Relaxed area of graphs of piecewise Lipschitz maps in the strict BV-convergence

Giovanni Bellettini [a,b], Simone Carano [c], Riccardo Scala [a,*]

[a] *Dipartimento di Ingegneria dell'Informazione e Scienze Matematiche, Università di Siena, 53100 Siena, Italy*
[b] *International Centre for Theoretical Physics ICTP, Mathematics Section, 34151 Trieste, Italy*
[c] *Area of Mathematical Analysis, Modelling, and Applications, Scuola Internazionale Superiore di Studi Avanzati "SISSA", Via Bonomea, 265 34136 Trieste, Italy*

ARTICLE INFO

ABSTRACT

We compute the relaxed Cartesian area in the strict BV-convergence on a class of piecewise Lipschitz maps from the plane to the plane, having jump made of several curves allowed to meet at a finite number of junction points. We show that the domain of this relaxed area is strictly contained in the domain of the classical L^1-relaxed area.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be an open bounded set. Given $v \in C^1(\Omega; \mathbb{R}^2)$, the area functional is defined as

$$\mathcal{A}(v, \Omega) := \int_\Omega \sqrt{1 + |\nabla v|^2 + |Jv|^2} \, dx = \int_\Omega |\mathcal{M}(\nabla v)| \, dx, \tag{1.1}$$

where $\mathcal{M}(\nabla v) = (1, \nabla v_1, \nabla v_2, Jv)$ and $Jv = \frac{\partial v_1}{\partial x_1} \frac{\partial v_2}{\partial x_2} - \frac{\partial v_2}{\partial x_1} \frac{\partial v_1}{\partial x_2}$ is the Jacobian determinant of v. The value $\mathcal{A}(v, \Omega)$ is the 2-dimensional Hausdorff measure of the graph

$$G_v := \{(x, y) \in \Omega \times \mathbb{R}^2 : y = v(x)\}$$

of v. In order to extend the area functional to a more general class of maps one is led to consider the relaxation of (1.1): Namely, for all $u \in L^1(\Omega; \mathbb{R}^2)$ one chooses a convergence, for instance the L^1-convergence [12], and sets

$$\overline{\mathcal{A}}_{L^1}(u, \Omega) := \inf \left\{ \liminf_{k \to +\infty} \mathcal{A}(u_k, \Omega), \ u_k \in C^1(\Omega; \mathbb{R}^2), \ u_k \to u \text{ in } L^1(\Omega; \mathbb{R}^2) \right\}. \tag{1.2}$$

* Corresponding author.
  E-mail addresses: bellettini@diism.unisi.it (G. Bellettini), scarano@sissa.it (S. Carano), riccardo.scala@unisi.it (R. Scala).

In contrast with the case of real valued maps, for which the L^1-relaxed area is well-understood [11], in higher dimension, including the case of $\mathbb{R}^2$-valued maps considered here, the analysis of $\overline{\mathcal{A}}_{L^1}$ has been shown to be very challenging and a lot of questions remain open. For instance, it is known that the domain $\text{Dom}(\overline{\mathcal{A}}_{L^1}(\cdot, \Omega))$ of $\overline{\mathcal{A}}_{L^1}(\cdot, \Omega)$ is strictly included in $BV(\Omega; \mathbb{R}^2)$, but its complete description is, so far, not available. The main difficulty to treat $\overline{\mathcal{A}}_{L^1}$ is due to its non-local behavior: Indeed, for general maps u with the only exception of very trivial cases, the set function $E \subseteq \Omega \mapsto \overline{\mathcal{A}}_{L^1}(u, E)$ is not subadditive, and this excludes to represent (1.2) in integral form. As a consequence, the explicit value of $\overline{\mathcal{A}}_{L^1}(u, \Omega)$ is, at the moment, known only for very specific non-smooth maps u enjoying a high degree of symmetry [5–7,25] (see also [4] for partial results without symmetry assumptions).

A useful simplification in the relaxation analysis of $\mathcal{A}$ is to consider some variants of (1.2), for example modifying the convergence of u_k to u (see [8,9,13,17]). Even if the L^1-convergence seems to be natural also with respect to the application to the non-parametric Plateau problem, one can replace the L^1-topology with different ones. In some recent works [3,22], instead of relaxing with respect to the L^1-topology, the authors have considered relaxation with respect to the strict convergence in $BV(\Omega; \mathbb{R}^2)$ (shortly BV-relaxed area). Namely, one defines

$$\overline{\mathcal{A}}_{BV}(u, \Omega) := \inf \left\{ \liminf_{k \to +\infty} \mathcal{A}(u_k, \Omega), \ u_k \in C^1(\Omega; \mathbb{R}^2), \ u_k \to u \text{ strictly } BV(\Omega; \mathbb{R}^2) \right\}. \tag{1.3}$$

Although the analysis of $\overline{\mathcal{A}}_{BV}$ seems quite more treatable, a complete picture and description of its behavior is still missing. It is straightforward that for any $u \in BV(\Omega; \mathbb{R}^2)$

$$\overline{\mathcal{A}}_{BV}(u, \Omega) \geq \overline{\mathcal{A}}_{L^1}(u, \Omega),$$

so $\text{Dom}(\overline{\mathcal{A}}_{BV}(\cdot, \Omega)) \subset \text{Dom}(\overline{\mathcal{A}}_{L^1}(\cdot, \Omega))$, and the inclusion is strict as Example 4.1 below shows.

Strictly related to the area functional is the Jacobian total variation functional, namely

$$TVJ(v, \Omega) := \int_\Omega |Jv| \, dx,$$

valid for all $v \in C^1(\Omega; \mathbb{R}^2)$. Also in this case, to extend TVJ to a larger class of functions, a relaxation procedure is in order. However, the choice of the L^1-convergence is in some cases not interesting: for instance, if $u \in W^{1,1}(\Omega; \mathbb{S}^1)$, with Ω simply connected, the corresponding relaxed functional trivializes and becomes constantly null (see [10, Cor. 5]). On the other hand, the notion of strict convergence in BV gives rise to a nontrivial relaxed functional which shows to play a crucial role in the analysis of $\overline{\mathcal{A}}_{BV}$. Specifically, for $u \in BV(\Omega; \mathbb{R}^2)$ we consider

$$\overline{TVJ}_{BV}(u, \Omega) := \inf \left\{ \liminf_{k \to +\infty} TVJ(v_k, \Omega), \ v_k \in C^1(\Omega; \mathbb{R}^2), \ v_k \to u \text{ strictly } BV(\Omega; \mathbb{R}^2) \right\}. \tag{1.4}$$

In the present paper we compute the value of $\overline{\mathcal{A}}_{BV}(u, \Omega)$ for some particular piecewise Lipschitz maps u which are allowed to jump on curves in turn meeting at junction points. We refer to Definition 5.2 for the details on these maps, and we summarize here their features: Let $\Omega \subset \mathbb{R}^2$ be a bounded open set of class C^1 and $\{\Omega_k\}_{k=1,\dots,N}$ a finite partition of Ω made of Lipschitz sets. Suppose that $\Sigma := \cup_{k=1}^N \partial \Omega_k$ is the support of a finite family of C^2-curves $\alpha_\ell : \overline{I}_\ell \to \overline{\Omega}$, $\ell = 1, \dots, n$, $I_\ell = (a_\ell, b_\ell)$. We suppose that the curves α_ℓ, arc-length parametrized on $\overline{I}_\ell$, are injective on I_ℓ, $\alpha_\ell(I_\ell) \subset \Omega$, and of class C^2 up to a_ℓ and b_ℓ (namely $\dot\alpha_\ell$ and $\ddot\alpha_\ell$ are continuous on $\overline{I}_\ell$). Furthermore, we assume that $\alpha_\ell(I_\ell)$ and $\alpha_h(I_h)$, for $\ell \neq h$, may intersect only at the endpoints. Endpoints of α_l are allowed to belong to $\partial \Omega$, and we assume such endpoints to be distinct for different curves.

A map $u \in BV(\Omega; \mathbb{R}^2)$ is called piecewise Lipschitz if its restriction to any Ω_k is Lipschitz. Notice that if p_i is a junction point and Ω_k^i ($k = 1, \dots, N_i$) are the connected components of $\Omega \setminus \Sigma$ having p_i as boundary point, then there exists the limit $\beta_k^i := \lim_{\substack{x \to p_i \\ x \in \Omega_k^i}} u(x)$. For the sake of simplicity, we assume that the enumeration $k = 1, \dots, N_i$ respects the counterclockwise order of Ω_k^i's around p_i.

To introduce our main result, we have to consider also a planar Plateau problem for Lipschitz curves: Given a Lipschitz curve $\varphi : \mathbb{S}^1 = \partial B_1 \to \mathbb{R}^2$ we consider the quantity

$$P(\varphi) := \inf \left\{ \int_{B_1} |Jv| \, dx : \ v \in \text{Lip}(B_1; \mathbb{R}^2) : \ v_{|\partial B_1} = \varphi \right\}. \tag{1.5}$$

For all $i = 1, \dots, m$ we denote by $\widetilde{\gamma}^i$ a Lipschitz curve which parametrizes on $\mathbb{S}^1$ the polygon in $\mathbb{R}^2$ with vertices $\beta_1^i, \beta_2^i, \dots, \beta_{N_i}^i$, in the order (see Fig. 3). Notice carefully that this curve may self-intersect. Also, $P(\varphi)$ is invariant under reparametrizations of φ (Proposition 2.9). Finally, set $I = [0, 1]$. The main result of the paper is the following

Theorem 1.1 (Relaxation for Piecewise Lipschitz Maps). *Let $u : \Omega \to \mathbb{R}^2$ be a piecewise Lipschitz map. Then*

$$\overline{\mathcal{A}}_{BV}(u, \Omega) = \int_{\Omega \setminus \Sigma} |\mathcal{M}(\nabla u)| \, dx + \sum_{\ell=1}^n \int_{[a_\ell, b_\ell] \times I} |\partial_t X_{(\ell)}^{\text{aff}} \wedge \partial_s X_{(\ell)}^{\text{aff}}| dt ds + \sum_{i=1}^m P(\widetilde{\gamma}^i), \tag{1.6}$$

where, for any $\ell = 1, \dots, n$,

$$X_{(\ell)}^{\text{aff}}(t, s) := (t, s u_\ell^+(t) + (1-s)u_\ell^-(t)) \qquad \forall (t, s) \in [a_\ell, b_\ell] \times I, \tag{1.7}$$

and $u_\ell^\pm$ are the traces of u on the support $\alpha_\ell(I_\ell)$ of α_ℓ.

One of the main features of expression (1.6) is the presence of *two* singular contributions: a 0-dimensional term due to the concentration of the Jacobian determinants of a recovery sequence (v_k) for $\overline{\mathcal{A}}_{BV}(u, \Omega)$ around the junction points (namely, the term involving the minimum of the Plateau problems $P(\widetilde{\gamma}^i)$), and a 1-dimensional term, which essentially takes into account the concentration of the gradients and of the Jacobian determinants of v_k along the jump set Σ. So, we can interpret (1.6) as a non-trivial generalization of [3, Theorem 1.3], valid for the triple point map u_T (see also Theorem 4.4), and of [3, Theorem 1.1], valid for 0-homogeneous maps of the form $\varphi(x/|x|)$ with $\varphi : \mathbb{S}^1 \to \mathbb{S}^1$ Lipschitz. Indeed, in the first case the 1-dimensional term was simply the total variation of u_T (consisting of the area of three vertical walls over Σ) and the 0-dimensional one was the area of the target triangle, which is a trivial minimum of (1.5), while in the second case we had no 1-dimensional contribution and the 0-dimensional one was the solution of (1.5) with this special φ, that reduces to $P(\varphi) = \pi|\deg(\varphi)|$. In other words, the relaxed area of a more general map u as in 1.1 is still a measure (if we regard it as a function of Ω), which has the same dimensional structure, but with a more involved and rich expression.

We observe that for this special kind of maps it always holds $\overline{\mathcal{A}}_{BV}(u, \Omega) < +\infty$, because the contributions of the Plateau problem $P(\widetilde{\gamma}^i)$ is always finite, since one can construct a Lipschitz competitor for (1.5). On the other hand, the presence of a finite number of junction points is crucial, because, as Example 4.1 shows, we can build a piecewise constant map whose BV-relaxed area is infinite. It is here remarkable that the same map can be seen to have finite L^1-relaxed area (compare with (4.2)). This in particular shows the proper inclusion

$$\mathrm{Dom}(\overline{\mathcal{A}}_{BV}(\cdot, \Omega)) \subsetneq \mathrm{Dom}(\overline{\mathcal{A}}_{L^1}(\cdot, \Omega)).$$

We divide the proof of Theorem 1.1 in several steps, and in particular we first focus on the relaxation on piecewise Lipschitz maps u without junction points. In this case we show in Corollary 3.12 (consequence of Theorems 3.7 and 3.11) that the relaxation provides as singular contribution the integral over the jump set S_u of u of the area spanned by the affine map X^{aff}. The main issue is the proof of Proposition 3.4, the lower bound for the relaxed area of maps jumping on the central horizontal segment of the rectangle $R = [a, b] \times [-1, 1]$. Here, we need to use some tools from the theory of integer multiplicity currents, in particular slicing arguments and the isoperimetric inequality, in order to show that over the jump segment the graph of the elements of an approximating smooth sequence (v_k) have area bounded below by the area of X^{aff}. The properties of the strict convergence (Lemmas 2.3 and 2.7) enter at the level of vertical slices of the graph of v_k in a neighborhood of the jump segment, but these results only are not enough to pass to the limit in the area of the graph of v_k. For this purpose, the idea is to make a decomposition of the graph of v_k and of the surface X^{aff} in several tiny strips, and notice that, when the number of these strips is very high, the boundaries of these two little pieces of surfaces are uniformly close together, as a consequence of the strict convergence and, at the same time, the strips which decompose X^{aff} are very close to a minimal mass current having the same boundary.

In [9], the authors compute the relaxed area $\overline{\mathcal{A}}_{L^\infty}(u, \Omega)$ with respect to the local uniform convergence out of the jump, for u as in Proposition 3.4. They obtain, as singular contribution, the area of the minimal semicartesian[1] surface spanning the graphs of the two traces. In particular, since X^{aff} is semicartesian and spans $\mathrm{graph}(u^\pm)$ as well (see [9, Definition 2.4]), we have $\overline{\mathcal{A}}_{L^\infty}(u, R) \leq \overline{\mathcal{A}}_{BV}(u, R)$. In general, this inequality holds strictly, even if $\mathrm{graph}(u^\pm)$ are coplanar. We can find an example in [9, Remark 8.5], where one can notice that in order to minimize the area of the spanning surface, the approximating sequence needs not keep the total variation of the limit map, which instead is forced to be preserved under strict convergence. Moreover, it is important to notice that $\overline{\mathcal{A}}_{L^\infty}(u, \cdot)$ is not subadditive (see [9, Thm. 8.1]), while $\overline{\mathcal{A}}_{BV}(u, \cdot)$ is clearly a measure.

In a second step we instead consider the case of maps u which are piecewise constant but whose jump might have junction points. Specifically, in Theorem 4.4 we see that the relaxation on a n-uple point map (i.e., whose jump consists of n radii of the same ball $B_r(0)$) provides as singular contribution, besides the total variation of u, the number $P(\widetilde{\gamma})$, where $\widetilde{\gamma}$ is the piecewise affine curve which parametrizes the perimeter of the polygon whose vertices are the values of u around 0.

Finally, in Section 5, we use Corollary 3.12 and Theorem 4.4 to complete the proof of Theorem 1.1.

We point out that, to our best knowledge, it is not yet known, in general, whether the BV-relaxed area is subadditive if considered as a set function and, further, if it gives rise to a measure. We expect such a subadditivity for BV-maps u from the plane to the plane, being motivated by relevant examples with explicit computations, and also because of the presence of a unique cartesian current with minimal completely vertical lifting associated to u (as recently shown in [23]). Unfortunately, this uniqueness result fails in higher codimension, where in addition we have less explicit examples.

2. Preliminaries

We start by collecting some tools needed in the proof of the main theorems. For an integer $M \geq 2$, set $\mathbb{S}^{M-1} := \{x \in \mathbb{R}^M : |x| = 1\}$. In what follows, $\Omega \subset \mathbb{R}^2$ is a bounded open set.

2.1. Some consequences of the strict convergence

Theorem 2.1 (Reshetnyak). *Let μ_h, μ be (finite) Radon measures in Ω, taking values in $\mathbb{R}^M$. Suppose that $\mu_h \overset{*}{\rightharpoonup} \mu$ and $|\mu_h|(\Omega) \to |\mu|(\Omega)$. Then*

$$\lim_{h \to +\infty} \int_\Omega f\left(x, \frac{\mu_h}{|\mu_h|}(x)\right) d|\mu_h|(x) = \int_\Omega f\left(x, \frac{\mu}{|\mu|}(x)\right) d|\mu|(x)$$

for any continuous bounded function $f : \Omega \times \mathbb{S}^{M-1} \to \mathbb{R}$.

---

[1] A map having the identity as the first component.

Proof. See for instance [2, Theorem 2.39]. □

For any $u \in BV(\Omega; \mathbb{R}^2)$, we recall that the distributional derivative Du is a Radon measure valued in $\mathbb{R}^{2 \times 2}$. $SBV(\Omega)$ stands for the space of special functions of bounded variation on Ω [2]. The symbol $|Du|(\Omega)$ stands for the total variation of Du (see [2, Definition 3.4, pag. 119]) with $|\cdot|$ the Frobenius norm. We denote by S_u the jump set of u.

Definition 2.2 (*Strict Convergence*). Let $u \in BV(\Omega; \mathbb{R}^2)$ and $(u_k) \subset BV(\Omega; \mathbb{R}^2)$. We say that (u_k) converges to u strictly BV, if

$$u_k \xrightarrow{L^1} u \qquad \text{and} \qquad |Du_k|(\Omega) \to |Du|(\Omega).$$

Let $R = [a, b] \times [-1, 1]$. For $(t, \sigma) \in R$, set

$$R_t^{x_1} := \{(x_1, x_2) \in R : x_1 = t\}, \qquad R_\sigma^{x_2} := \{(x_1, x_2) \in R : x_2 = \sigma\}.$$

If $u \in BV(R; \mathbb{R}^2)$, by Lebesgue differentiation theorem and Fubini theorem, for almost every $t \in [a, b]$, the restriction $u \llcorner R_t^{x_1}$ of u on the vertical segment $R_t^{x_1}$ coincides with the trace of u at $\mathcal{H}^1$-almost every point of $R_t^{x_1}$. So, for almost every $t \in [a, b]$, the map $u \llcorner R_t^{x_1}$ is well defined because it is independent of the representative of u. The same argument holds in $R_\sigma^{x_2}$ for almost every $\sigma \in [-1, 1]$.

