Conjunctive reducibilities and completeness

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Abstract

In this article we study the notion of completeness for conjunctive reducibilities. We investigate the relationship between c-completeness and r-completeness of computably enumerable (c.e.) sets with respect to various strong reducibilities $\leq_r$. By using simplicity properties of sets, we prove that there exist c.e. sets that are simultaneously Q-complete and bd-complete, yet fail to be c-complete. Similarly, there exist c.e. sets that are simultaneously Q-complete and $bwtt$-complete (respectively, btt-complete) but not c-complete. Furthermore, we study two restrictions of c-reducibility, namely c_1- and $c_{1,N}$-reducibility, and show that they are distinct on the c.e. sets. Nevertheless, we prove that the notions of completeness for c, c_1, and $c_{1,N}$ coincide.

Keywords: c-reducibility; c_1-reducibility; $c_{1,\mathrm{N}}$-reducibility

1 Introduction

In this paper we study conjunctive reducibility (c-reducibility), first proposed by Jockusch [10], and its subreducibility (known as $c_{1,N}$-reducibility), suggested by Bulitko [2]. Conjunctive reducibility is a *positive reducibility*, i.e. a notion of computability relative to an external oracle in which the computing agent makes use only of positive information about the oracle: informally, a set A is *c-reducible* (denoted by $A \leq_c B$) to a set B if membership in A is determined by a finite conjunction of positive questions of membership in B. In this sense, it is a restricted version of enumeration reducibility, although it provides a degree structure whose least degree consists exactly of the decidable sets. This marks a fundamental difference from the most commonly studied positive reducibilities, whose least degree consists instead of the computably enumerable (c.e.) sets.

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Conjunctive reducibility is usually formalized as follows. If A, B are sets of numbers, then $A \leq_c B$ if there exists a computable function f such that for all $x, y \in \omega$:

$$x \in A \iff D_{f(x)} \subseteq B.$$

(Throughout the paper the symbol ω denotes the set of natural numbers, and the symbol D_u denotes the finite set with canonical index u.) Interest in c-reducibility, its weaker variant known as Q-reducibility (see Definition 2.4), and the algebraic structure of their respective degrees grew substantially following a series of unexpected results by Dobritsa, Belegradek, and Ziegler, which revealed surprisingly close connections between these reducibilities and combinatorial group theory. Indeed, by extending from Q- to c-reducibility a famous theorem due to Dobritsa [8], Ziegler [24, Satz 1] showed that every c-degree contains the word problem of a 2-generated group. Extending previous work of McIntyre [13], Belegradek [1, Theorem 1.1] showed that Q-reducibility of word problems of finitely generated c.e. groups corresponds to relative embeddability into algebraically closed groups, namely if H_1, H_2 are finitely generated c.e. groups with H_2 non-trivial, then the word problem of H_1 is Q-reducible to the word problem of H_2 if and only if H_1 is embeddable in any algebraically closed group in which H_2 is embeddable. Degtev [6] proved that the upper semilattice R_c of the c.e. c-degrees is not distributive and is not elementarily equivalent to the structure R_r of the c.e. r-degrees, with $r \in \{tt, l, p, d\}$ (cf. [6], to which the reader is referred also for the meaning of the acronyms). Moreover, Degtev [7] proved that every maximal set has minimal c-degree. If, in addition, a c-reduction f from A to B satisfies

$$x \neq y \implies D_{f(x)} \cap D_{f(y)} = \emptyset,$$

then we say that *A is c_1-reducible to B* (notation $A \leq_{c_1} B$) and f is a *c_1-reduction from A to B*. This reducibility can naturally be regarded as the injective version of c-reducibility. The current state of knowledge about conjunctive reducibilities, and in particular the algebraic structures of c- and c_1-c.e. degrees, can be found in [3–5]. A further restriction of c-reducibility, more precisely of c_1-reducibility, is defined as follows. A set A is *$c_{1,N}$-reducible* to a set B (denoted $A \leq_{c_{1,N}} B$) if there is a c_1-reduction f from A to B such that the set

$$\bigcup_{x \in \omega} D_{f(x)}$$

is computable. In this case we say that f is a *$c_{1,N}$-reduction from A to B*. It is obvious that $\leq_{c_{1,N}} \subseteq \leq_{c_1}$. This restriction on c_1-reducibility, according to which only those c_1-reductions whose set of all possible queries is decidable are allowed to perform oracle computations, is in line with similar restrictions proposed for other classical reducibilities by Bulitko [2], and even by Rogers [21, Exercise 7.34].

This article is devoted to the study of completeness notions for conjunctive reducibilities. Section 2 contains some remarks on the relationships between c-completeness and r-completeness of c.e. sets, where $\leq_r$ ranges over various strong reducibilities. In particular, by using simplicity properties of sets, we show (Theorem 2.11) that there exists a c.e. set that is simultaneously Q-complete and *bd*-complete, but not c-complete; moreover, Corollary 2.13 shows that there exists a c.e. set that is simultaneously Q-complete and *bwtt*- (*btt*-) complete but not c-complete. In Section 3, we prove that on c.e. sets $c_{1,N}$-reducibility is strictly stronger than c_1-reducibility. Finally, in Section 4 we show that the notions of c-, c_1-, and $c_{1,N}$-completeness coincide.

