:

"For any 1 rank κ name σ, if $\mathbb{P} \Vdash \varphi(\sigma)$ then there is a filter g on $\mathbb{P}$ such that $\varphi(\sigma^g)$ holds in V."

In particular, we shall consider the axioms for the formulas $\mathsf{club}(x)$, $\mathsf{stat}(x)$, $\mathsf{ub}(x)$ and ω-$\mathsf{ub}(x)$ given in Definition 2.4:

(1) The *club name principle* club-$\mathsf{N}_{\mathbb{P},\kappa}$.
(2) The *stationary name principle* stat-$\mathsf{N}_{\mathbb{P},\kappa}$.
(3) The *unbounded name principle* ub-$\mathsf{N}_{\mathbb{P},\kappa}$.
(4) The *ω-unbounded name principle* ω-ub-$\mathsf{N}_{\mathbb{P},\kappa}$.

As usual, we also define similar axioms with $\mathcal{C}$ in place of $\mathbb{P}$. Note that we could also express ω-ub-N as an axiom about two names, one of which is forced to be an unbounded subset of κ while the other is forced to be equal to ω.

Remark 2.15. The axioms $\mathsf{club}\text{-}\mathsf{FA}_{\mathbb{P},\kappa}$, $\mathsf{stat}\text{-}\mathsf{FA}_{\mathbb{P},\kappa}$, $\mathsf{ub}\text{-}\mathsf{FA}_{\mathbb{P},\kappa}$ and ω-$\mathsf{ub}\text{-}\mathsf{FA}_{\mathbb{P},\kappa}$ in Definition 2.4 can be understood as a more general form of name principles for two formulas $\varphi(x)$ and $\psi(x)$:

"For any 1 rank κ name σ, if $\mathbb{P} \Vdash \varphi(\sigma)$ then there is a filter g on $\mathbb{P}$ such that $\psi(\sigma^g)$ holds in V,"

For instance, $\mathsf{stat}\text{-}\mathsf{FA}_{\mathbb{P},\kappa}$ is equivalent to the statement:

"If σ is a rank 1 name for ω_1, then there is a filter $g \in V$ such that σ^g is stationary."

We can also generalise the ideas here: rather than simply working with a single statement like "σ is unbounded" or "σ is some particular set in V", we could ask to be able to find a filter to correctly interpret every reasonable statement.

In the following definition, we allow bounded quantifiers in our Σ_0 formulas.

Definition 2.16. Let α be an ordinal and κ a cardinal. The simultaneous name principle $\Sigma_0^{(\mathrm{sim})}\text{-}\mathsf{N}_{\mathbb{P},X,\kappa}(\alpha)$ says the following:

"Whenever $\sigma_0, \ldots, \sigma_n$ are κ small $\leq\alpha$ rank X names, we can find a filter g in V such that $\varphi(\sigma_0^g, \ldots, \sigma_n^g)$ holds for *every* Σ_0 formula φ such that $\mathbb{P} \Vdash \varphi(\sigma_0, \ldots, \sigma_n)$."

Moreover:

- The *simultaneous name principle* $\Sigma_0^{(\mathrm{sim})}\text{-}\mathsf{N}_{\mathbb{P},\kappa}(\infty)$ is the same statement, except that the names are X names for some $X \in H_{\kappa^+}$ and there is no restriction on their rank.
- $\Sigma_0^{(\mathrm{sim})}\text{-}\mathsf{N}_{\mathcal{C},X,\kappa}(\alpha)$ is the statement that $\Sigma_0^{(\mathrm{sim})}\text{-}\mathsf{N}_{\mathbb{Q},X,\kappa}(\alpha)$ holds for all $\mathbb{Q} \in \mathcal{C}$.
- $\Sigma_0^{(\mathrm{sim})}\text{-}\mathsf{N}_{\mathcal{C},\kappa}(\infty)$ is defined similarly.
- The bounded name principles $\Sigma_0^{(\mathrm{sim})}\text{-}\mathsf{BN}^{\lambda}_{\mathbb{P},X,\kappa}(\alpha)$ are defined similarly.

The Σ_0 requirement on φ is necessary, because otherwise the axiom would say that any sentence which is forced to be true by $\mathbb{P}$ is already true in V. This would make the axiom trivially false for almost all interesting forcings. Again we will suppress X, κ and α as described earlier.

All of these name principles also have bounded variants:

Definition 2.17. Let α be an ordinal and κ, λ cardinals. The bounded name principle $\mathsf{BN}^{\lambda}_{\mathbb{P},X,\kappa}(\alpha)$ says the following:

"Whenever σ is a κ small λ bounded $\leq \alpha$ rank X name, and A is a set such that $\mathbb{P} \Vdash \sigma = A$, we can find a filter $g \in V$ such that $\sigma^g = A$."

We define similar bounded forms of all the other name principles we have introduced so far. Again, we will suppress λ when $\lambda = \kappa$ and will suppress other notation as described above.

2.3. Hybrid axioms

There is one more group of axioms which are worth mentioning, because of their frequent use in the literature. They are a hybrid of forcing axiom and name principle. The axioms MA^+ and PFA^+ were introduced by Baumgartner in [5, Section 8].

Definition 2.18. The forcing axiom $\mathsf{FA}^+_{\mathbb{P},\kappa}$ says:

"Suppose $\vec{D} = \langle D_\gamma : \gamma < \kappa \rangle$ is a sequence of dense subsets of $\mathbb{P}$ and let σ be a 1 rank κ name such that $\mathbb{P} \Vdash$ "σ is stationary". Then there is a filter g such that
(1) For all γ, $D_\gamma \cap g \neq \emptyset$; and
(2) σ^g is stationary."

The forcing axiom $\mathsf{FA}^{++}_{\mathbb{P},\kappa}$ says:

"Let $\langle D_\gamma : \gamma < \kappa \rangle$ be dense subsets of $\mathbb{P}$ and let $\langle \sigma_\gamma : \gamma < \kappa \rangle$ be 1 rank κ names such that $\mathbb{P} \Vdash$ "σ_γ is stationary" for every γ. Then we can find a filter g such that
(1) For all γ, $D_\gamma \cap g \neq \emptyset$; and
(2) For all γ, σ_γ^g is stationary."

As usual, we will also use versions of the above with $\mathcal{C}$ in place of $\mathbb{P}$, and bounded versions.

We have actually gone against convention slightly here: the literature generally uses the quasi-interpretation $\sigma^{(g)}$ when defining FA^+ and FA^{++} style axioms. However, our version is in fact equivalent, as the following theorem shows:

Theorem 2.19. *Let $\mathsf{FA}^{(+)}$ and $\mathsf{FA}^{(++)}$ be defined in the same way as FA^+ and FA^{++} above, but with $\sigma^{(g)}$ and $\sigma_\gamma^{(g)}$ in place of σ^g and σ_γ^g respectively. Then $\mathsf{FA}^+_{\mathbb{P},\kappa} \iff \mathsf{FA}^{(+)}_{\mathbb{P},\kappa}$ and $\mathsf{FA}^{++}_{\mathbb{P},\kappa} \iff \mathsf{FA}^{(++)}_{\mathbb{P},\kappa}$.*

Proof. We will prove the FA^+ case; the FA^{++} version is similar. The $\Leftarrow$ direction is trivial.

$\Rightarrow$: Let $\langle D_\gamma : \gamma < \kappa \rangle$ be a collection of κ many dense subsets of $\mathbb{P}$. Let σ be a rank 1 name with $\mathbb{P} \Vdash$ "σ is stationary".

For $\gamma \in \kappa$, let

$$E_\gamma := \{p \in \mathbb{P} : p \Vdash \check{\gamma} \notin \sigma \text{ or } \exists q \geq p\, (\check{\gamma}, q) \in \sigma\}$$

We can see that E_γ is dense: given $p \in \mathbb{P}$, either we can find some $q \parallel p$ with $\langle \check{\gamma}, q \rangle \in \sigma$ and we're done, or $p \Vdash \check{\gamma} \notin \sigma$ since all the elements of σ are check names.

Claim 2.20. *If g is any filter which meets all the E_γ, then $\sigma^g = \sigma^{(g)}$*

Proof. $\subseteq$: Let $\gamma \in \sigma^g$. Then there is a $q \in g$ with $(\check{\gamma}, q) \in \sigma$. Clearly $q \Vdash \check{\gamma} \in \sigma$, so $\gamma \in \sigma^{(g)}$.

$\supseteq$: Let $\gamma \in \sigma^{(g)}$. Then we can find $r \in g$ with $r \Vdash \check{\gamma} \in \sigma$. Certainly, then, there is no $p \in g$ with $p \Vdash \check{\gamma} \notin \sigma$. Since nonetheless g meets E_γ, there must be some $q \in g$ with $(\check{\gamma}, q) \in \sigma$. Hence $\gamma \in \sigma^g$. $\square$

Now we simply use our forcing axiom to take a filter g which meets all the D_γ, all the E_γ, and which is such that $\sigma^{(g)}$ is stationary. $\square$

In defining the E_γ in the above proof, we used a technique which we will be invoking many times. It will save us a lot of time if we give it a name now.

Definition 2.21. Let τ and σ be names, and $p \in \mathbb{P}$. We say p strongly forces $\tau \in \sigma$, and write $p \Vdash^+ \tau \in \sigma$, if there exists $q \geq p$ with $(\tau, q) \in \sigma$.

The value of this definition is shown in the following two propositions.

Proposition 2.22. *Let σ and τ be names, and $p \in \mathbb{P}$.*

(1) *If $p \Vdash \tau \in \sigma$, then there exist densely many $r \leq p$ such that for some name $\tilde{\tau}$, $r \Vdash \tilde{\tau} = \tau$ and $r \Vdash^+ \tilde{\tau} \in \sigma$.*
(2) *If $p \Vdash^+ \tau \in \sigma$ then $p \Vdash \tau \in \sigma$.*

Proof. (1): Assume $p \Vdash \tau \in \sigma$. Let $q \leq p$, and let G be a generic filter containing q. Then we know that $\tau^G \in \sigma^G$. Hence there is some pair $(\tilde{\tau}, s) \in \sigma$ such that $s \in G$ and $\tilde{\tau}^G = \tau^G$. Since $\tilde{\tau}^G = \tau^G$, there exists some condition $t \in G$ such that $t \Vdash \tilde{\tau} = \tau$. Now choose $r \leq q, s, t$, which exists by compatibility of elements of G. It is immediate that $r \Vdash \tilde{\tau} = \tau$ and that $r \Vdash^+ \tilde{\tau} \in \sigma$.

(2): Trivial. $\square$

Proposition 2.23. *Let σ and τ be names, let $p \in \mathbb{P}$ and let g be any filter containing p.*

(1) *If $p \Vdash^+ \tau \in \sigma$ then $\tau^g \in \sigma^g$.*
(2) *If for all $\tilde{\tau}$ with $(\tilde{\tau}, q) \in \sigma$ (for some $q \in \mathbb{P}$) we either know $\tau^g \neq \tilde{\tau}^g$ or have $p \Vdash \tilde{\tau} \notin \sigma$ then $\tau^g \notin \sigma^g$.*

3. Results for rank 1

We will start by looking at the positive results we can prove in general about forcing axioms and rank 1 name principles. We again take $\mathbb{P}$ to be an arbitrary forcing. We also take κ to be an uncountable cardinal, although we're mostly interested in the case where $\kappa = \omega_1$. Since $\mathbb{P}$ is arbitrary, we could just as easily replace it with a class $\mathcal{C}$ of forcings in all our results.

3.1. Basic implications

All the positive results expressed in Fig. 1 are proved in this section. The negative results will be proved later, when we look at the specific forcings that provide counterexamples. We will not need that κ is regular. In the case of $\mathrm{cof}(\kappa) = \omega$, a club is

Lemma 3.1. $\mathsf{FA}_{\mathbb{P},\kappa} \iff \mathsf{N}_{\mathbb{P},\kappa}$

Proof. $\Rightarrow$: Assume FA_κ. (That is, $\mathsf{FA}_{\mathbb{P},\kappa}$, recall that we said we'd suppress the $\mathbb{P}$ whenever it was clear.) Let σ be a rank 1 name for a subset of κ, and suppose that $1 \Vdash \sigma = A$ for some $A \subseteq \kappa$. For $\gamma \in A$, let

$$D_\gamma = \{p \in \mathbb{P} : p \Vdash^+ \check{\gamma} \in \sigma\}$$

It is clear that D_γ is dense by Proposition 2.22. For $\gamma \in \kappa \setminus A$, let $D_\gamma = \mathbb{P}$.

Using FA_κ, take a filter g that meets every D_γ. We claim that $\sigma^g = A$. For $\gamma \in A$, we know that some $p \in g$ strongly forces $\check{\gamma} \in \sigma$. By 2.23 then, $\gamma \in \sigma^g$. Conversely, if $\gamma \notin A$ then $1 \Vdash \check{\gamma} \notin \sigma$ and by the same proposition $\gamma \notin \sigma$.

$\Leftarrow$: Assume N_κ. Let $\langle D_\gamma : \gamma < \kappa \rangle$ be a collection of dense subsets of $\mathbb{P}$. Let

$$\sigma = \{(\check{\gamma}, p) : \gamma < \kappa, p \in D_\gamma\}$$

It is easy to see that $1 \Vdash \sigma = \check{\kappa}$. Take a filter g such that $\sigma^g = \kappa$, and then for all $\gamma < \kappa$ $D_\gamma \cap g \neq \emptyset$. $\square$

Lemma 3.2. $\mathsf{FA}_{\mathbb{P},\kappa}$ *holds if and only if for every rank* 1 *name* σ *for a subset of* κ*, there is some* g *with* $\sigma^{(g)} = \sigma^g$.

Proof. First suppose that $\mathsf{FA}_{\mathbb{P},\kappa}$ holds and σ is a rank 1 $\mathbb{P}$-name for a subset of κ. Note that $\sigma^g \subseteq \sigma^{(g)}$ holds for all filters g on $\mathbb{P}$. For each $\alpha < \omega_1$,

$$D_\alpha = \{p \in \mathbb{P} : p \Vdash \check{\alpha} \notin \sigma \vee p \Vdash^+ \check{\alpha} \in \sigma\}$$

is dense. By $\mathsf{FA}_{\mathbb{P},\kappa}$, there is a filter g with $g \cap D_\alpha$ for all $\alpha < \omega_1$. To see that $\sigma^{(g)} \subseteq \sigma^g$ holds, suppose that $\alpha \in \sigma^{(g)}$. Thus there is some $p \in g$ which forces $\check{\alpha} \in \sigma$. Take any $q \in g \cap D_\alpha$. Since $p \parallel q$, we have $p \Vdash^+ \check{\alpha} \in \sigma$ by the definition of D_α and thus $\alpha \in \sigma^g$.

On the other hand, $\mathsf{N}_{\mathbb{P},\kappa}$ and thus $\mathsf{FA}_{\mathbb{P},\kappa}$ (by Lemma 3.1) follows trivially from this principle, since for any rank 1 name σ with $\Vdash \sigma = \check{A}$, we have $\sigma^{(g)} = A$ for any filter g. □

Lemma 3.3.

(1) $\mathsf{FA}_{\mathbb{P},\kappa} \implies \text{club-}\mathsf{FA}_{\mathbb{P},\kappa} \implies \text{ub-}\mathsf{FA}_{\mathbb{P},\kappa}$
(2) $\mathsf{FA}_{\mathbb{P},\kappa} \implies \text{stat-}\mathsf{FA}_{\mathbb{P},\kappa} \implies \text{ub-}\mathsf{FA}_{\mathbb{P},\kappa}$
(3) $\mathsf{FA}_{\mathbb{P},\kappa} \implies \omega\text{-ub-}\mathsf{FA}_{\mathbb{P},\kappa} \implies \text{ub-}\mathsf{FA}_{\mathbb{P},\kappa}$
(4) *If* $\mathrm{cof}(\kappa) > \omega$*, then* $\text{club-}\mathsf{FA}_{\mathbb{P},\kappa} \implies \text{stat-}\mathsf{FA}_{\mathbb{P},\kappa}$

Proof. Follows immediately from the definitions of the axioms. □

Lemma 3.4. $\text{club-}\mathsf{FA}_{\mathbb{P},\kappa} \iff \mathsf{FA}_{\mathbb{P},\mathrm{cof}(\kappa)}$.

Proof. For $\mathrm{cof}(\kappa) = \omega$, the statements are both provably true. So assume $\mathrm{cof}(\kappa) > \omega$.

$\Longleftarrow$: Let $\pi\colon \mathrm{cof}(\kappa) \to \kappa$ be a continuous cofinal function. Let $\vec{D} = \langle D_\alpha : \alpha < \kappa\rangle$ be a sequence of dense open subsets of $\mathbb{P}$. Let $\vec{E} = \langle E_\beta : \beta < \lambda\rangle$, where $E_\alpha = D_{\pi(\alpha)}$ for $\alpha < \mathrm{cof}(\kappa)$. By $\mathsf{FA}_{\mathbb{P},\mathrm{cof}(\kappa)}$, there is a filter g with $g \cap E_\alpha$ for $\alpha < \mathrm{cof}(\kappa)$. Thus for all $\beta = \pi(\alpha) \in \mathrm{ran}(\pi)$, $g \cap D_\alpha = g \cap E_\beta \neq \emptyset$. This suffices since $\mathrm{ran}(\pi)$ is club in κ.

$\Longrightarrow$: We first claim that $\text{club-}\mathsf{FA}_{\mathbb{P},\kappa}$ implies $\text{club-}\mathsf{FA}_{\mathbb{P},\mathrm{cof}(\kappa)}$. To see this, let $\pi\colon \mathrm{cof}(\kappa) \to \kappa$ be a continuous cofinal function. Let $\vec{D} = \langle D_\alpha : \alpha < \mathrm{cof}(\kappa)\rangle$ be a sequence of dense open subsets of $\mathbb{P}$. Let $E_{\pi(\alpha)} = D_\alpha$ and $E_\gamma = \mathbb{P}$ for all $\gamma \notin \mathrm{ran}(\pi)$. Since $C \cap \mathrm{ran}(\pi)$ is club in κ and π is continuous, $\pi^{-1}(C)$ is club in $\mathrm{cof}(\kappa)$ and $g \cap D_\alpha = g \cap E_{\pi(\alpha)} \neq \emptyset$ for all $\alpha \in \pi^{-1}(C)$ as required.

It now suffices to prove $\text{club-}\mathsf{FA}_{\mathbb{P},\lambda} \Longrightarrow \mathsf{FA}_{\mathbb{P},\lambda}$ for regular λ. Given a sequence $\vec{D} = \langle D_\alpha : \alpha < \lambda\rangle$ of dense open subsets, partition λ into disjoint stationary sets S_α for $\alpha < \kappa$. Let $\vec{E} = \langle E_\beta : \beta < \lambda\rangle$, where $E_\beta = D_\alpha$ for $\beta \in S_\alpha$. By $\text{club-}\mathsf{FA}_\lambda$, there is a filter g and a club C in λ with $g \cap E_\beta$ for $\beta \in C$. Since C is club, $S_\alpha \cap C \neq \emptyset$ for all $\alpha < \lambda$. Thus $g \cap D_\alpha = g \cap E_\beta \neq \emptyset$. □

Lemma 3.5.

(1) $\mathsf{FA}_\kappa \implies \text{club-}\mathsf{N}_\kappa$
(2) $\text{club-}\mathsf{N}_\kappa \implies \text{club-}\mathsf{FA}_\kappa$

Proof. (1): Let σ be a rank 1 name such that $1 \Vdash$ "σ contains a club in κ". Then we can find a rank 1 name τ such that $1 \Vdash \tau \subseteq \sigma$ and $1 \Vdash$ "τ is a club in κ". For $\gamma < \kappa$, let D_γ denote the set of $p \in \mathbb{P}$ such that either

(a) $p \Vdash^+ \check{\gamma} \in \tau$, or
(b) for all sufficiently large $\alpha < \gamma$, $p \Vdash \check{\alpha} \notin \tau$.

We claim D_γ is dense. Let $p \in \mathbb{P}$. If $p \Vdash \check{\gamma} \in \tau$ then by Proposition 2.22 we can find $q \leq p$ strongly forcing this, and then $q \in D_\gamma$. Otherwise, take $q \leq p$ with $q \Vdash \check{\gamma} \notin \tau$. Then $q \Vdash$ "$\tau \cap \gamma$ is bounded in γ". Take $r \leq q$ deciding that bound, and then r satisfies condition b above.

For any filter g with $g \cap D_\gamma \neq \emptyset$, τ^g is closed at γ by Proposition 2.23.

Let E_γ denote the set of $p \in \mathbb{P}$ such that for some $\delta \geq \gamma$, $p \Vdash^+ \check{\delta} \in \tau$. Again, this is dense since τ is forced to be unbounded. For any filter g with $g \cap E_\gamma \neq \emptyset$ for all $\gamma < \kappa$, τ^g is unbounded.