Lemma 2.3 (*Inheritance of Strict Convergence to Slices*). *Let* $u \in BV(R; \mathbb{R}^2)$. *Suppose that* $(v_k) \subset C^1(R; \mathbb{R}^2)$ *is a sequence converging to* u *strictly* $BV(R; \mathbb{R}^2)$. *Then for almost every* $(t, \sigma) \in R$, *there exists a subsequence* $(v_{k_h}) \subset (v_k)$, *depending on* t *and* σ, *such that*

$$v_{k_h} \llcorner R_t^{x_1} \to u \llcorner R_t^{x_1} \quad \text{strictly } BV(R_t^{x_1}; \mathbb{R}^2), \tag{2.1}$$

$$v_{k_h} \llcorner R_\sigma^{x_2} \to u \llcorner R_\sigma^{x_2} \quad \text{strictly } BV(R_\sigma^{x_2}; \mathbb{R}^2). \tag{2.2}$$

Proof. For almost every $t \in [a, b]$, in view of the definition of $R_t^{x_1}$, we can define the total variation of $u \llcorner R_t^{x_1}$ as

$$|D(u \llcorner R_t^{x_1})|(R_t^{x_1}) = \sup \left\{ -\int_{-1}^{1} u(t, x_2) \cdot g'(x_2) dx_2; \, g \in C_c^1((-1, 1); \overline{B}_1(0)) \right\}, \tag{2.3}$$

where $\overline{B}_1(0) = \{(\xi, \eta) \in \mathbb{R}^2 : \xi^2 + \eta^2 \le 1\}$. Let us show that

$$|D_2 u|(R) = \int_a^b |D(u \llcorner R_t^{x_1})|(R_t^{x_1}) dt, \tag{2.4}$$

where $D_2 u := Du\, e_2$ is a Radon measure on R valued in $\mathbb{R}^2$ with finite total variation. Since, for almost every $t \in [a, b]$, $v_k \llcorner R_t^{x_1} \to u \llcorner R_t^{x_1}$ in $L^1(R_t^{x_1}; \mathbb{R}^2)$, we have, using (2.3),

$$|D(u \llcorner R_t^{x_1})|(R_t^{x_1}) \le \liminf_{k \to +\infty} \int_{R_t^{x_1}} |\partial_2 v_k(t, x_2)| dx_2. \tag{2.5}$$

Then, using Fatou lemma and Fubini theorem,

$$\int_a^b |D(u \llcorner R_t^{x_1})|(R_t^{x_1}) dt \le \int_a^b \liminf_{k \to +\infty} \int_{R_t^{x_1}} |\partial_2 v_k(t, x_2)| dx_2 dt \tag{2.6}$$

$$\le \liminf_{k \to +\infty} \int_R |\partial_2 v_k(t, x_2)| dt dx_2 = |D_2 u|(R),$$

where in the last equality we used Theorem 2.1 with $f(x, v) = \sqrt{v_3^2 + v_4^2}$, for every $x \in R$, $v \in \mathbb{S}^3 \subset \mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$, with

$$v = \begin{pmatrix} v_1 & v_3 \\ v_2 & v_4 \end{pmatrix}.$$

The converse inequality in (2.4) is standard.[2] So, (2.4) is proved and (2.6) holds as an equality, which implies that also (2.5) holds as an equality, namely

$$|D(u \llcorner R_t^{x_1})|(R_t^{x_1}) = \liminf_{k \to +\infty} \int_{R_t^{x_1}} |\partial_2 v_k(t, x_2)| dx_2.$$

Extracting a subsequence $(v_{k_h}) \subset (v_k)$ depending on t, we get

$$v_{k_h} \llcorner R_t^{x_1} \to u \llcorner R_t^{x_1} \quad \text{strictly } BV(R_t^{x_1}; \mathbb{R}^2).$$

Finally, repeating the same argument for v_{k_h} on the horizontal slices $\{R_\sigma^{x_2}\}$, we get (2.1) for a (not relabeled) sub-subsequence. □

---

[2] We recall that

$$|D_2 u|(R) = \sup \left\{ -\int_R u \cdot \partial_{x_2} g \, dx : \, g \in C_c^1(R; \overline{B}_1(0)) \right\}.$$

Now, for $g \in C_c^1(R; \overline{B}_1(0))$, $\int_R u \cdot \partial_{x_2} g \, dx = \int_a^b \left(\int_{-1}^1 u(t, x_2) \cdot \partial_{x_2} g(t, x_2) dx_2 \right) dt \le \int_a^b |D(u \llcorner R_t^{x_1})|(R_t^{x_1}) dt$, so $|D_2 u|(R) \le \int_a^b |D(u \llcorner R_t^{x_1})|(R_t^{x_1}) dt$.

Now, let B_l be the disk of $\mathbb{R}^2$ centered at the origin of radius $l > 0$. We want to prove the analogue of Lemma 2.3 in B_l, by slicing with concentric circumferences. If $u \in BV(B_l; \mathbb{R}^2)$, as in the previous case, for almost every $r \in (0, l)$ the restriction $u \llcorner \partial B_r$ is well-defined and independent of the representative of u. In particular, for almost every $r \in (0, l)$, we can define the total variation of $u \llcorner \partial B_r$ as

$$|D(u \llcorner \partial B_r)|(\partial B_r) := \sup \left\{ -\int_0^{2\pi} \bar{u}(r, \theta) \cdot f'(\theta) d\theta; f \in C^1([0, 2\pi]; \overline{B}_1(0)), f(0) = f(2\pi), f'(0) = f'(2\pi) \right\} \tag{2.7}$$

which turns out to be finite (see Lemma 2.5), giving that $u \llcorner \partial B_r \in BV(\partial B_r; \mathbb{R}^2)$, for almost every $r \in (0, l)$. Here

$$\bar{u}(r, \theta) := u(r \cos \theta, r \sin \theta), \qquad r \in (0, l], \ \theta \in [0, 2\pi).$$

We want to relate this quantity with the notion of tangential total variation.

Definition 2.4. For $x = (x_1, x_2) \in \mathbb{R}^2 \setminus \{(0, 0)\}$, set $\tau(x) = \frac{1}{|x|}(-x_2, x_1)$. Let $0 < l < L$ and $A_{L,l} := B_L(0) \setminus \overline{B_l(0)}$ be an annulus around 0. We define the tangential total variation of $u \in BV(A_{L,l}; \mathbb{R}^2)$ as the total variation of the Radon measure $D_\tau u := Du\tau$, namely

$$|D_\tau u|(A_{L,l}) = |Du\tau|(A_{L,l}) = \sup \left\{ -\int_{A_{L,l}} u \cdot (\nabla g \tau) \, dx : g \in C_c^1(A_{L,l}; \overline{B}_1(0)) \right\}. \tag{2.8}$$

The last equality in (2.8) is justified since $\tau \in C^\infty(A_{L,l}; \mathbb{R}^2)$ satisfies $\mathrm{div}\,\tau = 0$ everywhere, so for any $g = (g^1, g^2) \in C_c^1(A_{L,l}; \mathbb{R}^2)$ we have

$$-\int_{A_{L,l}} u \cdot (\nabla g \tau) \, dx = -\int_{A_{L,l}} u^1 \nabla g^1 \cdot \tau \, dx - \int_{A_{L,l}} u^2 \nabla g^2 \cdot \tau \, dx$$

$$= -\int_{A_{L,l}} u^1 \mathrm{div}(g^1 \tau) \, dx - \int_{A_{L,l}} u^2 \mathrm{div}(g^2 \tau) \, dx$$

$$= \int_{A_{L,l}} g^1 \tau \cdot d\,Du^1 + \int_{A_{L,l}} g^2 \tau \cdot d\,Du^2 = \int_{A_{L,l}} g \cdot (d\,Du)\tau = \langle Du\tau, g \rangle.$$

This computation shows that $|D_\tau u|(A_{L,l}) \le |Du|(A_{L,l})$, since $|\tau| \le 1$, and also that (2.8) is compatible with the case $u \in W^{1,1}(A_{L,l}; \mathbb{R}^2)$, where simply $|D_\tau u|(A_{L,l}) = \int_{A_{L,l}} |\nabla u\tau| \, dx$. Moreover, $Du = \frac{Du}{|Du|}|Du|$ by polar decomposition, so that

$$\langle Du\tau, g \rangle = \int_{A_{L,l}} g \cdot (d\,Du)\tau = \int_{A_{L,l}} g \cdot \left(\frac{Du}{|Du|} d|Du| \right) \tau = \int_{A_{L,l}} g \cdot \left(\frac{Du}{|Du|} \tau \right) d|Du| \qquad \forall g \in C_c^1(B_l; \mathbb{R}^2),$$

giving that

$$D_\tau u = Du\tau = \frac{Du}{|Du|} \tau |Du|. \tag{2.9}$$

Lemma 2.5 (Inheritance of Strict Convergence to Circumferences). Let $u \in BV(B_R; \mathbb{R}^2)$ and $(v_k) \subset C^1(B_R; \mathbb{R}^2)$ be a sequence converging to u strictly $BV(B_R; \mathbb{R}^2)$. Then, for almost every $r \in (0, R)$, there exists a subsequence $(v_{k_h}) \subset (v_k)$, depending on r, such that

$$v_{k_h} \llcorner \partial B_r \to u \llcorner \partial B_r \quad \text{strictly } BV(\partial B_r; \mathbb{R}^2) \qquad \text{as } h \to +\infty. \tag{2.10}$$

Proof. For almost every $r \in (0, R)$, by Fatou lemma and Fubini theorem, up to extracting a subsequence, we may assume that the restriction $v_k \llcorner \partial B_r$ has equi-bounded variation w.r.t. k. Moreover, we may also assume that (v_k) converges to u almost everywhere in B_R, so that, for almost every $r \in (0, R)$,

$$v_k \llcorner \partial B_r \to u \llcorner \partial B_r \quad \mathscr{H}^1\text{-a.e. in } \partial B_r. \tag{2.11}$$

Now, let $r \in (0, R)$ be such that $v_k \llcorner \partial B_r$ has equi-bounded variation and (2.11) holds. Then, there exists a subsequence $(v_{k_h}) \subset (v_k)$ depending on r such that

$$v_{k_h} \llcorner \partial B_r \overset{*}{\rightharpoonup} u \llcorner \partial B_r \quad \text{w*} - BV(\partial B_r; \mathbb{R}^2).$$

By lower semicontinuity of the variation, we infer that for almost every $r \in (0, R)$, $u \llcorner \partial B_r \in BV(\partial B_r; \mathbb{R}^2)$ and

$$|D(u \llcorner \partial B_r)|(\partial B_r) \le \liminf_{h \to +\infty} \int_{\partial B_r} \left| \nabla v_{k_h} \tau \right| d\mathcal{H}^1. \tag{2.12}$$

Let $0 < l < L \le R$ be such that $v_k \to u$ strictly $BV(A_{L,l}; \mathbb{R}^2)$ where, as in Definition 2.4, $A_{L,l} := B_L(0) \setminus \overline{B_l(0)}$ (notice that this holds for a.e. l and L); by integration, we get

$$\int_l^L |D(u \llcorner \partial B_r)|(\partial B_r) \, dr \le \int_l^L \left(\liminf_{h \to +\infty} \int_{\partial B_r} \left| \nabla v_{k_h} \tau \right| d\mathcal{H}^1 \right) \, dr$$

$$\le \liminf_{h \to +\infty} \int_l^L \int_{\partial B_r} \left| \nabla v_{k_h} \tau \right| d\mathcal{H}^1 dr = \liminf_{h \to +\infty} \int_{A_{L,l}} \left| \nabla v_{k_h} \tau \right| \, dx. \tag{2.13}$$

Thanks to Theorem 2.1, with the choices $M = 4$, $\mathbb{S}^3 \subset \mathbb{R}^4 = \mathbb{R}^{2 \times 2}$, $f \in C_b(A_{L,l} \times \mathbb{S}^3)$,

$$f(x, v) := \sqrt{|v_{\text{hor}} \cdot \tau(x)|^2 + |v_{\text{vert}} \cdot \tau(x)|^2},$$

where $v \in \mathbb{S}^3$ and $v_{\text{hor}} := (v_1, v_3)$, $v_{\text{vert}} := (v_2, v_4)$, we obtain

$$\lim_{k \to +\infty} \int_{A_{L,l}} |\nabla v_k \tau| \, dx = \int_{A_{L,l}} \left| \frac{Du}{|Du|} \tau \right| d|Du| = |D_\tau u|(A_{L,l}), \tag{2.14}$$

where in the last equality we have used (2.9). So we get

$$|D_\tau u|(A_{L,l}) \geq \int_l^L |D(u \llcorner \partial B_r)|(\partial B_r) \, dr.$$

In order to prove the converse inequality, let $g \in C_c^1(A_{L,l}; \overline{B}_1(0))$. Then, in polar coordinates, by definition (2.7),

$$\int_{A_{L,l}} u \cdot \nabla g \tau \, dx = \int_l^L \int_0^{2\pi} \bar{u}(\rho, \theta) \cdot \partial_\theta \bar{g}(\rho, \theta) \, d\rho d\theta \leq \int_l^L |D(u \llcorner \partial B_\rho)|(\partial B_\rho) \, d\rho,$$

where $\bar{g}(\rho, \theta) := g(\rho \cos \theta, \rho \sin \theta)$, for any $\rho \in (0, l]$, $\theta \in [0, 2\pi)$. So, we have proved that

$$|D_\tau u|(A_{L,l}) = \int_l^L |D(u \llcorner \partial B_r)|(\partial B_r) \, dr.$$

In particular, we deduce that (2.13) is a chain of equalities. Then, (2.12) holds as an equality and there exists a subsequence $(v_{k_h}) \subset (v_k)$, depending on r, which achieves the full limit. Since l and L are arbitrary, we get the thesis. $\square$

2.2. Further properties in dimension 1

In [3, Proposition 2.4] the following is proved:

Lemma 2.6. *Let $(\gamma_k) \subset W^{1,1}((a, b); \mathbb{R}^2)$ be a sequence converging strictly $BV((a, b); \mathbb{R}^2)$ to $\gamma \in W^{1,1}((a, b); \mathbb{R}^2)$. Then $\gamma_k \to \gamma$ uniformly in (a, b).*

For our purposes, we need an improvement of Lemma 2.6, where discontinuous functions γ at a single point, or at a finite number of points, are allowed; we start with one point discontinuity.

Lemma 2.7. *Let $I^- := [-1, 0)$, $I^+ := (0, 1]$. Suppose that $(\gamma_k) \subset W^{1,1}([-1, 1]; \mathbb{R}^2)$ is a sequence converging strictly $BV([-1, 1]; \mathbb{R}^2)$ to $\gamma \in BV([-1, 1]; \mathbb{R}^2) \cap W^{1,1}(I^-; \mathbb{R}^2) \cap W^{1,1}(I^+; \mathbb{R}^2)$, with $\gamma^+(0) \neq \gamma^-(0)$. Let $S : [-1/3, 1/3] \to \mathbb{R}^2$ be defined by*

$$S(\tau) := \frac{3}{2} \left((1/3 + \tau) \gamma^+(0) + (1/3 - \tau) \gamma^-(0) \right), \quad \tau \in [-1/3, 1/3].$$

Let $\widetilde{\gamma}^-$ (resp. $\widetilde{\gamma}^+$) be the reparametrization of $\gamma_{|I^-}$ (resp. $\gamma_{|I^+}$) on $(-1, -\frac{1}{3})$ (resp. $(\frac{1}{3}, 1)$) defined by the composition with the increasing linear function taking $[-1, -1/3]$ onto $[-1, 0]$ (resp. $[1/3, 1]$ onto $[0, 1]$). Define

$$\widetilde{\gamma} : [-1, 1] \to \mathbb{R}^2, \qquad \widetilde{\gamma} := \begin{cases} \widetilde{\gamma}^- & \text{in } [-1, -1/3) \\ S & \text{in } [-1/3, 1/3] \\ \widetilde{\gamma}^+ & \text{in } (1/3, 1]. \end{cases} \tag{2.15}$$

Then there exist:

(a) a Lipschitz strictly increasing surjective function $h : [-1, 1] \to [-1, 1]$,

(b) a subsequence (k_j) and Lipschitz strictly increasing surjective functions $h_{k_j} : [-1, 1] \to [-1, 1]$ for any $j \in \mathbb{N}$, with $\sup_j \|\dot{h}_{k_j}\|_\infty < +\infty$,

such that

$$\lim_{j \to +\infty} \gamma_{k_j} \circ h_{k_j} = \widetilde{\gamma} \circ h \quad \text{uniformly in } [-1, 1]. \tag{2.16}$$

Proof. The lengths L_k of γ_k and L of γ are given by

$$L_k = \int_{-1}^1 |\dot{\gamma}_k| \, d\tau,$$

$$L = |\dot{\gamma}|([-1, 1]) = \int_{-1}^0 |\dot{\gamma}| \, d\tau + |\gamma^+(0) - \gamma^-(0)| + \int_0^1 |\dot{\gamma}| \, d\tau.$$

Since, by assumption, $\gamma_k \to \gamma$ strictly $BV([-1, 1]; \mathbb{R}^2)$, we have that $L_k \to L$ as $k \to +\infty$. Fix $\eta > 0$ and for all $k \in \mathbb{N}$ define the function[3]

$$s_k : [-1, 1] \to [0, L + \eta], \qquad s_k(t) := \frac{L + \eta}{L_k + \eta} \int_{-1}^t \left(|\dot{\gamma}_k(\tau)| + \frac{\eta}{2} \right) d\tau, \tag{2.17}$$

---

[3] We need η, since in principle $\dot{\gamma}_k$ could vanish somewhere.

with Lipschitz inverse $\alpha_k := s_k^{-1} : [0, L + \eta] \to [-1, 1]$. Define

$$\widehat{\gamma}_k : [0, L + \eta] \to \mathbb{R}^2, \qquad \widehat{\gamma}_k(s) := \gamma_k(\alpha_k(s)) \qquad \forall s \in [0, L + \eta]. \tag{2.18}$$

Since from (2.17)

$$\left| \frac{d\widehat{\gamma}_k}{ds}(s) \right| \le \frac{|\dot{\gamma}_k(\alpha_k(s))|}{|\dot{s}_k(\alpha_k(s))|} \le \frac{L_k + \eta}{L + \eta} \le C \qquad \text{for a.e. } s \in [0, L + \eta],$$

for some constant $C > 0$ independent of k, the sequence $(\widehat{\gamma}_k)$ is bounded in $W^{1,\infty}([0, L + \eta]; \mathbb{R}^2)$. Thus, up to a (not relabeled) subsequence, we may assume that there exists $\widehat{\gamma} \in W^{1,\infty}([0, L + \eta]; \mathbb{R}^2)$ such that

$$\widehat{\gamma}_k \rightharpoonup \widehat{\gamma} \text{ weakly* in } W^{1,\infty}([0, L + \eta]; \mathbb{R}^2) \text{ and uniformly in } [0, L + \eta]. \tag{2.19}$$

We observe that for any open interval $J \subseteq [0, L + \eta]$,

$$\int_J |\dot{\widehat{\gamma}}| \, ds \le \liminf_{k \to +\infty} \int_J |\dot{\widehat{\gamma}}_k| \, ds \le |J| \liminf_{k \to +\infty} \frac{L_k + \eta}{L + \eta} = |J|,$$

and thus

$$|\dot{\widehat{\gamma}}| \le 1 \text{ a.e. in } [0, L + \eta]. \tag{2.20}$$

Now, in order to conclude the proof, we need to show that $\widehat{\gamma}$ is a reparametrization of $\widetilde{\gamma}$. Then the thesis of the lemma will follow by reparametrizing both $\widehat{\gamma}_k$ and $\widehat{\gamma}$ on $[-1, 1]$.

Using that (γ_k) strictly converges $BV([-1, 1]; \mathbb{R}^2)$ to $\gamma \in W^{1,1}(I^-; \mathbb{R}^2) \cap W^{1,1}(I^+; \mathbb{R}^2)$, by Lemma 2.6 and a diagonal process, we can find an infinitesimal sequence $(\delta_{k_j}) \subset (0, 1]$ such that

$$\|\gamma_{k_j} - \gamma\|_{L^\infty([-1,1] \setminus (-\delta_{k_j}, \delta_{k_j}))} \to 0 \tag{2.21}$$

and

$$\int_{-1}^{-\delta_{k_j}} |\dot{\gamma}_{k_j}(\tau)| \, d\tau \to \int_{-1}^{0} |\dot{\gamma}(\tau)| \, d\tau, \qquad \int_{\delta_{k_j}}^{1} |\dot{\gamma}_{k_j}(\tau)| \, d\tau \to \int_{0}^{1} |\dot{\gamma}(\tau)| \, d\tau$$

as $j \to +\infty$. In particular,

$$\lim_{j \to +\infty} \gamma_{k_j}(\pm\delta_{k_j}) = \gamma^{\pm}(0) \tag{2.22}$$

and, setting

$$r_{k_j}^- := s_{k_j}(-\delta_{k_j}) = \frac{L + \eta}{L_{k_j} + \eta} \int_{-1}^{-\delta_{k_j}} \left(|\dot{\gamma}_{k_j}| + \frac{\eta}{2} \right) d\tau,$$

$$r_{k_j}^+ := s_{k_j}(\delta_{k_j}) = \frac{L + \eta}{L_{k_j} + \eta} \left[\int_{-1}^{1} \left(|\dot{\gamma}_{k_j}| + \frac{\eta}{2} \right) d\tau - \int_{\delta_{k_j}}^{1} \left(|\dot{\gamma}_{k_j}| + \frac{\eta}{2} \right) d\tau \right],$$

we have

$$\begin{aligned} \lim_{j \to +\infty} r_{k_j}^- &= \frac{\eta}{2} + \int_{-1}^{0} |\dot{\gamma}| \, d\tau =: r^-, \\ \lim_{j \to +\infty} r_{k_j}^+ &= \frac{\eta}{2} + \int_{-1}^{0} |\dot{\gamma}| \, d\tau + |\gamma^+(0) - \gamma^-(0)| =: r^+. \end{aligned} \tag{2.23}$$

As a consequence of (2.19), (2.22), and (2.23) we get

$$\gamma_{k_j}(\alpha_{k_j}(r_{k_j}^\pm)) = \widehat{\gamma}_{k_j}(r_{k_j}^\pm) \to \widehat{\gamma}(r^\pm) = \gamma^\pm(0).$$

Therefore the curve $\widehat{\gamma}$ maps the segment $[r^-, r^+]$ into a curve joining $\gamma^-(0)$ and $\gamma^+(0)$. Now, since $r^+ - r^- = |\gamma^+(0) - \gamma^-(0)|$, from (2.20) we conclude that $\widehat{\gamma}$ coincides with the unit-speed parametrization of the segment joining $\gamma^-(0)$ and $\gamma^+(0)$ on $[r^-, r^+]$. Hence we have shown that

$$\gamma_{k_j} \circ \alpha_{k_j} \to S \circ \widetilde{\alpha} \text{ uniformly in } [r^-, r^+] \text{ as } j \to +\infty, \tag{2.24}$$

for the affine increasing reparametrization $\widetilde{\alpha} : [r^-, r^+] \to [-1/3, 1/3]$.