Unless otherwise specified, our notations and terminology concerning computability theory are standard and can be found in the classical textbooks [21] and [23]. The complement $\omega \setminus X$ of a set $X \subseteq \omega$ will be denoted by $\overline{X}$, and its cardinality by $|X|$. For a given reducibility r, and a c.e. set A,

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we say that A is r-complete if $B \leq_r A$ for every c.e. set B. The familiar symbols $\leq_m$ and $\leq_T$ denote respectively *m-reducibility* ($A \leq_m B$ if there exists a computable function f, called an *m-reduction from A to B*, such that $x \in A$ if and only if $f(x) \in B$ for every number x) and *T-reducibility* ($A \leq_T B$ if the characteristic function of A can be computed by some oracle Turing machine with oracle B). Finally, $A \leq_1 B$ if there exists a one-to-one m-reduction from A to B.

2 Relationships between c-, Q- and bd-complete sets

This section investigates the relationships between the notions of r-completeness for various reducibilities r. In this section, we prove that there exists a c.e. set that is simultaneously Q-complete and bd-complete, but not c-complete. Nowhere simple sets, introduced by Shore [22], are an important tool for the investigation carried out in this section.

DEFINITION 2.1.
A set A is *nowhere simple* if for every c.e. set B with $B \setminus A$ infinite, there is an infinite c.e. set $W \subseteq B \setminus A$.

Shore [22, Theorem 4] proved that every c.e. T-degree contains a nowhere simple set.

The following theorem, which is of independent interest, is useful for proving the main result of the section. To state the theorem, let us recall that a set A is *simple* if A is c.e., $\overline{A}$ is infinite and contains no infinite c.e. set. If A, B are sets of numbers then the *join* $A \oplus B$ of A, B is defined by

$$A \oplus B = \{2x : x \in A\} \cup \{2x+1 : x \in B\}.$$

THEOREM 2.2.
Let A be a simple set, B be an arbitrary set, C be a nowhere simple set and

$$A \leq_c B \oplus C.$$

Then $A \leq_c B$.

Proof. We first prove the following lemma, along the lines of a corresponding result for Q-reducibility, proved in [20, Lemma 3.3].

LEMMA 2.3.
Let A be a simple set, B be an arbitrary set, C be a nowhere simple set and $A \leq_c B \oplus C$ via f. Then, for every c.e. set W for which $W \setminus A$ is infinite, we have the following:

(1) $\left|\{2x+1 : 2x+1 \in \bigcup_{y \in W} D_{f(y)}\} \cap \overline{B \oplus C}\right| < \infty$;

(2) $\left|\{2x : 2x \in \bigcup_{y \in W} D_{f(y)}\} \cap \overline{B \oplus C}\right| = \infty$.

Proof of Lemma 2.3. First of all, notice that Claim (2) obviously follows from Claim (1). So, it is enough to prove Claim (1). Let $U = \bigcup_{y \in W} D_{f(y)}$, and let $V = U \cap (2\omega + 1)$. Throughout the proof of this lemma, for any set of numbers X, let $X^- = \{x : 2x+1 \in X\}$ and $X^+ = \{2x+1 : x \in X\}$. Assuming that the conditions of the lemma are satisfied, we first show that

$$\left|U \cap \overline{B \oplus C}\right| = \infty.$$

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Indeed, if this set were finite, then the set

$$\widetilde{W} = \left\{x \colon D_{f(x)} \cap \left(U \cap \overline{B \oplus C}\right) \neq \varnothing\right\}$$

would be c.e. and $W \setminus A \subseteq \widetilde{W}$ while $\widetilde{W} \cap A = \varnothing$, providing an infinite c.e. subset of $\overline{A}$, which contradicts the simplicity of A.

Assume that $|V \setminus B \oplus C| = \infty$. Then $\left|V^- \setminus C\right| = \infty$. Since C is nowhere simple, there exists an infinite c.e. set V_1 such that

$$V_1 \subseteq V^- \setminus C.$$

Then V_1^+ is infinite, $V_1^+ \subseteq \overline{B \oplus C}$ and $V_1^+ \subseteq V$.

Consider

$$V_2 = \{x \colon D_{f(x)} \cap V_1^+ \neq \varnothing\}.$$

Then V_2 is a c.e. set such that $V_2 \subseteq \overline{A}$ (as $x \in A$ implies $D_{f(x)} \subseteq 2\omega$). Moreover, V_2 is infinite since $V_1^+ \subseteq V$ is infinite and one need infinitely many x such that $\bigcup_x D_{f(x)}$ covers V_1^+. Such a c.e. set contradicts the assumption that A is simple. □

Let us now go back to the proof of Theorem 2.2. Recall that f is a computable function such that

$$(\forall x)\left[x \in A \iff D_{f(x)} \subseteq B \oplus C\right].$$

Let

$$R = (2\omega + 1) \cap \left(\bigcup_{y \in \omega} D_{f(y)}\right) \cap \overline{B \oplus C}.$$

By (1) of Lemma 2.3, $|R| < \infty$. Let

$$D_{f_1(x)} = \{y \colon 2y \in D_{f(x)}\}$$
$$R_1 = \{x \colon D_{f(x)} \cap R \neq \varnothing\}.$$

Since R_1 is an infinite c.e. subset of $\overline{A}$, by simplicity of A we have that $|R_1| < \infty$. Let f_2 be a computable function such that

$$D_{f_2(x)} = \begin{cases} D_{f_1(x)}, & \text{if } x \notin R_1 \\ \{a\}, & \text{if } x \in R_1, \end{cases}$$

where a is a fixed element of $\overline{B}$.