Let F_γ denote the dense set of $p \in \mathbb{P}$ such that $p \Vdash^+ \check{\gamma} \in \sigma$ or $p \Vdash \check{\gamma} \notin \tau$. Once again, F_γ is dense: given $p \in \mathbb{P}$ take $q \leq p$ deciding whether $\gamma \in \tau$. If it decides $\gamma \notin \tau$ then we're done; otherwise $q \Vdash \check{\gamma} \in \sigma$ and we can find $r \leq q$ with $r \Vdash^+ \check{\gamma} \in \sigma$

For any filter g with $g \cap F_\gamma \neq \emptyset$, $\gamma \in \tau^g \Rightarrow \gamma \in \sigma^g$.

Putting things together, if we find a filter g which meets every D_γ, E_γ and F_γ then τ^g will be both a club and a subset of σ^g.

(2): This works much like the proof that $\mathsf{N} \Rightarrow \mathsf{FA}$ above. Let $\langle D_\gamma : \gamma < \kappa \rangle$ be a collection of dense sets. Let

$$\sigma = \{(\check{\gamma}, p) : \gamma < \kappa, p \in D_\gamma\}$$

Clearly $1 \Vdash \sigma = \check{\kappa}$, and hence that σ contains a club. Take a filter g where σ^g contains a club. Then $\sigma^g = \{\gamma < \kappa : D_\gamma \cap g \neq \emptyset\}$ so g meets a club of D_γ. □

Putting together the previous results, we complete the top left corner of Fig. 1.

Corollary 3.6. *The following are all equivalent for all uncountable regular cardinals κ:* FA_κ, N_κ, $\mathsf{club\text{-}FA}_\kappa$, $\mathsf{club\text{-}N}_\kappa$.

The second half of the previous lemma also applies for the other special name principles.

Lemma 3.7. $\mathsf{stat\text{-}N}_\kappa \implies \mathsf{stat\text{-}FA}_\kappa$

Proof. As for the club case, except that we just insist on σ^g being stationary. □

Lemma 3.8. $\mathsf{ub\text{-}N}_\kappa \implies \mathsf{ub\text{-}FA}_\kappa$

Proof. As for the club case, except that we insist on σ^g being unbounded. □

Lemma 3.9. $\omega\text{-}\mathsf{ub\text{-}N}_\kappa \implies \omega\text{-}\mathsf{ub\text{-}FA}_\kappa$

Proof. Define σ as in the club case. Define

$$\tau = \{(\check{n}, p) : n < \omega, p \in E_n\}$$

where we want to meet all of the dense sets $\langle E_n : n < \omega \rangle$ as well as unboundedly many of the dense sets D_γ. Take g such that $\tau^g = \omega$ and σ^g is unbounded. □

We can also get converses for these in the case of ub and ω-ub.

Lemma 3.10.

(1) $\mathsf{ub\text{-}FA}_\kappa \implies \mathsf{ub\text{-}N}_\kappa$
(2) $\omega\text{-}\mathsf{ub\text{-}FA}_\kappa \implies \omega\text{-}\mathsf{ub\text{-}N}_\kappa$

Proof. (1): Assume $\mathsf{ub\text{-}FA}_\kappa$. Let σ be a rank 1 name for an unbounded subset of κ. For $\gamma < \kappa$ let D_γ be the set of all $p \in \mathbb{P}$ such that for some $\delta > \gamma$, $p \Vdash^+ \check{\delta} \in \sigma$. Let g be a filter meeting unboundedly many D_γ; then σ^g is unbounded.

(2): Let σ be a rank 1 name for an unbounded subset of κ and τ be a good name for ω. Define D_γ as above, and for $n < \omega$ let E_n be the set of all $p \in \mathbb{P}$ which strongly force $n \in \tau$. Find g meeting unboundedly many D_γ and every E_n; then σ^g is unbounded and $\tau^g = \omega$. □

This proves every implication in the left two columns of Fig. 1.

3.2. Extremely bounded name principles

Now, we address the right most column of Fig. 4. These axioms are more interesting if $\mathbb{P}$ is a complete Boolean algebra, since they can be trivial otherwise.

Lemma 3.11. BN^1_κ *is provable in* ZFC.

Proof. Let σ be a 1-bounded rank 1 name such that $1 \Vdash \sigma = \check{A}$ for some set A. Then for $\gamma \in \kappa \setminus A$, there is no $p \in \mathbb{P}$ such that $(\check{\gamma}, p) \in \sigma$. For $\gamma \in A$ there is a unique $p \in \mathbb{P}$ such that $(\check{\gamma}, p) \in \sigma$; and p is contained in every generic filter. Assuming $\mathbb{P}$ is atomless, it follows that $p = 1$ and hence that, if we let g be any filter at all, $\sigma^g = A$. It is also possible to adjust this proof to work for forcings with atoms; this is left as an exercise for the reader. □

All of these results also hold if we work with bounded name principles and forcing axioms, provided that the bound is at least κ.

For bounds below κ, we can almost get an equivalence between the different bounds for the stationary and unbounded name principles. A forcing is called *well-met* if any two compatible conditions p, q have a greatest lower bound $p \wedge q$.

The next result and proof is due to Hamkins for trees (see Corollary 3.13). We noticed that his proof shows a more general fact.

Lemma 3.12 *(with Hamkins). Suppose $\lambda < \kappa$ and $\mathbb{P}$ is well-met.*

(1) *If* $\mathsf{stat\text{-}BN}^\lambda_{\mathbb{P},\kappa}$ *fails, then there are densely many conditions $p \in \mathbb{P}$ such that* $\mathsf{stat\text{-}BN}^1_{\mathbb{P}_p,\kappa}$ *fails, where* $\mathbb{P}_p := \{q \in \mathbb{P} : q \leq p\}$.
(2) *The same result holds with* ub *in place of* stat.

Proof. We prove the stat case; the ub case is identical. The key fact the proof uses is that if we partition a stationary/unbounded subset of κ into $\lambda < \kappa$ many parts, then one of those parts must be stationary/unbounded.

Let σ be a λ-bounded (rank 1) name for a stationary set, such that there is no $g \in V$ with σ^g stationary. Then, without loss of generality, we can enumerate the elements of σ:

$$\sigma = \{(\check{\gamma}, p_{\gamma,\delta}) : \gamma < \kappa, \delta < \lambda\}$$

For $\delta < \lambda$, we define:

$$\sigma_\delta = \{(\check{\gamma}, p_{\gamma,\delta}) : \gamma < \kappa\}$$

Clearly, σ_δ is 1-bounded.

For any generic filter G, $\bigcup \sigma_\delta^G = \sigma^G$ is stationary in $V[G]$. Hence, $\mathbb{P}$ forces "There is some $\delta < \lambda$ such that σ_δ is stationary." Now, let $p \in \mathbb{P}$ be one of the densely many conditions which decides which δ this is. Then

$$\sigma_{\delta,p} = \{(\check{\gamma}, p_{\gamma,\delta} \wedge p) : \gamma < \kappa\}$$

is a 1-bounded $\mathbb{P}_p$-name and $\mathbb{P}_p \Vdash \sigma_{\delta,p}$ is stationary. If $\mathsf{stat\text{-}BN}^1_{\mathbb{P}_p,\kappa}$ would hold, there would exist a filter g such that $\sigma^g_{\delta,p}$ is stationary. Then g generates a filter h in $\mathbb{P}$ such that $\sigma^h_{\delta,p} \supseteq \sigma^g_{\delta,p}$ is stationary. □

Corollary 3.13 *(Hamkins). Suppose that T is a tree, $\mathbb{P}_T$ is T with reversed order and $\lambda < \kappa$.*

(1) *If $\mathsf{stat\text{-}BN}^\lambda_{\mathbb{P}_T,\kappa}$ fails, then there are densely many conditions $p \in \mathbb{P}$ such that $\mathsf{stat\text{-}BN}^1_{(\mathbb{P}_T)_p,\kappa}$ fails, where $(\mathbb{P}_T)_p := \{q \in \mathbb{P}_T : q \leq p\}$.*
(2) *The same result holds with ub in place of stat.*

Corollary 3.14. *Suppose $\lambda < \kappa$ and $\mathbb{P}$ is a well-met forcing such that for every $p \in \mathbb{P}$, $\mathbb{P}_p$ embeds densely into $\mathbb{P}$. Then*

$$\mathsf{stat\text{-}BN}^\lambda_{\mathbb{P},\kappa} \iff \mathsf{stat\text{-}BN}^1_{\mathbb{P},\kappa}$$
$$\mathsf{ub\text{-}BN}^\lambda_{\mathbb{P},\kappa} \iff \mathsf{ub\text{-}BN}^1_{\mathbb{P},\kappa}$$

Proof. We show that a failure of $\mathsf{stat\text{-}BN}^\lambda_{\mathbb{P},\kappa}$ implies the failure of $\mathsf{stat\text{-}BN}^1_{\mathbb{P},\kappa}$. The converse direction is clear and the proof for the unbounded name principles is analogous.

By Lemma 3.12, there is some $p \in \mathbb{P}$ such that $\mathsf{stat\text{-}BN}^1_{\mathbb{P}_p,\kappa}$ fails. Let $i\colon \mathbb{P}_p \to \mathbb{P}$ be a dense embedding and $\mathbb{Q} := i(\mathbb{P}_p)$. Since $\mathsf{stat\text{-}BN}^1_{\mathbb{Q},\kappa}$ fails, let σ be a 1-bounded $\mathbb{Q}$-name witnessing this failure. We claim that there is no filter g on $\mathbb{P}$ such that σ^g is stationary. Assume otherwise. Using that $\mathbb{Q}$ is well-met, let h denote the set of all $q \geq p_0 \wedge_{\mathbb{Q}} \cdots \wedge_{\mathbb{Q}} p_n$ for some $p_0, \ldots, p_n \in g \cap \mathbb{Q}$. It is easy to check that h is a well-defined filter on $\mathbb{Q}$ and contains $g \cap \mathbb{Q}$. Then $\sigma^h \supseteq \sigma^g$ is stationary. But this contradicts the choice of σ. □

3.3. Extremely bounded forcing axioms

We next study forcing axioms for very small predense sets. The next lemmas show that $\mathsf{BFA}^\omega_{\mathbb{P},\omega_1}$ has some of the same consequences as BFA.

Lemma 3.15. *If $\mathbb{P}$ is a complete Boolean algebra such that $\mathsf{BFA}^\omega_{\mathbb{P},\omega_1}$ holds, then $1_\mathbb{P}$ does not force that ω_1 is collapsed.*

Proof. Suppose $\Vdash \dot{f}\colon \omega_1 \to \omega$ is injective. Let $A_\alpha = \{[\![\dot{f}(\alpha) = n]\!] \neq 0 : n \in \omega\}$. Since each A_α is a maximal antichain, there is a filter g with $g \cap A_\alpha \neq \emptyset$ for all $\alpha < \omega_1$. Define $f'\colon \omega_1 \to \omega$ by letting $f'(\alpha) = n$ if $[\![\dot{f}(\alpha) = n]\!] \in g$ for all $\alpha < \omega_1$. Since g is a filter, $f'\colon \omega_1 \to \omega$ is well-defined and injective. □

Lemma 3.16. *If $\mathbb{P}$ is a complete Boolean algebra such that $\mathsf{BFA}^\omega_{\mathbb{P},\omega_1}$ holds and $\mathbb{P}$ adds a real, then CH fails.*

Proof. Suppose CH holds and let $\langle x_\alpha : \alpha < \omega_1\rangle$ be an enumeration of all reals. Let σ be a name for the real added by $\mathbb{P}$. For $\alpha < \omega_1$, let

$$D_\alpha = \{[\![t^\frown\langle n\rangle \subseteq \sigma]\!] : t \in 2^{<\omega}, n \in 2, t \subseteq x_\alpha, t^\frown\langle n\rangle \nsubseteq x_\alpha\}$$

For $n < \omega$, let

$$E_n = \{[\![\sigma(n) = m]\!] : m \in 2\}$$

Then the D_α and E_n are all predense and countable. Take a filter g which meets every D_α and E_n. The E_n ensure that g defines a real x (by $x(n) = m$ where $[\![\sigma(n) = m]\!] \in g$). But if $x = x_\alpha$ then $g \cap D_\alpha = \emptyset$. □

There exist forcings $\mathbb{P}$ such that the implication $\mathsf{BFA}^{\omega}_{\mathbb{P},\omega_1} \Rightarrow \mathsf{BFA}^{\omega_1}_{\mathbb{P},\omega_1}$ fails. To see this, suppose that $\mathbb{Q}$ is a forcing such that $\mathsf{BFA}^{\omega_1}_{\mathbb{Q},\omega_1}$ fails. Let $\mathbb{P}$ be a lottery sum of ω_1 many copies of $\mathbb{Q}$. Since $\mathsf{BFA}^{\omega_1}_{\mathbb{Q},\omega_1}$ fails, $\mathsf{BFA}^{\omega_1}_{\mathbb{P},\omega_1}$ fails as well. On the other hand, $\mathsf{BFA}^{\omega}_{\mathbb{P},\omega_1}$ holds trivially since any countable predense subset of $\mathbb{P}$ contains $0_{\mathbb{P}}$.

Question 3.17. Does the implication $\mathsf{BFA}^{\omega}_{\mathbb{P},\omega_1} \Rightarrow \mathsf{BFA}^{\omega_1}_{\mathbb{P},\omega_1}$ hold for all complete Boolean algebras $\mathbb{P}$?

By the previous lemmas, any forcing which is a counterexample cannot force that ω_1 is collapsed, and if it adds reals then CH holds.

3.4. Basic results on ub-FA

In this section, we collect some observations about weak forcing axioms. We aim to prove some consequences of these axioms. We first consider $\mathsf{ub\text{-}FA}$ and $\mathsf{stat\text{-}FA}$. How strong is $\mathsf{ub\text{-}FA}$? The next lemmas show that is has some of the same consequences as FA.

Lemma 3.18. *If $\mathsf{ub\text{-}FA}_{\mathbb{P},\omega_1}$ holds, then $\mathbb{P}$ does not force that ω_1 is collapsed.*

Proof. Towards a contradiction, suppose $\mathbb{P}$ forces that ω_1 is collapsed. Let $\dot{f}$ be a $\mathbb{P}$-name for an injective function $\omega_1 \to \omega$. For $\alpha < \omega_1$, let $D_\alpha = \{p \in \mathbb{P} : \exists n \in \omega \ p \Vdash \dot{f}(\alpha) = n\}$. By $\mathsf{ub\text{-}FA}_{\mathbb{P},\omega_1}$, there is a filter g and an unbounded subset A of ω_1 such that $g \cap D_\alpha \neq \emptyset$ for all $\alpha \in A$. Define $f\colon A \to \omega$ by letting $f(\alpha) = n$ if there is some $p \in g \cap D_\alpha$ with $p \Vdash \dot{f}(\alpha) = n$. Since g is a filter, f is injective. □

Lemma 3.19. *If $\mathsf{ub\text{-}FA}_{\mathbb{P},\omega_1}$ holds and $\mathbb{P}$ does not add reals, then for each stationary subset S of ω_1, $\mathbb{P}$ does not force that S is nonstationary.*

Proof. Suppose that $\dot{C}$ is a name for a club such that $\Vdash_{\mathbb{P}} S \cap \dot{C} = \emptyset$. Let $\dot{f}$ be a name for the characteristic function of $\dot{C}$. For each $\alpha < \omega_1$,

$$D_\alpha = \{p \in \mathbb{P} : \exists t \in 2^\alpha \ t \subseteq \dot{f}\}$$

is dense in $\mathbb{P}$, since $\mathbb{P}$ does not add reals. By $\mathsf{ub\text{-}FA}_{\mathbb{P},\omega_1}$, there is a filter g and an unbounded subset A of ω_1 such that $g \cap D_\alpha \neq \emptyset$ for all $\alpha \in A$. Since g is a filter, $C := \{\alpha < \omega_1 : \exists p \in g \ p \Vdash \alpha \in \dot{C}\}$ is a club in ω_1 with $S \cap C \neq \emptyset$. □

The previous lemma also follows from Theorem 4.17 and Lemma 4.25 below via an absoluteness argument, assuming $\mathbb{P}$ is a homogeneous complete Boolean algebra. It is open whether the lemma holds for forcings $\mathbb{P}$ which add reals.

What is the relationship between $\mathsf{ub\text{-}FA}_{\mathbb{P},\omega_1}$ and other forcing axioms? We find two opposite situations. For any σ-centred forcing, $\mathsf{ub\text{-}FA}_{\mathbb{P},\omega_1}$ and $\mathsf{stat\text{-}FA}_{\mathbb{P},\omega_1}$ are provable in ZFC by Lemma 5.7 below. For many other forcings though, $\mathsf{ub\text{-}FA}_{\mathbb{P},\omega_1}$ implies nontrivial axioms such as $\mathsf{FA}_{\mathbb{P},\omega_1}$ or $\mathsf{BFA}^{\omega_1}_{\mathbb{P},\omega_1}$. For instance, the implication $\mathsf{ub\text{-}FA}_{\mathbb{P},\omega_1} \Rightarrow \mathsf{FA}_{\mathbb{P},\omega_1}$ holds for all σ-distributive forcing by Lemma 5.1 below. We will further see in Lemma 4.25 below that for any complete Boolean algebra $\mathbb{P}$ which does not add reals, $(\forall q \in \mathbb{P} \ \mathsf{ub\text{-}FA}_{\mathbb{P}_q,\omega_1})$

implies $\mathsf{BFA}^{\omega_1}_{\mathbb{P},\omega_1}$. Moreover, the implication $\mathsf{ub\text{-}FA}_{\mathbb{P},\omega_1} \Rightarrow \mathsf{FA}_{\mathbb{P},\omega_1}$ also holds for some forcings that add reals, for instance for random forcing by Lemma 5.20.

We do not have any examples of forcings where $\mathsf{ub\text{-}FA}_{\mathbb{P},\omega_1}$ and $\mathsf{stat\text{-}FA}_{\mathbb{P},\omega_1}$ sit between these two extremes: strictly weaker than $\mathsf{FA}_{\mathbb{P},\omega_1}$, but not provable in ZFC.

In particular, we have not been able to separate the two axioms:

Question 3.20. Can forcings $\mathbb{P}$ exist such that $\mathsf{ub\text{-}FA}_{\mathbb{P},\kappa}$ holds, but $\mathsf{stat\text{-}FA}_{\mathbb{P},\kappa}$ fails?

For instance, we would like to know if these axioms hold for the following forcings:

Question 3.21. Do Baumgartner's forcing to add a club in ω_1 with finite conditions [5, Section 3] and Abraham's and Shelah's forcing for destroying stationary sets with finite conditions [1, Section 2] satisfy $\mathsf{ub\text{-}FA}_{\mathbb{P},\omega_1}$ and $\mathsf{stat\text{-}FA}_{\mathbb{P},\omega_1}$?

3.5. Characterisations of FA^+ *and* FA^{++}

The proof of the equivalence of FA and N still goes through fine if we change the axioms slightly, demanding some extra property to be true of the filter g we're looking for. This gives us a nice way to express FA^+ and FA^{++}.

Lemma 3.22. $\mathsf{FA}^+_{\mathcal{C},\kappa}$ *is equivalent to the following statement:*

For all $\mathbb{P} \in \mathcal{C}$, for all rank 1 names σ and τ for subsets of κ such that $\mathbb{P}$ forces "$\sigma = \check{A}$" for some A and "τ is stationary", there is some filter g with $\sigma^g = A$ and τ^g stationary.

Similarly, $\mathsf{FA}^{++}_{\mathcal{C},\kappa}$ is equivalent to being able to correctly interpret κ many stationary rank 1 names and a single rank 1 name for a specific set A.

Proof. Analogous to the proof of 3.1 in the previous section. □

In the case of FA^{++} this result can be sharpened further, getting rid of the name for A:

Lemma 3.23. $\mathsf{FA}^{++}_{\mathcal{C},\kappa}$ *is equivalent to the statement:*

For all $\mathbb{P} \in \mathcal{C}$ and all collections of κ many rank 1 names $\langle \sigma_\gamma : \gamma < \kappa \rangle$ with $\mathbb{P} \Vdash$ "σ_γ is stationary for all γ", there is a filter $g \in V$ such that for all γ, σ^g_γ is stationary.

Proof. $\Rightarrow$: By the previous lemma.