We now check that $\widehat{\gamma} = \gamma \circ \alpha$ on $[0, r^-]$ for some increasing bijection $\alpha : [0, r^-] \to [-1, 0]$, and similarly $\widehat{\gamma} = \gamma \circ \beta$ on $[r^+, L + \eta]$ for some increasing bijection $\beta : [r^+, L + \eta] \to [0, 1]$.

Indeed, the functions $\alpha_k : [0, L + \eta] \to [-1, 1]$ are strictly increasing and satisfy

$$|\dot{\alpha}_k(s_k(t))| = \frac{L_k + \eta}{(L + \eta)(|\dot{\gamma}_k(t)| + \frac{\eta}{2})} \le \frac{C}{\eta},$$

so that we may assume (up to extracting a further not relabeled subsequence) that

$$\alpha_{k_j} \rightharpoonup \alpha \text{ weakly* in } W^{1,\infty}([0, L + \eta]) \text{ and uniformly in } [0, L + \eta],$$

for some nondecreasing map $\alpha \in W^{1,\infty}([0, L + \eta])$. Hence, using (2.21), we find out

$$\widehat{\gamma}_{k_j}(s) = \gamma_{k_j}(\alpha_{k_j}(s)) \to \gamma(\alpha(s)) \quad \text{for all } s \in [0, r^-).$$

This, together with (2.19), implies

$$\widehat{\gamma}(s) = \gamma \circ \alpha(s) \quad \text{for all } s \in [0, r^-).$$

A similar argument shows that this also holds for all $s \in (r^+, L + \eta]$.

Finally, we observe that α is strictly increasing on $[0, r^-) \cup (r^+, L + \eta]$. For, if α is constant on some interval $[s_1, s_2] \subset [0, r^-)$, we have $\lim_{j \to +\infty} \alpha_{k_j}(s_1) = \lim_{h \to +\infty} \alpha_{k_j}(s_2)$ and hence

$$0 = \lim_{j \to +\infty} \int_{s_1}^{s_2} \dot{\alpha}_{k_j}(s) ds = \lim_{j \to +\infty} \int_{t_{k_j,1}}^{t_{k_j,2}} d\tau = \lim_{j \to +\infty} (t_{k_j,2} - t_{k_j,1}), \tag{2.25}$$

where $t_{k_j,i}$ are defined by $s_{k_j}(t_{k_j,1}) = s_1$ and $s_{k_j}(t_{k_j,2}) = s_2$. By definition (2.17) of s_{k_j} we have

$$0 < s_2 - s_1 = \int_{t_{k_j,1}}^{t_{k_j,2}} \left(|\dot{\gamma}_{k_j}(\tau)| + \frac{\eta}{2} \right) d\tau. \tag{2.26}$$

Possibly passing to a (not relabeled) subsequence and using (2.25), let $\bar{t} \in [-1, 0]$ be the limit of $(t_{k_j,1})$ and $(t_{k_j,2})$. If $\bar{t} \neq 0$, for any open neighborhood $J \subset (-1, 0)$ of $\bar{t}$, using (2.26), we get

$$\int_J |\dot{\gamma}| \, d\tau = \lim_{h \to +\infty} \int_J |\dot{\gamma}_{k_j}| \, d\tau \geq s_2 - s_1,$$

which contradicts the inclusion $\dot{\gamma} \in L^1((-1, 0); \mathbb{R}^2)$. The same argument holds if $\bar{t} = 0$, for J a left neighborhood of 0 in $(-1, 0)$. We conclude that α is strictly increasing.

Let h_{k_j} be a rescaling of α_{k_j} on $[-1, 1]$; rescaling also α from $[0, r^-]$ to $[-1, -1/3]$, and then from $[r^+, L + \eta]$ to $[1/3, 1]$, using also $\widetilde{\alpha}$ in (2.24), we construct a reparametrization $h : [-1, 1] \to [-1, 1]$ such that (2.16) holds, and the lemma is proved. $\quad\square$

Lemma 2.7 can be readily extended to curves γ with finitely many jump points:

Corollary 2.8. *Let $(\gamma_k) \subset W^{1,1}([0, 2\pi]; \mathbb{R}^2)$ be a sequence converging strictly $BV([0, 2\pi]; \mathbb{R}^2)$ to a map $\gamma \in SBV([0, 2\pi]; \mathbb{R}^2)$ having finitely many jump points $0 < z_1 < z_2 < \cdots < z_n < 2\pi$. Let $\theta_0 > 0$ be such that the intervals $(z_i - \theta_0, z_i + \theta_0) \subset (0, 2\pi)$ are disjoint, and for all $i = 1, \ldots, n$ let $S_i : [z_i - \theta_0, z_i + \theta_0] \to \mathbb{R}^2$ be defined by*

$$S_i(\tau) := \frac{1}{2\theta_0} \left((\tau - z_i + \theta_0) \gamma^+(z_i) + (z_i + \theta_0 - \tau) \gamma^-(z_i) \right), \quad \tau \in [z_i - \theta_0, z_i + \theta_0].$$

Setting $z_0 := 0$ and $z_{n+1} := 2\pi$, for all $i = 0, \ldots, n$ let $\widetilde{\gamma}_i : [z_i + \theta_0, z_{i+1} - \theta_0] \to \mathbb{R}^2$ be a rescaled reparametrization of $\gamma : [z_i, z_{i+1}] \to \mathbb{R}^2$. Finally, let $\widetilde{\gamma} : [0, 2\pi] \to \mathbb{R}^2$ be the Lipschitz curve defined as

$$\widetilde{\gamma} := \widetilde{\gamma}_0 \star S_1 \star \widetilde{\gamma}_1 \star S_2 \star \widetilde{\gamma}_2 \star \ldots \star S_n \star \widetilde{\gamma}_n, \tag{2.27}$$

where $\star$ denotes the arc composition. Then there exist a subsequence (k_j) and Lipschitz increasing surjective functions $h, h_{k_j} : [0, 2\pi] \to [0, 2\pi]$ such that

$$\lim_{j \to +\infty} \gamma_{k_j} \circ h_{k_j} = \widetilde{\gamma} \circ h \quad \text{uniformly in } [0, 2\pi]. \tag{2.28}$$

Proof. We sketch the proof which is a direct consequence of the arguments used to prove Lemma 2.7. Choose points w_i, $i = 1, \ldots, n-1$ so that $z_i + \theta_0 < w_i < z_{i+1} - \theta_0$, and let $w_0 = 0$ and $w_n = 2\pi$. Then we can apply Lemma 2.7 to any interval $[w_i, w_{i+1}]$, and taking a suitable subsequence and concatenating the obtained maps one can easily construct the desired parametrizations. $\quad\square$

2.3. Planar Plateau-type problem

Let $\varphi : \mathbb{S}^1 \to \mathbb{R}^2$ be a possibly self-intersecting Lipschitz curve. Let us consider, as in [24] (see also [14]), the planar Plateau-type problem (1.5) spanning φ. Notice that the class of competitors is non-empty, since it contains the map $v(x) = |x| \varphi\left(\frac{x}{|x|}\right)$ for $x \neq 0$, and $v(0) = 0$. We first observe that P is independent of the radius of the domain of integration. Specifically, for any $r > 0$, let

$$\varphi_r(y) := \varphi\left(\frac{y}{r}\right) \quad \text{for all } y \in \partial B_r. \tag{2.29}$$

Setting $y := rx$, $y \in \overline{B_r}$ and $v_r(y) := v(\frac{y}{r})$, we have

$$\int_{B_1} |Jv| \, dx = \int_{B_r} |Jv_r| dy \qquad \forall v \in \text{Lip}(B_1; \mathbb{R}^2). \tag{2.30}$$

In particular, for any $r > 0$,

$$P(\varphi) = \inf \left\{ \int_{B_r} |Jv| \, dx : v \in \text{Lip}(B_r; \mathbb{R}^2), v_{|\partial B_r} = \varphi_r \right\}. \tag{2.31}$$

In the next proposition we show that $P(\cdot)$ is invariant under Lipschitz reparametrizations of φ.

Proposition 2.9 (*Invariance*). *Let* $\varphi \in \text{Lip}(\mathbb{S}^1; \mathbb{R}^2)$ *and* h *be a Lipschitz homeomorphism of* $\mathbb{S}^1$. *Then*

$$P(\varphi \circ h) = P(\varphi).$$

Proof. Since h and the identity map $\text{id} : \mathbb{S}^1 \to \mathbb{S}^1$ have the same degree, they are homotopic in $\mathbb{S}^1$ by Hopf Theorem (see [21], pag. 51]), namely there exists a Lipschitz map[4] $K : [0,1] \times \mathbb{S}^1 \to \mathbb{S}^1$ such that

$$K(0, \cdot) = \text{id}, \quad K(1, \cdot) = h.$$

Define $H : [0,1] \times \mathbb{S}^1 \to \mathbb{R}^2$ as $H(t, v) = \varphi(K(t, v))$. Then H is Lipschitz and

$$H(0, \cdot) = \varphi, \quad H(1, \cdot) = \varphi \circ h.$$

Now, suppose $v_k \in \text{Lip}(B_1; \mathbb{R}^2)$ is such that $v_k = \varphi$ on ∂B_1 and

$$\lim_{k \to +\infty} \int_{B_1} |Jv_k| \, dx \to P(\varphi).$$

Define the map $\widetilde{v}_k : B_1 \to \mathbb{R}^2$ as

$$\widetilde{v}_k(x) = \begin{cases} v_k(kx) & \text{for } x \in B_{\frac{1}{k}}, \\ H\left(k|x| - 1, \frac{x}{|x|}\right) & \text{for } x \in B_{\frac{2}{k}} \setminus B_{\frac{1}{k}}, \\ \varphi \circ h\left(\frac{x}{|x|}\right) & \text{for } x \in B_1 \setminus B_{\frac{2}{k}}. \end{cases} \tag{2.32}$$

Then $\widetilde{v}_k \in \text{Lip}(B_1; \mathbb{R}^2)$ and $\widetilde{v}_k = \varphi \circ h$ on ∂B_1. Moreover, since H and $\varphi \circ h$ take values in $\varphi(\mathbb{S}^1)$ which is 1-dimensional, by the area formula and (2.30) we have

$$\int_{B_1} |J\widetilde{v}_k(x)| \, dx = \int_{B_{\frac{1}{k}}} |Jv_k(kx)| \, dx = \int_{B_1} |Jv_k| \, dx \to P(\varphi)$$

as $k \to +\infty$. In particular $P(\varphi \circ h) \leq P(\varphi)$. Exchanging the role of φ and $\varphi \circ h$, we obtain the converse inequality. $\square$

Lemma 2.10. *Let* $\varphi_1, \varphi_2 \in \text{Lip}(\mathbb{S}^1; \mathbb{R}^2)$. *Then*

$$|P(\varphi_1) - P(\varphi_2)| \leq 2\|\varphi_1 - \varphi_2\|_\infty \left(\|\dot{\varphi}_1\|_1 + \|\dot{\varphi}_2\|_1\right). \tag{2.33}$$

Proof. Let $v \in \text{Lip}(B_1; \mathbb{R}^2)$ be such that $v = \varphi_2$ on $\mathbb{S}^1$. We define

$$w(x) = \begin{cases} v_{\frac{1}{2}}(x) = v(2x) & \text{if } |x| < \frac{1}{2}, \\ 2(1 - |x|)\varphi_2\left(\frac{x}{|x|}\right) + 2\left(|x| - \frac{1}{2}\right)\varphi_1\left(\frac{x}{|x|}\right) & \text{if } \frac{1}{2} \leq |x| \leq 1. \end{cases} \tag{2.34}$$

Then $w \in \text{Lip}(B_1; \mathbb{R}^2)$, $w(x) = \varphi_2(x/|x|)$ if $x \in \partial B_{\frac{1}{2}}$ and $w = \varphi_1$ on ∂B_1. Let us estimate

$$\int_{B_1 \setminus \overline{B_{\frac{1}{2}}}} |Jw| \, dx.$$

Writing w in polar coordinates in the annulus $B_1 \setminus \overline{B_{\frac{1}{2}}}$, $\rho \in (\frac{1}{2}, 1)$, $\theta \in [0, 2\pi)$,

$$\bar{w}(\rho, \theta) := w(\rho \cos \theta, \rho \sin \theta) = 2(1 - \rho)\bar{\varphi}_2(\theta) + 2\left(\rho - \frac{1}{2}\right)\bar{\varphi}_1(\theta),$$

where $\bar{\varphi}_i(\theta) := \varphi_i(\cos \theta, \sin \theta)$, $i = 1, 2$. Then

$$|\partial_\rho \bar{w} \wedge \partial_\theta \bar{w}| = 4 \left|(\bar{\varphi}_1(\theta) - \bar{\varphi}_2(\theta)) \wedge \left((1 - \rho)\dot{\bar{\varphi}}_2(\theta) + \left(\rho - \frac{1}{2}\right)\dot{\bar{\varphi}}_1(\theta)\right)\right|$$

$$\leq 4 \left|\bar{\varphi}_1(\theta) - \bar{\varphi}_2(\theta)\right| \left|(1 - \rho)\dot{\bar{\varphi}}_2(\theta) + \left(\rho - \frac{1}{2}\right)\dot{\bar{\varphi}}_1(\theta)\right|$$

$$\leq 4\|\varphi_1 - \varphi_2\|_\infty \left(|\dot{\bar{\varphi}}_2(\theta)| + |\dot{\bar{\varphi}}_1(\theta)|\right).$$

Thus, integrating on $B_1 \setminus \overline{B_{\frac{1}{2}}}$, by the change of variable formula,

$$\int_{B_1 \setminus \overline{B_{\frac{1}{2}}}} |Jw(x)| \, dx = \int_{\frac{1}{2}}^1 \int_0^{2\pi} \rho \left|\partial_\rho \bar{w} \wedge \frac{\partial_\theta \bar{w}}{\rho}\right| \, d\rho d\theta \tag{2.35}$$

---

[4] The construction of a Lipschitz homotopy between h and id can be done at the level of liftings, by considering the affine interpolation map (for more details, see for instance [3, Proposition 3.4]).

$$\leq 2\|\varphi_1 - \varphi_2\|_\infty \int_0^{2\pi} \left(|\dot{\tilde{\varphi}}_2(\theta)| + |\dot{\tilde{\varphi}}_1(\theta)|\right) d\theta \tag{2.36}$$

$$= 2\|\varphi_1 - \varphi_2\|_\infty \left(\|\dot{\varphi}_1\|_1 + \|\dot{\varphi}_2\|_1\right). \tag{2.37}$$

Hence

$$P(\varphi_1) \leq \int_{B_1} |Jw| \, dx \leq \int_{B_{\frac{1}{2}}} |Jv_{\frac{1}{2}}| \, dx + 2\|\varphi_1 - \varphi_2\|_\infty \left(\|\dot{\varphi}_1\|_1 + \|\dot{\varphi}_2\|_1\right). \tag{2.38}$$

Since v is a Lipschitz map such that (with the notation in (2.29)) $v_{\frac{1}{2}} = (\varphi_2)_{\frac{1}{2}}$ on $\partial B_{\frac{1}{2}}$, using (2.31) with $r = \frac{1}{2}$ we can take the infimum in (2.38) on these maps v and get

$$P(\varphi_1) - P(\varphi_2) \leq 2\|\varphi_1 - \varphi_2\|_\infty \left(\|\dot{\varphi}_1\|_1 + \|\dot{\varphi}_2\|_1\right).$$

Exchanging the role of φ_1 and φ_2 we find that also $P(\varphi_2) - P(\varphi_1)$ is bounded by the right-hand side of the previous expression. This concludes the proof. $\square$

Remark 2.11. With a similar argument used in the proof of Lemma 2.10 it is immediate to obtain that if $[a, b] \subset \mathbb{R}$ is a bounded interval and $\gamma_1, \gamma_2 : [a, b] \to \mathbb{R}^2$ are Lipschitz curves, then the following holds: Let $\Phi : [a, b] \times [0, 1] \to \mathbb{R}^2$ be the affine interpolation map $\Phi(t, s) := s\gamma_1(t) + (1 - s)\gamma_2(t)$. Then, as in (2.35),

$$\int_{[a,b]\times[0,1]} |\Phi_t \wedge \Phi_s| \, dt ds \leq \|\gamma_1 - \gamma_2\|_\infty (\|\dot{\gamma}_1\|_1 + \|\dot{\gamma}_2\|_1). \tag{2.39}$$

Using Lemma 2.10 we readily obtain the following continuity property for the minimum of the Plateau-type problem (1.5).

Corollary 2.12 (*Continuity of P*). *Let* $\varphi \in \mathrm{Lip}(\mathbb{S}^1; \mathbb{R}^2)$ *and suppose that* $(\varphi_k)_k \subset \mathrm{Lip}(\mathbb{S}^1; \mathbb{R}^2)$ *is such that*

$$\varphi_k \to \varphi \quad \text{uniformly} \quad \text{and} \quad \sup_{k\in\mathbb{N}} \|\dot{\varphi}_k\|_1 < +\infty.$$

Then $P(\varphi_k) \to P(\varphi)$ *as* $k \to +\infty$.

In what follows it is convenient to consider the relaxation

$$\overline{P}(\gamma) := \inf \left\{ \liminf_{k\to+\infty} P(\varphi_k) : \varphi_k \in \mathrm{Lip}(\mathbb{S}^1; \mathbb{R}^2), \varphi_k \to \gamma \text{ strictly } BV(\mathbb{S}^1; \mathbb{R}^2) \right\} \quad \forall \gamma \in BV(\mathbb{S}^1; \mathbb{R}^2) \tag{2.40}$$

of P with respect to the strict convergence in BV of the boundary datum. It is well known that the infimum in (2.40) is taken on a non-empty class of approximation maps. Moreover, by (2.30), also $\overline{P}$ is invariant by rescaling, i.e. $\overline{P}(\gamma) = \overline{P}(\gamma_r)$.

Lemma 2.13. *Let* $\varphi \in \mathrm{Lip}(\mathbb{S}^1; \mathbb{R}^2)$. *Then* $\overline{P}(\varphi) = P(\varphi)$.