If $x \in A$, then $D_{f(x)} \subseteq B \oplus C$. Since $A \cap R_1 = \varnothing$, $D_{f_1(x)} \subseteq B$. If $x \notin A$, then either $x \in R_1$ (so $D_{f_2(x)} = \{a\} \nsubseteq B$) or $x \notin R_1$ (so $D_{f(x)} \nsubseteq B \oplus C$ being $x \notin A$, and $D_{f(x)} \cap R = \varnothing$, implying $D_{f_2(x)} = D_{f_1(x)} \nsubseteq B$). In all cases,

$$(\forall x)\left[x \in A \iff D_{f_2(x)} \subseteq B\right].$$

Thus, $A \leq_c B$. □

For the following lemma we need the following definition.

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DEFINITION 2.4.
A set A is said to be *Q-reducible* to a set B (in symbols: $A \leq_Q B$) if there exists computable function f such that for all x:

$$x \in A \iff W_{f(x)} \subseteq B.$$

If in addition f satisfies $W_{f(x)} \cap W_{f(y)} = \emptyset$ whenever $x \neq y$, then we say that *A is Q_1-reducible to B* (notation $A \leq_{Q_1} B$).

Clearly, Q-reducibility extends c-reducibility (and, analogously, Q_1 extends c_1), since $D_x = W_{\rho(x)}$ for a suitable computable function ρ.

LEMMA 2.5.
Let A_0 and A_1 be c.e. sets. Then $A_0 \oplus A_1$ is a nowhere simple set if and only if A_0 and A_1 are both nowhere simple.

PROOF. In [20] it is proved that if A_0, A_1 are nowhere simple, then $A_0 \oplus A_1$ is nowhere simple. Conversely, if $A_0 \oplus A_1$ is nowhere simple, since $A_0 \leq_Q A_0 \oplus A_1$ and $A_1 \leq_Q A_0 \oplus A_1$, it follows by Proposition 3.5 in [20] that A_0 and A_1 are nowhere simple. □

DEFINITION 2.6.
A set A is said to be *bd-reducible to a set B* (in symbols: $A \leq_{bd} B$) if there exist a computable function f and $n \in \omega$ such that for all x:

$$x \in A \iff D_{f(x)} \cap B \neq \varnothing$$

and

$$\left|D_{f(x)}\right| \leq n.$$

The acronym *bd* stands for *bounded disjunctive reducible*: a set A is *disjunctive reducible* to a set B if there exists a computable function f such that $x \in A \iff D_{f(x)} \cap B \neq \varnothing$, for every x. Notice that $A \leq_d B$ if and only if $\overline{A} \leq_c \overline{B}$. Analogously to bounded disjunctive reducibility $\leq_{bd}$ one could define also *bounded conjunctive reducibility* $\leq_{bc}$, i.e. $A \leq_{bc} B$ if $A \leq_c B$ via some computable function f such that there exists n with $|D_{f(x)}| \leq n$, for every x.

PROPOSITION 2.7.
There exists a *bd*-complete nowhere simple set.

PROOF. Let A be a c.e. *bd*-complete set. By [22, Theorem 2], A can be split into two disjoint nowhere simple sets A_0, A_1. Clearly, $A \equiv_{bd} A_0 \oplus A_1$, and by the previous Lemma, $A_0 \oplus A_1$ is nowhere simple. □

For the following proposition, and its proof, we recall the following two definitions.

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DEFINITION 2.8.
A set of numbers A is *semicomputable* if there is a two-variable computable function f such that for every $x, y \in \omega$, we have $f(x,y) \in \{x,y\}$ and if $\{x,y\} \cap A \neq \emptyset$ then $f(x,y) \in A$.

And

DEFINITION 2.9.
A c.e. set A is *hypersimple* if $\overline{A}$ is infinite, and for every computable function f, such that $D_{f(x)} \cap D_{f(y)} = \emptyset$ whenever $x \neq y$, we have that $D_{f(x)} \subseteq A$ for at least one x.

(Any collection of finite sets $\{D_{g(x)}\}_{x\in\omega}$, where g is computable and the sets are pairwise disjoint is called a *disjoint strong array*.)

PROPOSITION 2.10.
There exists a Q-complete hypersimple set.

PROOF.Dekker (see Theorem 9.XVI in [21]) proved there exists a T-complete semicomputable hypersimple set. Marchenkov [14, Lemma 1] showed that if B is a semicomputable c.e. set such that $B \neq \varnothing, \omega$ and A is a c.e. set then $A \leq_T B \Rightarrow A \leq_Q B$. Thus, Q-complete hypersimple sets do exist. □

As an application of Theorem 2.2, Proposition 2.7 and Proposition 2.10, we get:

THEOREM 2.11.
There exists a c.e. set that is simultaneously Q-complete and *bd*-complete, but not *c*-complete.