$\Leftarrow$: Let σ be a rank 1 name, such that $\mathbb{P} \Vdash \sigma = \check{A}$ for some $A \subseteq \kappa$. We claim there is a collection $\langle \tau_\gamma : \gamma < \kappa \rangle$ of rank 1 names, which are forced to be stationary in κ, such that any filter g which interprets every τ_γ as stationary will interpret σ as A. Once we have proved this claim, the lemma follows immediately from the second part of Lemma 3.22. For $\gamma \in A$, let $\tau_\gamma = \{(\check{\alpha}, p) : \alpha \in \kappa, p \Vdash^+ \check{\gamma} \in \sigma\}$. For $\gamma \notin A$, let $\tau_\gamma = \check{\kappa}$. We will see that $\mathbb{P} \Vdash$ "$\tau_\gamma = \kappa$" for $\gamma \in A$. Note that $\mathbb{P} \Vdash \sigma = \check{A}$ by assumption. So for $\gamma \in A$, every generic filter will contain some p with $p \Vdash^+ \check{\gamma} \in \sigma$. Hence $\mathbb{P} \Vdash \tau_\gamma = \check{\kappa}$. There is a filter g such that τ^g_γ is stationary for all $\gamma < \kappa$ by assumption. If $\gamma \in A$, then in particular $\tau^g_\gamma \neq \emptyset$. Hence $\gamma \in \sigma^g$. If a filter interprets all the τ_γ as stationary sets, then $\sigma^g \supseteq A$. If $\gamma \in \sigma^g \setminus A$, then there is some $p \in \mathbb{P}$ with $\langle \check{\gamma}, p \rangle \in \sigma$, which is impossible as $\mathbb{P} \Vdash \check{\gamma} \notin \sigma$. □

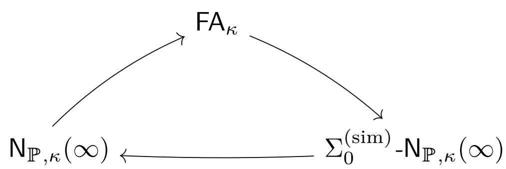
4. A correspondence for arbitrary ranks

We now move on to discuss higher ranked name principles, including those of the ranked or unranked simultaneous variety. It turns out that even at high ranks, a surprising variety of these are equivalent to one another and to a suitable forcing axiom. These are summarised in the following theorems.

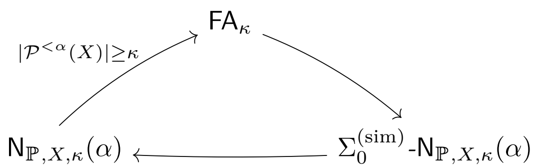
4.1. The correspondence

Theorem 4.1. *Let $\mathbb{P}$ be a forcing and let κ be a cardinal. The following implications hold, given the assumptions noted at the arrows:*

(1)



(2) *For any ordinal $\alpha > 0$, and any transitive set X of size at most κ*[2]*:*

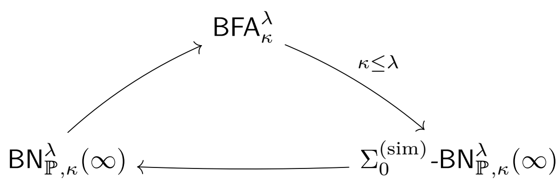


As usual, we can generally think of X as being a cardinal.

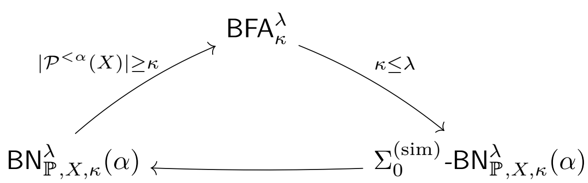
There is also a bounded version of this theorem.

Theorem 4.2. *Let $\mathbb{P}$ be a complete Boolean algebra, and let κ, λ be cardinals. The following implications hold, given the assumptions noted at the arrows:*

(1)



(2) *For any ordinal $\alpha > 0$, and transitive set X of size at most κ:*



[2] Recall that $\mathsf{N}_{\mathbb{P},X,\kappa}(\alpha)$ is only defined if X has size at most κ.

Remark 4.3. For the ∞ case it suffices to look only at $\emptyset$ names, as we discussed after Definition 2.13. Moreover, for the implication $\mathsf{N}_{\mathbb{P},\kappa}(\infty) \Rightarrow \mathsf{FA}_{\mathbb{P},\kappa}$ (and the corresponding ones in the other diagrams), we need only rank 1 κ-names for κ. These can be understood as rank κ $\emptyset$-names for κ. For $\mathsf{N}_{\mathbb{P},X,\kappa}(\alpha) \Rightarrow \mathsf{FA}_{\mathbb{P},\kappa}$, rank 1 Y-names for a fixed set Y of size κ suffice. These can be understood as rank $\leq\alpha$ X-names. These remarks are also true for the bounded versions. Note that for $\mathsf{N}_{\mathbb{P},\kappa}(1) \Rightarrow \mathsf{FA}_{\mathbb{P},\kappa}$, rank 1 κ-names for κ suffice by Lemma 3.1.

We give some simple instances of Theorem 4.1 (2) and postpone the proofs to Section 4.2. The variant for bounded forcing axioms has similar consequences. The next result follows by letting $\kappa = X$ and $\alpha = 1$.

Corollary 4.4. *For any forcing* $\mathbb{P}$*,* $\mathsf{FA}_{\mathbb{P},\kappa} \iff \Sigma_0^{(\mathrm{sim})}\text{-}\mathsf{N}_{\mathbb{P},\kappa} \iff \mathsf{N}_{\mathbb{P},\kappa}$*.*

To illustrate this, we note how some concrete forcing axioms can be characterized by name principles. For example, we can characterize PFA as follows:

$$\mathsf{PFA} \iff \Sigma_0^{(\mathrm{sim})}\text{-}\mathsf{N}_{\mathrm{proper},\omega_1} \iff \mathsf{N}_{\mathrm{proper},\omega_1}.$$

In other words, rank 1 names for ω_1 can be interpreted correctly.

For higher ranks, it is useful to choose α, κ and X such that $|\mathcal{P}^{<\alpha}(X)| \geq \kappa$ holds to get an equivalence in Theorem 4.1 (2). This condition holds for $\kappa \geq 2^\omega$, $X = \omega$ and $\alpha = 2$.

Corollary 4.5. *For any cardinal* $\kappa \leq 2^\omega$ *and any forcing* $\mathbb{P}$*, we have* $\mathsf{FA}_{\mathbb{P},\kappa} \iff \Sigma_0^{(\mathrm{sim})}\text{-}\mathsf{N}_{\mathbb{P},\omega,\kappa}(2) \iff \mathsf{N}_{\mathbb{P},\omega,\kappa}(2)$*.*

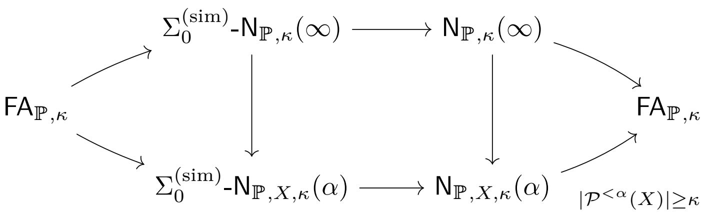
For example, we can characterize PFA as follows:

$$\mathsf{PFA} \iff \Sigma_0^{(\mathrm{sim})}\text{-}\mathsf{N}_{\mathrm{proper},\omega,\omega_1}(2) \iff \mathsf{N}_{\mathrm{proper},\omega,\omega_1}(2).$$

In other words, rank 2 names for sets of reals can be interpreted correctly. We leave open how to characterise higher rank (e.g. rank 2) principles for names for reals.

4.2. The proofs

Proof of Theorem 4.1. We prove both parts of the theorem simultaneously, by fixing X and α and proving all the implications in the following diagram:



Of these, the first $\mathsf{FA}_{\mathbb{P},\kappa} \Rightarrow \Sigma_0^{(\mathrm{sim})}\text{-}\mathsf{N}_{\mathbb{P},\kappa}(\infty)$ is the hardest to prove, and the main work on the theorem. We'll leave it to the end, and prove the other implications first. Note that $\mathsf{FA}_{\mathbb{P},\kappa} \Rightarrow \Sigma_0^{(\mathrm{sim})}\text{-}\mathsf{N}_{\mathbb{P},X,\kappa}(\alpha)$ follows from the rest of the diagram.

Proof of $\Sigma_0^{(\mathrm{sim})}$-$\mathsf{N}_{\mathbb{P},\kappa}(\infty) \Rightarrow \Sigma_0^{(\mathrm{sim})}$-$\mathsf{N}_{\mathbb{P},\kappa,X}(\alpha)$. The latter is a special case of the former. □

Proof of $\mathsf{N}_{\mathbb{P},\kappa}(\infty) \Rightarrow \mathsf{N}_{\mathbb{P},X,\kappa}(\alpha)$. Again, this is a special case. □

Proof of $\Sigma_0^{(\mathrm{sim})}$-$\mathsf{N}_{\mathbb{P},X,\kappa}(\alpha) \Rightarrow \mathsf{N}_{\mathbb{P},X,\kappa}(\alpha)$. Given a κ-small name σ of rank α or less, and a set A as called for by $\mathsf{N}_{\mathbb{P},\kappa}(\alpha)$, we know $A \in \mathcal{P}^\alpha(X) \cap H_{\kappa^+}$. Hence $\check{A}$ is a κ small α rank X name, so "$\sigma = \check{A}$" is one of the formulas discussed by the simultaneous name principle. □

Proof of $\Sigma_0^{(\mathrm{sim})}$-$\mathsf{N}_{\mathbb{P},\kappa}(\infty) \Rightarrow \mathsf{N}_{\mathbb{P},\kappa}(\infty)$. Similar to the previous proof: if σ is any κ-small name, and $A \in H_{\kappa^+}$ is such that $\mathbb{P} \Vdash \sigma = \check{A}$, then since $\check{A}$ is κ-small we know from $\Sigma_0^{(\mathrm{sim})}$-$\mathsf{N}(\infty)$ that we can find a filter g such that $\sigma^g = \check{A}^g = A$. □

Proof of $\mathsf{N}_{\mathbb{P},X,\kappa}(\alpha) \Rightarrow \mathsf{FA}_{\mathbb{P},\kappa}$. We assume $|P^{<\alpha}(X)| \geq \kappa$. The idea is similar to the proof of $\mathsf{N}_{\mathbb{P},\kappa} \Rightarrow \mathsf{FA}_{\mathbb{P},\kappa}$ from Lemma 3.1, but first we must prove a technical claim.

Claim 4.6. *$\mathcal{P}^{<\alpha}(X)$ contains at least κ many elements whose check names are κ-small $<\alpha$-rank X-names.*

Proof (Claim). Let $\alpha' \leq \alpha$ be minimal such that $|\mathcal{P}^{<\alpha'}(X)| \geq \kappa$.

Let $A \in \mathcal{P}^{<\alpha'}(X)$. Then $A \in \mathcal{P}^\epsilon(X)$ for some $\epsilon < \alpha'$. We show by induction on ϵ that $\check{A}$ is in fact a κ-small ϵ-rank X-name. From this and the assumption on the size of κ, it of course follows that there are at least κ many elements of $\mathcal{P}^{<\alpha'}(X) \subseteq \mathcal{P}^{<\alpha}(X)$ whose check names are κ-small $< \alpha$-rank X-names.

The case $\epsilon = 0$ is trivial. Suppose $\epsilon > 0$. By inductive hypothesis, we know that all the names which are contained in $\check{A}$ are κ-small $< \epsilon$-rank X-names. It remains to check that there are at most κ many of them; that is, that $|A| \leq \kappa$. But this is obvious, since $A \subseteq \mathcal{P}^{<\epsilon}(X)$ and $|\mathcal{P}^{<\epsilon}(X)| < \kappa$ by our choice of α'. □

Given the claim, we can now take a set of κ many distinct sets $A := \{A_\gamma : \gamma < \kappa\} \subseteq \mathcal{P}^{<\alpha}(X)$, such that for all γ, the name $\check{A}_\gamma$ is a κ small $<\alpha$ rank X-name.

Let $\langle D_\gamma \rangle_{\gamma<\kappa}$ be a sequence of dense sets in $\mathbb{P}$. We define a name σ:

$$\sigma = \{\langle \check{A}_\gamma, p \rangle : \gamma < \kappa, p \in D_\gamma\}$$

Then σ is a κ-small $\leq\alpha$-rank X-name, and $\mathbb{P} \Vdash \sigma = \check{A}$. Hence, if we assume $\mathsf{N}_{\mathbb{P},X,\kappa}(\alpha)$ we can choose a filter g such that $\sigma^g = A$. It is easy to see that g must meet every D_γ. □

Proof of $\mathsf{N}_{\mathbb{P},\kappa}(\infty) \Rightarrow \mathsf{FA}_{\mathbb{P},\kappa}$. Essentially the same as the previous proof, but since we're no longer required to make sure σ has rank α we can omit the technical claim and just take $A_\gamma := \gamma$ for all $\gamma < \kappa$. □

Proof of $\mathsf{FA}_{\mathbb{P},\kappa} \Rightarrow \Sigma_0^{(\mathrm{sim})}$-$\mathsf{N}_{\mathbb{P},\kappa}(\infty)$. This is the main work of the theorem. By a delicate series of inductions, we will prove the following lemma:

Lemma 4.7. *Let $\varphi(\vec{\sigma})$ be a Σ_0 formula where $\vec{\sigma}$ is a tuple of κ-small names. Then there is a collection $\mathcal{D}_{\varphi(\vec{\sigma})}$ of at most κ many dense sets, which has the following property: if g is any filter meeting every set in $\mathcal{D}_{\varphi(\vec{\sigma})}$ and g contains some p such that $p \Vdash \varphi(\vec{\sigma})$, then in fact $\varphi(\vec{\sigma}^g)$ holds in V.*

The result we're trying to show follows easily from this lemma: Fix a tuple $\vec{\sigma} = \langle \sigma_0, \ldots, \sigma_n \rangle$ of κ small names, and let $\mathcal{D} := \bigcup\{\mathcal{D}_{\varphi(\vec{\sigma})} : \varphi(v_0, \ldots, v_n) \text{ is } \Sigma_0\}$. $\mathcal{D}$ is a collection of at most κ many dense sets. Using $\mathsf{FA}_{\mathbb{P},\kappa}$, take a filter g meeting every dense set in $\mathcal{D}$. If $\varphi(v_0, \ldots, v_n)$ is a Σ_0 formula and $1 \Vdash \varphi(\vec{\sigma})$ then since $1 \in g$ we know that $\varphi(\vec{\sigma}^g)$ holds.

We will work our way up to proving the lemma, by first proving it in simpler cases. We opt for a direct proof of the name principle $\mathsf{N}_{\mathbb{P},\kappa}(\infty)$ in the next Claim 4.8. This and Claim 4.11 could be replaced by shorter arguments for κ-small $\emptyset$-names, since it suffices to deal with $\Sigma_0^{(\mathrm{sim})}$-$\mathsf{N}_{\mathbb{P},\emptyset,\kappa}(\infty)$ as discussed after Definition 2.13.

Claim 4.8. *The lemma holds when φ is of the form $\sigma = \check{A}$ for some set $A \in H_{\kappa^+}$ and (κ-small) name σ.*

Note that since $A \in H_{\kappa^+}$, we know that $\check{A}$ is a κ small name. So the statement in the claim does make sense.

Proof. We use induction on the rank of σ. If σ is rank 0 then it is a check name, and so the lemma is trivial: we can just take $\mathcal{D}_{\sigma=\check{A}} = \emptyset$. So say σ is rank $\alpha > 0$ and the lemma is proved for all names of rank $<\alpha$. Since σ is κ-small, we can write $\sigma = \{(\sigma_\gamma, p) : \gamma < \kappa, p \in S_\gamma\}$ for some κ-small names σ_γ and sets $S_\gamma \subseteq \mathbb{P}$.

First, let $B \in A$. We shall define a set D_B, whose "job" is to ensure B ends up in σ^g.

$$D_B = \{p \in \mathbb{P} : (p \Vdash \sigma \neq \check{A}) \vee (\exists \gamma < \kappa\, (p \Vdash \sigma_\gamma = \check{B}) \wedge (p \Vdash^+ \sigma_\gamma \in \sigma))\}$$

D_B is dense: if we take $p \in \mathbb{P}$ then either we can find $r \leq p$ with $r \Vdash \sigma \neq \check{A}$, or else $p \Vdash \sigma = \check{A}$. In the first case, we're done. In the second, given any (truly) generic filter G containing p, there will be some $\gamma < \kappa$ and $q \in G$ such that[3] $\sigma_\gamma^G = B$ and $(\sigma_\gamma, q) \in \sigma$, so $q \Vdash^+ \sigma_\gamma \in \sigma$. Take $r \in G$ such that $r \Vdash \sigma_\gamma = \check{B}$, and take s below p, q and r by compatibility; then $s \in D_B$.

Now let $\gamma < \kappa$. In a similar way, we define a set E_γ, which is designed to ensure that σ_γ ends up in A if it's going to be in σ.

$$E_\gamma = \{p \in \mathbb{P} : (p \Vdash \sigma \neq \check{A}) \vee (p \Vdash \sigma_\gamma \notin \sigma) \vee (\exists B \in A, p \Vdash \sigma_\gamma = \check{B})\}$$

Again, E_γ is dense: Let $p \in \mathbb{P}$. We can assume that $p \Vdash \sigma = \check{A}$ and $p \Vdash \sigma_\gamma \in \sigma$; otherwise we're done immediately. But now we can strengthen p to some $r \leq p$ which forces $\sigma_\gamma \in \check{B}$ for some $B \in A$ and again we're done.

We define

$$\mathcal{D}_{\sigma=\check{A}} := \{D_B : B \in A\} \cup \{E_\gamma : \gamma < \kappa\} \cup \bigcup_{\gamma<\kappa} \bigcup_{B\in A} \mathcal{D}_{\sigma_\gamma=\check{B}}$$

Every σ_γ is a κ-small name of rank less than α, and every $B \in H_{\kappa^+}$, so this is well defined by inductive hypothesis. By assumption, $|A| \leq \kappa$. Hence $\mathcal{D}_{\sigma=\check{A}}$ contains at most κ many dense sets. Fix a filter g which meets every element of $\mathcal{D}_{\sigma=\check{A}}$, and which contains some p forcing $\sigma = \check{A}$. We must verify that $\sigma^g = A$.

First, let $B \in A$. Find $q \in g \cap D_B$, and without loss of generality say $q \leq p$. Then clearly $q \Vdash \sigma = \check{A}$, so (by definition of D_B) we can find γ such that $q \Vdash \sigma_\gamma = \check{B}$ and $q \Vdash^+ \sigma_\gamma \in \sigma$. The latter means that $\sigma_\gamma^g \in \sigma^g$. Since g also meets every element of $\mathcal{D}_{\sigma_\gamma=\check{B}}$, the fact that $q \in g$ forces $\sigma_\gamma = \check{B}$ implies that $\sigma_\gamma^g = \check{B}^g = B$. Hence $B \in \sigma^g$.

Now let $B \in \sigma^g$. Then we can find $\gamma < \kappa$ such that $B = \sigma_\gamma^g$ and such that for some $q \in g$ we have $q \Vdash^+ \sigma_\gamma \in \sigma$. Without loss of generality, say $q \leq p$. Then $q \Vdash \sigma = \check{A}$. Let $r \in g \cap E_\gamma$, and again without loss of generality say $r \leq q$. Then for some $B' \in A$, $r \Vdash \sigma_\gamma = \check{B}'$. Since g meets every element of $\mathcal{D}_{\sigma_\gamma=\check{B}'}$, this tells us that $\sigma_\gamma^g = B'$. But then $B = \sigma_\gamma^g = B' \in A$.

Hence $\sigma^g = A$ as required. □

Next, we go up one step in complexity, by allowing both sides of the equality to be nontrivial.

Claim 4.9. *The lemma holds when φ has the form $\sigma = \tau$ for two (κ-small) names σ and τ.*

[3] Note the somewhat delicate nature of this statement: we cannot first take an arbitrary γ such that $\sigma_\gamma^G = B$ then try to find q such that $q \Vdash^+ \sigma_\gamma \in \sigma$.

Proof. We use induction on the ranks of σ and τ. Without loss of generality, let us assume the rank of σ is α, and the rank of τ is $\leq \alpha$. If $\mathrm{rank}(\tau) = 0$ then τ is a check name. Since τ is κ-small, it can only be a check name for some $A \in H_{\kappa^+}$, so we are already done by the previous claim. So suppose $\mathrm{rank}(\sigma) = \alpha \geq \mathrm{rank}(\tau) > 0$, and the result is proven for all τ', σ' where $\mathrm{rank}(\sigma') < \mathrm{rank}(\sigma)$ and $\mathrm{rank}(\tau') < \mathrm{rank}(\tau)$.

Let us write $\sigma = \{(\sigma_\gamma, p) : \gamma < \kappa, p \in S_\gamma\}$ and $\tau = \{(\tau_\delta, q) : \delta < \kappa, q \in T_\delta\}$.

For $\gamma \in \kappa$, we define a set D_γ, whose job is to ensure that if σ_γ ends up being put in σ by g, then it will also be equal to some element of τ.

$$D_\gamma = \Big\{p \in \mathbb{P} :(p \Vdash \sigma \neq \tau) \vee (p \Vdash \sigma_\gamma \notin \sigma) \\ \vee \exists \delta < \kappa\Big((p \Vdash \sigma_\gamma = \tau_\delta) \wedge (p \Vdash^+ \tau_\delta \in \tau)\Big)\Big\}$$

We claim D_γ is dense: Let $p \in \mathbb{P}$. If $p \nVdash \sigma_\gamma \in \sigma$ or $p \nVdash \sigma = \tau$ then take some $q \leq p$ forcing the converse of one of these statements, and we are done. If $p \Vdash \sigma_\gamma \in \sigma \wedge \sigma = \tau$ then take a generic filter G containing p. We know $\sigma_\gamma^G \in \tau^G$, so $\sigma_\gamma^G = \tau_\delta^G$ for some τ_δ which is strongly forced to be in τ by some $q \in G$. Then take $r \in G$ below p and q, and we know $r \Vdash \sigma_\gamma = \tau_\delta$ and $r \Vdash^+ \tau_\delta \in \tau$. Hence $r \in D_\gamma$.