Proof. If $(\varphi_k) \subset \mathrm{Lip}(\mathbb{S}^1; \mathbb{R}^2)$ is a sequence converging to φ strictly $BV(\mathbb{S}^1; \mathbb{R}^2)$, then by Lemma 2.6 $\varphi_k \to \varphi$ uniformly on $\mathbb{S}^1$ as $k \to +\infty$. Moreover, the strict convergence guarantees that the total variations of φ_k are equibounded. So, thanks to Corollary 2.12,

$$P(\varphi_k) \to P(\varphi) \tag{2.41}$$

as $k \to +\infty$. Since this holds for any sequence (φ_k) as above, the thesis follows. $\square$

Lemma 2.14. *Let* $\gamma \in SBV(\mathbb{S}^1; \mathbb{R}^2)$ *have a finite number of jump points* $z_i \in \mathbb{S}^1$, $i = 1, \dots, n$. *Let* $\widetilde{\gamma} : \mathbb{S}^1 \to \mathbb{R}^2$ *be the Lipschitz map in* (2.27) *(with* $\mathbb{S}^1$ *identified with* $[0, 2\pi]$*). Then*

$$\overline{P}(\gamma) = P(\widetilde{\gamma}). \tag{2.42}$$

Proof. Let $(\varphi_k)_k \subset \mathrm{Lip}(\mathbb{S}^1; \mathbb{R}^2)$ be a sequence converging strictly to γ. Let us consider a not-relabeled subsequence of $(\varphi_k)_k$; by Corollary 2.8 there are a further subsequence $(\varphi_{k_j})_j$ and Lipschitz reparametrizations $\gamma_{k_j} = \varphi_{k_j} \circ h_{k_j} \in \mathrm{Lip}(\mathbb{S}^1; \mathbb{R}^2)$ of φ_{k_j} such that $\gamma_{k_j} \to \widetilde{\gamma} \circ h$ uniformly as $j \to +\infty$, for some Lipschitz homeomorphism $h : \mathbb{S}^1 \to \mathbb{S}^1$. Moreover, since by Lemma 2.7(b) the reparametrization maps h_{k_j} can be chosen with uniformly bounded Lipschitz constants, it follows that γ_{k_j} have uniformly bounded total variations. Hence it follows from Corollary 2.12 that $P(\gamma_{k_j}) \to P(\widetilde{\gamma} \circ h)$ as $j \to +\infty$. On the other hand, by Proposition 2.9 we also have $P(\varphi_{k_j}) \to P(\widetilde{\gamma})$ as $j \to +\infty$. Finally, since this argument holds for any subsequence of (φ_k), we conclude that the whole sequence satisfies $P(\varphi_k) \to P(\widetilde{\gamma})$, and therefore $\overline{P}(\gamma) = P(\widetilde{\gamma})$. $\square$

As a consequence of the argument in the proof of Lemma 2.14, we easily infer the following continuity property:

Corollary 2.15. *Let* $\gamma \in SBV(\mathbb{S}^1; \mathbb{R}^2)$ *and* $\widetilde{\gamma}$ *be as in Corollary 2.8, and assume that* $(\varphi_k)_k \subset \mathrm{Lip}(\mathbb{S}^1; \mathbb{R}^2)$ *is a sequence converging strictly to* γ. *Then*

$$\lim_{k\to+\infty} P(\varphi_k) = \overline{P}(\gamma) = P(\widetilde{\gamma}).$$

Furthermore, we can refine the previous corollary as follows:

Corollary 2.16. *Let $\gamma, \gamma_k \in SBV(\mathbb{S}^1; \mathbb{R}^2)$, $k \geq 1$, be maps as in Corollary 2.8. Assume that (γ_k) converges to γ strictly $BV(\mathbb{S}^1; \mathbb{R}^2)$. Then*

$$\lim_{k \to +\infty} \overline{P}(\gamma_k) = \overline{P}(\gamma).$$

Proof. By Corollary 2.15 and the density of $\text{Lip}(\mathbb{S}^1; \mathbb{R}^2)$ in $BV(\mathbb{S}^1; \mathbb{R}^2)$ with respect to the strict convergence, for all $k \geq 1$ we can find $\varphi_k \in \text{Lip}(\mathbb{S}^1; \mathbb{R}^2)$ such that

$$\|\gamma_k - \varphi_k\|_1 + |\dot{\varphi}_k|(\mathbb{S}^1) - |\dot{\gamma}_k|(\mathbb{S}^1)| + \left| P(\varphi_k) - \overline{P}(\gamma_k) \right| \leq \frac{1}{k}.$$

Hence the sequence (φ_k) converges to γ strictly $BV(\mathbb{S}^1; \mathbb{R}^2)$, and by the triangle inequality and Corollary 2.15 we conclude

$$\lim_{k \to +\infty} \overline{P}(\gamma_k) = \overline{P}(\gamma). \quad \square$$

3. Relaxation on piecewise Lipschitz maps jumping on a curve

Recalling that $R = [a, b] \times [-1, 1]$, consider $R^+ = \{(x_1, x_2) \in R : x_2 > 0\}$ and $R^- = \{(x_1, x_2) \in R : x_2 < 0\}$.

Definition 3.1 (*Piecewise Lipschitz Map*). We say that a map $u : R \to \mathbb{R}^2$ is piecewise Lipschitz if $u \in BV(R; \mathbb{R}^2)$ and $u \in \text{Lip}(R^-; \mathbb{R}^2) \cap \text{Lip}(R^+; \mathbb{R}^2)$.

Thus $S_u \subseteq [a, b] \times \{0\}$; we denote $u^\pm : [a, b] \times \{0\} \to \mathbb{R}^2$ the traces of $u_{|R^\pm}$, which are Lipschitz maps. Set $I = [0, 1]$ and define $X^{\text{aff}} : [a, b] \times I \to \mathbb{R}^3$ to be the affine interpolation surface spanning $\text{graph}(u^\pm) = \{(t, u^\pm(t)) : t \in [a, b]\} \subset \mathbb{R} \times \mathbb{R}^2 = \mathbb{R}^3$, namely

$$X^{\text{aff}}(t, s) = (t, su^+(t) + (1 - s)u^-(t)) =: (t, \hat{X}(t, s)) \qquad \forall (t, s) \in [a, b] \times I. \tag{3.1}$$

Remark 3.2. For a (semicartesian) map $\Phi : [a, b] \times [c, d] \to \mathbb{R}^3$ of the form $\Phi(t, \sigma) = (t, \phi(t, \sigma)) = (t, \phi_1(t, \sigma), \phi_2(t, \sigma))$, the area integrand is given by

$$|\partial_t \Phi \wedge \partial_\sigma \Phi| = \sqrt{|\partial_\sigma \phi_1|^2 + |\partial_\sigma \phi_2|^2 + (\partial_t \phi_1 \partial_\sigma \phi_2 - \partial_\sigma \phi_1 \partial_t \phi_2)^2} = \sqrt{|\partial_\sigma \phi|^2 + |J\phi|^2}.$$

The main result of this section is the following:

Theorem 3.3 (*Relaxed Area of Piecewise Lipschitz Maps: Straight Jump*). *Let $u : R \to \mathbb{R}^2$ be a piecewise Lipschitz map. Then*

$$\overline{\mathcal{A}}_{BV}(u, R) = \mathcal{A}(u, R^+) + \mathcal{A}(u, R^-) + \int_{[a,b] \times I} |\partial_t X^{\text{aff}} \wedge \partial_s X^{\text{aff}}| \, dt \, ds. \tag{3.2}$$

Notice that the Lipschitz regularity of u on $R^\pm$ ensures that the area functional has the classical expression

$$\mathcal{A}(u, R^\pm) = \int_{R^\pm} \sqrt{1 + |\nabla u|^2 + |\det \nabla u|^2} \, dx;$$

therefore, the singular contribution produced by the relaxation in (3.2) is given by the area of X^{aff}.

We divide the proof of (3.2) in two parts: the lower bound (Proposition 3.4) and the upper bound (Proposition 3.5).

Proposition 3.4 (*Lower Bound for* (3.2)). *Let $u : R \to \mathbb{R}^2$ be a piecewise Lipschitz map, and $(v_k) \subset C^1(R; \mathbb{R}^2) \cap BV(R; \mathbb{R}^2)$ be a sequence converging to u strictly $BV(R; \mathbb{R}^2)$. Then*

$$\liminf_{k \to +\infty} \mathcal{A}(v_k, R) \geq \mathcal{A}(u, R^+) + \mathcal{A}(u, R^-) + \int_{[a,b] \times I} |\partial_t X^{\text{aff}} \wedge \partial_s X^{\text{aff}}| \, dt \, ds. \tag{3.3}$$

Proof. Fix $\varepsilon \in (0, 1)$. We have

$$\liminf_{k \to +\infty} \mathcal{A}(v_k, R) \geq \liminf_{k \to +\infty} \mathcal{A}(v_k, R \setminus ([a, b] \times [-\varepsilon, \varepsilon])) + \liminf_{k \to +\infty} \mathcal{A}(v_k, [a, b] \times [-\varepsilon, \varepsilon])$$
$$\geq \mathcal{A}(u, R \setminus ([a, b] \times [-\varepsilon, \varepsilon])) + \liminf_{k \to +\infty} \mathcal{A}(v_k, [a, b] \times [-\varepsilon, \varepsilon]),$$

where in the last inequality we used [1, Theorem 3.7]. Sending ε to 0^+, by dominated convergence it follows $\mathcal{A}(u, R \setminus ([a, b] \times [-\varepsilon, \varepsilon])) \to \mathcal{A}(u, R^+) + \mathcal{A}(u, R^-)$, so (3.3) will be proven provided we show that

$$\lim_{\varepsilon \to 0^+} \liminf_{k \to +\infty} \mathcal{A}(v_k, [a, b] \times [-\varepsilon, \varepsilon]) \geq \int_{[a,b] \times I} |\partial_t X^{\text{aff}} \wedge \partial_s X^{\text{aff}}| \, dt \, ds. \tag{3.4}$$

Consider the maps

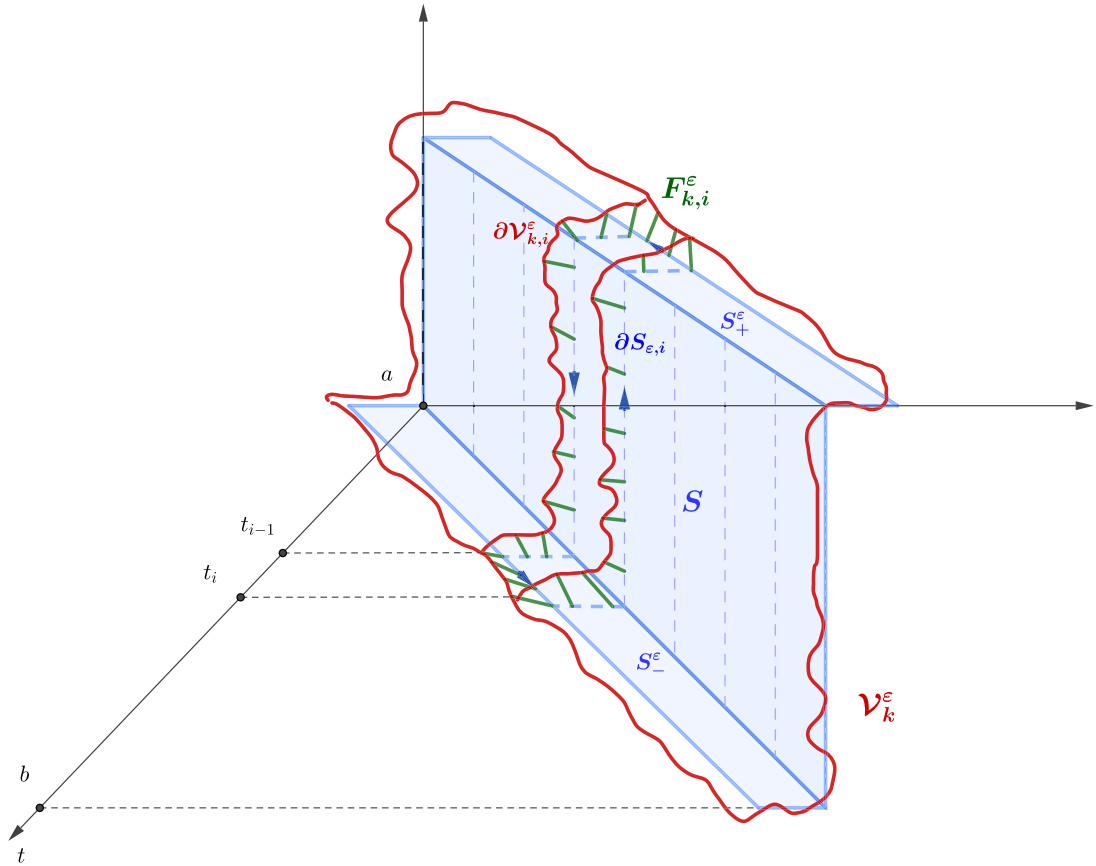$$V_k^\varepsilon : R \to \mathbb{R}^3, \quad V_k^\varepsilon(t, \sigma) = (t, v_k(t, \varepsilon \sigma)),$$

Fig. 1. Here $S = X^{\mathrm{aff}}_\sharp [\![[a,b] \times I]\!]$, $S^\varepsilon_\pm = U^\varepsilon_{\pm\sharp}[\![R^\pm]\!]$. The horizontal and vertical axes span the target space $\mathbb{R}^2$. The approximating current $\mathcal{V}^\varepsilon_k$ is depicted in bold, as well as the boundary of its restriction to R_i, i.e. the current $\partial \mathcal{V}^\varepsilon_{k,i}$. The current $\partial S_{\varepsilon,i}$ is depicted with the oriented dotted straight segments, while $F^\varepsilon_{k,i}$ is the oriented surface obtained as the union of the short segments connecting $\partial \mathcal{V}^\varepsilon_{k,i}$ and $\partial S_{\varepsilon,i}$. Finally, for simplicity, we depict with straight segments the graph of $u^\pm$ and the (semi)graph of u on $\{(t,\sigma) : \sigma = \pm\varepsilon\}$, but it is worth to remember that they are graph of Lipschitz maps.

and the associated integer multiplicity 2-currents in $\mathbb{R}^3$

$$\mathcal{V}^\varepsilon_k = V^\varepsilon_{k\,\sharp}[\![R]\!].$$

Notice that, neglecting the term $1 + |\partial_{x_1} v_k|^2$, we get

$$
\begin{aligned}
\mathcal{A}(v_k, [a,b] \times [-\varepsilon, \varepsilon]) &\geq \int_{[a,b] \times [-\varepsilon, \varepsilon]} \sqrt{|\partial_{x_2} v_k|^2 + |Jv_k|^2} \, dx \\
&= \int_R |\partial_t V^\varepsilon_k \wedge \partial_\sigma V^\varepsilon_k| \, dt d\sigma = |\mathcal{V}^\varepsilon_k|,
\end{aligned}
\tag{3.5}
$$

where we used Remark 3.2, and $|\cdot|$ stands for the mass current. Consider also the maps

$$U^\varepsilon_\pm : R^\pm \to \mathbb{R}^3, \quad U^\varepsilon_\pm(t,\sigma) = (t, u(t, \varepsilon\sigma)), \tag{3.6}$$

and the current

$$S_\varepsilon = X^{\mathrm{aff}}_\sharp [\![[a,b] \times I]\!] + U^\varepsilon_{+\sharp}[\![R^+]\!] + U^\varepsilon_{-\sharp}[\![R^-]\!], \tag{3.7}$$

see Fig. 1. We want now prove the following crucial inequality:

$$\liminf_{k \to +\infty} |\mathcal{V}^\varepsilon_k| \geq |S_\varepsilon|. \tag{3.8}$$

To show (3.8) we prove that $\mathcal{V}^\varepsilon_k$ are close to suitable currents $\mathcal{M}^\varepsilon_n$ independent of k (see (3.19)) which converge to S_ε as $n \to +\infty$.

For any $n \in \mathbb{N}$, $n \geq 1$, consider a partition $\{t_0 = a, t_1, \ldots, t_{n+1} = b\}$ of $[a,b]$ in $(n+1)$ intervals $[t_{i-1}, t_i)$, with

$$t_i - t_{i-1} \in \left(\frac{b-a}{2n}, 2\frac{(b-a)}{n} \right). \tag{3.9}$$

Moreover, set

$$R_i = [t_{i-1}, t_i) \times [-1, 1], \quad R_i^+ = [t_{i-1}, t_i) \times (0, 1], \quad R_i^- = [t_{i-1}, t_i) \times [-1, 0),$$

and define the currents

$$\mathcal{V}_{k,i}^\varepsilon = V_{k\,\sharp}^\varepsilon [\![R_i]\!], \quad S_{\varepsilon,i} = X_\sharp^{\mathrm{aff}} [\![[t_{i-1}, t_i) \times I]\!] + U_{+\sharp}^\varepsilon [\![R_i^+]\!] + U_{-\sharp}^\varepsilon [\![R_i^-]\!], \tag{3.10}$$

see Fig. 1. By definition, we have

$$\mathcal{V}_k^\varepsilon = \sum_{i=1}^{n+1} \mathcal{V}_{k,i}^\varepsilon \quad \text{and} \quad \mathcal{H}^2(\mathrm{spt}\,\mathcal{V}_{k,i}^\varepsilon \cap \mathrm{spt}\,\mathcal{V}_{k,j}^\varepsilon) = 0 \quad \text{for } i \neq j,$$

$$\tag{3.11}$$

$$S_\varepsilon = \sum_{i=1}^{n+1} S_{\varepsilon,i} \quad \text{and} \quad \mathcal{H}^2(\mathrm{spt}\,S_{\varepsilon,i} \cap \mathrm{spt}\,S_{\varepsilon,j}) = 0 \quad \text{for } i \neq j.$$

Furthermore,

$$\begin{aligned}
\partial S_{\varepsilon,i} = &- \Big(U_{-\sharp}^\varepsilon [\![\{t_{i-1}\} \times [-1, 0)]\!] + X_\sharp^{\mathrm{aff}} [\![\{t_{i-1}\} \times I]\!] + U_{+\sharp}^\varepsilon [\![\{t_{i-1}\} \times (0, 1]]\!] \Big) \\
&- U_{+\sharp}^\varepsilon [\![(t_{i-1}, t_i) \times \{1\}]\!] \\
&+ \Big(U_{-\sharp}^\varepsilon [\![\{t_i\} \times [-1, 0)]\!] + X_\sharp^{\mathrm{aff}} [\![\{t_i\} \times I]\!] + U_{+\sharp}^\varepsilon [\![\{t_i\} \times (0, 1]]\!] \Big) \\
&+ U_{-\sharp}^\varepsilon [\![(t_{i-1}, t_i) \times \{-1\}]\!].
\end{aligned} \tag{3.12}$$

Now, for fixed $i \in \{1, \dots, n\}$, set

$$\begin{aligned}
\gamma_{-,i}^{u,\varepsilon}(\sigma) &= u(t_i, \varepsilon\sigma) & \forall \sigma \in [-1, 0), \\
\gamma_{+,i}^{u,\varepsilon}(\sigma) &= u(t_i, \varepsilon\sigma) & \forall \sigma \in (0, 1], \\
\gamma_i^0(s) &= su^+(t_i) + (1 - s)u^-(t_i) & \forall s \in I, \\
\Lambda_{u,i}^{\pm,\varepsilon}(t) &= (t, u(t, \pm\varepsilon)) & \forall t \in [t_{i-1}, t_i],
\end{aligned}$$

and define $\gamma_i^{u,\varepsilon} : [-1, 1] \to \mathbb{R}^2$ as in (2.15) where $\widetilde{\gamma}^-$, S, and $\widetilde{\gamma}^+$ are replaced by $\gamma_{-,i}^{u,\varepsilon}$, γ_i^0 and $\gamma_{+,i}^{u,\varepsilon}$ in the order, after a rescaling on $\left[-1, -\frac{1}{3}\right]$, $\left[-\frac{1}{3}, \frac{1}{3}\right]$, and $\left[\frac{1}{3}, 1\right]$, respectively, as in the statement of Lemma 2.7. Also, define $\Gamma_i^{u,\varepsilon} : [-1, 1] \to (\{t_i\} \times \mathbb{R}^2)$ as

$$\Gamma_i^{u,\varepsilon}(\sigma) := (t_i, \gamma_i^{u,\varepsilon}(\sigma)) \quad \forall \sigma \in [-1, 1].$$

Using the definition of $U_\pm^\varepsilon$ and X^{aff}, by (3.12) we infer

$$\partial S_{\varepsilon,i} = -\Gamma_{i-1\,\sharp}^{u,\varepsilon} [\![[-1, 1]]\!] - \Lambda_{u,i\,\sharp}^{+,\varepsilon} [\![(t_{i-1}, t_i)]\!] + \Gamma_{i\,\sharp}^{u,\varepsilon} [\![[-1, 1]]\!] + \Lambda_{u,i\,\sharp}^{-,\varepsilon} [\![(t_{i-1}, t_i)]\!]. \tag{3.13}$$

Moreover, set

$$\begin{aligned}
\gamma_{k,i}^\varepsilon(\sigma) &= v_k(t_i, \varepsilon\sigma), \quad \Gamma_{k,i}^\varepsilon(\sigma) = (t_i, \gamma_{k,i}^\varepsilon(\sigma)) & \forall \sigma \in [-1, 1], \\
\Lambda_{k,i}^{\pm,\varepsilon}(t) &= (t, v_k(t, \pm\varepsilon)) & \forall t \in [t_{i-1}, t_i].
\end{aligned}$$

By definition of $\mathcal{V}_{k,i}^\varepsilon$ in (3.10), we also have

$$\partial \mathcal{V}_{k,i}^\varepsilon = -\Gamma_{k,i-1\,\sharp}^\varepsilon [\![[-1, 1]]\!] - \Lambda_{k,i\,\sharp}^{+,\varepsilon} [\![(t_{i-1}, t_i)]\!] + \Gamma_{k,i\,\sharp}^\varepsilon [\![[-1, 1]]\!] + \Lambda_{k,i\,\sharp}^{-,\varepsilon} [\![(t_{i-1}, t_i)]\!]. \tag{3.14}$$

We now define $F_{k,i}^\varepsilon \in \mathcal{D}_2(\mathbb{R}^3)$ as a suitable affine interpolation between $\partial \mathcal{V}_{k,i}^\varepsilon$ and $\partial S_{\varepsilon,i}$, see Fig. 1. First observe that by Lemma 2.3, we can suppose that, for our choice of ε and $\{t_1, \dots, t_n\}$, there exists a (not relabeled) subsequence of $(v_k)_k$, such that

$$v_k(t_i, \varepsilon\cdot) \to u(t_i, \varepsilon\cdot) \quad \text{strictly } BV([-1, 1]; \mathbb{R}^2) \quad \forall i = 1, \dots, n, \tag{3.15}$$

$$v_k(\cdot, \pm\varepsilon) \to u(\cdot, \pm\varepsilon) \quad \text{strictly } BV([a, b]; \mathbb{R}^2). \tag{3.16}$$

In particular, by Lemma 2.7, we know that there are increasing Lipschitz bijections $h_{k,i}^\varepsilon, h_i^\varepsilon : [-1, 1] \to [-1, 1]$ such that $\gamma_{k,i}^\varepsilon \circ h_{k,i}^\varepsilon \to \gamma_i^{u,\varepsilon} \circ h_i^\varepsilon$ uniformly in $[-1, 1]$ as $k \to +\infty$.