PROOF. Let A be a Q-complete hypersimple set (such set exists by Proposition 2.10), and let B be a *bd*-complete nowhere simple set (such a set exists by Proposition 2.7). Consider $A \oplus B$. This set is Q-complete and *bd*-complete. Suppose $A \oplus B$ is *c*-complete. Let S be a *c*-complete simple set (such a simple set exists by Theorem 8.VIII in [21]). Then $S \leq_c A \oplus B$. By Theorem 2.2, $S \leq_c A$, so A is *c*-complete. However, no hypersimple set can be *wtt*-complete, let alone *c*-complete [10]. Contradiction. □

DEFINITION 2.12.
A set A is *weak truth-table reducible* (*wtt*-reducible) to a set B (in symbols: $A \leq_{wtt} B$) [21], if $(\exists z)[\chi_A = \varphi_z^B]$ and there is a computable f such that, for every x, the finite set $D_{f(x)}$ consists exactly of the numbers y queried during the computation $\varphi_z^B(x)$, i.e. the numbers y for which the oracle is asked to determine the value $B(y)$. Here, χ_A denotes the characteristic function of A. If, in addition, there exists n such that $\left|D_{f(x)}\right| \leq n$ for all x, then $A \leq_{bwtt} B$. The notion of *bwtt*-reducibility was introduced by Lachlan [12].

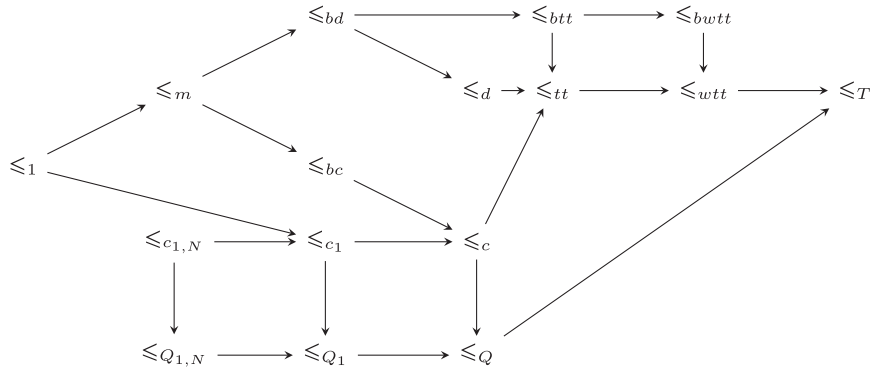
Lachlan [12] proved that *bwtt*-complete sets are *btt*-complete (even *bd*-complete). Therefore, from Theorem 2.11, we have the following:

COROLLARY 2.13.
There exists a c.e. set that is simultaneously Q-complete and *bwtt*- (*btt*-) complete but not *c*-complete.

PROOF. Immediate. □

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We conclude the section by summarizing the inclusion relationships on c.e. sets between the reducibilities considered in this section and the previous one .



The arrows indicate proper inclusions. Apart from those obtained by transitivity of $\subseteq$, no other inclusions hold. For a verification of this summary, the reader is referred to [11], [15], and [16]. The claim $\leq_{c_{1,N}} \subset \leq_{c_1}$ is proved in next section, Theorem 3.4. Notice that the inclusion $\leq_Q \subseteq \leq_T$ does not hold in general if the sets are not c.e. .

3 On the difference between c_1-reducibility and $c_{1,N}$-reducibility on c.e. sets

It is well known that, on c.e. sets, c-reducibility and c_1-reducibility do not coincide. In fact, there are c.e. c-degrees that contain infinitely many c_1-degrees (see, for instance, [3, 4] and [5]), and the poset of c.e. c-degrees is not elementarily equivalent to the poset of c_1-degrees (see [3]). In this section, we show that c_1-reducibility and $c_{1,N}$-reducibility do not coincide on c.e. sets.

Similarly to the way we define $c_{1,N}$-reducibility starting from c_1-reducibility, given two sets A, B we say that $A \leq_{Q_{1,N}} B$, if $A \leq_{Q_1} B$ via a computable function f such that $\bigcup_{x\in\omega} W_{f(x)}$ is a computable set.

THEOREM 3.1.
Let $A \leq_{Q_1} B$, where A and B are c.e. sets. Then there exists a c.e. set C such that

$$A \leq_{Q_{1,N}} C \leq_1 B.$$

PROOF. Assume that A and B are c.e. sets and $A \leq_{Q_1} B$ via f. Let g be a one-to-one computable function such that

$$\text{range}(g) = \bigcup_{x\in\omega} W_{f(x)}.$$

Consider the c.e. set $C = g^{-1}(B)$ and let h be a computable function such that for all x,

$$W_{h(x)} = g^{-1}(W_{f(x)}).$$

Then, for all x, y:

- $x \in A \Leftrightarrow W_{h(x)} \subseteq C$,

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- $x \neq y \Rightarrow W_{h(x)} \cap W_{h(y)} = \emptyset$,
- The set $\bigcup_{x\in\omega} W_{h(x)}$ is computable (in fact, $\bigcup_{x\in\omega} W_{h(x)} = \omega$).