Symmetrically, for $\delta < \kappa$ let

$$E_\delta = \Big\{p \in \mathbb{P} :(p \Vdash \sigma \neq \tau) \vee (p \Vdash \tau_\delta \notin \tau) \\ \vee \exists \gamma < \kappa\Big((p \Vdash \sigma_\gamma = \tau_\delta) \wedge (p \Vdash^+ \sigma_\gamma \in \sigma)\Big)\Big\}$$

Again, E_δ is dense.

We now let

$$\mathcal{D}_{\sigma=\tau} := \{D_\gamma : \gamma < \kappa\} \cup \{E_\delta : \delta < \kappa\} \cup \bigcup_{\gamma,\delta<\kappa} \mathcal{D}_{\sigma_\gamma=\tau_\delta}$$

Note that for all $\sigma, \delta < \kappa$, we know $\mathrm{rank}(\sigma_\gamma) < \mathrm{rank}(\sigma)$ and $\mathrm{rank}(\tau_\delta) < \mathrm{rank}(\tau)$, so $\mathcal{D}_{\sigma_\gamma=\tau_\delta}$ is already defined. Clearly, $\mathcal{D}_{\sigma=\tau}$ contains at most κ many dense sets. Let g be a filter meeting every element of it, and let $p \in g$ force $\sigma = \tau$.

Suppose $B \in \sigma^g$. Then for some $q \in g$ and $\gamma < \kappa$, $B = \sigma_\gamma^g$ and $q \Vdash^+ \sigma_\gamma \in \sigma$ (and hence $q \Vdash \sigma_\gamma \in \sigma$). We can also find some $r \in g \cap D_\gamma$. Without loss of generality, say r is below both p and q. Certainly r cannot force $\sigma \neq \tau$, nor that $\sigma_\gamma \notin \sigma$. Hence, for some $\delta < \kappa$, we know $r \Vdash \sigma_\gamma = \tau_\delta$ and $r \Vdash^+ \tau_\delta \in \tau$. But then $\tau_\delta^g \in \tau^g$, and since g meets every element of $\mathcal{D}_{\sigma_\gamma=\tau_\delta}$, we also know that $B = \sigma_\gamma^g = \tau_\delta^g$. Hence $B \in \tau$.

Hence $\sigma^g \subseteq \tau^g$, and by a symmetrical argument $\tau^g \subseteq \sigma^g$. □

Claim 4.10. *The lemma holds when φ has the form $\tau \in \sigma$.*

Proof. Write $\sigma = \{(\sigma_\gamma, p) : \gamma < \kappa, p \in S_\gamma\}$ as usual. Let

$$D = \Big\{p \in \mathbb{P} : (p \Vdash \tau \notin \sigma) \vee \exists \gamma < \kappa\Big((p \Vdash \tau = \sigma_\gamma) \wedge (p \Vdash^+ \sigma_\gamma \in \sigma)\Big)\Big\}$$

As usual, D is dense. Let

$$\mathcal{D}_{\tau\in\sigma} := \{D\} \cup \bigcup_{\gamma<\kappa} \mathcal{D}_{\tau=\sigma_\gamma}$$

Let g meet every element of $\mathcal{D}_{\tau\in\sigma}$ and contain some p forcing $\tau \in \sigma$. Let $q \in g \cap D$, and assume $q \leq p$. Then for some γ, $q \Vdash \tau = \sigma_\gamma$ and $q \Vdash^+ \sigma_\gamma \in \sigma$, so $\sigma_\gamma^g \in \sigma^g$. Since g meets every element of $\mathcal{D}_{\tau=\sigma_\gamma}$ we know $\tau^g = \sigma_\gamma^g \in \sigma^g$. □

We next need to prove similar claims about the negations of all these formulas.

Claim 4.11. *The lemma holds when φ is of the form $\sigma \neq \check{A}$ for $A \in H_\kappa$.*

Proof. As before, this is trivial is σ is rank 0. Otherwise, let us write $\sigma = \{(\sigma_\gamma, p) : \gamma < \kappa, p \in S_\gamma\}$ and let

$$D = \Big\{p \in \mathbb{P} :(p \Vdash \sigma = \check{A}) \vee \Big(\exists \gamma < \kappa (p \Vdash^+ \sigma_\gamma \in \sigma) \wedge (p \Vdash \sigma_\gamma \notin \check{A})\Big) \\ \vee (\exists B \in A : p \Vdash \check{B} \notin \sigma)\Big\}$$

As usual, D is dense.

We then let

$$\mathcal{D}_{\sigma \neq \check{A}} := \{D\} \cup \bigcup_{\gamma<\kappa} \bigcup_{B \in A} \mathcal{D}_{\sigma_\gamma \neq \check{B}}$$

By induction, this is well defined, and since A is in H_{κ^+} it has cardinality at most κ. Let g be a filter meeting all of $\mathcal{D}_{\sigma \neq \check{A}}$ with $p \in g$ forcing $\sigma \neq \check{A}$. Take $q \in g \cap D$ below p. There are two cases to consider.

(1) For some γ, $q \Vdash^+ \sigma_\gamma \in \sigma$ and $q \Vdash \sigma_\gamma \notin \check{A}$. Then certainly $\sigma_\gamma^g \in \sigma^g$. Let $B \in A$. Then $q \Vdash \sigma_\gamma \neq \check{B}$. Since g meets all of $\mathcal{D}_{\sigma_\gamma \neq B}$, we know $\sigma_\gamma^g \neq B$. Hence $\sigma_\gamma^g \in \sigma^g \setminus A$ so $\sigma^g \neq A$.
(2) For some $B \in A$, $q \Vdash \check{B} \notin \sigma$. Let $B' \in \sigma^g$. Then for some $\gamma < \kappa$ and $r \leq q$ in g, $\sigma_\gamma^g = B'$ and $r \Vdash^+ \sigma_\gamma \in \sigma$. Hence $r \Vdash \sigma_\gamma \in \sigma$. But also $r \Vdash \check{B} \notin \sigma$ since $r \leq q$. Therefore $r \Vdash \sigma_\gamma \neq \check{B}$, and so $B' = \sigma_\gamma^g \neq B$ since g meets $\mathcal{D}_{\sigma_\gamma \neq \check{B}}$. Hence $B \in A \setminus \sigma^g$, so again $\sigma^g \neq A$. □

Claim 4.12. *The lemma holds when φ is of the form $\sigma \neq \tau$.*

Proof. The dense sets we need to use are very similar to the ones in the previous lemma. We assume $\operatorname{rank}(\sigma) \geq \operatorname{rank}(\tau)$ and note that if $\operatorname{rank}(\tau) = 0$ we're looking at the previous case. So let us assume $\operatorname{rank}(\sigma) \geq \operatorname{rank}(\tau) > 0$ and that we have proved the statement for all σ' and τ' with lower ranks than σ and τ respectively. As usual, write $\sigma = \{(\sigma_\gamma, p) : \gamma < \kappa, p \in S_\gamma\}$ and $\tau = \{(\tau_\delta, q) : \delta < \kappa, q \in T_\gamma\}$.

Let

$$D = \Big\{p \in \mathbb{P} :(p \Vdash \sigma = \tau) \vee \Big(\exists \gamma < \kappa (p \Vdash^+ \sigma_\gamma \in \sigma) \wedge (p \Vdash \sigma_\gamma \notin \tau)\Big) \\ \vee \Big(\exists \delta < \kappa (p \Vdash^+ \tau_\delta \in \tau) \wedge (p \Vdash \tau_\delta \notin \sigma)\Big)\Big\}$$

Once again D is dense. We define

$$\mathcal{D}_{\sigma \neq \tau} := \{D\} \cup \bigcup_{\gamma, \delta < \kappa} \mathcal{D}_{\sigma_\gamma \neq \tau_\delta}$$

Letting g be our usual filter meeting all of $\mathcal{D}_{\sigma \neq \tau}$ and containing some p forcing $\sigma \neq \tau$, we can find $q \in g \cap D$ below p. Without loss of generality, there exists $\gamma < \kappa$ such that $q \Vdash^+ \sigma_\gamma \in \sigma$ and $q \Vdash \sigma_\gamma \notin \tau$. As always, the first statement implies $\sigma_\gamma^g \in \sigma^g$. If $\sigma_\gamma^g \in \tau^g$ then for some $\delta < \kappa$ and $r \in g$ (which we can take to be below q), $\sigma_\gamma^g = \tau_\delta^g$ and $r \Vdash^+ \tau_\delta \in \tau$. But then we know $r \Vdash \sigma_\gamma \neq \tau_\delta$. Since g meets all of $\mathcal{D}_{\sigma_\gamma \neq \tau_\delta}$ this implies $\sigma_\gamma^g \neq \tau_\gamma^g$. Contradiction. Hence $\sigma_\gamma^g \in \sigma^g \setminus \tau^g$, so $\sigma^g \neq \tau^g$. □

Claim 4.13. *The lemma holds when φ has the form $\tau \notin \sigma$.*

Proof. Write $\sigma = \{(\sigma_\gamma, p) : \gamma < \kappa, p \in S_\gamma\}$ as usual. Let

$$\mathcal{D}_{\tau \notin \sigma} := \bigcup_{\gamma<\kappa} \mathcal{D}_{\tau \neq \sigma_\gamma}$$

Suppose g meets all of $\mathcal{D}_{\tau \notin \sigma}$ and contains some p forcing $\tau \notin \sigma$. Let $B \in \sigma^g$. For some $\gamma < \kappa$ and some $q \in g$ below p, $B = \sigma^g_\gamma$ and $q \Vdash^+ \sigma_\gamma \in \sigma$. Then $q \Vdash \tau \neq \sigma_\gamma$, so $\tau^g \neq \sigma^g_\gamma = B$. Hence $\tau^g \notin \sigma^g$. □

We can now finally prove the full lemma.

Claim 4.14. *The lemma holds in all cases.*

Proof. We use induction on the length of the formula φ. By rearranging φ, we can assume that all the $\neg$'s in φ are in front of atomic formulas. Throughout this proof, we will suppress the irrelevant variables $\vec{\sigma}$ of formulas $\psi(\vec{\sigma})$, and will write ψ^g to denote $\psi(\vec{\sigma}^g)$.

The base case, where φ is either atomic or the negation of an atomic formula, was covered in the previous lemmas.

$\varphi = \psi \wedge \chi$: We let $\mathcal{D}_\varphi := \mathcal{D}_\psi \cup \mathcal{D}_\chi$. If $p \in g$ forces φ then it also forces ψ and χ, so if also g meets all of $\mathcal{D}_\varphi$ then ψ^g and χ^g hold.

$\varphi = \psi \vee \chi$: We let $D = \{p \in \mathbb{P} : (p \Vdash \neg\varphi) \vee (p \Vdash \psi) \vee (p \Vdash \chi)\}$, and let $\mathcal{D}_\varphi := \{D\} \cup \mathcal{D}_\psi \cup \mathcal{D}_\chi$. If g meets all of $\mathcal{D}_\varphi$ and contains some p which forces φ then take $q \leq p$ in $g \cap D$. Then $q \Vdash \psi$ or $q \Vdash \chi$, and by definition of $\mathcal{D}_\psi$ and $\mathcal{D}_\chi$ this implies ψ^g or χ^g respectively.

$\varphi = \forall x \in \sigma\ \psi(x)$: Write $\sigma = \{(\sigma_\gamma, p) : \gamma < \kappa, p \in S_\gamma\}$, and let $\mathcal{D}_\varphi := \bigcup_{\gamma<\kappa} \mathcal{D}_{\psi(\sigma_\gamma)}$. Suppose, as usual, that g meets all of $\mathcal{D}_\varphi$ and contains some p forcing φ. Let $B \in \sigma^g$. Then we have some $\gamma < \kappa$ and $q \in g$ such that $\sigma^g_\gamma = B$ and $q \Vdash^+ \sigma_\gamma \in \sigma$. Taking (without loss of generality) $q \leq p$, we then have that $q \Vdash \psi(\sigma_\gamma)$. Hence $\psi^g(\sigma^g_\gamma)$ holds. But we know $\sigma^g_\gamma = B$. Hence $\psi^g(B)$ holds for all $B \in \sigma^g$, so φ^g holds.

$\varphi = \exists x \in \sigma\ \psi(x)$: Again we write $\sigma = \{(\sigma_\gamma, p) : \gamma < \kappa, p \in S_\gamma\}$. Let D be the dense set $\{p \in \mathbb{P} : (p \Vdash \neg\varphi) \vee \exists \gamma < \kappa\ (p \Vdash^+ \sigma_\gamma \in \sigma \wedge p \Vdash \psi(\sigma_\gamma))\}$, and let $\mathcal{D}_\varphi := \{D\} \cup \bigcup_{\gamma<\kappa} \mathcal{D}_{\psi(\sigma_\gamma)}$. If g meets all of $\mathcal{D}_\varphi$ and contains p forcing φ then we can take some element q of $g \cap D$ below p. Then for some $\gamma < \kappa$, we know $q \Vdash \psi(\sigma_\gamma)$ and $q \Vdash^+ \sigma_\gamma \in \sigma$. Then $\psi^g(\sigma^g_\gamma)$ holds, and $\sigma^g_\gamma \in \sigma^g$. □

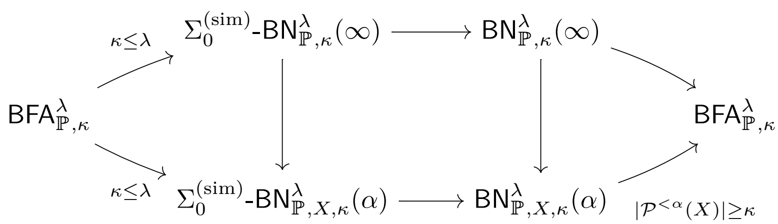
This completes the proof of Lemma 4.7. Hence $\mathsf{FA}_{\mathbb{P},\kappa}$ implies $\Sigma^{(\mathrm{sim})}_0$-$\mathsf{N}_{\mathbb{P},\kappa}(\infty)$, as discussed earlier. □

This completes the proof of Theorem 4.1. □

In fact, this proof works even if we allow formulas to have conjunctions and disjunctions of κ many formulas (and accordingly let formulas have κ many variables).

The proof of Theorem 4.2 is essentially the same:

Proof of Theorem 4.2. We prove all the implications in the following diagram.



Note that $\mathsf{BFA}^\lambda_{\mathbb{P},\kappa} \Rightarrow \Sigma^{(\mathrm{sim})}_0$-$\mathsf{BN}^\lambda_{\mathbb{P},X,\kappa}(\alpha)$ for $\kappa \leq \lambda$ follows from the rest of the diagram.

Proof of $\Sigma_0^{(\mathrm{sim})}$-$\mathsf{BN}^{\lambda}_{\mathbb{P},\kappa}(\infty) \Rightarrow \Sigma_0^{(\mathrm{sim})}$-$\mathsf{BN}^{\lambda}_{\mathbb{P},\kappa}(\alpha)$ and $\mathsf{BN}^{\lambda}_{\mathbb{P},\kappa}(\infty) \Rightarrow \mathsf{BN}^{\lambda}_{\mathbb{P},\kappa}(\alpha)$.
The latter are special cases of the former. □

Proof of $\Sigma_0^{(\mathrm{sim})}$-$\mathsf{BN}^{\lambda}_{\mathbb{P},X,\kappa}(\alpha) \Rightarrow \mathsf{BN}^{\lambda}_{\mathbb{P},X,\kappa}(\alpha)$ and $\Sigma_0^{(\mathrm{sim})}$-$\mathsf{BN}^{\lambda}_{\mathbb{P},\kappa}(\infty) \Rightarrow \mathsf{BN}^{\lambda}_{\mathbb{P},\kappa}(\infty)$.
As before, similar to the proofs in Theorem 4.1. □

Proof of $\mathsf{BN}^{\lambda}_{\mathbb{P},X,\kappa}(\alpha) \Rightarrow \mathsf{BFA}^{\lambda}_{\mathbb{P},\kappa}$ and $\mathsf{BN}^{\lambda}_{\mathbb{P},\kappa}(\infty) \Rightarrow \mathsf{BFA}^{\lambda}_{\mathbb{P},\kappa}$. Letting $\langle D_\gamma : \gamma < \kappa\rangle$ be a sequence of predense sets of cardinality at most λ, we define a name σ exactly as in the corresponding proof from Theorem 4.1. Since the D_γ have cardinality at most λ, and all the names that appear in σ are 1 bounded check names, σ is λ-bounded.

As in the earlier proof, a filter g such that $\sigma^g = A$ will meet all of the D_γ. □

Proof of $\mathsf{BFA}^{\lambda}_{\mathbb{P},\kappa} \Rightarrow \Sigma_0^{(\mathrm{sim})}$-$\mathsf{BN}^{\lambda}_{\mathbb{P},\kappa}$. Assume $\lambda \geq \kappa$. We prove the following lemma (very similar to Lemma 4.7).

Lemma 4.15. *Let $\varphi(\vec{\sigma})$ be a Σ_0 formula where $\vec{\sigma}$ is a tuple of κ-small λ-bounded names. Then there is a collection $\mathcal{D}_{\varphi(\vec{\sigma})}$ of at most κ many predense sets each of cardinality at most λ, which has the following property: if g is any filter meeting every set in $\mathcal{D}_{\varphi(\vec{\sigma})}$ and g contains some p such that $p \Vdash \varphi(\vec{\sigma})$, then in fact $\varphi(\vec{\sigma^g})$ holds in V.*

We use the same proof as in Theorem 4.1, adjusting the dense sets we work with. Whenever a dense set appears, we will replace it with a predense set of size at most λ which fulfils all the same functions. To obtain these sets, we use a few techniques.

First, whenever the original proof calls for an arbitrary condition which forces some desirable property, we replace it with the supremum of all such conditions (exploiting the fact that we are in a complete Boolean algebra).

For example, in place of

$$E_\gamma = \Big\{p \in \mathbb{P} : (p \Vdash \sigma \neq \check{A}) \vee (p \Vdash \sigma_\gamma \notin A) \vee \Big(\exists B \in A, p \Vdash \sigma_\gamma = \check{B}\Big)\Big\}$$

in Claim 4.8, we would take the set

$$E^*_\gamma := \{q_0, q_1\} \cup \{q_B : B \in A\}$$

where

$$q_0 = \sup\{p \in \mathbb{P} : p \Vdash \sigma \neq \check{A}\}$$
$$q_1 = \sup\{p \in \mathbb{P} : p \Vdash \sigma_\gamma \notin \check{A}\}$$

and for $B \in A$,

$$q_B = \sup\{p \in \mathbb{P} : p \Vdash \sigma_\gamma = \check{B}\}.$$

E^*_γ has cardinality at most λ, since $|A| \leq \kappa \leq \lambda$.

When the original set calls for a condition which strongly forces $\tau \in \sigma$ for some τ and σ, simply taking suprema won't work. Instead, we ask for a condition q such that $(\tau, q) \in \sigma$. Since all the names σ we deal with in the proof are λ-bounded, there will be at most λ many such conditions.

For example, in the same claim,

$$D_B := \Big\{p \in \mathbb{P} : (p \Vdash \sigma \neq \check{A}) \vee \Big(\exists \gamma < \kappa\, (p \Vdash \sigma_\gamma = \check{B}) \wedge (p \Vdash^+ \sigma_\gamma \in \sigma)\Big)\Big\}$$

will be replaced by

$$D^*_B := \{r\} \cup \{r_{\gamma,q} : \gamma < \kappa, q \in \mathbb{P}, (\sigma_\gamma, q) \in \sigma, r_{\gamma,q} \neq 0\}$$

where

$$r = \sup\{p \in \mathbb{P} : p \Vdash \sigma \neq \check{A}\}$$

and for $\gamma < \kappa$, $q \in \mathbb{P}$,

$$r_{\gamma,q} = \sup\{p \leq q : p \Vdash \sigma_\gamma = \check{B}\}.$$

Checking that we can indeed apply these techniques to turn all the dense sets in the proof into predense sets of cardinality at most λ is left as an exercise for the particularly thorough reader. □

This completes the proof of Theorem 4.2. □

4.3. Generic absoluteness

In this section, we derive generic absoluteness principles from the above correspondence.

Fix a cardinal κ. We start by defining the class of $\Sigma^1_1(\kappa)$-formulas. To this end, work with a two-sorted logic with two types of variables, interpreted as ranging over ordinals below κ and over subsets of κ, respectively. The language contains a binary relation symbol $\in$ and a binary function symbol p for a pairing function $\kappa \times \kappa \to \kappa$. Thus, atomic formulas are of the form $\alpha = \beta$, $x = y$, $\alpha \in x$ and $p(\alpha, \beta) = \gamma$, where α, β, γ denote ordinals and x, y denote subsets of κ.