For $i = 1, \dots, n$, we define

$$\begin{aligned}
\Phi_{k,i}^\varepsilon(\sigma, s) &:= s(\Gamma_{k,i}^\varepsilon \circ h_{k,i}^\varepsilon(\sigma)) + (1 - s)(\Gamma_i^{u,\varepsilon} \circ h_i^\varepsilon(\sigma)), & (\sigma, s) \in [-1, 1] \times I, \\
\Psi_{k,i}^{\pm,\varepsilon}(t, s) &:= s\Lambda_{k,i}^{\pm,\varepsilon}(t) + (1 - s)\Lambda_{u,i}^{\pm,\varepsilon}(t), & (t, s) \in [t_{i-1}, t_i] \times I.
\end{aligned}$$

Therefore we set

$$\begin{aligned}
F_{k,i}^\varepsilon = &- \Phi_{k,i-1\,\sharp}^\varepsilon [\![[-1, 1] \times I]\!] - \Psi_{k,i\,\sharp}^{+,\varepsilon} [\![[t_{i-1}, t_i] \times I]\!] \\
&+ \Phi_{k,i\,\sharp}^\varepsilon [\![[-1, 1] \times I]\!] + \Psi_{k,i\,\sharp}^{-,\varepsilon} [\![[t_{i-1}, t_i] \times I]\!].
\end{aligned} \tag{3.17}$$

In particular, from (3.13) and (3.14), a direct check shows that

$$\partial F_{k,i}^{\varepsilon} = \partial \mathcal{V}_{k,i}^{\varepsilon} - \partial S_{\varepsilon,i}. \tag{3.18}$$

Eventually, we let $M_{\varepsilon,i}$ be an integer multiplicity 2-current of $\mathbb{R}^3$ with minimal mass and boundary $\partial S_{\varepsilon,i}$ (the existence of $M_{\varepsilon,i}$ is guaranteed, for instance, by [19, Theorem 8.3.3]) and set

$$\mathcal{M}_n^{\varepsilon} := \sum_{i=2}^{n} M_{\varepsilon,i}. \tag{3.19}$$

Note carefully that we do not sum over i from 1 to $n+1$, but only from 2 to n. In particular, setting $S_{\varepsilon}^n = S_{\varepsilon} - S_{\varepsilon,1} - S_{\varepsilon,n+1}$, we have

$$\partial \mathcal{M}_n^{\varepsilon} = \partial S_{\varepsilon}^n = -\Gamma_1^{u,\varepsilon}{}_{\sharp}[\![-1,1]\!] + \Gamma_n^{u,\varepsilon}{}_{\sharp}[\![-1,1]\!] - \Lambda_u^{+,\varepsilon}{}_{\sharp}[\![t_1,t_n]\!] + \Lambda_u^{-,\varepsilon}{}_{\sharp}[\![t_1,t_n]\!], \tag{3.20}$$

where

$$\Lambda_u^{\pm,\varepsilon}(t) := (t, u(t, \pm\varepsilon)), \qquad t \in (t_1, t_n).$$

Thus, we have

$$|\mathcal{V}_{k,i}^{\varepsilon}| \geq |\mathcal{V}_{k,i}^{\varepsilon} - F_{k,i}^{\varepsilon}| - |F_{k,i}^{\varepsilon}| \geq |M_{\varepsilon,i}| - |F_{k,i}^{\varepsilon}| \quad \text{for } i = 2, \ldots, n,$$

where we used the minimality of $M_{\varepsilon,i}$ and (3.18). By summing up, using (3.11), we get[5]

$$|\mathcal{V}_k^{\varepsilon}| = \sum_{i=1}^{n+1} |\mathcal{V}_{k,i}^{\varepsilon}| \geq \sum_{i=2}^{n} |\mathcal{V}_{k,i}^{\varepsilon}| \geq \sum_{i=2}^{n} |M_{\varepsilon,i}| - \sum_{i=2}^{n} |F_{k,i}^{\varepsilon}| \geq |\mathcal{M}_n^{\varepsilon}| - \sum_{i=2}^{n} |F_{k,i}^{\varepsilon}|. \tag{3.21}$$

Therefore,

$$\liminf_{k \to +\infty} |\mathcal{V}_k^{\varepsilon}| \geq |\mathcal{M}_n^{\varepsilon}| - \sum_{i=2}^{n} \limsup_{k \to +\infty} |F_{k,i}^{\varepsilon}|. \tag{3.22}$$

In order to obtain (3.8), we have to prove that:

(i) $|F_{k,i}^{\varepsilon}| \to 0$ as $k \to +\infty$ for every $i = 2, \ldots, n$;

(ii) $\mathcal{M}_n^{\varepsilon} \rightharpoonup S_{\varepsilon}$ as $n \to +\infty$,

so that (3.8) would follow by lower semicontinuity of the mass and (3.22).

(i). Since $\gamma_{k,i}^{\varepsilon} \circ h_{k,i}^{\varepsilon} \to \gamma_i^{u,\varepsilon} \circ h_i^{\varepsilon}$ uniformly in $[-1, 1]$ as $k \to +\infty$, also $\Gamma_{k,i}^{\varepsilon} \circ h_{k,i}^{\varepsilon} \to \Gamma_i^{u,\varepsilon} \circ h_i^{\varepsilon}$ uniformly; moreover, by Lemma 2.6 and thanks to (3.16), $v_k(\cdot, \pm\varepsilon) \to u(\cdot, \pm\varepsilon)$ uniformly in $[t_{i-1}, t_i]$, and the same holds for $\Lambda_{k,i}^{\pm,\varepsilon}$ and $\Lambda_{u,i}^{\pm,\varepsilon}$. Finally, by (3.15) and (3.16), and recalling also Lemma 2.7(b), the L^1-norm of the derivative of $\Gamma_{k,i}^{\varepsilon} \circ h_{k,i}^{\varepsilon}$ and of $\Lambda_{k,i}^{\pm,\varepsilon}$ is uniformly bounded with respect to k. Hence (i) readily follows from the definition of $F_{k,i}^{\varepsilon}$ in (3.17) and Remark 2.11.

(ii). First observe that $\partial \mathcal{M}_n^{\varepsilon}$ has mass uniformly bounded with respect to n. Indeed by (3.20)

$$|\partial \mathcal{M}_n^{\varepsilon}| = |\partial S_{\varepsilon}^n| \leq |\dot{\gamma}_1^{u,\varepsilon}|([-1,1]) + |\dot{\gamma}_n^{u,\varepsilon}|([-1,1]) + \int_a^b \sqrt{1 + |\partial_t u(t, \varepsilon)|^2}\, dt + \int_a^b \sqrt{1 + |\partial_t u(t, -\varepsilon)|^2}\, dt$$

$$\leq C(\varepsilon, \|u\|_{\infty}, \mathrm{lip}(u_{|R^+}), \mathrm{lip}(u_{|R^-})).$$

Moreover, by minimality of $\mathcal{M}_n^{\varepsilon}$ and (3.11), $|\mathcal{M}_n^{\varepsilon}| \leq |S_{\varepsilon}^n| \leq |S_{\varepsilon}|$, hence the sequence $(\mathcal{M}_n^{\varepsilon})_n$ is compactly supported in $\mathbb{R}^3$ and has bounded mass and bounded boundary mass. Then, by [19, Theorem 8.2.1], we have

$$\mathcal{M}_n^{\varepsilon} \rightharpoonup S_{\varepsilon} \iff \|\mathcal{M}_n^{\varepsilon} - S_{\varepsilon}\|_F \to 0 \quad \text{as } n \to +\infty,$$

where $\|\cdot\|_F$ stands for the flat norm. Then, we are reduced to show that $\|\mathcal{M}_n^{\varepsilon} - S_{\varepsilon}\|_F \to 0$ as $n \to +\infty$. Notice that

$$\|\mathcal{M}_n^{\varepsilon} - S_{\varepsilon}\|_F \leq \sum_{i=2}^{n} \|M_{\varepsilon,i} - S_{\varepsilon,i}\|_F + \|S_{\varepsilon,1}\|_F + \|S_{\varepsilon,n+1}\|_F, \tag{3.23}$$

where, by definition of flat norm (see [15, Sec. 5.1.3]),

$$\|M_{\varepsilon,i} - S_{\varepsilon,i}\|_F \leq \inf\{|G_i^{\varepsilon}| : G_i^{\varepsilon} \text{ integer multiplicity 3-current s.t. } \partial G_i^{\varepsilon} = M_{\varepsilon,i} - S_{\varepsilon,i}\}.$$

Observe that the class of competitors in the above minimum problem is non empty, since it contains the affine interpolation current between $M_{\varepsilon,i}$ and $S_{\varepsilon,i}$. So, pick a 3-current G_i^{ε} such that $\partial G_i^{\varepsilon} = M_{\varepsilon,i} - S_{\varepsilon,i}$; then

$$|G_i^{\varepsilon}| \leq C|\partial G_i^{\varepsilon}|^{\frac{3}{2}}$$

---

[5] In (3.21) we had to remove the first and last term of the sum, because condition (i) can be false for $i = 1$ and $i = n+1$, since the strict convergence is inherited only on almost every line, as stated in Lemma 2.3.

by the isoperimetric inequality [19, Theorem 7.9.1], for an absolute positive constant $C > 0$. For $i = 2, \ldots, n$, we have

$$\|M_{\varepsilon,i} - S_{\varepsilon,i}\|_F \le |G_i^\varepsilon| \le C|\partial G_i^\varepsilon|^{\frac{3}{2}} = C|M_{\varepsilon,i} - S_{\varepsilon,i}|^{\frac{3}{2}} \le C\left(|M_{\varepsilon,i}|^{\frac{3}{2}} + |S_{\varepsilon,i}|^{\frac{3}{2}}\right) \le 2C|S_{\varepsilon,i}|^{\frac{3}{2}}, \tag{3.24}$$

where in the last inequality we used the minimality of $M_{\varepsilon,i}$. Now let us prove that $|S_{\varepsilon,i}| \le \frac{C}{n}$ for every $i = 1, \ldots, n+1$, where C is a constant independent of n. We start observing that

$$
\begin{aligned}
|X_\sharp^{\mathrm{aff}} [\![(t_{i-1}, t_i) \times I]\!]| &= \int_{[t_{i-1}, t_i] \times I} |\partial_t X^{\mathrm{aff}} \wedge \partial_s X^{\mathrm{aff}}| \, dt ds \\
&= \int_{t_{i-1}}^{t_i} \int_I |(1, s\dot{u}^+ + (1-s)\dot{u}^-) \wedge (0, u^+ - u^-)| \, dt ds \\
&\le \int_{t_{i-1}}^{t_i} \int_I \left(|u^+ - u^-| + \left|(s\dot{u}_1^+ + (1-s)\dot{u}_1^-)(u_2^+ - u_2^-) - (s\dot{u}_2^+ + (1-s)\dot{u}_2^-)(u_1^+ - u_1^-)\right|\right) \, dt ds \\
&\le \frac{C_1}{n}\|u^+ - u^-\|_{L^\infty(a,b)} + \frac{C_2}{n}\|u^+ - u^-\|_{L^\infty(a,b)} \left(\|\dot{u}^+\|_{L^\infty(a,b)} + \|\dot{u}^-\|_{L^\infty(a,b)}\right) \\
&= \frac{C}{n},
\end{aligned}
$$

where we used (3.9). Moreover, recalling (3.6), we have

$$
\begin{aligned}
|U_{\pm\sharp}^\varepsilon [\![R_i^\pm]\!]| &= \int_{R_i^\pm} |\partial_t U_\pm^\varepsilon \wedge \partial_\sigma U_\pm^\varepsilon| \, dt d\sigma \\
&= \int_{R_i^\pm} |(1, \partial_t u(t, \varepsilon\sigma)) \wedge (0, \varepsilon \partial_\sigma u(t, \varepsilon\sigma))| \, dt d\sigma \\
&\le \varepsilon \int_{R_i^\pm} |\partial_\sigma u(t, \varepsilon\sigma)| \, dt d\sigma + \varepsilon \int_{R_i^\pm} |\partial_t u_1(t, \varepsilon\sigma) \partial_\sigma u_2(t, \varepsilon\sigma) - \partial_t u_2(t, \varepsilon\sigma) \partial_\sigma u_1(t, \varepsilon\sigma)| \, dt d\sigma \\
&\le \varepsilon \frac{C_3}{n} \left(\|\nabla u\|_{L^\infty(R^\pm)} + \|\nabla u\|_{L^\infty(R^\pm)}^2\right) \\
&= \frac{C\varepsilon}{n}.
\end{aligned}
\tag{3.25}
$$

Thus,

$$|S_{\varepsilon,i}| \le |X_\sharp^{\mathrm{aff}} [\![(t_{i-1}, t_i) \times I]\!]| + |U_{+\sharp}^\varepsilon [\![R_i^+]\!]| + |U_{-\sharp}^\varepsilon [\![R_i^-]\!]| \le \frac{C}{n},$$

as claimed. Finally, by definition of flat norm and the isoperimetric inequality, $\|S_{\varepsilon,i}\|_F \le |S_{\varepsilon,i}|^{\frac{3}{2}}$ for $i = 1, \ldots, n+1$, so that, from (3.24) and (3.23), we obtain

$$\|\mathcal{M}_n^\varepsilon - S_\varepsilon\|_F \le C(n-1)\frac{1}{n^{\frac{3}{2}}} + \frac{C}{n^{\frac{3}{2}}} \le \frac{C}{n^{\frac{1}{2}}} + \frac{C}{n^{\frac{3}{2}}} \to 0$$

as $n \to +\infty$. This concludes the proof of (ii) and hence of (3.8).

We are now in a position to show (3.4). From (3.5) and (3.8),

$$\liminf_{k \to +\infty} \mathcal{A}(v_k, [a,b] \times [-\varepsilon, \varepsilon]) \ge \liminf_{k \to +\infty} |\mathcal{V}_k^\varepsilon| \ge |S_\varepsilon|. \tag{3.26}$$

As in (3.25), we have

$$|U_{\pm\sharp}^\varepsilon [\![R^\pm]\!]| \le \varepsilon \left(\|\nabla u\|_{L^\infty(R^\pm)} + \|\nabla u\|_{L^\infty(R^\pm)}^2\right) \to 0 \quad \text{as } \varepsilon \to 0^+,$$

so, from (3.26) and (3.7), we conclude

$$\lim_{\varepsilon \to 0^+} \liminf_{k \to +\infty} \mathcal{A}(v_k, [a,b] \times [-\varepsilon, \varepsilon]) \ge \lim_{\varepsilon \to 0^+} |S_\varepsilon| = |X_\sharp^{\mathrm{aff}} [\![(a,b) \times I]\!]| = \int_{[a,b] \times I} |\partial_t X^{\mathrm{aff}} \wedge \partial_s X^{\mathrm{aff}}| \, dt ds. \quad \square$$

Proposition 3.5 (*Upper Bound for* (3.2)). *Let* $u : R \to \mathbb{R}^2$ *be a piecewise Lipschitz map. Then there exists a sequence* $(v_k)_k \subset C^1(R; \mathbb{R}^2)$ *converging to* u *strictly* $BV(R; \mathbb{R}^2)$ *such that*

$$\limsup_{k \to +\infty} \mathcal{A}(v_k, R) \le \mathcal{A}(u, R^+) + \mathcal{A}(u, R^-) + \int_{[a,b] \times I} |\partial_t X^{\mathrm{aff}} \wedge \partial_s X^{\mathrm{aff}}| \, dt ds. \tag{3.27}$$

Proof. Although v_k needs to be of class C^1, we claim that it suffices to build v_k just Lipschitz continuous. Indeed, assume that $(v_k)_k \subset W^{1,\infty}(R; \mathbb{R}^2)$ converges to u strictly $BV(R; \mathbb{R}^2)$ and (3.27) holds. Consider, for all $k \in \mathbb{N}$, a sequence $(v_h^k)_h \subset C^1(R; \mathbb{R}^2)$ approaching v_k in $W^{1,2}(R; \mathbb{R}^2)$ as $h \to +\infty$. In particular, we get the L^1-convergence of all minors of ∇v_h^k to the corresponding ones of ∇v_k. Then, by dominated convergence,

$$\lim_{h \to +\infty} \mathcal{A}(v_h^k, R) = \mathcal{A}(v_k, R). \tag{3.28}$$

Hence, by a diagonal argument, we find a sequence $(v_{h_k}^k)_k$ converging to u strictly $BV(R; \mathbb{R}^2)$ such that (3.27) holds for $v_{h_k}^k$ in place of v_k.