Furthermore, $C \leq_1 B$ via g. Thus, $A \leq_{Q_{1,N}} C \leq_1 B$. □

The following proposition is a direct corollary of Proposition 3.1 of [19]. Recall that a c.e. set A is *r-maximal* if A is infinite and there is no computable set R such that $|R \cap A| = |\overline{R} \cap A| = \infty$.

PROPOSITION 3.2.
Let A and B be c.e. sets. If B is an r-maximal set and $A \leq_{Q_{1,N}} B$, then A is an r-maximal set.

PROOF. Immediate from Proposition 3.1 of [19]. □

THEOREM 3.3.
There exist c.e. sets A, B, C such that B is r-maximal, A is not r-maximal, C is not r-maximal, and $A \leq_{Q_{1,N}} C <_1 B$.

PROOF. Let B be a Q_1-complete r-maximal set (such a set exists by Corollary 2 of [18] and the fact that Q-completeness and Q_1-completeness coincide [9]) and let A be a non-r-maximal c.e. set. Then $A \leq_{Q_1} B$. By Theorem 3.1, there exists a c.e. set C such that $A \leq_{Q_{1,N}} C \leq_1 B$. Since A is not r-maximal and $A \leq_{Q_{1,N}} C$, the set C cannot be r-maximal by Proposition 3.2. Since C is not r-maximal, but B is r-maximal, they are not recursively isomorphic, which means that $B \not\leq_1 C$. Thus $A \leq_{Q_{1,N}} C <_1 B$. □

THEOREM 3.4.
The reducibilities c_1 and $c_{1,N}$ do not coincide on the c.e. sets.

PROOF. By Theorem 3.3, there exist c.e. sets C and B such that C is not r-maximal, B is r-maximal, and $C <_1 B$. Since 1-reducibility implies c_1-reducibility, it follows that $C \leq_{c_1} B$. On the other hand, by Proposition 3.2, since B is r-maximal and C is not r-maximal, we have that $C \not\leq_{c_{1,N}} B$. Therefore, there is a clear difference between c_1 and $c_{1,N}$ on the class of c.e. sets. □

4 c_1-completeness and $c_{1,N}$-completeness coincide

In this section we show that the notions of c-completeness, c_1-completeness and $c_{1,N}$-completeness coincide. Since we know already [17] that c-completeness implies c_1 completeness, to prove our claim it is enough to show (see Theorem 4.3) that c_1-completeness implies $c_{1,N}$-completeness. However, we prefer to precede Theorem 4.3 with a new proof (namely, the proof of Lemma 4.1) of the fact that c-completeness implies c_1-completeness. We do this mainly for pedagogical reasons. First, the arguments in the two cases are similar and complement each other. Moreover, as far as we know, our proof of Lemma 4.1 gives a novel derivation of the fact that c-completeness implies c_1-completeness, introducing a technique that may be adaptable to new similar contexts in which, for a given reducibility r and its "injective" version r_1, one aims to prove that r-completeness implies r_1-completeness.

LEMMA 4.1.
[17] If A is c-complete then A is c_1-complete.

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PROOF. Let A be a given c.e. c-complete set, and let B be any c.e. set: we want to show that $B \leq_{c_1} A$.

Let us construct a family $\{V_e : e \in \omega\}$ of uniformly c.e. sets (i.e. $V_e = W_{k(e)}$ for some computable function k) and consider

$$W = \bigoplus_{e\in\omega} V_e \left(= \bigcup_e \{e\} \times V_e \right).$$

Since W is c.e. and A is c-complete, we have that $W \leq_c A$. For each e, we describe a strategy $\mathcal{I}_e$ that uniformly builds a c.e set V_e and a corresponding partial (in fact finite or total) computable function $\widehat{f_e}$, such that if φ_e is a c-reduction from W to A then $\widehat{f_e}$ is a c_1-reduction from B to A. □

REMARK 4.2.
We will work with computable approximations $\{\varphi_{e,s}\}_{e,s\in\omega}$ to the partial computable functions, such that $\varphi_{e,0}(x)$ is undefined for every x; see [23, p.17]. To simplify notation, we use the symbol f_e to denote the partial function $f_e(y) = \varphi_e(\langle e, y\rangle)$, and accordingly $f_{e,s}(y) = \varphi_{e,s}(\langle e, y\rangle)$. Notice that if φ_e is a c-reduction from W to A, then f_e is a c-reduction from V_e to A. We will also use computable finite approximations $\{A_s\}_{s\in\omega}$ and $\{B_s\}_{s\in\omega}$ to the c.e. sets A, B (in the usual meaning of [23]). Sets of numbers will often be identified with their characteristic functions. Therefore, if X is a set of numbers and y is a number, then $X(y)$ denotes the value of the characteristic function of X at y.