Definition 4.16. A $\Sigma^1_1(\kappa)$ formula is of the form

$$\exists x_0, \ldots, x_m \ \varphi(x_0, \ldots, x_m, y_0, \ldots, y_n),$$

where the x_i are variables for subsets of κ, the y_i are either type of variables, and φ is a formula which only quantifies over variables for ordinals.

As a corollary to the results in Section 4.1, we obtain Bagaria's characterisation of bounded forcing axioms [4, Theorem 5] as the equivalence (1) $\Leftrightarrow$ (4) of the next theorem. It also shows that the principles $\Sigma_0^{(\mathrm{sim})}$-$\mathsf{BN}^\lambda_{\mathbb{P},\kappa}$ for $\lambda < \kappa$ are all equivalent to $\mathsf{BFA}^\kappa_{\mathbb{P},\kappa}$.

Theorem 4.17. *Suppose that κ is a cardinal with $\mathrm{cof}(\kappa) > \omega$, $\mathbb{P}$ is a complete Boolean algebra and $\dot{G}$ is a $\mathbb{P}$-name for the generic filter. Then the following conditions are equivalent*[4]*:*

(1) $\mathsf{BFA}^\kappa_{\mathbb{P},\kappa}$
(2) $\Sigma_0^{(\mathrm{sim})}$-$\mathsf{BN}^1_{\mathbb{P},\kappa}(1)$[5]
(3) $\Vdash_{\mathbb{P}} V \prec_{\Sigma^1_1(\kappa)} V[\dot{G}]$
(4) $\Vdash_{\mathbb{P}} H^V_{\kappa^+} \prec_{\Sigma_1} H^{V[\dot{G}]}_{\kappa^+}$

[4] The equivalence (1) $\Leftrightarrow$ (4) is equivalent to Bagaria's version, since his definition of BFA refers to Boolean completions.
[5] The version $\Sigma_0 - \mathsf{BN}^1_{\mathbb{P},\kappa}(1)$ for single Σ_0-formulas is also equivalent by the proof below.

Proof. The implication (1) ⇒ (2) holds since $\mathsf{BFA}^{\kappa}_{\mathbb{P},\kappa} \Leftrightarrow \Sigma_0^{(\mathrm{sim})}\text{-}\mathsf{BN}^{\kappa}_{\mathbb{P},\kappa}(1)$ by Theorem 4.2 and $\Sigma_0^{(\mathrm{sim})}\text{-}\mathsf{BN}^{\kappa}_{\mathbb{P},\kappa}(1)$ implies $\Sigma_0^{(\mathrm{sim})}\text{-}\mathsf{BN}^{1}_{\mathbb{P},\kappa}(1)$.

(2) ⇒ (3): To simplify the notation, we will only work with $\Sigma^1_1(\kappa)$-formulas of the form $\exists x\ \varphi(x, y)$, where x and y range over subsets of κ. Suppose that y is a subset of κ and $p \Vdash \exists x\ \varphi(x, \check{y})$. Let σ be a $\mathbb{P}$-name with $p \Vdash_{\mathbb{P}} \varphi(\sigma, \check{y})$. Since the variables of φ are interpreted as subsets of κ, this means that $p \Vdash \sigma \subseteq \check{\kappa}$. Let τ be defined by

$$\tau := \{(\check{\alpha}, [\![\check{\alpha} \in \sigma]\!]) : \alpha < \kappa, [\![\check{\alpha} \in \sigma]\!] \neq 0\}.$$

Then τ is a 1-bounded 1 rank κ name with $p \Vdash_{\mathbb{P}} \sigma = \tau$. Note that $\check{y}$ is a 1-bounded rank 1 name, too. By $\Sigma_0^{(\mathrm{sim})}\text{-}\mathsf{BN}^{\kappa}_{\mathbb{P},\kappa}(1)$, there exists a filter $g \in V$ on $\mathbb{P}$ such that $V \models \varphi(\sigma^g, y)$. Hence $V \models \exists x\ \varphi(x, y)$.

The implication (3) ⇒ (1) works just like in the proof of [4, Theorem 5]. In short, the existence of the required filter is equivalent to a $\Sigma^1_1(\kappa)$-statement.

For (3) ⇒ (4), suppose that $\psi = \exists x\ \varphi(x, y)$ is a Σ_1-formula with a parameter $y \in H_{\kappa^+}$. Then

$$H_{\kappa^+} \models \psi \iff H_{\kappa^+} \models \text{“}\exists M \text{ transitive s.t. } y \in M \wedge M \models \psi\text{”}.$$

We express the latter by a $\Sigma^1_1(\kappa)$-formula θ with a parameter $A \subseteq \kappa$ which codes y in the sense that $f(0) = y$ for the transitive collapse f of $(\kappa, p^{-1}[A])$.

θ states the existence of a subset B of κ such that $\in_M := p^{-1}[B]$ has the following properties:

- $\in_M$ is wellfounded and extensional
- For all $\alpha < \beta < \kappa$, $2 \cdot \alpha \in_M 2 \cdot \beta$ and for all $\alpha, \beta < \kappa$, $2 \cdot \alpha + 1 \notin_M 2 \cdot \beta$.
- There is some $\hat{\kappa} < \kappa$ with $\{\alpha < \kappa : \alpha \in_M \hat{\kappa}\} = \{2 \cdot \alpha : \alpha < \kappa\}$
- There exists some $\hat{A} < \kappa$ such that for all $\beta < \kappa$, $\beta \in_M \hat{A} \Leftrightarrow \exists \alpha \in A\ 2 \cdot \alpha = \beta$
- There exists some $\hat{y} < \kappa$ such that in $(\kappa, \in_M)$, $\hat{A}$ codes $\hat{y}$
- $\varphi(\hat{y})$ holds in $(\kappa, \in_M)$

The transitive collapse f of $(\kappa, \in_M)$ to a transitive set M will satisfy $f(2 \cdot \alpha) = \alpha$ for all $\alpha < \kappa$, $f(\hat{\kappa}) = \kappa$, $f(\hat{A}) = A$, $f(\hat{y}) = y$ and $M \models \psi(y)$.

All the above conditions apart from wellfoundedness of $\in_M$ are first order over $(\kappa, \in, p, A, \in_M)$. It remains to express wellfoundedness of $\in_M$ in a $\Sigma^1_1(\kappa)$ way.[6] To see that we can do this, suppose that R is a binary relation on κ. Since $\mathrm{cof}(\kappa) > \omega$, R is wellfounded if and only if for all $\gamma < \kappa$, $R{\restriction}\gamma$ is wellfounded. Since $\gamma < \kappa$, $R{\restriction}\gamma$ is wellfounded if and only if there exists a map $f : \gamma \to \kappa$ such that for all $\alpha, \beta < \gamma$, $(\alpha, \beta) \in R \Rightarrow f(\alpha) < f(\beta)$. The existence of such a map f is a $\Sigma^1_1(\kappa)$ statement.

Finally, (4) ⇒ (3) holds since every $\Sigma^1_1(\kappa)$-formula is equivalent to a Σ_1-formula over H_{κ^+} with parameter κ. □

Remark 4.18. Note that for rank 1, $\Sigma_0^{(\mathrm{sim})}\text{-}\mathsf{BN}^{\lambda}_{\mathbb{P},\kappa}(1)$ implies the simultaneous λ-bounded rank 1 name principle for all $\Sigma^1_1(\kappa)$-formulas (see Definition 2.16) by picking 1-bounded names for witnesses.

Remark 4.19. The previous results cannot be extended to higher complexity. To see this, recall that a $\Pi^1_1(\kappa)$-formula is the negation of a $\Sigma^1_1(\kappa)$-formula. We claim that there exists a $\Pi^1_1(\omega_1)$-formula $\varphi(x)$ such that the 1-bounded rank 1 $\Pi^1_1(\omega_1)$-name principle for the class of c.c.c. forcings fails. Otherwise MA_{ω_1} would hold by (2) ⇒ (1) of Theorem 4.17. So in particular, there are no Suslin trees. Since adding a Cohen real adds a

[6] $\mathrm{cof}(\kappa) > \omega$ is in fact necessary to ensure that the set of codes on κ for elements of H_{κ^+} is $\Sigma^1_1(\kappa)$-definable with parameters in $\mathcal{P}(\kappa)$. If $\mathrm{cof}(\kappa) = \omega$ and κ is a strong limit, then this set is $\Pi^1_1(\kappa)$-complete and hence not $\Sigma^1_1(\kappa)$ by a result of Dimonte and Motto Ros [8].

Suslin tree, let σ be a 1-bounded rank 1 $\mathbb{P}$-name for it, where $\mathbb{P}$ denotes the Boolean completion of Cohen forcing, and apply the name principle to the statement "σ is a Suslin tree". But then we would have a Suslin tree in V.

Remark 4.20. Fuchs and Minden show in [11, Theorem 4.21] assuming CH that the bounded subcomplete forcing axiom BSCFA can be characterised by the preservation of $(\omega_1, \leq\omega_1)$-Aronszajn trees. The latter can be understood as the 1-bounded name principle for statements of the form "σ is an ω_1-branch in T", where T is an $(\omega_1, \leq\omega_1)$-Aronszajn tree. (See [11,15] for more about subcomplete forcing.)

We now consider forcing axioms at cardinals κ of countable cofinality. To our knowledge, these have not been studied before. $\mathsf{BFA}^{\kappa}_{c.c.c.,\kappa} = \mathsf{MA}_\kappa$ is an example of a consistent forcing axiom of this form. We fix some notation. If κ is uncountable cardinals with $\mathrm{cof}(\kappa) = \mu$, we fix a continuous strictly increasing sequence $\langle \kappa_i : i \in \mu \rangle$ of ordinals with $\kappa_0 = 0$ and $\sup_{i\in\mu} \kappa_i = \kappa$. We assume that each κ_i is closed under the pairing function p.[7] For each $x \in 2^\kappa$, we define a function $f_x \colon \mu \to 2^{<\kappa}$ by letting $f_x(i) = x{\upharpoonright}\kappa_i$.

Lemma 4.21. *Suppose that κ is an uncountable cardinal with $\mathrm{cof}(\kappa) = \mu$. Suppose that $\varphi(x, y)$ is a formula with quantifiers ranging over κ and $y \in 2^\kappa$ is fixed. Then there is a subtree $T \in V$ of $((2^{<\kappa})^{<\mu})^2$ such that in all generic extensions $V[G]$ of V[8] which do not add new bounded subsets of κ,*

$$\varphi(x, y) \Longleftrightarrow \exists g \in (2^{<\kappa})^\mu \ (f_x, g) \in [T]$$

holds for all $x \in (2^\kappa)^{V[G]}$. Moreover, for any branch $(\vec{s}, \vec{t}) \in [T]$ in $V[G]$ with $\vec{s} = \langle s_i : i \in \mu \rangle$, $\bigcup_{i\in\mu} s_i = f_x$ for some $x \in (2^\kappa)^{V[G]}$.

Proof. We construct the i-th levels $\mathrm{Lev}_i(T)$ by induction on $i \in \mu$. Let $\mathrm{Lev}_0(T) = \{(\emptyset, \emptyset)\}$. If $j \in \mu$ is a limit, let $(\vec{s}, \vec{t}) \in \mathrm{Lev}_j(T)$ if $(\vec{s}{\upharpoonright}i, \vec{t}{\upharpoonright}i) \in \mathrm{Lev}_i(T)$ for all $i < j$.

For the successor step, suppose that $\mathrm{Lev}_j(T)$ has been constructed. Write $\vec{s} = \langle s_i : i \leq j \rangle$ and $\vec{t} = \langle t_i : i \leq j \rangle$. Let $(\vec{s}, \vec{t}) \in \mathrm{Lev}_{j+1}(T)$ if the following conditions hold:

(1) $(\vec{s}{\upharpoonright}j, \vec{t}{\upharpoonright}j) \in \mathrm{Lev}_j(T)$.
(2) $s_j \in 2^{\kappa_j}$ and $\forall i < j \ s_j{\upharpoonright}\kappa_i = s_i$.
(3) $t_j \in 2^{\kappa_j}$ codes the following two objects.
 (i) A truth table p_j which assigns to each formula $\psi(\xi_0, \dots, \xi_k)$ and parameters $\alpha_0, \dots, \alpha_k < \kappa_j$ a truth value 0 or 1.
 (ii) A function q_j which assigns a value in ω to each existential formula $\exists\eta\ \psi(\xi_0, \dots, \xi_k, \eta)$ and associated parameters $\alpha_0, \dots, \alpha_k < \kappa_j$.

 They satisfy $p_i \subseteq p_j$, $q_i \subseteq q_j = q_i$ for all $i < j$ and the following conditions:
 (a) $p_j(\varphi) = 1$.
 (b) p_j satisfies the equality axioms:

$$p_j((\psi(\vec{\xi})), \vec{\alpha}) = 1 \wedge \vec{\alpha} = \vec{\beta} \Longleftrightarrow p_j((\psi(\vec{\xi})), \vec{\beta}) = 1$$

 (c) p_j is correct about atomic formulas $\psi(\vec{\xi})$ which do not mention $\dot{x}$ and $\dot{y}$:

$$p_j((\psi(\vec{\xi})), \vec{\alpha}) = 1 \Longleftrightarrow \psi(\vec{\alpha})$$

[7] If κ_i is multiplicatively closed, i.e. $\forall\alpha < \kappa\alpha \cdot \alpha < \kappa_i$, then this holds for Gödel's pairing function.
[8] This includes the case $V[G] = V$.

(d) The truth in p_j of all atomic formulas of the form $\xi \in \dot{x}$, $\xi \in \dot{y}$ is fixed according to s_j and y, respectively:

$$p_j((\xi \in \dot{x}), \alpha) = 1 \Longleftrightarrow \alpha \in s_j$$
$$p_j((\xi \in \dot{y}), \alpha) = 1 \Longleftrightarrow \alpha \in y$$

(e) p_j respects propositional connectives:

$$p_j(\psi \wedge \theta, \vec{\alpha}) = 1 \Longleftrightarrow p_j(\psi, \vec{\alpha}) = 1 \wedge p_j(\theta, \vec{\alpha}) = 1$$
$$p_j(\neg\psi, \vec{\alpha}) = 1 \Longleftrightarrow p_j(\psi, \vec{\alpha}) = 0$$

(f) p_j respects witnesses of existential formulas $\exists\eta\ \psi(\vec{\xi}, \eta), \vec{\alpha})$ which it has identified:

$$\exists\beta < \kappa_j\ p_j(\psi(\vec{\xi}, \eta), \vec{\alpha}, \beta) = 1 \Longrightarrow p_j(\exists\eta\ \psi(\vec{\xi}, \eta), \vec{\alpha}) = 1.$$

(g) q_j promises the existence of existential witnesses: for any existential formula $\exists\eta\ \psi(\vec{\xi}, \eta)$ and any tuple $\vec{\alpha}$ of parameters, if $p_j(\exists\eta\ \psi(\vec{\xi}, \eta), \vec{\alpha}) = 1$ and $q_j(\exists\eta\ \psi(\vec{\xi}, \eta), \vec{\alpha}) \leq n$, then there exists some $\beta < \kappa_j$ such that $p_j(\psi(\vec{\xi}, \eta), \vec{\alpha}, \beta) = 1$.

Let $V[G]$ be a generic extension of V with no new bounded subsets of κ. Work in $V[G]$.

$\Rightarrow$: Suppose that $\varphi(x, y)$ holds. We define $s_j = x{\upharpoonright}\kappa_j$ for each $j \in \mu$ and $p_j(\psi(\vec{\xi}), \vec{\alpha}) = 1$ if $(\kappa, \in, p, x, y) \models \psi(\vec{\alpha})$. We further define $q_j(\exists\eta\ \psi(\vec{\xi}, \eta), \vec{\alpha}) = 0$ if $p_j(\exists\eta\ \psi(\vec{\xi}, \eta), \vec{\alpha}) = 0$. Otherwise, $q_j(\exists\eta\ \psi(\vec{\xi}, \eta), \vec{\alpha})$ is defined as the least $l \in \mu$ such that for some $\beta < \kappa_l$, $(\kappa, \in, p, x, y) \models \psi(\vec{\alpha}, \beta)$. Let t_j code p_j and q_j (via the pairing function p). Note that s_j, p_j and q_j are in V, since $V[G]$ has no new bounded subsets of κ. Hence $\langle (s_j, t_j) : j \in \mu \rangle$ is a branch through T.

$\Leftarrow$: Suppose that $\langle (s_j, t_j) : j \in \mu \rangle$ is a branch through T. Let $x = \bigcup_{j \in \mu} s_j$. By induction on complexity of formulas, p_j and q_j are correct about x and y. Therefore $(\kappa, \in, p, x, y) \models \varphi(x, y)$. □

Theorem 4.22. *Suppose that κ is an uncountable cardinal with $\mathrm{cof}(\kappa) = \omega$, $\mathbb{P}$ is a complete Boolean algebra and $\dot{G}$ is a $\mathbb{P}$-name for the generic filter. Then the following conditions are equivalent:*

(1) $\mathsf{BFA}^{\kappa}_{\mathbb{P},\kappa}$
(2) $\Sigma_0^{(\mathrm{sim})}$-$\mathsf{BN}^1_{\mathbb{P},\kappa}$
(3) $\Vdash_{\mathbb{P}} V \prec_{\Sigma^1_1(\kappa)} V[\dot{G}]$

If moreover $2^{<\kappa} = \kappa$ holds,[9] *then the next condition is equivalent to* (1), (2) *and* (3)*:*

(4) $1_{\mathbb{P}}$ *forces that no new bounded subset of κ are added.*

If there exists no inner model with a Woodin cardinal,[10] *then the next condition is equivalent to* (1), (2) *and* (3)*:*

(5) $\Vdash_{\mathbb{P}} H^V_{\kappa^+} \prec_{\Sigma_1} H^{V[\dot{G}]}_{\kappa^+}$

Proof. The proofs of (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftarrow$ (5) in Theorem 4.17 work for all uncountable cardinals κ.

9 The assumption $2^{<\kappa} = \kappa$ is not needed for (4) $\Rightarrow$ (3).
10 The assumption that there is no inner model with a Woodin cardinal is not used for (5) $\Rightarrow$ (3).

(3) ⇒ (4): We assume $2^{<\kappa} = \kappa$. Towards a contradiction, suppose that $V[G]$ is a generic extension that adds a new subset of $\gamma < \kappa$. Note that $2^\gamma \leq \kappa$. Let $\vec{y} = \langle y_i : i < 2^\gamma \rangle$ list all subsets of γ. We define $x \subseteq \gamma \cdot 2^\gamma \subseteq \kappa$ by letting $\gamma \cdot i + j \in x \Leftrightarrow j \in y_i$. The next formula expresses "there is a new subset of $\gamma < \kappa$" as a $\Sigma^1_1(\kappa)$-statement in parameters coding the + and · operations:

$$\exists z\ [z \subseteq \gamma \wedge \neg \exists i\ \forall j < \gamma\ (j \in z \Leftrightarrow \gamma \cdot i + j \in x)].$$

This contradicts $\Sigma^1_1(\kappa)$-absoluteness.

(4) ⇒ (3): Suppose that $\exists x\ \psi(x, y)$ is a $\Sigma^1_1(\kappa)$-formula and $y \in (2^\kappa)^V$. Let T be a subtree of $((2^{<\kappa})^{<\omega})^2$ as in Lemma 4.21. Let G be $\mathbb{P}$-generic over V with $V[G] \vDash \exists x\ \psi(x, y)$. $V[G]$ does not have new bounded subsets of κ by assumption. Then $[T]$ has a branch in $V[G]$ by the property of T in Lemma 4.21. Since wellfoundedness is absolute, T has a branch $\langle s_n, t_n : n \in \omega \rangle$ in V. Then $\bigcup_{n\in\omega} s_n = f_x$ for some $x \in 2^\kappa$ by the properties of T. We have $V \models \psi(x, y)$, since

$$\psi(x, y) \Longleftrightarrow \exists g\ (f_x, g) \in [T].$$

(3) ⇒ (5): Note that the implication holds vacuously if κ is collapsed in some $\mathbb{P}$-generic extension of V. In this case, both (3) and (5) fail, since the statement "κ is not a cardinal" is $\Sigma^1_1(\kappa)$.

We next show: if $q \in \mathbb{P}$ forces that κ^+ is preserved, then $q \Vdash H^V_{\kappa^+} \prec_{\Sigma_1} H^{V[\dot{G}]}_{\kappa^+}$ holds. To see this, let G be $\mathbb{P}$-generic over V with $q \in G$. Suppose $\psi = \exists x\ \varphi(x, y)$ is a Σ_1-formula with a parameter $y \in H_{\kappa^+}$. We follow the proof of (3) ⇒ (4) in Corollary 4.17 to construct a $\Sigma^1_1(\kappa)$-formula θ that is equivalent to ψ. However, we replace the first condition by:

- $\in_M$ is extensional and wellfounded of rank γ

for a fixed $\gamma < (\kappa^+)^V = (\kappa^+)^{V[G]}$. If ψ is true, then for sufficiently large γ, θ will be true. Now we only need to modify the last step of the above proof. Let C be a subset of κ such that $(\kappa, p^{-1}[C]) \cong (\gamma, <)$. Suppose R is a binary relation on κ. The condition "R is wellfounded of rank $\leq\gamma$" is $\Sigma^1_1(\kappa)$ in C, since it is equivalent to the existence of a function $f : \kappa \to \gamma$ such that for all $\alpha, \beta < \kappa$, $(\alpha, \beta) \in R \Rightarrow f(\alpha) < f(\beta)$.