Set for simplicity $\varepsilon = \varepsilon_k = \frac{1}{k}$, and define the sequence $(v_\varepsilon) \subset \mathrm{Lip}(R; \mathbb{R}^2)$ as

$$v_\varepsilon(t, \sigma) := \begin{cases} u(t, \sigma) & (t, \sigma) \in R \setminus ([a,b] \times [-\varepsilon, \varepsilon]), \\ \frac{\varepsilon + \sigma}{2\varepsilon} u(t, \varepsilon) + \frac{\varepsilon - \sigma}{2\varepsilon} u(t, -\varepsilon) & (t, \sigma) \in [a,b] \times (-\varepsilon, \varepsilon). \end{cases} \tag{3.29}$$

First, let us check that $v_\varepsilon \to u$ strictly $BV(R; \mathbb{R}^2)$ as $\varepsilon \to 0^+$. Clearly, $v_\varepsilon \to u$ in $L^1(R; \mathbb{R}^2)$. Hence, by lower semicontinuity of the total variation, it is enough to show that

$$\limsup_{\varepsilon \to 0^+} \int_R |\nabla v_\varepsilon| dt d\sigma \le |Du|(R),$$

which in turn reduces to prove

$$\limsup_{\varepsilon \to 0^+} \int_{[a,b] \times [-\varepsilon, \varepsilon]} |\nabla v_\varepsilon| dt d\sigma \le |Du|([a,b] \times \{0\}),$$

since

$$\int_{R \setminus ([a,b] \times [-\varepsilon, \varepsilon])} |\nabla v_\varepsilon| dt d\sigma = \int_{R \setminus ([a,b] \times [-\varepsilon, \varepsilon])} |\nabla u| dt d\sigma \to \int_{R^+} |\nabla u| dt d\sigma + \int_{R^-} |\nabla u| dt d\sigma \quad \text{as } \varepsilon \to 0^+.$$

For almost every $t \in [a,b]$ and every $\sigma \in [-\varepsilon, \varepsilon]$, one has

$$\partial_t v_\varepsilon(t, \sigma) = \frac{\varepsilon + \sigma}{2\varepsilon} \partial_t u(t, \varepsilon) + \frac{\varepsilon - \sigma}{2\varepsilon} \partial_t u(t, -\varepsilon), \quad \partial_\sigma v_\varepsilon(t, \sigma) = \frac{1}{2\varepsilon}(u(t, \varepsilon) - u(t, -\varepsilon)).$$

Thus, setting $M := \max\{\mathrm{lip}(u_{|R^-}), \mathrm{lip}(u_{|R^+})\}$, we get

$$\int_{[a,b] \times [-\varepsilon, \varepsilon]} |\nabla v_\varepsilon| \, dt d\sigma \le \int_{[a,b] \times [-\varepsilon, \varepsilon]} |\partial_t v_\varepsilon(t, \sigma)| \, dt d\sigma + \int_{[a,b] \times [-\varepsilon, \varepsilon]} |\partial_\sigma v_\varepsilon(t, \sigma)| \, dt d\sigma$$

$$\le M \int_{[a,b] \times [-\varepsilon, \varepsilon]} dt d\sigma + \int_{[a,b] \times [-\varepsilon, \varepsilon]} \frac{1}{2\varepsilon} |u(t, \varepsilon) - u(t, -\varepsilon)| \, dt d\sigma$$

$$= M(b-a)2\varepsilon + \int_a^b |u(t, \varepsilon) - u(t, -\varepsilon)| \, dt$$

$$\xrightarrow{\varepsilon \to 0^+} \int_a^b |u^+(t) - u^-(t)| \, dt = |Du|([a,b] \times \{0\}).$$

Furthermore, since u is piecewise Lipschitz, we have

$$\mathcal{A}(v_\varepsilon, R \setminus [a,b] \times [-\varepsilon, \varepsilon]) = \mathcal{A}(u, R \setminus [a,b] \times [-\varepsilon, \varepsilon]) \to \mathcal{A}(u, R^+) + \mathcal{A}(u, R^-) \quad \text{as } \varepsilon \to 0^+.$$

So it remains to prove that

$$\limsup_{\varepsilon \to 0^+} \mathcal{A}(v_\varepsilon, [a,b] \times [-\varepsilon, \varepsilon]) \le \int_{[a,b] \times I} |\partial_t X^{\mathrm{aff}} \wedge \partial_s X^{\mathrm{aff}}| \, dt ds. \tag{3.30}$$

Let us linearly reparametrize X^{aff} on $R = [a,b] \times [-1,1]$, namely consider Y, having the same image as X^{aff}, given by

$$Y(t, \sigma) = (t, \widehat{Y}(t, \sigma)) = \left(t, \frac{1+\sigma}{2} u^+(t) + \frac{1-\sigma}{2} u^-(t)\right), \qquad (t, \sigma) \in R.$$

Now, using the trivial inequality $\sqrt{1 + a^2 + b^2 + c^2} \le 1 + |a| + \sqrt{b^2 + c^2}$, we find

$$\mathcal{A}(v_\varepsilon, [a,b] \times [-\varepsilon, \varepsilon]) \le \int_{[a,b] \times [-\varepsilon, \varepsilon]} dt d\sigma + \int_{[a,b] \times [-\varepsilon, \varepsilon]} |\partial_t v_\varepsilon| \, dt d\sigma + \int_{[a,b] \times [-\varepsilon, \varepsilon]} \sqrt{|\partial_\sigma v_\varepsilon|^2 + |J v_\varepsilon|^2} dt d\sigma$$

$$= 2\varepsilon(b-a) + 2\varepsilon \int_R |\partial_t \tilde{v}_\varepsilon| \, dt d\sigma + \int_R \sqrt{|\partial_\sigma \tilde{v}_\varepsilon|^2 + |J \tilde{v}_\varepsilon|^2} \, dt d\sigma, \tag{3.31}$$

where $\tilde{v}_\varepsilon : R \to \mathbb{R}^2$ is defined as $\tilde{v}_\varepsilon(t, \sigma) = v_\varepsilon(t, \varepsilon\sigma)$. A direct computation based in (3.29) gives

$$\partial_t \tilde{v}_\varepsilon(t, \sigma) = \frac{1+\sigma}{2} \partial_t u(t, \varepsilon) + \frac{1-\sigma}{2} \partial_t u(t, -\varepsilon) \quad \text{for a.e. } t \in [a,b] \quad \forall \sigma \in [-1, 1]$$

$$\partial_\sigma \tilde{v}_\varepsilon(t, \sigma) = \varepsilon \partial_\sigma v_\varepsilon(t, \varepsilon\sigma) = \frac{u(t, \varepsilon) - u(t, -\varepsilon)}{2} \quad \text{for a.e. } t \in [a,b] \quad \forall \sigma \in [-1, 1].$$

Then we have

$$\partial_t \tilde{v}_\varepsilon(t, \sigma) \to \frac{1+\sigma}{2} \dot{u}^+(t) + \frac{1-\sigma}{2} \dot{u}^-(t) = \partial_t \widehat{Y}(t, \sigma) \quad \text{a.e. in } R,$$

$$\partial_\sigma \tilde{v}_\varepsilon(t, \sigma) \to \frac{u^+(t) - u^-(t)}{2} = \partial_\sigma \widehat{Y}(t, \sigma) \quad \text{a.e. in } R.$$

Since $\partial_\sigma \widehat{Y}$ and $\partial_t \widehat{Y}$ are in $L^\infty(R; \mathbb{R}^2)$, by dominated convergence we can pass to the limit in (3.31) as $\varepsilon \to 0^+$, so that, using Remark 3.2, we obtain (3.30). $\square$

Remark 3.6. After having proved the upper bound inequality in Proposition 3.5, we readily infer that $\overline{\mathcal{A}}_{BV}(u, R) < +\infty$. Hence Proposition 3.4 can be deduced from an argument independently developed in [23], based on the theory of Cartesian currents [16]. Indeed, consider $T_u := G_u + S$, where G_u is the 2-current on $R \times \mathbb{R}^2$ carried by the graph of u and S is the 2-current on $R \times \mathbb{R}^2$ given by $S := \tilde{X}_\sharp [\![[a, b] \times I]\!]$, where

$$\tilde{X}(t, s) := (t, 0, \hat{X}(t, s)) = (t, 0, su^+(t) + (1 - s)u^-(t)), \qquad t \in [a, b], s \in I.$$

Clearly, the mass ot T_u is given by

$$|T_u| = |G_u| + |S| = \mathcal{A}(u, R^+) + \mathcal{A}(u, R^-) + \int_{[a,b] \times I} |\partial_t \tilde{X} \wedge \partial_s \tilde{X}| \, dt ds$$

$$= \mathcal{A}(u, R^+) + \mathcal{A}(u, R^-) + \int_{[a,b] \times I} |\partial_t X^{\mathrm{aff}} \wedge \partial_s X^{\mathrm{aff}}| \, dt ds.$$

Now we claim that T_u is the unique Cartesian current on $R \times \mathbb{R}^2$ with minimal completely vertical lifting associated to u, according to [23, Definition 3.1]. Borrowing the notation from [23], this definition is given by imposing that the mixed components of T_u are the minimal lifting measures $\mu_i^j[u]$ associated to u in the sense of Jerrard and Jung [18]. Once the claim is proven, by the lower semicontinuity of the mass and the continuity of the lifting measures with respect to the strict convergence (see [18, Theorem 1.1]), we deduce

$$|T_u| \le \overline{\mathcal{A}}_{BV}(u, R),$$

i.e., inequality (3.3).

In order to show the claim, we start to prove that $T_u \in \mathrm{cart}(R, \mathbb{R}^2)$. For this, it is enough to see that $(\partial T_u) \llcorner (R \times \mathbb{R}^2) = 0$: We get

$$(\partial G_u) \llcorner (R \times \mathbb{R}^2) = \hat{X}_\sharp^- [\![[a, b]]\!] - \hat{X}_\sharp^+ [\![[a, b]]\!] = -\partial \tilde{X}_\sharp [\![[a, b] \times I]\!] = -(\partial S) \llcorner (R \times \mathbb{R}^2),$$

where $\hat{X}^\pm(t) := (t, 0, u^\pm(t))$, $t \in [a, b]$. Next, what remains to prove is that the vertical component of T_u is the minimal completely vertical lifting associated to u. To this purpose, denote by $x = (x^1, x^2)$ the (horizontal) variable of R, $y = (y^1, y^2)$ the vertical variable of $\mathbb{R}^2$ and $u = (u^1, u^2)$ the components of u. We have to check that

$$\mu_i^j[T_u] = \mu_i^j[u] \quad \forall i, j = 1, 2, \tag{3.32}$$

where $\mu_i^j[T_u] := T_u \llcorner ((-1)^i dx^{\bar{i}} \wedge dy^j)$. By [18, Theorem 2.2], for every $f \in C_c^\infty(R \times \mathbb{R}^2)$,

$$\int_{R \times \mathbb{R}^2} f(x, y) d\mu_j^i[u] = \int_{R^+ \cup R^-} f(x, u(x))\partial_i u^j dx + \int_a^b \left(\int_0^1 f(t, 0, \hat{X}(t, s))ds \right)(u^{j^+} - u^{j^-})\delta_{i2} \, dt,$$

where δ_{ij} is the Kronecker symbol. On the other hand, setting $\omega(x, y) := (-1)^i f(x, y)dx^{\bar{i}} \wedge dy^j$, we have

$$\int_{R \times \mathbb{R}^2} f(x, y) d\mu_j^i[T_u] = \int_{R^+ \cup R^-} f(x, u(x))\partial_i u^j dx + \int_{\tilde{X}([a,b] \times I)} \omega$$

$$= \int_{R^+ \cup R^-} f(x, u(x))\partial_i u^j dx + \int_{[a,b] \times I} \omega(\tilde{X}(t, s))d\tilde{X}^{\bar{i}j},$$

where, if $\tilde{X} = (\tilde{X}_1^1, \tilde{X}_2^1, \tilde{X}_1^2, \tilde{X}_2^2)$, then $d\tilde{X}^{\bar{i}j} = d\tilde{X}_1^{\bar{i}} \wedge d\tilde{X}_2^j$. Notice that $d\tilde{X}^{\bar{i}j} = 0$ if $\bar{i} = 2$ and $d\tilde{X}^{1j} = (u^{j^+} - u^{j^-}) \, dt \wedge ds$, so we get

$$\int_{[a,b] \times I} \omega(\tilde{X}(t, s))d\tilde{X}^{\bar{i}j} = \int_{[a,b] \times I} (-1)^i f(\tilde{X}(t, s))(u^{j^+} - u^{j^-})\delta_{i2} \, dt \wedge ds$$

$$= \int_a^b \left(\int_0^1 f(t, 0, \hat{X}(t, s))ds \right)(u^{j^+} - u^{j^-})\delta_{i2} \, dt,$$

and (3.32) follows.

3.1. Extension of Theorem 3.3

The validity of Theorem 3.3 is guaranteed also when the two traces $u^\pm$ of u on $[a, b] \times \{0\}$ coincide on some subset of $[a, b] \times \{0\}$. In particular, (3.2) extends to maps u whose jump set S_u is a subset of $[a, b] \times \{0\}$. However, the situation is different when the jump set is curvilineous. Specifically, assume $\Omega \subset \mathbb{R}^2$ is a bounded open and connected set, and:

(H1) $\Sigma = \alpha([a, b]) \subset \Omega$ is a simple curve, arc-length parametrized by $\alpha : [a, b] \to \Omega$ of class C^2 and injective in $[a, b)$;

(H2) If $\alpha(a) = \alpha(b)$, then $\dot{\alpha}(a^+) = \dot{\alpha}(b^-)$ and $\ddot{\alpha}(a^+) = \ddot{\alpha}(b^-)$;

(H3) $u \in W^{1,\infty}(\Omega \setminus \Sigma; \mathbb{R}^2)$; as usual, we denote by $u^\pm$ the traces of u on Σ, satisfying $u^\pm \in \mathrm{Lip}(\Sigma; \mathbb{R}^2)$.

Again, we introduce the affine interpolation surface $X^{\mathrm{aff}} : [a, b] \times I \to \mathbb{R}^3$ spanning $\mathrm{graph}(u^\pm) = \{(t, u^\pm(\alpha(t))) : t \in [a, b]\} \subset \mathbb{R} \times \mathbb{R}^2 = \mathbb{R}^3$, namely

$$X^{\mathrm{aff}}(t, s) = (t, su^+(\alpha(t)) + (1 - s)u^-(\alpha(t))) \qquad \forall(t, s) \in [a, b] \times I. \tag{3.33}$$

Theorem 3.7 (*Relaxed Area of Piecewise Lipschitz Maps: Curved Jump*). *Suppose (H1)-(H3). Then*

$$\overline{\mathcal{A}}_{BV}(u, \Omega) = \int_{\Omega \setminus \Sigma} |\mathcal{M}(\nabla u)| \, dx + \int_{[a,b] \times I} |\partial_t X^{\mathrm{aff}} \wedge \partial_s X^{\mathrm{aff}}| \, dt ds. \tag{3.34}$$

Remark 3.8. The image of the map X^{aff} sits in $\mathbb{R}^3$ and it is not exactly the interpolation surface which closes the holes in the graph of u, which is instead given by

$$\Psi(t, s) = (\alpha(t), su^+(\alpha(t)) + (1-s)u^-(\alpha(t))) \in \mathbb{R}^4 \qquad \forall t \in [a, b] \times I. \tag{3.35}$$

However, since $|\dot{\alpha}| = 1$,

$$\int_{[a,b] \times I} |\partial_t \Psi \wedge \partial_s \Psi| \, dt ds = \int_{[a,b] \times I} |\partial_t X^{\mathrm{aff}} \wedge \partial_s X^{\mathrm{aff}}| \, dt ds. \tag{3.36}$$

To prove Theorem 3.7, we borrow from [8] some notation. We denote by $x = (x_1, x_2)$ coordinates in Ω and by (t, σ) coordinates in $R = [a, b] \times [-1, 1]$. Since Σ is simple and of class C^2, we can find $\delta > 0$ and a C^1-diffeomorphism $\Lambda : R_\delta \to \Lambda(R_\delta)$, where $R_\delta = [a, b] \times [-\delta, \delta]$ and $\Lambda(R_\delta) \subset \Omega$ is a curvilineous strip containing Σ of width 2δ. Explicitly we have

$$\Lambda(t, \sigma) = \alpha(t) + \sigma \dot{\alpha}(t)^\perp \quad \forall (t, \sigma) \in R_\delta, \tag{3.37}$$

with $\dot{\alpha}(t)^\perp$ the counter-clockwise $\frac{\pi}{2}$-rotation of $\dot{\alpha}(t)$. For $(x_1, x_2) \in \Lambda(R_\delta)$, we can write the inverse $\Lambda^{-1}(x_1, x_2) = (t(x_1, x_2), \sigma(x_1, x_2))$, where:

- $\sigma(x_1, x_2) = d_\Sigma(x_1, x_2)$ is the signed distance[6] of (x_1, x_2) from Σ;
- $t(x_1, x_2)$ is the unique number in $[a, b]$ such that $\alpha(t(x_1, x_2)) = \pi_\Sigma(x_1, x_2)$, where $\pi_\Sigma(x_1, x_2) = (x_1, x_2) - d_\Sigma(x_1, x_2) \nabla d_\Sigma(x_1, x_2)$ is the orthogonal projection on Σ.

Since α is of class C^2, we have that σ is of class C^2 as well and t is of class C^1 on $\overline{\Lambda(R_\delta)}$. Moreover, for $(x_1, x_2) \in \overline{\Lambda(R_\delta)}$, we have

$$|\nabla \sigma(x_1, x_2)| = |\nabla d_\Sigma(x_1, x_2)| = 1, \tag{3.38}$$

$$|\nabla t(x_1, x_2)| = 1 + \delta \|\nabla d_\Sigma\|_\infty \le 1 + C\delta. \tag{3.39}$$

We divide the proof of Theorem 3.3 in two parts, the lower and the upper bound inequalities.

Proposition 3.9 (*Lower Bound for* (3.34)). *Let $u : \Omega \to \mathbb{R}^2$ as in Theorem 3.7 and $(v_k) \subset C^1(\Omega; \mathbb{R}^2)$ be a sequence converging to u strictly $BV(\Omega; \mathbb{R}^2)$. Then* (3.3) *holds with X^{aff} in* (3.33).

Proof. It is enough to show that

$$\lim_{\varepsilon \to 0^+} \liminf_{k \to +\infty} \mathcal{A}(v_k, \Lambda([a, b] \times [-\varepsilon, \varepsilon])) \ge \int_{[a,b] \times I} |\partial_t X^{\mathrm{aff}} \wedge \partial_s X^{\mathrm{aff}}| \, dt ds. \tag{3.40}$$

We start by defining the maps $\Psi_k^\varepsilon : R \to \mathbb{R}^4$ and $\Psi_\pm^\varepsilon : R^\pm \to \mathbb{R}^4$ given by

$$\Psi_k^\varepsilon(t, \sigma) = (\Lambda(t, \varepsilon\sigma), v_k(\Lambda(t, \varepsilon\sigma))), \qquad \Psi_\pm^\varepsilon(t, \sigma) = (\Lambda(t, \varepsilon\sigma), u(\Lambda(t, \varepsilon\sigma))).$$

Introduce the following integer multiplicity 2-currents in $\mathbb{R}^4$:

$$\mathcal{V}_k^\varepsilon = \Psi_{k\sharp}^\varepsilon [\![R]\!], \quad S^\varepsilon = \Psi_\sharp [\![a, b] \times I]\!] + \Psi_{-\sharp}^\varepsilon [\![R^-]\!] + \Psi_{+\sharp}^\varepsilon [\![R^+]\!],$$

where Ψ is defined in (3.35). Using that $Av \wedge Aw = \det A \, v \wedge w$ for any $A \in \mathbb{R}^{2 \times 2}$ and $v, w \in \mathbb{R}^2$, by direct computation, we have

$$|\partial_t \Psi_k^\varepsilon \wedge \partial_\sigma \Psi_k^\varepsilon|^2 = \varepsilon^2 |\partial_t \Lambda(t, \varepsilon\sigma) \wedge \partial_\sigma \Lambda(t, \varepsilon\sigma)|^2 \left[1 + |\nabla v_k(\Lambda(t, \varepsilon\sigma))|^2 + |Jv_k(\Lambda(t, \varepsilon\sigma))|^2\right].$$

Hence, making the change of variable $x = \Lambda(t, \varepsilon\sigma)$, we obtain

$$\mathcal{A}(v_k, \Lambda([a, b] \times [-\varepsilon, \varepsilon])) = \int_{\Lambda([a,b] \times [-\varepsilon, \varepsilon])} |\mathcal{M}(\nabla v_k)| \, dx = \int_R |\partial_t \Psi_k^\varepsilon \wedge \partial_\sigma \Psi_k^\varepsilon| \, dt d\sigma = |\mathcal{V}_k^\varepsilon|.$$

We notice that $|\Psi_{\pm\sharp}^\varepsilon [\![R^\pm]\!]| \to 0$ as $\varepsilon \to 0^+$, as in (3.25), where $\|\nabla u\|_{L^\infty(R^\pm)}$ is replaced with $\|u\|_{W^{1,\infty}(\Omega)}$ and it is used that $|\ddot{\alpha}| \le C$. Therefore, recalling also (3.36),

$$\lim_{\varepsilon \to 0^+} |S^\varepsilon| = |\Psi_\sharp [\![a, b] \times I]\!]| = \int_{[a,b] \times I} |\partial_t \Psi \wedge \partial_s \Psi| \, dt ds = \int_{[a,b] \times I} |\partial_t X^{\mathrm{aff}} \wedge \partial_s X^{\mathrm{aff}}| \, dt ds.$$

So it is enough to show $\liminf_{k \to +\infty} |\mathcal{V}_k^\varepsilon| \ge |S^\varepsilon|$, which can be proved proceeding as in the proof of Proposition 3.4, once we have checked that $v_k \circ \Lambda(\cdot, \varepsilon \cdot) \to u \circ \Lambda(\cdot, \varepsilon \cdot)$ strictly $BV(R; \mathbb{R}^2)$. This is a straightforward computation, and we omit the details. $\square$

---

[6] The sign of d_Σ is determined by the orientation induced on Σ by α, so that $d_\Sigma > 0$ in the part of $\Lambda(R_\delta)$ which is pointed by $\dot{\alpha}^\perp$.