The strategy $\mathcal{I}_e$ and the definitions of V_e and $\widehat{f_e}$. The strategy $\mathcal{I}_e$ builds a c.e. set V_e, under the assumption that φ_e is a c-reduction from W to A and thus f_e is a c-reduction from V_e to A. By suitably coding B into V_e, we try to exploit this c-reduction to obtain a c_1-reduction $\widehat{f_e} : V_e \leq_{c_1} A$, so that by the coding of B into V_e we achieve $B \leq_{c_1} A$. If at some stage we see that our attempts at making $\widehat{f_e}$ a suitable c_1-reduction fail, then our definition of V_e will force φ_e not to be a c-reduction from W to A. In this case we say that we *abandon* the strategy $\mathcal{I}_e$, and we never again modify the parameters of the strategy. Since in the end there must be some e such that φ_e is a c-reduction from W to A, then, for this e, we have that $V_e \leq_{c_1} A$ and thus $B \leq_{c_1} A$.

The strategy $\mathcal{I}_e$ works with three parameters, namely $a_e(s)$ (a number), $\widehat{f}_{e,s}$ (a finite function), and $V_{e,s}$ (a finite set). The parameters $\widehat{f}_{e,s}$ and $V_{e,s}$ will be finite approximations in the usual way to a partial (in fact finite or total) computable function $\widehat{f_e}$ and a c.e. set V_e. If $\mathcal{I}_e$ has not as yet been abandoned by stage s, then at stage $s+1$ the strategy updates its parameters for which the following inductive assumptions hold:

(1) $\text{domain}(\widehat{f}_{e,s}) = [0, a_e(s))$, and $[0, a_e(s)) \subseteq \text{domain}(f_{e,s})$;
(2) $V_{e,s}(y) = B_s(y)$, for every $y < a_e(s)$.

At this stage $s+1$ we also decide whether or not to abandon $\mathcal{I}_e$.

Stage 0) Let $a_e(0) = 0$ and $V_{e,0} = \widehat{f}_{e,0} = \emptyset$.

Stage $s+1$) If we have already decided at a previous stage to abandon $\mathcal{I}_e$, then do nothing and leave the values of the parameters for $\mathcal{I}_e$ at $s+1$ to be the same as at stage s. Otherwise, take the first available action from the following list,

(1) if we see that there exists some $y < a_e(s)$ such that $y \notin V_{e,s}$ and $D_{\widehat{f}_{e,s}(y)} \subseteq A_s$, then put all $z < y$ into V_e by letting $V_{e,s+1} = V_{e,s} \cup [0, y)$, and abandon $\mathcal{I}_e$ (hence, $y \notin V_e$);
(2) if $f_{e,s}(a_e(s))$ is still undefined, then go to next stage, and leave the values of the parameters at $s+1$ to be the same as at stage s;
(3) define $a_e(s+1) = a_e(s) + 1$, and distinguish the following two cases:

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Case (i): $D_{f_e(a_e(s))} \subseteq \bigcup_{y<a_e(s)} D_{f_e(y)}$. In this case, put each $y < a_e(s)$ into V_e by letting $V_{e,s+1} = V_{e,s} \cup [0, a_e(s))$, abandon $\mathcal{I}_e$ (hence, $a_e(s) \notin V_e$).

Case (ii): Otherwise, let $\widehat{f_e}(a_e(s))$ be such that

$$D_{\widehat{f_e}(a_e(s))} = D_{f_e(a_e(s))} \setminus \bigcup_{y<a_e(s)} D_{f_e}(y).$$

(Updating of V_e.) If we have not abandoned $\mathcal{I}_e$, then at the end of stage $s+1$ update $V_{e,s+1}(y) = B_s(y)$ for every $y < a_e(s+1)$.

Verification. Since $W \leq_c A$, let e be such that φ_e is a total computable function that c-reduces W to A. We claim in this case that the strategy $\mathcal{I}_e$ is never abandoned. Otherwise, there is some y (coming from Case (1), or $y = a_e(s)$, for some s, coming from Case (3)(i) of the construction), such that $\langle e, y\rangle \notin W$ (since $y \notin V_e$) but $D_{\varphi_e(\langle e,y\rangle)} \subseteq A$, contradicting that f_e is a c-reduction. Indeed, to see that $D_{\varphi_e(\langle e,y\rangle)} \subseteq A$, we first notice that in both cases $\bigcup_{z<y} D_{\varphi_e(\langle e,z\rangle)} \subseteq A$ being $[0, y) \subseteq V_e$ and thus $\{e\} \times [0, y) \subseteq W$. On the other hand, in Case (1) since $D_{\widehat{f_e}(y)} = D_{f_e(y)} \setminus \bigcup_{z<y} D_{f_e(z)}$, we have

$$D_{\varphi_e(\langle e,y\rangle)} = D_{f_e(y)} \subseteq \bigcup_{z<y} D_{f_e(z)} \cup D_{\widehat{f_e}(y)} \subseteq A.$$

In Case (3)(i), we have

$$D_{\varphi_e(\langle e,y\rangle)} = D_{f_e(y)} \subseteq \bigcup_{z<y} D_{f_e(z)} \subseteq A.$$

We now show that $\widehat{f_e}$ is a c-reduction from B to A. If $y \in B$ then $y \in V_e$ by the step-by-step updating of $V_{e,s}$. Therefore, the pair $\langle e, y\rangle$ lies in W and thus $D_{\varphi_e(\langle e,y\rangle)} = D_{f_e(y)} \subseteq A$, but then $D_{\widehat{f_e}(y)} \subseteq A$ since $D_{\widehat{f_e}(y)} \subseteq D_{f_e(y)}$. If now $y \notin B$ then $y \notin V_e$ since $\mathcal{I}_e$ cannot force $y \in V_e$ without being abandoned, but $\mathcal{I}_e$ is never abandoned as we have already seen. On the other hand, it cannot be $D_{\widehat{f_e}(y)} \subseteq A$, otherwise at some stage y would be a candidate for abandoning $\mathcal{I}_e$.