Towards a contradiction, suppose that there is no inner model with a Woodin cardinal and in some $\mathbb{P}$-generic extension $V[G]$ of V, $H^V_{\kappa^+} \prec_{\Sigma_1} H^{V[G]}_{\kappa^+}$ fails. By the previous remarks, κ is preserved and κ^+ is collapsed in $V[G]$. Since there is no inner model with a Woodin cardinal, the Jensen-Steel core model K from [16] is generically absolute and satisfies $(\lambda^+)^K = \lambda^+$ for all singular cardinals λ by [16, Theorem 1.1]. Therefore any generic extension $V[G]$ of V which does not collapse λ satisfies $(\lambda^+)^V = (\lambda^+)^{V[G]}$. For $\lambda = \kappa$, this contradicts our assumption. □

Can one remove the assumption that there is no inner model with a Woodin cardinal? A forcing $\mathbb{P}$ that witnesses the failure of (3) ⇒ (5) must preserve κ and collapse κ^+ by the above proof. The existence of a forcing $\mathbb{P}$ with these two properties is consistent relative to the existence of a λ^+-supercompact cardinal λ by a result of Adolf, Apter and Koepke [2, Theorem 7]. Their forcing does not add new bounded subsets of κ as in (4) and thus also satisfies (1)-(3). However, we do not know if it satisfies (5).

Question 4.23. Is it consistent that there exist an uncountable cardinal κ with $\mathrm{cof}(\kappa) = \omega$ and a forcing $\mathbb{P}$ with the properties:

(a) $\mathbb{P}$ does not add new bounded subsets of κ and
(b) $\Vdash_{\mathbb{P}} H^V_{\kappa^+} \prec_{\Sigma_1} H^{V[\dot{G}]}_{\kappa^+}$ fails?

(Thus $\mathbb{P}$ necessarily collapses κ^+.)

4.4. Boolean ultrapowers

In this section, we translate the above correspondence to Boolean ultrapowers and use this to characterise forcing axioms via elementary embeddings.

The Boolean ultrapower construction generalises ultrapowers with respect to ultrafilters on the power set of a set to ultrafilters on arbitrary Boolean algebras. We recall the basic definitions from Hamkins' and Seabold's work on Boolean ultrapowers [12, Section 3]. Suppose that $\mathbb{P}$ is a forcing and $\mathbb{B}$ its Boolean completion. Fix an ultrafilter U on $\mathbb{B}$, which may or may not be in the ground model. We define two relations $=_U$ and $\in_U$ on $V^{\mathbb{B}}$:

$$\sigma =_U \tau :\Leftrightarrow [\![\sigma = \tau]\!] \in U$$
$$\sigma \in_U \tau :\Leftrightarrow [\![\sigma \in \tau]\!] \in U$$

Let $[\sigma]_U$ denote the equivalence class of $\sigma \in V^{\mathbb{B}}$ with respect to $=_U$. Let $V^{\mathbb{B}}/U = \{[\sigma]_U : \sigma \in V^{\mathbb{B}}\}$ denote the quotient with respect to $=_U$. $\in_U$ is well-defined on equivalence classes and $(V^{\mathbb{B}}/U, \in_U)$ is a model of ZFC [12, Theorem 3]. It is easy to see from these definitions that for any $\mathbb{P}$-generic filter G over V, $V^{\mathbb{B}}/G$ is isomorphic to the generic extension $V[G]$. Moreover, we can determine the truth of sentences in $V^{\mathbb{B}}/U$ via Łos' theorem [12, Theorem 10]: $V^{\mathbb{B}}/U \models \varphi([\sigma_0]_U, \dots [\sigma_n]_U) \iff [\![\varphi(\sigma_0, \dots, \sigma_n)]\!] \in U$. In other words, the forcing theorem holds.

The *Boolean ultrapower* is the subclass

$$\check{V}_U = \{[\sigma]_U : [\![\sigma \in \check{V}]\!] \in U\}$$

of $V^{\mathbb{B}}/U$. It is isomorphic to V if and only if U is generic over V. The *Boolean ultrapower embedding* is the elementary embedding

$$j_U \colon V \to \check{V}_U, \quad j_U(x) = [\check{x}]_U.$$

We are interested in the case that U is an ultrafilter in the ground model. In particular, U is not $\mathbb{P}$-generic over V. j_U has the following properties:

- If U is generic, then j_U is an isomorphism.
- If U is not generic, then $\check{V}_U$ is ill-founded and $\mathrm{crit}(j_U)$ equals the least size of a maximal antichain in $\mathbb{B}$ not met by U [12, Theorem 17]. For example, if $\mathbb{P}$ is c.c.c. then $\mathrm{crit}(j_U) = \omega$.

For any $x \in V^{\mathbb{B}}/U$, let $x^{\in_U} = \{y \in V^{\mathbb{B}}/U : y \in_U x\}$ denote the set of all $\in_U$-elements of x. If κ is a cardinal and σ is a name for a subset of κ, then $[\sigma]_U^{\in_U} \cap j[\kappa] = j[\sigma^{(U)}]$, since

$$V^{\mathbb{B}}/U \models j_U(\alpha) = [\check{\alpha}]_U \in [\sigma]_U \Leftrightarrow [\![\check{\alpha} \in \sigma]\!] \in U \Leftrightarrow \alpha \in \sigma^{(U)}$$

for all $\alpha < \kappa$.

Theorem 4.24. *The following statements are equivalent:*

(1) $\mathsf{FA}_{\mathbb{P},\kappa}$
(2) *For any transitive set $M \in H_{\kappa^+}$ and for every κ-small M-name σ, there is an ultrafilter $U \in V$ on $\mathbb{P}$ such that*

$$j_U{\restriction}M \colon M \to j_U(M)^{\in_U}$$

is an elementary embedding from $(M, \in, \sigma^U)$ *to* $(j_U(M)^{\in_U}, \in_U, [\sigma]_U)$.

(3) *For any transitive set* $M \in H_{\kappa^+}$ *and for any* κ*-small* M*-name* σ*, there is an ultrafilter* U *on* $\mathbb{P}$ *such that*

$$(M, \in, \sigma^U) \equiv (j_U(M)^{\in_U}, \in_U, [\sigma]_U),$$

i.e. these structures are elementarily equivalent.

Proof. (1) $\Rightarrow$ (2): Recall from Lemma 4.7 that for any finite sequence $\vec{\sigma} = \sigma_0, \ldots, \sigma_k$ of κ-small names and every Σ_0-formula $\varphi(x_0, \ldots, x_k)$, there is a collection $\mathcal{D}_{\varphi(\vec{\sigma})}$ of $\leq\kappa$ many dense subsets of $\mathbb{P}$ with the following property: if g is any filter meeting every set in $\mathcal{D}_{\varphi(\vec{\sigma})}$ and g contains some p such that $p \Vdash \varphi(\vec{\sigma})$, then in fact $\varphi(\vec{\sigma^g})$ holds in V. Let $\mathcal{D}$ be the union of all collections $\mathcal{D}_{\varphi(\vec{\sigma})}$, where $k \in \omega$, $\varphi(x_0, \ldots, x_k)$ is a Σ_0-formula and each σ_i is σ, $\check{M}$ or $\check{x}$ for some $x \in M$. By $\mathsf{FA}_{\mathbb{P},\kappa}$, there is a filter g which meets all sets in $\mathcal{D}$. We extend g to an ultrafilter U.

Suppose that $\psi(x_0, \ldots, x_k)$ is a formula such that $(j_U(M)^{\in_U}, \in_U, [\sigma]_U) \models \psi(j_U(y_0), \ldots, j_U(y_k))$. We obtain $\varphi(x_0, \ldots, x_{k+2})$ by replacing the unbounded quantifiers in ψ by quantifiers bounded by x_{k+1}, and any occurrence of $[\sigma]_U$ by x_{k+2}. Then

$$(V^{\mathbb{B}}/U, \in_U) \models \varphi(j_U(y_0), \ldots, j_U(y_k), j_U(M), [\sigma]_U).$$

Recall that $j_U(y) = [\check{y}]_U$ for all $u \in M$. Therefore by Łos' theorem, we have $[\![\varphi(\check{y}_0, \ldots, \check{y}_k, \check{M}, \sigma)]\!] \in U$. So there is some $p \in U$ with $p \Vdash \varphi(\check{y}_0, \ldots, \check{y}_k, \check{M}, \sigma)$. Since U meets all dense sets in $\mathcal{D}_{\varphi(\check{y}_0, \ldots, \check{y}_k, \check{M}, \sigma)}$,

$$(V, \in) \models \varphi(y_0, \ldots, y_k, M, \sigma^U).$$

Hence $(M, \in, \sigma^U) \models \psi(y_0, \ldots, y_k)$.

(2) $\Rightarrow$ (3): This is clear.

(3) $\Rightarrow$ (1): Let $M = \kappa$ and suppose that σ is a rank 1 M-name such that $\mathbb{P} \Vdash \sigma = \check{\kappa}$. Then $\sigma^{(g)} = \kappa$ for any filter g on $\mathbb{P}$. It suffices to find a filter g with $\sigma^g = \kappa$ by Lemma 3.2. Let U be an ultrafilter as in (3). Since $M = \kappa$ and $j_U(M) = j_U(\kappa) = [\check{\kappa}]_U = [\sigma]_U$, we have $(j_U(M)^{\in_U}, \in_U, [\sigma]_U) \models \forall x\ x \in_U [\sigma]_U$. Thus $(\kappa, \in, \sigma^U) \models \forall x\ x \in_U \sigma^U$ by elementary equivalence. Thus $\sigma^U = \kappa$. □

A version of Theorem 4.24 for $\mathsf{BFA}^{\lambda}_{\mathbb{P},\kappa}$ and λ-bounded names also holds for any cardinal $\lambda \geq \kappa$. The proof is essentially the same.

4.5. An application to ub-FA

Lemma 4.25. *If* $\mathbb{P}$ *is a complete Boolean algebra that does not add reals, then*

$$(\forall q \in \mathbb{P}\ \ \mathsf{ub\text{-}FA}_{\mathbb{P}_q,\omega_1}) \Longrightarrow \mathsf{BFA}^{\omega_1}_{\mathbb{P},\omega_1}.$$

More generally, if κ *an uncountable cardinal and* $\mathbb{P}$ *is a complete Boolean algebra that does not add bounded subsets of* κ*, then*

$$(\forall q \in \mathbb{P}\ \ \mathsf{ub\text{-}FA}_{\mathbb{P}_q,\kappa}) \Longrightarrow \mathsf{BFA}^{\kappa}_{\mathbb{P},\kappa}.$$

Proof. If $\mathrm{cof}(\kappa) = \omega$, then adding no new bounded subsets of κ already implies $\mathsf{BFA}^{\kappa}_{\mathbb{P},\kappa}$ by the proof of (4) $\Rightarrow$ (3) in Theorem 4.22. Now suppose that $\mathrm{cof}(\kappa) > \omega$. Towards a contradiction, suppose that $\mathsf{BFA}^{\kappa}_{\mathbb{P},\kappa}$ fails. Then $\Sigma^1_1(\kappa)$-absoluteness fails for some $\Sigma^1_1(\kappa)$-formula $\exists x\ \psi(x, y)$ and some $y \in (2^\kappa)^V$ by Theorem 4.17.

Take a subtree T of $(2^{<\kappa} \times \kappa^{<\kappa})^{<\mathrm{cof}(\kappa)}$ for ψ as in Lemma 4.21. Then $[T] \neq \emptyset$ in $V[G]$ in some $\mathbb{P}$-generic extension $V[G]$, but $[T] = \emptyset$ in V. Let σ denote a rank 1 T-name and let $q \in \mathbb{P}$ such that $q \Vdash_{\mathbb{P}} \sigma \in [T]$. Let

$$\tau = \{(\alpha, p) : p \leq q \wedge \exists s \in \mathrm{Lev}_\alpha(T)\ p \Vdash^+_{\mathbb{P}} \check{s} \in \sigma\}$$

Then $\Vdash_{\mathbb{P}_q} \tau = \kappa$. For any filter $g \in V$ on $\mathbb{P}_q$ we have $\tau^g = \mathrm{dom}(\sigma^g)$. But $\mathrm{dom}(\sigma^g) \in \kappa$, since $[T] = \emptyset$. Therefore $\mathsf{ub\text{-}N}_{\mathbb{P}_q,\kappa}$ fails and hence $\mathsf{ub\text{-}FA}_{\mathbb{P}_q,\kappa}$ fails by Lemma 3.10. □

We will see in Lemma 5.1 that for any $<\kappa$-distributive forcing $\mathbb{P}$, $\mathsf{ub\text{-}FA}_{\mathbb{P},\kappa}$ implies $\mathsf{FA}_{\mathbb{P},\kappa}$. In combination with the previous lemma, this begs the question:

Question 4.26. If $\lambda > \kappa$ is a cardinal and $\mathbb{P}$ is a complete Boolean algebra that does not add new elements of ${}^{<\kappa}\lambda$, then does the implication

$$(\forall q \in \mathbb{P}\ \ \mathsf{ub\text{-}FA}_{\mathbb{P}_q,\omega_1}) \implies \mathsf{BFA}^\lambda_{\mathbb{P},\omega_1}$$

hold?

5. Specific classes of forcings

5.1. Classes of forcings

We now move on to look, over the next few sections, at what further results we can prove if we assume $\mathbb{P}$ is some specific kinds of forcing. We shall mostly return to the rank 1 cases for this and discuss the club, stat, ub and ω-ub axioms in Fig. 1.

5.1.1. σ-distributive forcings

We begin with a relatively simple case, where $\mathbb{P}$ is $<\kappa$-distributive. In this case, several of our axioms turn out to be equivalent to one another. The implications for the class of $<\kappa$-distributive forcings are summarised in the next diagram.

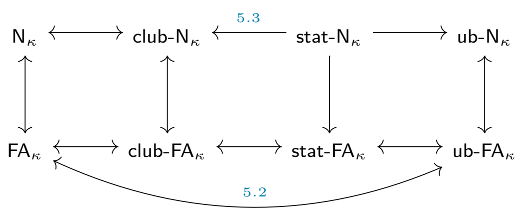


Fig. 5. Forcing axioms and name principles for any $<\kappa$-distributive forcing for regular κ. Lemma 5.3 shows that $\mathsf{stat\text{-}N}_{\mathbb{P},\omega_1}$ is strictly stronger than the remaining principles for some σ-closed forcing $\mathbb{P}$.

Lemma 5.1. *For any $<\kappa$-distributive forcing $\mathbb{P}$,* $\mathsf{ub\text{-}FA}_{\mathbb{P},\kappa} \implies \mathsf{FA}_{\mathbb{P},\kappa}$.

Proof. Given a sequence $\vec{D} = \langle D_i : i < \kappa\rangle$ of open dense subsets of $\mathbb{P}$, let $E_j = \bigcap_{i \leq j} D_i$ for $j < \kappa$. If for a filter g, $g \cap E_j \neq \emptyset$ for unboundedly many $j < \kappa$, then $g \cap D_i \neq \emptyset$ for all $i < \kappa$. □

Lemma 5.2. *Let $\mathbb{P}$ be $<\kappa$-distributive.* $\mathsf{stat\text{-}N}_{\mathbb{P},\kappa} \implies \mathsf{FA}^+_{\mathbb{P},\kappa}$

Proof. Suppose that $\vec{D} = \langle D_i : i < \kappa \rangle$ is a sequence of open dense subsets of $\mathbb{P}$ and $\sigma = \{(\check{\alpha}, p) : p \in S_\alpha\}$ is a name with $1 \Vdash_{\mathbb{P}}$ "σ is stationary". For each $j < \kappa$, let $E_j = \bigcap_{i \leq j} D_i$. For $j < \kappa$ and $p \in \mathbb{P}$, let $E_{j,p}$ denote a subset of $\{q \in E_j : q \leq p\}$ that is dense below p. Let

$$\tau = \{(\check{\alpha}, q) : \alpha < \kappa,\ \exists p \in S_\alpha\ q \in E_{j,p}\}.$$

$1 \Vdash_{\mathbb{P}}$ "τ is stationary", since $1 \Vdash_{\mathbb{P}} \sigma = \tau$. By $\mathsf{stat\text{-}N}_{\mathbb{P},\kappa}$, there is a filter g such that τ^g is stationary. By the definition of τ, $\tau^g \subseteq \sigma^g$. Thus σ^g is stationary. We further have $g \cap E_j$ for unboundedly many $j < \kappa$ and hence $g \cap D_i \neq \emptyset$ for all $i < \kappa$. $\square$

An equivalent argument can be made with names for unbounded sets, or for sets containing a club.

5.1.2. σ-closed forcings

Note that $\mathsf{FA}_{\mathbb{P},\omega_1}$ fails for some σ-distributive forcings, for instance for Suslin trees. But $\mathsf{FA}_{\sigma-\text{closed},\omega_1}$ is provable: if $\langle D_\alpha : \alpha < \omega_1 \rangle$ is a sequence of dense subsets of a σ-closed $\mathbb{P}$, let $\langle p_\alpha : \alpha < \omega_1 \rangle$ be a decreasing sequence of conditions in $\mathbb{P}$ with $p_\alpha \in D_\alpha$ and let $g = \{q \in \mathbb{P} : \exists \alpha < \omega_1\ p_\alpha \leq q\}$. Therefore, the other principles in Fig. 5 are provable, with the exception of $\mathsf{stat\text{-}N}_{\mathbb{P},\omega_1}$ by the next lemma. The lemma follows from known results.

Lemma 5.3. *It is consistent that there is a σ-closed forcing $\mathbb{P}$ such that $\mathsf{stat\text{-}N}_{\mathbb{P}}$ fails.*

Proof. It suffices to argue that $\mathsf{stat\text{-}N}_{\mathbb{P}}$ has large cardinal strength for some σ-closed forcing $\mathbb{P}$. Note that $\mathsf{stat\text{-}N}_{\mathbb{P}}$ implies $\mathsf{FA}^+_{\mathbb{P}}$ for any σ-closed forcing $\mathbb{P}$ by Lemma 5.2. There is a cardinal $\mu \geq \omega_2$ such that $\mathsf{FA}^+_{\mathrm{Col}(\omega_1,\mu)}$ implies the failure of $\square(\kappa)$ for all regular $\kappa \geq \omega_2$ by [9, Page 20 & Proposition 14] and [20, Theorem 2.1].[11] The proofs show that a single collapse suffices for the conclusion. The failure of $\square(\kappa^+)$ and thus Jensen's $\square_\kappa$ at a singular strong limit cardinal κ implies the existence of an inner model with a proper class of Woodin cardinals (and more) by [21, Theorem 0.1] and [23, Theorem 15.1]. $\square$

Presaturation of the nonstationary ideal on ω_1 is another interesting consequence of $\mathsf{stat\text{-}N}_{\sigma\text{-closed},\omega_1}$ (equivalently, of $\mathsf{FA}^+_{\sigma\text{-closed},\omega_1}$) [9, Theorem 25]. Even for very simple σ-closed forcings $\mathbb{P}$, $\mathsf{stat\text{-}N}_{\mathbb{P},\omega_1}$ is an interesting axiom. For instance, Sakai showed in [19, Section 3] that $\mathsf{FA}^+_{\mathrm{Add}(\omega_1),\omega_1}$ and thus $\mathsf{stat\text{-}N}_{\mathrm{Add}(\omega_1),\omega_1}$ is not provable in ZFC. We do not know much about the weakest stationary name principle for σ-closed forcing:

Question 5.4. Is $\mathsf{stat\text{-}BN}^1_{\sigma\text{-closed}}$ provable in ZFC?

5.1.3. c.c.c. forcings

The class of c.c.c. forcings is rather more interesting. It has also historically been a class where forcing axioms have been frequently used; for example $\mathsf{FA}_{\text{c.c.c.},\omega_1}$ is the well-known Martin's Axiom MA_{ω_1}. Note that $\mathsf{FA}_{\mathbb{P},\kappa}$ is equivalent to $\mathsf{BFA}^{\omega}_{\mathbb{P},\kappa}$.

[11] A more direct argument using [9, Page 20] and [26, Theorem 3.8] should be possible, but the required results are not explicitly mentioned there.

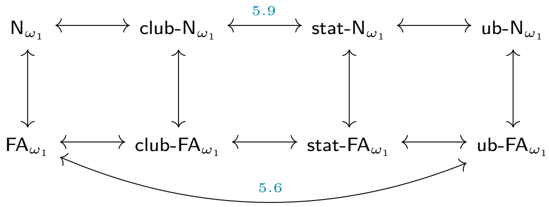


Fig. 6. Forcing axioms and name principles at ω_1 for the class of all c.c.c. forcings.