Proposition 3.10 (*Upper Bound for* (3.34)). *Let* $u : \Omega \to \mathbb{R}^2$ *be as in Theorem 3.7. Then, there exists a sequence* $(v_k) \subset C^1(\Omega; \mathbb{R}^2)$ *converging to u strictly* $BV(\Omega; \mathbb{R}^2)$ *and such that* (3.27) *holds with* X^{aff} *in* (3.33).

Proof. For simplicity, we assume that $\alpha(a) \neq \alpha(b)$ (the case of closed curves is simpler and the following proof can be straightforwardly adapted). We start by fixing $\eta > 0$ small enough and we extend the curve α to $[a - \eta, b + \eta]$ in a C^2-way, so that $\Sigma^\eta := \alpha([a - \eta, b + \eta]) \subset \Omega$, keeping the validity of (H1) on Σ^η. With this extension, we can assume (by choosing a different δ if necessary) that Λ in (3.37) is defined on $R^\eta := [a - \eta, b + \eta] \times [-\delta, \delta]$. We observe that

$$u^+(\alpha(t)) = u^-(\alpha(t)) \qquad \text{for all } t \in [a - \eta, a] \cup [b, b + \eta]. \tag{3.41}$$

Now, set $\varepsilon = \frac{1}{k}$ and, for k large enough,

$$\Delta_\varepsilon^a := \{x \in \Lambda([a - \varepsilon, a] \times [-\varepsilon, \varepsilon]) : |\sigma(x)| \leq t(x) - a + \varepsilon\},$$
$$\Delta_\varepsilon^b := \{x \in \Lambda([b, b + \varepsilon] \times [-\varepsilon, \varepsilon]) : |\sigma(x)| \leq b + \varepsilon - t(x)\}.$$

We define the recovery sequence $(v_\varepsilon) \subset \text{Lip}(\Omega; \mathbb{R}^2)$ as

$$v_\varepsilon(x) = \begin{cases} \frac{\varepsilon + \sigma(x)}{2\varepsilon} u\big(\Lambda(t(x), \varepsilon)\big) + \frac{\varepsilon - \sigma(x)}{2\varepsilon} u\big(\Lambda(t(x), -\varepsilon)\big) & \text{in } \Lambda([a, b] \times [-\varepsilon, \varepsilon]), \\ u(x) & \text{in } \Omega \setminus \big(\Lambda([a, b] \times [-\varepsilon, \varepsilon])) \cup \overline{\Delta}_\varepsilon^a \cup \overline{\Delta}_\varepsilon^b\big). \end{cases} \tag{3.42}$$

In order to define v_ε in $\Delta_\varepsilon^a \cup \Delta_\varepsilon^b$ it is sufficient to observe that, by (3.41), the restriction of v_ε on $\partial \Delta_\varepsilon^a$ and $\partial \Delta_\varepsilon^b$ is Lipschitz continuous with Lipschitz constant bounded by $\|u\|_{W^{1,\infty}}$. Hence, we can take a Lipschitz extension of v_ε in $\Delta_\varepsilon^a \cup \Delta_\varepsilon^b$ keeping the Lipschitz constant (up to a dimensional factor independent of ε). Thus

$$\int_{\Delta_\varepsilon^a \cup \Delta_\varepsilon^b} |\mathcal{M}(\nabla v_\varepsilon)| \, dx \to 0 \qquad \text{as } \varepsilon \to 0^+. \tag{3.43}$$

Let us check that $v_\varepsilon \to u$ strictly $BV(\Omega; \mathbb{R}^2)$ as $\varepsilon \to 0^+$. Clearly, $v_\varepsilon \to u$ in $L^1(\Omega; \mathbb{R}^2)$, since $|\Lambda([a, b] \times [-\varepsilon, \varepsilon])| \to 0$ and $|\Delta_\varepsilon^a \cup \Delta_\varepsilon^b| \to 0$. So, by (3.43), as in the proof of Proposition 3.5, it is enough to show that

$$\limsup_{\varepsilon \to 0^+} \int_{\Lambda([a,b] \times [-\varepsilon, \varepsilon])} |\nabla v_\varepsilon| \, dx \leq |Du|(\Sigma) = \int_a^b |u^+(\alpha(t)) - u^-(\alpha(t))| \, dt.$$

Almost everywhere in $\Lambda([a, b] \times [-\varepsilon, \varepsilon])$, we have

$$\nabla v_\varepsilon = \frac{\varepsilon + \sigma}{2\varepsilon} \nabla u(\Lambda(t, \varepsilon)) \partial_t \Lambda(t, \varepsilon) \otimes \nabla t + \frac{\varepsilon - \sigma}{2\varepsilon} \nabla u(\Lambda(t, -\varepsilon)) \partial_t \Lambda(t, -\varepsilon) \otimes \nabla t$$
$$+ \frac{1}{2\varepsilon} \nabla \sigma \otimes (u(\Lambda(t, \varepsilon)) - u(\Lambda(t, -\varepsilon))).$$

Therefore,

$$|\nabla v_\varepsilon| \leq \frac{1}{2\varepsilon} \Big[(\varepsilon + \sigma) \|\partial_t \Lambda\|_\infty |\nabla u(\Lambda(t, -\varepsilon))\| \nabla t| + (\varepsilon - \sigma) \|\partial_t \Lambda\|_\infty |\nabla u(\Lambda(t, \varepsilon))\| \nabla t|$$
$$+ |\nabla \sigma \| u(\Lambda(t, \varepsilon)) - u(\Lambda(t, -\varepsilon))| \Big]$$
$$\leq \frac{1}{2\varepsilon} \Big[2\varepsilon \|u\|_{W^{1,\infty}} \|\partial_t \Lambda\|_\infty (1 + C\varepsilon) + |u(\Lambda(t, \varepsilon)) - u(\Lambda(t, -\varepsilon))| \Big],$$

where we used (3.38) and (3.39) with ε in place of δ. Thus, we get

$$\int_{\Lambda([a,b] \times [-\varepsilon, \varepsilon])} |\nabla v_\varepsilon| \, dx \leq C(\delta)(1 + C\varepsilon)|\Lambda([a, b] \times [-\varepsilon, \varepsilon])|$$
$$+ \frac{1}{2\varepsilon} \int_{\Lambda([a,b] \times [-\varepsilon, \varepsilon])} |u(\Lambda(t, \varepsilon)) - u(\Lambda(t, -\varepsilon))| \, dx$$
$$= o_\varepsilon(1) + \frac{1}{2\varepsilon} \int_{\Lambda([a,b] \times [-\varepsilon, \varepsilon])} |u(\Lambda(t, \varepsilon)) - u(\Lambda(t, -\varepsilon))| dx,$$

where $o_\varepsilon(1)$ is infinitesimal as $\varepsilon \to 0^+$. Consider the last integral and perform the change of variable $x = (x_1, x_2) = \Lambda(t, \sigma)$, with

$$|\det \nabla \Lambda(t, \sigma)| = |\partial_t \Lambda \wedge \partial_\sigma \Lambda| = |1 + \sigma \ddot{\alpha} \wedge \dot{\alpha}| = |1 - \kappa_\Sigma \sigma|,$$

where κ_Σ is the curvature of Σ. We get

$$\frac{1}{2\varepsilon} \int_{\Lambda([a,b] \times [-\varepsilon, \varepsilon])} |u(\Lambda(t, \varepsilon)) - u(\Lambda(t, -\varepsilon))| dx$$
$$= \frac{1}{2\varepsilon} \int_{[a,b] \times [-\varepsilon, \varepsilon]} |u(\Lambda(t, \varepsilon)) - u(\Lambda(t, -\varepsilon)) \| 1 - \kappa_\Sigma \sigma| dt d\sigma$$
$$\leq \frac{1}{2\varepsilon} \int_a^b \int_{-\varepsilon}^\varepsilon |u(\Lambda(t, \varepsilon)) - u(\Lambda(t, -\varepsilon))| \, dt d\sigma + o_\varepsilon(1) = \int_a^b |u(\Lambda(t, \varepsilon)) - u(\Lambda(t, -\varepsilon))| \, dt + o_\varepsilon(1)$$

$$\longrightarrow \int_a^b |u^+(\alpha(t)) - u^-(\alpha(t))| \ dt \quad \text{as } \varepsilon \to 0^+.$$

It remains to prove (3.27) with X^{aff} in (3.33). To this purpose it is enough to show that

$$\liminf_{\varepsilon \to 0^+} \mathcal{A}(v_\varepsilon; \Lambda([a,b] \times [-\varepsilon, \varepsilon])) \leq \int_{[a,b] \times I} |\partial_t X^{\text{aff}} \wedge \partial_s X^{\text{aff}}| \ dt ds.$$

Let us define $\varphi_\varepsilon : R \to \mathbb{R}^2$ as

$$\varphi_\varepsilon(t, \sigma) := \frac{1 + \sigma}{2} u(\Lambda(t, \varepsilon)) + \frac{1 - \sigma}{2} u(\Lambda((t, -\varepsilon))).$$

Thus, for $x \in \Lambda([a,b] \times [-\varepsilon, \varepsilon])$

$$v_\varepsilon(x) = \varphi_\varepsilon\left(t(x), \frac{\sigma(x)}{\varepsilon}\right)$$

and, almost everywhere in $\Lambda([a,b] \times [-\varepsilon, \varepsilon])$,

$$\nabla v_\varepsilon = \partial_t \varphi_\varepsilon \nabla t + \frac{1}{\varepsilon} \partial_\sigma \varphi_\varepsilon \nabla \sigma, \qquad J v_\varepsilon = \frac{1}{\varepsilon} |\partial_t \varphi_\varepsilon \wedge \partial_\sigma \varphi_\varepsilon \| \nabla t \wedge \nabla \sigma|,$$

where from now on, ∇t and $\nabla \sigma$ are evaluated at x, while $\partial_t \varphi_\varepsilon$ and $\partial_\sigma \varphi_\varepsilon$ are evaluated at $\left(t(x), \frac{\sigma(x)}{\varepsilon}\right)$. Then, we get

$$|\mathcal{M}(\nabla v_\varepsilon)|^2 = 1 + |\partial_t \varphi_\varepsilon|^2 |\nabla t|^2 + \frac{2}{\varepsilon} \partial_t \varphi_\varepsilon \cdot \partial_\sigma \varphi_\varepsilon \nabla t \cdot \nabla \sigma + \frac{1}{\varepsilon^2} \left[|\partial_\sigma \varphi_\varepsilon|^2 |\nabla \sigma|^2 + |\partial_t \varphi_\varepsilon \wedge \partial_\sigma \varphi_\varepsilon|^2 |\nabla t \wedge \nabla \sigma|^2 \right]$$

$$\leq 1 + |\partial_t \varphi_\varepsilon|^2 (1 + o_\varepsilon(1)) + \frac{2}{\varepsilon} |\partial_t \varphi_\varepsilon \cdot \partial_\sigma \varphi_\varepsilon| (1 + o_\varepsilon(1))$$

$$+ \frac{1}{\varepsilon^2} \left[|\partial_\sigma \varphi_\varepsilon|^2 + |\partial_t \varphi_\varepsilon \wedge \partial_\sigma \varphi_\varepsilon|^2 (1 + o_\varepsilon(1)) \right],$$

where we used (3.38) and (3.39) with ε in place of δ. Now, since $o_\varepsilon(1) \sim \varepsilon$ and φ_ε is Lipschitz with Lipschitz constant independent of ε, we obtain

$$\mathcal{A}(v_\varepsilon, \Lambda([a,b] \times [-\varepsilon, \varepsilon]))$$

$$\leq \int_{\Lambda([a,b] \times [-\varepsilon, \varepsilon])} \sqrt{1 + |\partial_t \varphi_\varepsilon|^2 + \frac{2}{\varepsilon} |\partial_t \varphi_\varepsilon \cdot \partial_\sigma \varphi_\varepsilon| + \frac{1}{\varepsilon^2} \left[|\partial_\sigma \varphi_\varepsilon|^2 + |\partial_t \varphi_\varepsilon \wedge \partial_\sigma \varphi_\varepsilon|^2 (1 + o_\varepsilon(1)) \right]} \ dx + o_\varepsilon(1)$$

$$\leq \int_{[a,b] \times [-\varepsilon, \varepsilon]} \sqrt{1 + |\partial_t \varphi_\varepsilon|^2 + \frac{2}{\varepsilon} |\partial_t \varphi_\varepsilon \cdot \partial_\sigma \varphi_\varepsilon| + \frac{1}{\varepsilon^2} \left[|\partial_\sigma \varphi_\varepsilon|^2 + |\partial_t \varphi_\varepsilon \wedge \partial_\sigma \varphi_\varepsilon|^2 (1 + o_\varepsilon(1)) \right]} |1 - \kappa_\Sigma \sigma| \ dt d\sigma$$

$$+ o_\varepsilon(1),$$

where we made the change of variable $x = \Lambda(t, \sigma)$, and so $\partial_t \varphi_\varepsilon$ and $\partial_\sigma \varphi_\varepsilon$ are computed at $\left(t, \frac{\sigma}{\varepsilon}\right)$. Finally, by the change of variable $\frac{\sigma}{\varepsilon} \to \sigma$, we get

$$\mathcal{A}(v_\varepsilon, \Lambda([a,b] \times [-\varepsilon, \varepsilon]))$$

$$\leq \int_R \sqrt{o_\varepsilon(1) + |\partial_\sigma \varphi_\varepsilon(t, \sigma)|^2 + |\partial_t \varphi_\varepsilon(t, \sigma) \wedge \partial_\sigma \varphi_\varepsilon(t, \sigma)|^2 (1 + o_\varepsilon(1))} |1 - \kappa_\Sigma \varepsilon \sigma| \ dt d\sigma$$

$$+ o_\varepsilon(1) \longrightarrow \int_{[a,b] \times I} |\partial_t X^{\text{aff}} \wedge \partial_s X^{\text{aff}}| \ dt ds,$$

where, to pass to the limit as $\varepsilon \to 0^+$, we apply the dominated convergence theorem (as in the proof of Proposition 3.5). $\quad \square$

We observe that Theorem 3.7 can be easily extended to the case of curves with one endpoint or both endpoints on $\partial \Omega$. Write:

(H4) Ω is of class C^1, $\alpha : [a,b] \to \overline{\Omega}$ is injective, arc-length parametrized, of class C^2, $\alpha((a,b)) \subset \Omega$, and α hits $\partial \Omega$ transversally at $\alpha(a), \alpha(b)$.

Theorem 3.11. *Suppose (H3) and (H4). Then (3.34) holds with X^{aff} in (3.33).*

Proof. Lower bound: let $(v_k) \subset C^1(\Omega; \mathbb{R}^2)$ be a sequence converging to u strictly $BV(\Omega; \mathbb{R}^2)$. Fix $0 < \rho < \frac{b-a}{2}$ and notice that $\Lambda([a + \rho, b - \rho] \times [-\varepsilon, \varepsilon]) \subset \Omega$, for $\varepsilon > 0$ small enough. Then it is sufficient to show that

$$\lim_{\varepsilon \to 0^+} \liminf_{k \to +\infty} \mathcal{A}(v_k, \Lambda([a,b] \times [-\varepsilon, \varepsilon]) \cap \Omega) \geq \int_{[a+\rho, b-\rho] \times I} |\partial_t X^{\text{aff}} \wedge \partial_s X^{\text{aff}}| \ dt ds, \tag{3.44}$$

since the lower bound will follow by the arbitrariness of $\rho > 0$. After writing $\mathcal{A}(v_k, \Lambda([a,b] \times [-\varepsilon, \varepsilon]) \cap \Omega) \geq \mathcal{A}(v_k, \Lambda([a+\rho, b-\rho] \times [-\varepsilon, \varepsilon]))$, the proof of (3.44) is identical to that of (3.40).

Upper bound: let us fix $\eta > 0$ small enough so that $B_{2\eta}(\alpha(a))$ and $B_{2\eta}(\alpha(b))$ are disjoint, and consider $\Omega^\eta := \Omega \cup B_{2\eta}(\alpha(a)) \cup B_{2\eta}(\alpha(b))$. We extend the curve α (still calling α the extension) in $\Omega^\eta \setminus \Omega$ in such a way that it satisfies (H4) in Ω^η, and so that it reaches the boundary of $B_{2\eta}(\alpha(a)) \setminus \overline{\Omega}$ and of $B_{2\eta}(\alpha(b)) \setminus \overline{\Omega}$ splitting both $B_{2\eta}(\alpha(a)) \setminus \overline{\Omega}$ and $B_{2\eta}(\alpha(b)) \setminus \overline{\Omega}$ in two connected components. If α is

now defined on an interval of the form $[a-\delta, b+\delta]$ with $\delta = \delta(\eta) > \eta$, and if we set $\Sigma^\delta = \alpha([a-\delta, b+\delta])$, we prescribe the traces u^+ and u^- on Σ^δ in such a way that they are Lipschitz continuous and $u^+ \circ \alpha = u^- \circ \alpha$ on $[a-\delta, a-\eta] \cup [b+\eta, b+\delta]$. Finally we take a Lipschitz extension u^η of u on the four connected components of $B_{2\eta}(\alpha(a)) \setminus \overline{\Omega} \setminus \Sigma^\delta$ and of $B_{2\eta}(\alpha(b)) \setminus \overline{\Omega} \setminus \Sigma^\delta$. It turns out that $u^\eta \in W^{1,\infty}\big((B_{2\eta}(\alpha(a)) \cup B_{2\eta}(\alpha(b))) \setminus \Sigma^\eta; \mathbb{R}^2\big)$, where $\Sigma^\eta = \alpha([a-\eta, b+\eta]) \subset \Omega^\eta$. Since the definition of $(u^\eta)^\pm$ is arbitrary, we can assume that

$$(u^\eta)^\pm(\alpha(t)) = u^\pm(\alpha(a))\Big(1 - \frac{a-t}{\eta}\Big) \qquad \text{for } t \in [a-\eta, a],$$

$$(u^\eta)^\pm(\alpha(t)) = u^\pm(\alpha(b))\Big(1 - \frac{t-b}{\eta}\Big) \qquad \text{for } t \in [b, b+\eta].$$

For $\varepsilon > 0$ small enough, we see that $\Lambda_\varepsilon := \Lambda([a-\eta, b+\eta] \times [-\varepsilon, \varepsilon]) \subset \Omega^\eta$. Hence we define v_k as in the proof of Proposition 3.10 with Ω replaced by Ω^η and u replaced by u^η (in particular, $v_\varepsilon = u$ on $\Omega \setminus \Lambda_\varepsilon$). Finally, let us fix $\rho \in (0, \eta)$. We can write

$$\overline{\mathcal{A}}_{BV}(u, \Omega) \leq \liminf_{\varepsilon \to 0^+} \mathcal{A}(v_\varepsilon, \Omega)$$

$$\leq \lim_{\varepsilon \to 0^+} \int_{\Omega \setminus \Lambda_\varepsilon} |\mathcal{M}(\nabla u)| \, dx + \liminf_{\varepsilon \to 0^+} \int_{\Lambda([a-\rho, b+\rho] \times [-\varepsilon, \varepsilon])} |\mathcal{M}(\nabla v_\varepsilon)| \, dx$$

$$= \int_\Omega |\mathcal{M}(\nabla u)| \, dx + \int_{[a-\rho, b+\rho] \times I} |\partial_t X^{\text{aff}} \wedge \partial_s X^{\text{aff}}| \, dtds,$$

where we use that $\Omega \subset \big((\Omega \setminus \Lambda_\varepsilon) \cup \Lambda([a-\rho, b+\rho] \times [-\varepsilon, \varepsilon])\big)$ for $\varepsilon > 0$ small enough. The upper bound then follows by the arbitrariness of ρ. $\quad\square$

Finally, with straightforward modifications of the previous arguments one can show the following:

Corollary 3.12. *Let Ω have C^1-boundary, let $n \in \mathbb{N}$ and $\alpha_i : [a_i, b_i] \to \overline{\Omega}$, $i = 1, \ldots, n$, be curves satisfying either (H1)-(H2), or (H4). Assume that $\Sigma_i := \alpha_i([a_i, b_i]) \subset \overline{\Omega}$ are mutually disjoint, and let $u \in W^{1,\infty}(\Omega \setminus \Sigma; \mathbb{R}^2)$ satisfy (H3), where $\Sigma := \cup_{i=1}^n \Sigma_i$. Then*

$$\overline{\mathcal{A}}_{BV}(u, \Omega) = \int_\Omega |\mathcal{M}(\nabla u)| \, dx + \sum_{i=1}^n \int_{[a_i, b_i] \times I} |\partial_t X_{(i)}^{\text{aff}} \wedge \partial_s X_{(i)}^{\text{aff}}| \, dtds,$$

where $X_{(i)}^{\text{aff}} : [a_i, b_i] \times I \to \mathbb{R}^3$ is the map $X_{(i)}^{\text{aff}}(t, s) = (t, su^+(\alpha_i(t)) + (1-s)u^-(\alpha_i(t)))$.