In the end we need to show that $\widehat{f_e}$ is in fact a c_1-reduction, i.e. it satisfies $D_{\widehat{f_e}(x)} \cap D_{\widehat{f_e}(y)} = \emptyset$ if $x \neq y$. But this follows from the fact that by Case (3)(ii) of the construction, for every x we have $D_{\widehat{f_e}(x)} \subseteq D_{f_e(x)}$, but $D_{\widehat{f_e}(x)} \cap \bigcup_{y<x} D_{f_e(y)} = \emptyset$, and thus $D_{\widehat{f_e}(x)} \cap D_{\widehat{f_e}(y)} = \emptyset$ if $x > y$.

THEOREM 4.3.
If A is c_1-complete then it is $c_{1,N}$-complete.

PROOF. Let A be a c_1-complete c.e. set, and let B be any c.e. set. We keep the notations and the terminology of the previous proof.

Uniformly in e we will define a c.e. set V_e, and a partial (in fact, finite or total) computable function $\widehat{f_e}$.

The strategy $\mathcal{I}_e$ and the definitions of V_e and $\widehat{f_e}$. The strategy $\mathcal{I}_e$ defining V_e and $\widehat{f_e}$ is similar to the one in the proof of Lemma 4.1. A slight complication is the delayed coding of B into V_e. Suppose that the strategy has defined $\widehat{f_e}(y)$ for all $y < x$, and has already coded into V_e the values $B(y)$ for all $y < x$. In order to define $\widehat{f_e}(x)$ and code $B(x)$ into V_e, the strategy waits for a number a such that $\min(D_{f_e(a)}) > \max\left(\bigcup_{y<x} D_{\widehat{f_e}(y)}\right)$, defines $\widehat{f_e}(x) = f_e(a)$, and makes the number a to be the coding location $b_e(x)$ of x in V_e. If the coding is more complicated, the rest of the construction is simpler

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than the one in the proof of Lemma 4.1, since, as we will see, nothing goes wrong and there is ever no need to abandon $\mathcal{I}_e$, and make $b_e(y) \in V_e$ if $y \notin B$.

At stage s we have this time the following parameters for $\mathcal{I}_e$: $a_e(s), x_e(s), V_{e,s}, b_{e,s}$. As we see, with respect to the proof of the previous lemma, we have two additional parameters, namely a number $x_e(s) \leq a_e(s)$ and a finite function $b_{e,s}$.

Stage 0) Let $V_{e,0} = \widehat{f}_{e,0} = b_{e,0} = \emptyset$, $x_e(0) = a_e(0) = 0$.

Stage $s+1$) The strategy updates its parameters for which the following assumptions inductively hold:

(1) $\text{domain}(\widehat{f}_{e,s}) = \text{domain}(b_e(s)) = [0, x_e(s))$ and $[0, a_e(s)) \subseteq \text{domain}(f_{e,s})$;
(2) $V_{e,s}(b_{e,s}(y)) = B_s(y)$, for every $y < x_e(s)$, and $V_{e,s}(x) = 0$ otherwise.

We distinguish the following cases:

(1) If $f_{e,s}(a_e(s))$ is still undefined then go to next stage, and leave the values of the parameters for $\mathcal{I}_e$ at $s+1$ to be the same as at stage s.
(2) If $f_{e,s}(a_e(s))$ converges, and

$$\min(D_{f_e(a_e(s))}) > \max\left(\bigcup_{y<x_e(s)} D_{\widehat{f}_e(y)}\right), \qquad (*)$$

then let

$$\widehat{f}_e(x_e(s)) = f_e(a_e(s)),$$

and define $x_e(s+1) = x_e(s)+1$, $a_e(s+1) = a_e(s)+1$. Code $B_s(x)$ (where for simplicity we write $x = x_e(s)$) into V_e, by defining $V_{e,s+1}(a_e(s)) = B_s(x)$: we thus view $a_e(s)$ as the coding location of $B(x)$ in V_e, and we record this by defining $b_{e,s}(x) = a_e(s)$.

(3) Otherwise, let $a_e(s+1) = a_e(s)+1$, leave the values of the other parameters for $\mathcal{I}_e$ at $s+1$ to be the same as at stage s and go to the next stage.

(Updating of V_e.) Finally, update $V_{e,s+1}(b_{e,s}(y)) = B_s(y)$, for every $y < x_e(s)$.

Verification. Since $W \leq_{c_1} A$, let φ_e be a total computable function which c_1-reduces W to A. By injectivity of the c_1-reduction φ_e (i.e. $D_{\varphi_e(x)} \cap D_{\varphi_e(y)} = \emptyset$ if $x \neq y$) sooner or later our waits for desired numbers a satisfying $(*)$ will end. Therefore, $\widehat{f}_e$ and b_e are total.