All principles in Fig. 1 for $\kappa = \omega_1$ turn out to be equivalent to FA_{ω_1}. The implications are valid for the class of all c.c.c. forcings, but not for all single c.c.c. forcings. For instance, for the class of σ-centred forcings, the right side of Fig. 1 is provable in ZFC by Lemma 5.7, but the left side is not.

We first derive the implication $\mathsf{ub\text{-}FA}_{c.c.c.,\omega_1} \implies \mathsf{FA}_{c.c.c.,\omega_1}$ from well-known results. Note that this implication does not hold for individual c.c.c. forcings, for instance it fails for Cohen forcing by Lemma 5.7 and Remark 5.18. We need the following definition:

Definition 5.5. Suppose that $\mathbb{P}$ is a forcing.

(1) A subset A of $\mathbb{P}$ is *centred* if every finite subset of A has a lower bound in $\mathbb{P}$. A is *σ-centred* if it is a union of countably many centred sets.
(2) $\mathbb{P}$ is *precaliber* κ if, whenever $A \in [\mathbb{P}]^\kappa$, there is some $B \in [A]^\kappa$ that is centred.

The hard implications in the next lemma are due to Todorčević and Veličković [25].

Lemma 5.6. *The following conditions are equivalent:*

(1) $\mathsf{ub\text{-}FA}_{\mathrm{c.c.c.},\omega_1}$ *holds.*
(2) *Every c.c.c. forcing is precaliber* ω_1*.*
(3) *Every c.c.c. forcing of size* ω_1 *is* σ*-centred.*
(4) $\mathsf{FA}_{\mathrm{c.c.c.},\omega_1}$ *holds.*

Proof. (1)⇒(2): This follows immediately from the proof of [14, Theorem 16.21]. The proof only requires meeting unboundedly many dense sets.

(2)⇒(3): See [25, Corollary 2.7].

(3)⇒(4): See [25, Theorem 3.3].

(4)⇒(1): This is immediate. □

Given Lemma 5.6, one wonders whether the equivalence of (1) and (4) also holds for σ-centred forcings instead of c.c.c. forcings. The next lemma together with the fact that $\mathsf{FA}_{\sigma\text{-centred}}$ is equivalent to $\mathfrak{p} > \omega_1$ (see [25, Theorem 3.1]) shows that this is not the case.

Lemma 5.7. *For any cardinal* κ *with* $\mathrm{cof}(\kappa) > \omega$*,* $\mathsf{stat\text{-}N}_{\sigma\text{-centred},\kappa}$ *holds.*

Proof. Suppose that σ is name for a stationary subset of ω_1. Let $f \colon \mathbb{P} \to \omega$ witness that $\mathbb{P}$ is σ-centred. Let S be the stationary set of α such that $p \Vdash \alpha \in \sigma$ for some $p \in \mathbb{P}$. For each $\alpha \in S$, let p_α be such that $(\alpha, p_\alpha) \in \sigma$. There is a stationary subset R of S and $n \in \omega$ with $f(p_\alpha) = n$ for all $\alpha \in R$. Let g be a filter containing p_α for all $\alpha \in S$. Then $R \subseteq \sigma^g$, as required. □

This suggests to ask whether $\mathsf{FA}_{\sigma\text{-centred}}$ implies $\mathsf{FA}^+_{\sigma\text{-centred}}$ as well. A further, long-standing open question is whether one can replace precaliber ω_1 by Knaster in the implication (2)⇒(4) of Lemma 5.6. Recall that a subset of $\mathbb{P}$ is *linked* if it consists of pairwise compatible conditions. $\mathbb{P}$ is called *Knaster* if, whenever $A \in [\mathbb{P}]^{\omega_1}$, there is some $B \in [A]^{\omega_1}$ that is linked.

Question 5.8. [24, Problem 11.1] Does the statement that every c.c.c. forcing is Knaster imply $\mathsf{FA}_{\text{c.c.c.},\omega_1}$?

We now turn to the implication $\mathsf{FA}_{\text{c.c.c.},\omega_1} \implies \mathsf{stat\text{-}N}_{\text{c.c.c.},\omega_1}$. To this end, we reconstruct Baumgartner's unpublished result $\mathsf{FA}_{\text{c.c.c.},\kappa} \implies \mathsf{FA}^{+n}_{\text{c.c.c.},\kappa}$ that is mentioned without proof in [5, Section 8] and [6, Page 14]. Here $\mathsf{FA}^{+n}_{\kappa}$ denotes the version of FA^+ with n many names for stationary subsets of κ.

Lemma 5.9 *(Baumgartner). For any uncountable cardinal κ and for any $n \in \omega$, $\mathsf{FA}_{\text{c.c.c.},\kappa}$ implies $\mathsf{FA}^{+n}_{\text{c.c.c.},\kappa}$.*

Proof. Suppose that for each $i < n$, σ_i is a rank 1 $\mathbb{P}$-name for a stationary subset of ω_1. For each $\vec{\alpha} = \langle \alpha_i : i < n\rangle \in \kappa^n$, let $A_{\vec{\alpha}}$ be a maximal antichain of conditions which strongly decide $\alpha \in \sigma_i$ for each $i < k$. Let $A = \bigcup_{\vec{\alpha}\in\kappa^n} A_{\vec{\alpha}}$. Since $\mathbb{P}$ satisfies the c.c.c. and $|A| \leq \omega_1$, there exists a subforcing $\mathbb{Q} \subseteq \mathbb{P}$ with $A \subseteq \mathbb{Q}$ and $|\mathbb{Q}| \leq \omega_1$ such that compatibility is absolute between $\mathbb{P}$ and $\mathbb{Q}$. In particular, $\mathbb{Q}$ is c.c.c.

Since every c.c.c. forcing of size ω_1 is σ-centred by MA_{ω_1} (see [27, Theorem 4.5]), there is a sequence $\vec{g} = \langle g_k : k \in \omega\rangle$ of filters g_k on $\mathbb{P}$ with $\mathbb{Q} \subseteq \bigcup_{k\in\omega} g_k$. Moreover, it follows from the proof of [27, Theorem 4.5] (by a density argument) that we can choose the filters g_k such that $g_k \cap B_\alpha \neq \emptyset$ for all $(k, \alpha) \in \omega \times \kappa$, where $\vec{B} = \langle B_\alpha : \alpha < \kappa\rangle$ is any sequence of dense subsets of $\mathbb{P}$. (The conditions in the c.c.c. forcing consists of finite approximations to finitely many filters.)

It remains to find some $k \in \omega$ such that for all $i < n$, the set $\sigma_i^{g_k}$ is stationary. Let G be $\mathbb{P}$-generic over V. We claim that

$$\prod_{i<n} \sigma_i^G \subseteq \bigcup_{k\in\omega} \prod_{i<n} \sigma_i^{g_k}.$$

To see this, suppose that $\vec{\alpha} = \langle \alpha_i : i < n\rangle \in \prod_{i<n} \sigma_i^G$ and let $p \in A_{\vec{\alpha}} \cap G$. Then $p \Vdash^+ \bigwedge_{i<n} \alpha_i \in \sigma_i$. Since $p \in \mathbb{Q}$, we have $p \in g_k$ for some $k \in \omega$. Hence $\vec{\alpha} \in \prod_{i<n} \sigma_i^{g_k}$. Since σ_i^G is stationary for all $i < n$, the above inclusion easily yields that there is some $k \in \omega$ such that $\prod_{i<n} \sigma_i^{g_k}$ is stationary. □

Our proof of the previous lemma does not work for $\mathsf{MA}^{+\omega}$. In fact, Baumgartner asked in [5, Section 8]:

Question 5.10 *(Baumgartner 1984).* Does MA_{ω_1} imply $\mathsf{MA}^{+\omega_1}_{\omega_1}$?

We finally turn to bounded name principles for c.c.c. forcings.

Lemma 5.11.

(1) $\mathsf{club\text{-}BN}^1_{\text{c.c.c.}}$ *holds.*
(2) *For any c.c.c. forcing $\mathbb{P}$,* $\mathsf{ub\text{-}BN}^1_{\mathbb{P}}$ *implies* $\mathsf{ub\text{-}FA}_{\mathbb{P}}$.

Proof. (1) If σ is a $\mathbb{P}$-name for a set that contains a club, then by the c.c.c. there is a club C with $1 \Vdash C \subseteq \sigma$. Since σ is 1-bounded, $(\alpha, 1) \in \sigma$ for all $\alpha \in C$. Thus for every filter g, we have $C \subseteq \sigma^g$.

(2) Suppose that $\mathbb{P}$ satisfies the c.c.c. and $\vec{D} = \langle D_\alpha : \alpha < \omega_1\rangle$ is a sequence of dense subsets of $\mathbb{P}$. Let A_α be a maximal antichain in D_α and let $\vec{a}_\alpha = \langle a^n_\alpha : n \in \omega\rangle$ enumerate A_α. (For ease of notation, we assume for that each A_α is infinite.) Let $\sigma = \{(\omega \cdot \alpha + n, a^n_\alpha) : \alpha < \omega_1,\ n \in \omega\}$. By $\mathsf{ub\text{-}BN}^1_{\mathbb{P}}$, there is a filter g such that σ^g is unbounded. Hence $D_\alpha \cap g \neq \emptyset$ for unboundedly many $\alpha < \omega_1$. □

For any c.c.c. forcing $\mathbb{P}$, the principles $\mathsf{ub\text{-}BN}^1_{\mathbb{P}}$, $\mathsf{ub\text{-}N}_{\mathbb{P}}$ and $\mathsf{ub\text{-}FA}_{\mathbb{P}}$ are equivalent by Lemma 5.11 (2) and the implications in Fig. 6. We do not know what is their relationship with $\mathsf{stat\text{-}BN}^1_{\text{c.c.c.}}$. However, we will show in Lemma 5.24 below that $\mathsf{stat\text{-}BN}^1_{\text{random},\omega_1}$ is not provable in ZFC.

Regarding Lemma 5.11 (1), it is also easy to see that $\mathsf{club\text{-}BN}^1_{\sigma\text{-closed}}$ is provable. This suggests to ask:

Question 5.12. Is $\mathsf{club\text{-}BN}^1_{\mathbb{P}}$ is provable for any proper forcing $\mathbb{P}$?

5.2. Specific forcings

5.2.1. Cohen forcing

We will now drop down from classes of forcings, to forcing axioms on specific forcings. This is also where we prove most of the negative results in the diagram from earlier. We start with the simplest, Cohen forcing and let $\kappa = \omega_1$. For Cohen forcing, all principles in the right part of the next diagram (Fig. 7) are provable in ZFC by Lemma 5.7 (on σ-centred forcing) and the basic implications in Fig. 1. The left part is not provable by Remark 5.18 below.

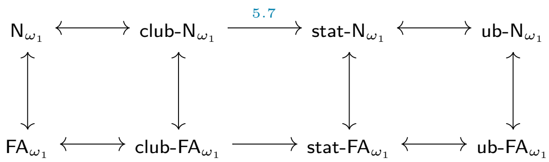


Fig. 7. Forcing axioms and name principles at ω_1 for Cohen forcing.

Our first result is an improvement to Lemma 5.7. It shows that a simultaneous version of the stationary forcing axiom for countably many sequences of dense sets holds.

Lemma 5.13. *Let $\mathbb{P}$ be Cohen forcing and κ a cardinal with $\operatorname{cof}(\kappa) > \omega$. For each $n \in \omega$, let $\vec{D}_n = \langle D^n_\alpha : \alpha < \kappa\rangle$ be a sequence of dense sets. Then there exists a filter $g \in V$ such that for all n, the trace $\operatorname{Tr}_{g,\vec{D}_n}$ is stationary in κ.*[12]

Proof. Suppose that there is no filter g as described. For $x \in 2^\omega$, let us write g_x to denote the filter $\{x{\restriction}n : n \in \omega\}$. Then for each $x \in 2^\omega$, the filter g_x does not have the required property. So there is a natural number n_x and a club $C_x \subseteq \kappa$ with $g_x \cap D^{n_x}_\alpha = \emptyset$ for all $\alpha \in C_x$. Then the sets $A_n := \{x \in 2^\omega : n_x = n\}$ partition 2^ω. By the Baire Category Theorem, not all A_n are nowhere dense. So there is some $n \in \omega$ and basic some open subset $N_t = \{x \in 2^\omega : t \subseteq x\}$ for some $t \in 2^{<\omega}$ such that $A_n \cap N_t$ is dense in N_t. Fix a countable set $D \subseteq A_n \cap U$ which is dense in U. Let α be an element of the club $\bigcap_{x\in D} C_x$. Let further $u \in D^n_\alpha$ with $u \le t$. Since D is dense in N_t, there is some $x \in D \cap N_u$. Then $u \in g_x \cap D^n_\alpha$ and hence $g_x \cap D^n_\alpha \neq \emptyset$. On the other hand, we have $x \in A_n$ and hence $n_x = n$. Since also $\alpha \in C_x$, we have $g_x \cap D^n_\alpha = \emptyset$. □

Using a variant of the previous proof, we can also improve $\mathsf{stat\text{-}N}_{\mathbb{P}}$ to work for finitely many names.

Lemma 5.14. *Let $\mathbb{P}$ be Cohen forcing and κ a cardinal with $\operatorname{cof}(\kappa) > \omega$. Suppose that $\vec{\sigma} = \langle \sigma_i : i \le n\rangle$ is a sequence of rank 1 $\mathbb{P}$-names such that for each $i \le n$, $\mathbb{P} \Vdash \sigma_i$ is stationary in κ. Then there is a filter g on $\mathbb{P}$ such that for all $i \le n$, σ^g_i is stationary in κ. In particular, $\mathsf{stat\text{-}N}_{\mathbb{P},\kappa}$ holds.*

[12] See Definition 2.2.

Proof. As in the previous proof, let $g_x = \{x \restriction n : n \in \omega\}$ for $x \in 2^\omega$. The result will follow from the next claim.

Claim 5.15. *If D is any dense subset of 2^ω, then there is some $x \in D$ such that $\sigma_i^{g_x}$ is stationary in κ for all $i \leq n$.*

Proof. We can assume that D is countable. If the claim fails, then for each $x \in D$, there is some $i \leq n$ and a club C_x such that $\sigma_i^{g_x} \cap C_x = \emptyset$. Then $C := \bigcap_{x \in D} C_x$ is a club. Moreover, for each $x \in D$, there is some $i \leq n$ such that $\sigma_i^{g_x} \cap C = \emptyset$. There is some $p \in \mathbb{P}$ such that for each $i \leq n$, there is some $\alpha_i \in C$ such that $p \Vdash \check{\alpha}_i \in \sigma_i$. By Lemma 2.22, we can assume that $p \Vdash^+ \check{\alpha}_i \in \sigma_i$ for all $i \leq n$. Now, since D is dense, we can find some $x \in D$ with $p \subseteq x$. Then $p \in g_x$, so by 2.23 we conclude $\alpha_i \in \sigma_i^{g_x}$ for all $i \leq n$. This contradicts the above property of C. □

This completes the proof of Lemma 5.14. □

Given the previous result about stat-FA, we might expect to be able to correctly interpret ω many names. But the above proof does not work: it breaks down where we introduce p. For each i, we can find p_i strongly forcing $\alpha_i \in \sigma_i$; but then we would want to take some p that was below every p_i and that is only possible in σ-closed forcings.

We can, however, apply the same technique in the presence of FA to prove FA^+.

Lemma 5.16. *Let $\mathbb{P}$ be Cohen forcing and κ a cardinal with $\mathrm{cof}(\kappa) > \omega$. Then $\mathsf{FA}_{\mathbb{P},\kappa}$ implies $\mathsf{FA}^+_{\mathbb{P},\kappa}$.*

Proof. We will in fact prove a stronger version for finitely many names. Suppose that $\vec{\sigma} = \langle \sigma_i : i \leq n \rangle$ is a sequence of rank 1 $\mathbb{P}$-names such that for each $i \leq n$, $\mathbb{P} \Vdash \sigma_i$ is stationary in κ. Suppose that $\vec{D} = \langle D_\alpha : \alpha < \kappa \rangle$ is a sequence of dense open sets. Then

$$D := \{x \in 2^\omega : \forall \alpha < \kappa\ \exists p \in D_\alpha\ p \subseteq x\}$$

consists of all reals x such that $g_x \cap D_\alpha \neq \emptyset$ for all $\alpha < \omega_1$.

The next claim suffices. By Claim 5.15, it implies that for some $x \in D$, $\sigma_i^{g_x}$ is stationary for all $i \leq n$.

Claim 5.17. *D is dense in 2^ω.*

Proof. Fix $q \in \mathbb{P}$; we will find some $x \in D$ with $q \subseteq x$. Since the forcing $\mathbb{P}_q := \{p \in \mathbb{P} : p \leq q\}$ is isomorphic to Cohen forcing via the map $r \mapsto q^\frown r$, $\mathsf{FA}_{\mathbb{P}_q}$ holds. Hence, we can find a filter g on $\mathbb{P}_q$ which meets $D_\alpha \cap \mathbb{P}_q$ for every $\alpha < \omega_1$. $\cup g$ is an element of $2^{\leq\omega}$ with $q \subseteq \cup g$ by compatibility of elements of a filter. Then any real x with $\cup g \subseteq x$ satisfies $x \in D$ and $q \subseteq x$. □

Lemma 5.16 follows. □

Remark 5.18. Note that $\mathsf{FA}_{\mathrm{Cohen},\omega_1}$ also has a well known characterisation via sets of reals: it is equivalent to the statement that the union of ω_1 many meagre sets does not cover 2^ω. In particular, $\mathsf{FA}_{\mathrm{Cohen},\omega_1}$ is not provable in ZFC.

5.2.2. Random forcing

The product topology on 2^ω is induced by the basic open sets $N_t = \{x \in 2^\omega : t \subseteq x\}$ for $t \in 2^{<\omega}$. *Lebesgue measure* is by definition the unique measure μ on the Borel subsets of 2^ω with $\mu(N_t) = \frac{2}{2^n}$.

Definition 5.19. *Random forcing* $\mathbb{P}$ *is the set of Borel subsets of* 2^ω *with positive Lebesgue measure.* $\mathbb{P}$ *is quasi-ordered by inclusion, i.e.* $p \leq q :\Leftrightarrow p \subseteq q$ *for* $p, q \in \mathbb{P}$.

Strictly speaking, random forcing is the partial order obtained by taking the quotient of the preorder, where two conditions are equivalent if their symmetric difference has measure 0. To simplify notation, we will talk about Borel sets of positive measure as if they were conditions in random forcing.

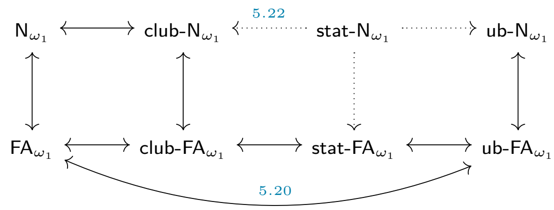


Fig. 8. Forcing axioms and name principles at ω_1 for random forcing.

We have seen in Lemma 5.7 and the following remark that ub-$\mathsf{FA}_{\mathbb{P}}$ implies $\mathsf{FA}_{\mathbb{P}}$ for σ-centred forcings. However, random forcing is not σ-centred by [7, Lemma 3.7]. The implication still holds:

Lemma 5.20. *Let* $\mathbb{P}$ *denote random forcing. The following are equivalent:*

(1) $\mathsf{FA}_{\mathbb{P},\omega_1}$
(2) ub-$\mathsf{FA}_{\mathbb{P},\omega_1}$
(3) 2^ω *is not the union of* ω_1 *many null sets*

The equivalence of (1) and (3) is a well-known fact, but we really interested in the equivalence of (1) and (2). The proof of (2)⇒(3) also works for certain forcings of the form $\mathbb{P}_I$. $\mathbb{P}_I$ consists of all Borel subsets $B \notin I$ of 2^ω, where I is a σ-ideal on the Borel subsets of the Cantor space, ordered by inclusion up to sets in I. For (2)⇒(3), it suffices that the set of closed $p \in \mathbb{P}$ is dense in $\mathbb{P}$ and $N_t \notin I$ for all $t \in 2^{<\omega}$. If additionally (3)⇒(1) holds, then ub-$\mathsf{FA}_{\mathbb{P}_I,\omega_1}$ implies $\mathsf{FA}_{\mathbb{P}_I,\omega_1}$.

Proof. (1)⇒(2): Immediate.

(2)⇒(3): We prove the contrapositive. Suppose $2^\omega = \bigcup_{\alpha<\omega_1} S_\alpha$, where $S_\alpha \subseteq 2^\omega$ has measure 0. Without loss of generality, we may assume that $\langle S_\alpha : \alpha < \omega_1 \rangle$ is an increasing sequence; i.e. $\alpha < \beta < \omega_1 \Rightarrow S_\alpha \subseteq S_\beta$. Then

$$D_\alpha = \{B \in \mathbb{P} : B \subseteq 2^\omega \setminus S_\alpha \text{ and } B \text{ is closed}\}$$

is dense.

Let $g \in V$ be a filter. Without loss of generality, assume g is an ultrafilter. Then for any $n \in \omega$, there is some $t \in 2^n$ with $N_t \in g$. It follows that there is a unique $x \in 2^\omega$ such that $N_t \in g$ for all $t \subseteq x$. It is easy to check that x is in the closure of any element of g.