If $y \in B$, then $b_e(y) \in V_e$ and therefore the pair $\langle e, b_e(y)\rangle$ lies in W, so that $D_{f_e(b_e(y))} \subseteq A$. Since $D_{\widehat{f}_e(y)} = D_{f_e(b_e(y))}$, we have $D_{\widehat{f}_e(y)} \subseteq A$.

Suppose now that $y \notin B$. In this case we have that $D_{\widehat{f}_e(y)} \not\subseteq A$, since $D_{\widehat{f}_e(y)} = D_{f_e(b_e(y))}$ and $b_e(y) \notin V_e$, so that $D_{f_e(b_e(y))} \not\subseteq A$.

It follows that $\widehat{f}_e$ is a total computable function that c-reduces B to A. Injectivity of the reduction (making $\widehat{f}_e$ a c_1-reduction) is immediate since it follows by $(*)$ that

$$x > y \Rightarrow \min(D_{\widehat{f}(x)}) > \max(D_{\widehat{f}(y)})$$

since $D_{\widehat{f}_e(y)}$ is one of the addenda of the union in $(*)$.

Finally, we show that the range of $\widehat{f}_e$ is decidable. To decide whether $z \in \bigcup_x D_{\widehat{f}_e(x)}$, compute the least x such that $z < \min(D_{\widehat{f}_e(x)})$. Then $z \in \bigcup_x D_{\widehat{f}_e(x)}$ if and only if $z \in \bigcup_{y<x} D_{\widehat{f}_e(y)}$. □

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References

[1] Belegradek OVC. Algebraically closed groups. *Algebra Log* 1974;**13**:135–43.
[2] Bulitko VK. On ways of characterizing complete sets. *Math USSR-Izv* 1992;**38**:225–49.
[3] Chitaia I, Omanadze R. Degree structures of conjunctive reducibility. *Arch Math Log* 2022;**61**:19–31.
[4] Chitaia I, Omanadze R, Sorbi A. Notes on conjunctive and quasi degrees. *J Log Comput* 2021;**31**:1317–29.
[5] Chitaia I, Omanadze R, Sorbi A. Conjunctive degrees and cylinders. *J Log Comput* 2025;**35**:81–94.
[6] Degtev AN. Several results on upper semilattices and m-degrees. *Algebra Log* 1979;**18**:420–30.
[7] Degtev AN. Comparison of linear reducibility with other reducibilities of tabular type. *Algebra Log* 1982;**21**:511–29.
[8] Dobritsa VN. On word problem of recursively definable groups. In: Palyutin EA (ed), *Proc. 3rd All-Union Conf. on Mathematical Logic*. Novosibirsk: Institute of Mathematics, Siberian Branch of the USSR Academy of Sciences, 1974, 63–65.
[9] Gill III JT, Morris PH. On subcreative sets and s-reducibility. *J Symb Log* 1974;**39**:669–77.
[10] Jockusch Jr CG. Relationships between reducibilities. *Trans Amer Math Soc* 1969;**142**:229–37.
[11] Kobzev GN. Recursively enumerable bw-degrees. *Mat Zametki* 1977;**21**:839–46.
[12] Lachlan AH. wtt-complete sets are not necessarily tt-complete. *Proc Amer Math Soc* 1975;**48**:429–34.
[13] Macintyre A. On algebraically closed groups. *Ann of Math* 1972;**96**:53–97.
[14] Marchenkov SS. A certain class of incomplete sets. *Math Notes* 1976;**20**:823–5.
[15] Odifreddi P. Strong reducibilities. *Bull Amer Math Soc (N.S.)* 1981;**4**:37–86.
[16] Odifreddi P. Classical recursion theory (volume I). In: *Studies in Logic and the Foundations of Mathematics*, vol. 125. Amsterdam: North-Holland, 1989.
[17] Omanadze RS. On the completeness of recursively enumerable sets. *Bull Acad Sci Georgian SSR* 1976;**81**:529–32. in Russian.
[18] Omanadze RS. Major sets, classes of simple sets, and Q-complete sets. *Math Notes* 2002;**71**: 90–7.
[19] Omanadze RS. Some properties of r-maximal sets and $Q_{1,N}$-reducibility. *Arch Math Log* 2015;**54**:941–59.
[20] Omanadze RS. On the connections between wtt- and Q-reducibility. *J Log Comput* 2019;**29**: 37–51.
[21] Rogers Jr H. *Theory of Recursive Functions and Effective Computability*, 2nd edn. Cambridge: MIT Press, 1987.

[22] Shore RA. Nowhere simple sets and the lattice of recursively enumerable sets. *J Symb Log* 1978;**43**:322–30.
[23] Soare RI. Recursively enumerable sets and degrees. In: *Perspectives in Mathematical Logic*, Omega Series. Heidelberg: Springer, 1987. https://doi.org/10.1007/978-3-662-02460-7.
[24] Ziegler M. Gruppen mit vorgeschriebenem Wortproblem. *Math Ann* 1976;**219**:43–51.

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