Towards a contradiction, suppose that for unboundedly many α we can find $B_\alpha \in D_\alpha \cap g$. Then B_α is closed, so $x \in B_\alpha \subseteq 2^\omega \setminus S_\alpha$ so $x \notin S_\alpha$. This contradicts the assumptions that $2^\omega = \bigcup S_\alpha$ and the S_α are increasing.

(3)⇒(1): Again we prove the contrapositive. Let $\langle D_\alpha : \alpha < \omega_1 \rangle$ be a sequence of predense sets such that there is no filter in V meeting all of them. $\mathbb{P}$ has the c.c.c., so without loss of generality we may assume every D_α is countable.

Fix the following notation. Recall that $x \in 2^\omega$ is a density point of B if $\frac{\mu(B \cap N_{(x|k)})}{\mu(N_{(x|k)})}$ tends to 1 as k tends to infinity. For $B \in \mathbb{P}$, let $D(B)$ be the set of density points of B. For $\alpha < \omega_1$, let

$$T_\alpha = \bigcup_{B \in D_\alpha} D(B) \quad \text{and} \quad S_\alpha = 2^\omega \setminus T_\alpha.$$

We first show that S_α is a null set. To see this, suppose that S_α has positive measure. Then we can find a closed subset $C \subseteq S_\alpha$ with positive measure. Since D_α is predense, we can find some $B \in D_\alpha$ with $\mu(B \cap C) > 0$. Since $B \triangle D(B)$ is null by Lebesgue's Density Theorem, we have $\mu(D(B) \cap C) > 0$. This contradicts $D(B) \cap C \subseteq T_\alpha \cap C = \emptyset$.

We now show $\bigcup_{\alpha<\omega_1} S_\alpha = 2^\omega$. To see this, take any $x \in 2^\omega$ and let

$$g_x = \{B \in \mathbb{P} : x \in D(B)\}$$

denote the filter generated by x. Take $\alpha < \omega_1$ such that $g_x \cap D_\alpha = \emptyset$. We show that $x \in S_\alpha$, as required. Otherwise $x \in T_\alpha$, so we can find $B \in D_\alpha$ with $x \in D(B)$. But then $B \in g_x \cap D_\alpha$. This contradicts $g_x \cap D_\alpha = \emptyset$. □

Combining the proofs of (2)⇒(3) and (3)⇒(1), we can obtain the following refinement:

Lemma 5.21. *Let $\mathbb{P}$ be random forcing. Let $\langle D_\alpha : \alpha < \omega_1 \rangle$ be a collection of predense sets. There exists another collection $\langle D'_\alpha : \alpha < \omega_1 \rangle$ of dense sets, such that if a filter g meets unboundedly many D'_α, then it can be extended to a filter g' which meets every D_α.*

Proof. Define S_α as in the proof of (3)⇒(1). Then for any $x \in 2^\omega$, we have $g_x \cap D_\alpha \neq \emptyset$ or $x \in S_\alpha$. Consider the null sets $S'_\alpha = \bigcup_{\beta<\alpha} S_\beta$. Then define D'_α from S'_α in the same way we defined D_α from S_α in the proof of (2)⇒(3). As in the proof of (2)⇒(3), we obtain the following for any $x \in 2^\omega$ and $\alpha < \omega_1$: if $g_x \cap D'_\alpha \neq \emptyset$, then $x \notin S'_\alpha$. Let g be a filter which meets unboundedly many D'_α. Then $g \subseteq g_x$ for some $x \in 2^\omega$. We have seen that $x \notin S'_\alpha$ for unboundedly many α. Therefore x misses all S'_α and all S_α. By the choice of the S_α, we have $g_x \cap D_\alpha \neq \emptyset$ for all $\alpha < \omega_1$. □

This then allows us to prove that $\mathsf{stat\text{-}N}$ alone gives us the full FA^+.

Lemma 5.22. *Let $\mathbb{P}$ be random forcing. Then* $\mathsf{stat\text{-}N}_{\mathbb{P}} \implies \mathsf{FA}^+_{\mathbb{P}}$.

Proof. Suppose that $\langle D_\alpha : \alpha < \omega_1 \rangle$ is a sequence of dense subsets of $\mathbb{P}$. Suppose further that σ is a rank 1 name which is forced to be stationary. Let $\langle D'_\alpha : \alpha < \omega_1 \rangle$ be a sequence as in Lemma 5.21 and

$$\tau = \{(\check{\alpha}, p) : p \in D'_\alpha \wedge p \Vdash^+ \check{\alpha} \in \sigma\}.$$

Note that $\mathbb{P} \Vdash \sigma = \tau$. By $\mathsf{stat\text{-}N}_{\mathbb{P}}$, we obtain a filter g such that τ^g is stationary. Since $\tau^h \subseteq \sigma^h$ for all filters h, σ^g is stationary as well. Moreover, $g \cap D'_\alpha \neq \emptyset$, for stationarily many α. By the choice of $\langle D'_\alpha : \alpha < \omega_1 \rangle$, we can extend g to a filter g' such that $g' \cap D_\alpha \neq \emptyset$ for all $\alpha < \omega_1$. Moreover, $\sigma^g \subseteq \sigma^{g'}$, so $\sigma^{g'}$ is stationary. □

The missing link in Fig. 8 is:

Question 5.23. If $\mathbb{P}$ denotes random forcing, does $\mathsf{FA}_{\mathbb{P},\omega_1}$ imply $\mathsf{stat\text{-}N}_{\mathbb{P},\omega_1}$?

We finally show that the 1-bounded stationary name principle for random forcing is non-trivial, as we discussed at the end of Section 5.1.3.

Lemma 5.24. *Let $\kappa = 2^{\aleph_0}$ and assume that every set of size $<\kappa$ is null.*[13] *Then* $\mathsf{stat\text{-}BN}^1_{\mathbb{P},\kappa}$ *fails for random forcing* $\mathbb{P}$. *In particular,* CH *implies that* $\mathsf{stat\text{-}BN}^1_{\mathbb{P},\omega_1}$ *fails.*

Proof. It suffices to show that $\mathsf{stat\text{-}BN}^{\omega}_{\mathbb{P},\kappa}$ fails. To see this, apply Corollary 3.14 and use the fact that random forcing is well-met and for any $q \in \mathbb{P}$, the forcing $\mathbb{P}_q$ is isomorphic to $\mathbb{P}$ by [17, Theorem 17.41]. Let $\vec{x} = \langle x_\alpha : \alpha < \kappa\rangle$ enumerate all reals. Then $C_\beta := \{x_\alpha : \alpha < \beta\}$ is null for all $\beta < \kappa$. For each $\alpha < \kappa$, let A_α be a countable set of approximations to the complement of C_α in the following sense:

(a) Each element of A_α is a closed set disjoint from C_α, and
(b) For all $\epsilon > 0$, A_α contains a set C with $\mu(C) \geq 1 - \epsilon$.

Let $\sigma = \{(\check{\alpha}, p) : p \in A_\alpha\}$. Then $\Vdash_{\mathbb{P}} \sigma$ is stationary, since each A_α is predense by (b). We claim that there is no filter g in V such that σ^g is unbounded. If g were such a filter, then we could assume that for every $n \in \omega$, g contains N_{t_n} for some (unique) $t_n \in 2^n$ by extending g. (Clearly σ^g will remain unbounded.) Let $x = \bigcup_{n\in\omega} t_n$ and suppose that $x = x_\alpha$. Since σ^g is unbounded, there is some $\gamma > \alpha$ in σ^g. Find some $p \in A_\gamma$ with $p \in g$. By the definition of A_γ, p is a closed set with $x_\alpha \notin p$. Hence $p \cap N_{t_n} = \emptyset$ for some $n \in \omega$. But this contradicts the fact that both p and N_{t_n} are in g. □

5.2.3. Hechler forcing

For σ-centred forcings $\mathbb{P}$, the principles on the right side of Fig. 1 are provable in ZFC (see Lemma 5.7). A subtle difference appears when we add the requirement that the filter has to meet countably many fixed dense sets. We write ω-ub-FA for this axiom (see Definition 2.4). For some forcings, this axiom is stronger that ub-FA. To see this, we will make use of the fact that for Hechler forcing, a filter that meets certain countably many dense sets corresponds to a real. Recall that a subset $A \subseteq \omega^\omega$ is *unbounded* if no $y \in \omega^\omega$ eventually strictly dominates all $x \in A$, i.e. $\exists m\ \forall n \geq m\ x(n) < y(n)$. The next result shows that ω-ub-FA_{ω_1} for Hechler forcing implies the negation of the continuum hypothesis.

Lemma 5.25. *Let $\mathbb{P}$ denote Hechler forcing. If* ω-ub-$\mathsf{FA}_{\mathbb{P}}$ *holds, then the size of any unbounded family is at least ω_2.*

Proof. Towards a contradiction, suppose ω-ub-$\mathsf{FA}_{\mathbb{P}}$ holds and A is an unbounded family of size ω_1. Let us enumerate its elements as $\vec{x} = \langle x_\alpha : \alpha < \omega_1\rangle$. We define the following dense sets: For $\alpha < \omega_1$, we define a real y_α by taking a sort of "diagonal maximum" of $\vec{x}$. Let $\pi : \alpha \to \omega$ be a bijection and let

$$y_\alpha(n) = \max\{x_\gamma(n) : \pi(\gamma) \leq n\}.$$

It is easy to check that y_α is well defined, and that it eventually dominates x_γ for all $\gamma < \alpha$. We now define

$$D_\alpha = \{(s, x) \in \mathbb{P} : x \text{ eventually strictly dominates } y_\gamma\}$$

For $n < \omega$, let

$$E_n = \{(s, x) \in \mathbb{P} : \text{length}(s) \geq n\}$$

Now let $g \in V$ be a filter meeting unboundedly many D_α and all E_n. Since g meets all E_n, the first components of its conditions are arbitrarily long. Since all its elements are compatible, this means that the

[13] This assumption is equivalent to non(null) $= 2^{\aleph_0}$. It follows from MA, but not from $\mathsf{FA}_{\text{random}}$ by known facts about Cichon's diagram.

union $\cup\{s : (s,x) \in g\}$ is a real y. And y must eventually strictly dominate x for every $(s,x) \in g$. But there are unboundedly many α such that g meets D_α. For any such D_α, then, we have $(s,x) \in g$ where x eventually strictly dominates y_α. Hence, y must eventually strictly dominate unboundedly many y_α and hence every $x \in A$. But A was assumed to be unbounded. □

5.2.4. Suslin trees

A Suslin tree is a tree of height ω_1, with no uncountable branches or antichains. The existence of Suslin trees is not provable from ZFC, but follows from $\diamondsuit_{\omega_1}$. We can of course think of a Suslin tree T as a forcing; it will add a cofinal branch through the tree. We use Suslin trees as test cases for the weakest principles defined above. As expected, we can show that $\mathsf{stat\text{-}BN}^1_{T,\omega_1}$ fails in most cases.

Lemma 5.26. *Suppose T is a Suslin tree. Then $\mathsf{stat\text{-}BN}^\omega_{T,\omega_1}$ fails.*

Proof. Let $\sigma = \{\langle\alpha, p\rangle : \alpha < \omega_1, p \in T, \mathrm{height}(p) = \alpha\}$. It is easy to see that σ is ω bounded, and is forced to be equal to ω_1. But any filter $g \in V$ is a subset of a branch in V, and therefore countable. So σ^g is not stationary, or even unbounded. □

Corollary 5.27. *Suppose that a Suslin tree exists. Then there exists a Suslin tree T such that $\mathsf{stat\text{-}BN}^1_{T,\omega_1}$ fails.*

Proof. Let T be any Suslin tree. By the previous lemma we know that $\mathsf{stat\text{-}BN}^\omega_{T,\omega_1}$ fails. But then by Corollary 3.13, T contains a subtree S such that $\mathsf{stat\text{-}BN}^1_{S,\omega_1}$ fails. □

This also tells us that $\mathsf{stat\text{-}BN}^1_{\mathbb{P},\omega_1}$ is not equivalent to $\mathsf{stat\text{-}BFA}^1_{\mathbb{P},\omega_1}$, since the latter is trivially provable for any forcing in ZFC.

In fact, if we assume $\diamondsuit_{\omega_1}$ (which is somewhat stronger than the existence of a Suslin tree, see [18, Section 3]) then we can do better than this: we can show that $\mathsf{stat\text{-}BN}^1_{\omega_1}$ fails for every Suslin tree.

Lemma 5.28. *Suppose $\diamondsuit_{\omega_1}$ holds. If T is a Suslin tree, then $\mathsf{stat\text{-}BN}^1_{T,\omega_1}$ fails.*

Proof. Let (A_γ) be the sequence given by $\diamondsuit_{\omega_1}$. That is, let it be such that $A_\gamma \subseteq \gamma$ and for any $S \subseteq \omega_1$, the set $\{\gamma < \omega_1 : S \cap \gamma = A_\gamma\}$ is stationary. We build up a rank 1 name $\sigma = \{(\check{\alpha}, p) : \alpha < \gamma, p \in B_\alpha\}$ recursively as follows.

Suppose we have defined B_γ for all $\gamma < \alpha$. Consider $\bigcup_{\gamma\in A_\alpha} B_\gamma$. If this union is predense, then we let $B_\alpha = \emptyset$. Otherwise, choose a condition $p \in T$, sitting beyond level α of the Suslin tree, such that p is incompatible with every element of that union. Let $B_\alpha = \{p\}$.

If G is a generic filter, then every club $C' \subseteq \omega_1$ in $V[G]$ contains a club $C \in V$. Hence, to show that $T \Vdash$ "σ is stationary" we only need to show that for every club $C \in V$, the set $\bigcup_{\alpha\in C} B_\alpha$ is predense. Suppose for some club C that is not the case. For stationarily many α, we have that $C \cap \alpha = S_\alpha$ and hence the union we are looking at in defining B_α is $\bigcup_{\gamma\in A_\alpha} B_\gamma = \bigcup_{\gamma\in C\cap\alpha} B_\gamma$. Hence, the union is not predense, and B_α contains an element that is incompatible with every element of $\bigcup_{\gamma\in C\cap\alpha} B_\gamma$. But this is true for unboundedly many such α, so this gives us an ω_1 long sequence of pairwise incompatible conditions, i.e. an uncountable antichain. Since a Suslin tree is by definition c.c.c., this is a contradiction. Hence $T \Vdash$ "σ is stationary".

But now let $g \in V$ be a filter. By extending it if necessary, without loss of generality we can assume g is a maximal branch of the tree. Since $g \in V$, we know that g is countable, so let the supremum of the heights of its elements be γ. Let $\alpha > \gamma$, and let $q \in g$. Since B_α is at most a singleton $\{p\}$ with $\mathrm{ht}(p) \geq \alpha > \gamma > \mathrm{ht}(q)$, and since T is atomless, we know there is some $r \leq q$ with $r \Vdash \alpha \notin \sigma$. Hence $q \nVdash \alpha \in \sigma$. Since this is true for all $q \in g$, it follows that $\alpha \notin \sigma^{(g)}$. Hence far from being stationary, $\sigma^{(g)}$ is not even unbounded! □

So (assuming the existence of Suslin trees) there are certainly some Suslin trees in which $\mathsf{stat\text{-}BN}^1$ fails. And with strong enough assumptions, we can show that $\mathsf{stat\text{-}BN}^1$ fails for every tree. So it's natural to ask:

Question 5.29. Can we show in ZFC that $\mathsf{stat\text{-}BN}^1_{T,\omega_1}$ fails for every Suslin tree T?

Note that we can show the failure of $\mathsf{ub\text{-}BN}^1_{T,\omega_1}$ for any Suslin tree. Enumerate its level α elements as $\{p_{\alpha,n} : n \in \omega\}$. Now let

$$\sigma = \{(\check{\beta}, p_{\alpha,n}) : \alpha < \omega_1, n \in \omega, \beta = \omega \cdot \alpha + n\}$$

Then σ is forced to be unbounded but if $g \in V$ is such that σ^g is unbounded, then g defines an uncountable branch through T.

5.2.5. Club shooting

The next lemma is a counterexample to the implication $\mathsf{club\text{-}BFA}^\lambda_\kappa \Rightarrow \mathsf{club\text{-}BN}^\lambda_\kappa$ in Fig. 3. It is open whether there is such a counterexample for complete Boolean algebras.

Suppose that S is a stationary and co-stationary subset of ω_1. Let $\mathbb{P}_S$ denote the forcing that shoots a club through S. Its conditions are closed bounded subsets of S, ordered by end extension.

Lemma 5.30.

(1) $\mathsf{BFA}^\omega_{\mathbb{P}_S,\omega_1}$ *holds.*
(2) $\mathsf{club\text{-}BN}^1_{\mathbb{P}_S,\omega_1}$ *fails.*

In particular, for no $1 \leq \lambda \leq \omega$ *does* $\mathsf{BFA}^\lambda_{\mathbb{P}_S,\omega_1}$ *imply* $\mathsf{club\text{-}BN}^\lambda_{\mathbb{P}_S,\omega_1}$.

Proof. (1): We claim that every maximal antichain $A \neq \{1_{\mathbb{P}_S}\}$ is uncountable. (This shows that $\mathsf{BFA}^\omega_{\mathbb{P}_S,\omega_1}$ holds vacuously.) To see this, suppose that A is countable. Let $\alpha = \sup\{\min(p) : p \in A\}$ and find some $\beta > \alpha$ in S. Then $q = \{\beta\}$ is incompatible with all $p \in A$, so A cannot be maximal.

(2): $\sigma = \check{S}$ is 1-bounded and $\mathbb{P}_S \Vdash$ "σ contains a club". But for every filter g, $\sigma^g = S$ does not contain a club, since S is co-stationary. $\square$

6. Conclusion

The above results show that often, name principles are equivalent to forcing axioms. This provides an understanding of basic name principles $\mathsf{N}_{\mathbb{P},\kappa}$ and of simultaneous name principles for Σ_0-formulas. For bounded names, the results provide new characterisations of the bounded forcing axioms BFA^λ for $\lambda \geq \kappa$. Name principles are closely related with generic absoluteness and can be used to reprove Bagaria's equivalence between bounded forcing axioms of the form BFA^κ and generic absoluteness principles. Bagaria's result has been recently extended by Fuchs [10]. He introduced a notion of $\Sigma^1_1(\kappa,\lambda)$-absoluteness for cardinals $\lambda \geq \kappa$ and proved that it is equivalent to $\mathsf{BFA}^\lambda_\kappa$. It remains to see if this can be derived from our results.

Several problems about the unbounded name principle $\mathsf{ub\text{-}FA}_\kappa$ remain unclear. The results in Lemmas 4.25 and 5.1 about obtaining (bounded) forcing axioms from $\mathsf{ub\text{-}FA}_\kappa$ for forcings that do not add reals or $<\kappa$-sequences, respectively, hint at possible generalisations (see Question 4.26). For forcings which add reals, we have that $\mathsf{ub\text{-}FA}_{\omega_1}$ is trivial for all σ-linked forcings and implies FA_{ω_1} for random forcing. In all these cases, $\mathsf{ub\text{-}FA}_{\omega_1}$ and $\mathsf{stat\text{-}FA}_{\omega_1}$ are either both trivial or both equivalent to FA_{ω_1}. Can we separate $\mathsf{ub\text{-}FA}_{\omega_1}$ from $\mathsf{stat\text{-}FA}_{\omega_1}$ (see Question 3.20)? Can $\mathsf{ub\text{-}FA}_{\omega_1}$ be nontrivial but not imply FA_{ω_1}? It remains to study other forcings adding reals and Baumgartner's forcing [5, Section 3] (see Question 3.21).

The stationary name principle $\mathsf{stat\text{-}N}_{\omega_1}$ follows from the forcing axiom FA_{ω_1} for some classes of forcings. For example, for the class of c.c.c. forcings both $\mathsf{stat\text{-}N}_{\omega_1}$ and $\mathsf{FA}^+_{\omega_1}$ are equivalent to FA_{ω_1} by results of Baumgartner (see Lemma 5.9), Todorčević and Veličković [25] (see Lemma 5.6). In general, FA^+ goes beyond FA, since being stationary is not first-order over $(\kappa, \in)$. For example, for the class of proper forcings, PFA^+ is strictly stronger that PFA by results of Beaudoin [6, Corollary 3.2] and Magidor (see [22]). So FA^+ and BFA^+ do not fall in the scope of generic absoluteness principles, unless one artificially adds a predicate for the nonstationary ideal. Can one formulate PFA^+ as a generic absoluteness or name principle for a logic beyond first order? Some questions remain about the weak variant $\mathsf{stat\text{-}BN}^1_{\mathbb{P},\omega_1}$ of $\mathsf{stat\text{-}N}_{\omega_1}$. It is nontrivial for random forcing (see Lemma 5.24) and for Suslin trees (see Corollary 5.27). What is its relation with other principles? Does $\mathsf{stat\text{-}BN}^1_{\text{c.c.c.},\omega_1}$ imply MA_{ω_1}?

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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