4. Piecewise constant maps

In this section we study the relaxed area (1.3) and the relaxed total variation (1.4), on certain piecewise constant maps. We start by exhibiting a BV map taking three values having infinite relaxed total variation of the Jacobian (and hence infinite BV-relaxed area), but finite L^1-relaxed area.

Example 4.1 (*BV-Relaxed Area and L^1-Relaxed Area*). Let $\alpha, \beta, \gamma \in \mathbb{R}^2$ be three non-collinear vectors. Consider the map $u : B_1(0) \subset \mathbb{R}^2 \to \{\alpha, \beta, \gamma\}$ in Fig. 2, obtained by the following procedure: divide the source equilateral triangle $T_{A_0 O B_0}$ in two regions with a vertical segment connecting A_1 and B_1, the middle points of the oblique sides of the triangle; assign the value β and γ on the right and on the left as in the figure, and repeat this construction on the equilateral triangle $T_{A_1 O B_1}$, and then repeat the argument iteratively on all smaller triangles; finally set $u = \alpha$ in $B_1(0) \setminus T_{A_0 O B_0}$. In this way we get an infinite collection of triple points located at $\{A_i, B_i\}_{i \geq 1}$. Then, $u \in BV(B_1(0); \{\alpha, \beta, \gamma\})$, since

$$|Du|(B_1(0)) = \left(1 + 2(1 - \sum_{i=1}^{+\infty} 2^{-2i})\right)|\beta - \alpha| + 2\sum_{i=1}^{+\infty} 2^{-2i}|\alpha - \gamma| + \sum_{i=1}^{+\infty} 2^{-i}|\beta - \gamma|$$

$$= \frac{7}{3}|\beta - \alpha| + \frac{2}{3}|\alpha - \gamma| + |\beta - \gamma|.$$

On the other hand, consider an infinitesimal sequence $(r_i)_{i \geq 1}$ of radii with $0 < r_i < 2^{-(i+1)}$. With an argument similar to [3, Theorem 1.3], we have

$$\overline{TVJ}_{BV}(u, B_{r_i}(A_i)) = |T_{\alpha\beta\gamma}|,$$

$|T_{\alpha\beta\gamma}|$ denoting the Lebesgue measure of the target triangle with vertices α, β, γ, and thus, for every $N \in \mathbb{N}$,

$$\overline{TVJ}_{BV}(u, B_1(0)) \geq \overline{TVJ}_{BV}(u, \cup_{i=1}^N B_{r_i}(A_i)) \geq \sum_{i=1}^N |T_{\alpha\beta\gamma}| = N|T_{\alpha\beta\gamma}|.$$

Whence

$$\overline{\mathcal{A}}_{BV}(u, B_1(0)) \geq \overline{TVJ}_{BV}(u, B_1(0)) = +\infty. \tag{4.1}$$

On the other hand, we claim that

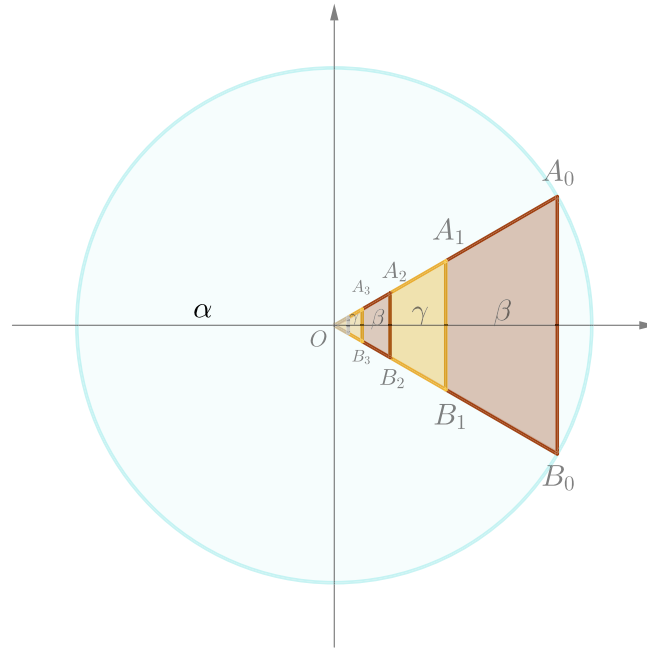$$\overline{\mathcal{A}}_{L^1}(u, B_1(0)) < +\infty. \tag{4.2}$$

Fig. 2. The source disk $B_1(0)$ and the values $\{\alpha, \beta, \gamma\}$ of u, with infinitely many triple points.

Indeed, we can construct a sequence (v_ε) of piecewise constant maps on $B_1(0)$, taking values in $\{\alpha, \beta, \gamma\}$, with uniformly bounded L^1-relaxed area and converging to u in $L^1(B_1(0); \mathbb{R}^2)$: Let $\varepsilon \in (0, 1)$ and consider the intersection with $T_{A_0 0 B_0}$ of a tubular neighborhood of the segment $\overline{A_i B_i}$ of diameter $\varepsilon 2^{-(i+1)}$, for every $i \in \mathbb{N}$. Then, the map v_ε is obtained by modifying u on these strips in the triangle, by assigning the value α. Now, v_ε is a piecewise constant map valued in $\{\alpha, \beta, \gamma\}$ without triple points, hence, by [1, Theorem 3.14],

$$\overline{\mathcal{A}}_{L^1}(v_\varepsilon, B_1(0)) = |B_1(0)| + |Dv_\varepsilon|(B_1(0))$$

$$\leq \pi + \frac{7}{3}|\beta - \alpha| + \frac{2}{3}|\alpha - \gamma| + \left(1 + \frac{\varepsilon}{2}\right) \sum_{i=1}^{+\infty} 2^{-i}(|\beta - \alpha| + |\alpha - \gamma|)$$

$$\leq \pi + \frac{23}{6}|\beta - \alpha| + \frac{13}{6}|\alpha - \gamma|.$$

Clearly, $v_\varepsilon \to u$ in $L^1(B_1(0); \mathbb{R}^2)$ as $\varepsilon \to 0^+$, so by lower semicontinuity

$$\overline{\mathcal{A}}_{L^1}(u, B_1(0)) \leq \pi + \frac{23}{6}|\beta - \alpha| + \frac{13}{6}|\alpha - \gamma| < +\infty.$$

In particular

$$\text{Dom}\left(\overline{\mathcal{A}}^{BV}(\cdot, B_1(0))\right) \subsetneq \text{Dom}\left(\overline{\mathcal{A}}^{L^1}(\cdot, B_1(0))\right).$$

Remark 4.2. Following the notation of [23], one can show (4.1) also by considering the measure μ_w^J defined for every $w \in BV(B_1(0); \mathbb{R}^2)$ as

$$\langle \mu_w^J, g \rangle = \frac{1}{2} \int_{S_w} (w^{1^-} w^{2^+} - w^{1^+} w^{2^-}) \partial_\tau g d\mathcal{H}^1 \quad \forall g \in C_c^\infty(B_1(0)),$$

where $\tau = \nu^\perp$ and ν is the unit normal to S_w, so that $Dw \, \llcorner \, S_w = (w^+ - w^-) \otimes \nu \mathcal{H}^1 \, \llcorner \, S_w$.

If $\overline{\mathcal{A}}_{BV}(w, B_1(0))$ is finite, we can consider the unique cartesian current $T_w \in \text{cart}(B_1(0); \mathbb{R}^2)$ associated to w defined in [23, Theorem 3.5], whose vertical part is by definition equal to the minimal completely vertical lifting $\mu_v[w]$ associated to w, according to [23, Definition 3.1]. Then, since $|\mu_v[w]|$ is lower semicontinuous with respect to the weak convergence of measures and, for v smooth, $|\mu_v[v]|(B_1(0) \times \mathbb{R}^2) = TVJ(v, B_1(0))$ (see [23, Theorem 6.2]), we get

$$|\mu_v[w]|(B_1(0) \times \mathbb{R}^2) \leq \overline{TVJ}_{BV}(w, B_1(0)).$$

In particular, if $w \in BV(B_1(0); \mathbb{R}^2)$ is piecewise constant, we have

$$|\mu_w^J|(B_1(0)) \leq |\mu_v[w]|(B_1(0) \times \mathbb{R}^2) \leq \overline{TVJ}_{BV}(w, B_1(0)), \tag{4.3}$$

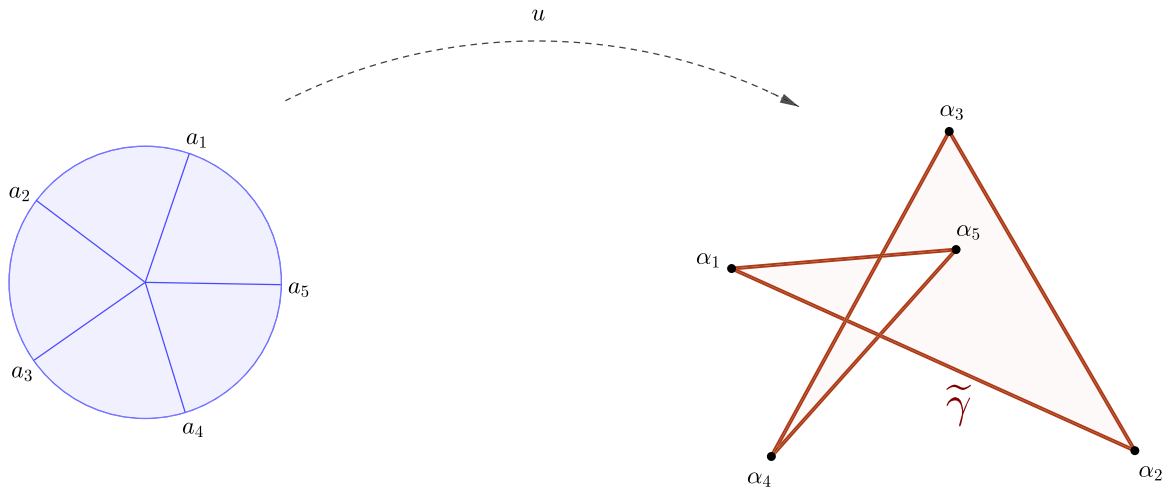where the first inequality is a consequence of [23, Corollary 4.3].

Fig. 3. An n-uple point map and the corresponding curve γ, for $n = 5$.

Now, if by contradiction $\overline{\mathcal{A}}_{BV}(u, B_1(0))$ is finite for the map u in Example 4.1 we have

$$\mu_u^J = \sum_{i=1}^{+\infty} |T_{\alpha\beta\gamma}|(\delta_{A_i} - \delta_{B_i}).$$

In particular $|\mu_u^J|(B_1(0)) = +\infty$, and (4.1) follows from (4.3). In Example 4.6, we construct a piecewise constant map $u \in BV(B_1(0); \mathbb{R}^2)$ taking only five values in $\mathbb{R}^2$ with $\overline{TVJ}_{BV}(u, B_1(0)) = +\infty$ and $\mu_u^J = 0$. In that case, one can see even that $\mu_v[u] = 0$, whence a maximal gap phenomenon occurs between the mass of the current T_u (which is finite and without a vertical contribution) and $\overline{\mathcal{A}}_{BV}(u, B_1(0))$ (which is infinite as well).

4.1. Piecewise constant homogeneous maps

We need some tools that allow us to characterize (and compute in some cases) the relaxed functionals for $n-$uple point maps with $n \geq 3$. Thus, for $r > 0$, we consider maps $u : B_r := B_r(0) \to \mathbb{R}^2$ of the form

$$u(x) = \gamma\left(\frac{x}{|x|}\right) \quad \text{for a.e. } x \in B_r, \tag{4.4}$$

where $\gamma : \mathbb{S}^1 \to \{\alpha_1, \dots, \alpha_n\}$ is piecewise constant and takes the (not necessarily distinct) values $\alpha_1, \dots, \alpha_n \in \mathbb{R}^2$ on the arcs $C_1, \dots, C_n$ in the order (see Fig. 3 for $n = 5$). So, u is an $n-$uple point map with one $n-$uple junction at the origin. Now, we can consider the broken line curve $\widetilde{\gamma} \subset \mathbb{R}^2$ (an example of which is in Fig. 3) made of the segments connecting α_1 to α_2, α_2 to α_3 and so on, closing up by connecting α_n to α_1. The curve $\widetilde{\gamma}$ can be parametrized as in (2.27), and the curves $\widetilde{\gamma}_i$ are constant. Denoting by $L(\gamma)$ the length of $\widetilde{\gamma}$, we have

$$L(\gamma) = \sum_{i=1}^{n} |\alpha_{i+1} - \alpha_i| = |\dot{\gamma}|(\mathbb{S}^1) = \sup\left\{\sum_{i=1}^{m-1} |\gamma(v_{i+1}) - \gamma(v_i)| : m \in \mathbb{N}, \{v_1, \dots, v_m\} \subset \mathbb{S}^1\right\}, \tag{4.5}$$

with the convention $\alpha_{n+1} := \alpha_1$, Clearly, by definition of u, we have

$$|Du|(B_r) = r|\dot{\gamma}|(\mathbb{S}^1) = rL(\gamma).$$

Thanks to Lemma 2.14, for $\overline{P}(\gamma)$ as in (2.40) we know that

$$\overline{P}(\gamma) = P(\widetilde{\gamma}). \tag{4.6}$$

For a general γ the computation of $\overline{P}(\gamma)$ seems not immediate. For the configuration in Fig. 3, we expect it to be the area of the region enclosed by $\widetilde{\gamma}$, with the small internal quadrilateral counted twice.

Theorem 4.3 (*Relaxation of TVJ on Piecewise Constant Maps*). *Let $\{\alpha_1, \dots, \alpha_n\} \subset \mathbb{R}^2$, $\gamma \in BV(\mathbb{S}^1; \{\alpha_1, \dots, \alpha_n\})$ be a function with a finite number of jump points, and let u be as in* (4.4). *Then*

$$\overline{TVJ}_{BV}(u, B_r) = \overline{P}(\gamma).$$

Proof. Lower bound: Assume that $(v_k) \subset C^1(B_r; \mathbb{R}^2)$ converges to u strictly $BV(B_r; \mathbb{R}^2)$ and

$$\lim_{k \to +\infty} \int_{B_r} |J v_k| \, dx = \overline{TVJ}_{BV}(u, B_r).$$

By Lemma 2.5, we can fix $\varepsilon \in (0, r)$ and a not-relabeled subsequence depending on ε, such that $v_k \llcorner \partial B_\varepsilon \to u \llcorner \partial B_\varepsilon$ strictly $BV(\partial B_\varepsilon; \mathbb{R}^2)$. Thus, using Corollary 2.15 and the rescaling invariance of (2.40), we can estimate

$$\overline{TVJ}_{BV}(u, B_r) \geq \liminf_{k \to +\infty} \int_{B_\varepsilon} |J v_k| \, dx \geq \liminf_{k \to +\infty} P(v_k \llcorner \partial B_\varepsilon) = \overline{P}(u \llcorner \partial B_\varepsilon) = \overline{P}(\gamma). \tag{4.7}$$

Upper bound: By an argument similar to the one at the beginning of the proof of Proposition 3.5, it will be enough to construct a recovery sequence $(u_k) \subset \mathrm{Lip}(B_r; \mathbb{R}^2)$. Let $\widetilde{\gamma}$ be as above. We start by building a sequence $(\gamma_k)_k$ of Lipschitz reparameterizations of $\widetilde{\gamma}$ which converges strictly $BV(\mathbb{S}^1; \mathbb{R}^2)$ to γ. Let us denote by $a_1, \ldots, a_n \in [0, 2\pi]$ the angular coordinates of the extremal points of $C_1, \ldots, C_n$, and assume without loss of generality $0 = a_1 < a_2 < \cdots < a_n$. Then

$$\bigcup_{i=1}^{n} [a_i, a_{i+1}] = [0, 2\pi],$$

with the convention $a_{n+1} = 2\pi$. Let $(\delta_k)_k$ be an infinitesimal sequence with $0 < \delta_k < \max\{|a_{i+1} - a_i|, i = 1, \ldots, n\}$, for instance $\delta_k = \frac{2}{k}$, k large enough. We define the piecewise affine map $\gamma_k : [0, 2\pi] \to \mathbb{R}^2$ as

$$\gamma_k(t) = \begin{cases} \alpha_i & \text{if } t \in [a_i + \delta_k/2, a_{i+1} - \delta_k/2], \\ \dfrac{a_{i+1} + \delta_k/2 - t}{\delta_k} \alpha_i + \dfrac{t - a_{i+1} + \delta_k/2}{\delta_k} \alpha_{i+1} & \text{if } t \in [a_{i+1} - \delta_k/2, a_{i+1} + \delta_k/2], \end{cases} \quad i = 1, \ldots, n. \tag{4.8}$$

Then $\gamma_k \to \gamma$ strictly $BV(\mathbb{S}^1; \mathbb{R}^2)$ (actually a direct computation shows that $|\dot{\gamma}_k|(\mathbb{S}^1) = |\dot{\gamma}|(\mathbb{S}^1)$), γ_k are uniformly bounded in L^∞, and converge almost everywhere to γ. As a consequence, from Corollary 2.15,

$$P(\gamma_k) \to \overline{P}(\gamma) \qquad \text{as } k \to +\infty. \tag{4.9}$$

Therefore, by (1.5) we choose, for all $k > 1$ large enough, a map $v_k \in \mathrm{Lip}(B_1; \mathbb{R}^2)$ such that

$$v_k \llcorner \mathbb{S}^1 = \gamma_k, \qquad \left| P(\gamma_k) - \int_{B_1} |J v_k| \, dx \right| \leq \frac{1}{k}. \tag{4.10}$$

Let $c_k > 0$ be the Lipschitz constant of v_k. Defining $v_{k,\rho} \in \mathrm{Lip}(B_\rho; \mathbb{R}^2)$ as $v_{k,\rho}(y) := v_k(\frac{y}{\rho})$ for any $\rho > 0$, it is straightforward that the Lipschitz constant of $v_{k,\rho}$ is c_k/ρ.

We now choose an infinitesimal sequence $(\rho_k) \subset (0, r)$ in such a way that $\lim_{k \to +\infty} c_k \rho_k = 0$. As a consequence we get

$$\int_{B_{\rho_k}} |\nabla v_{k,\rho_k}| \, dx \leq \pi c_k \rho_k \to 0 \qquad \text{as } k \to +\infty. \tag{4.11}$$

We are now in a position to introduce our recovery sequence: We define $u_k \in \mathrm{Lip}(B_r; \mathbb{R}^2)$ as

$$u_k(x) := \begin{cases} \gamma_k \left(\dfrac{x}{|x|} \right) & \forall x \in B_r \setminus B_{\rho_k}, \\ v_{k,\rho_k}(x) & \forall x \in B_{\rho_k}. \end{cases} \tag{4.12}$$

Using that $\gamma_k \to \gamma$ strictly $BV(\mathbb{S}^1; \mathbb{R}^2)$ and (4.11) we see that $u_k \to u$ strictly $BV(B_r; \mathbb{R}^2)$. Finally, since in $B_r \setminus B_{\rho_k}$ the map u_k depends only on the angular coordinate, its Jacobian determinant vanishes in $B_r \setminus B_{\rho_k}$. Hence

$$\liminf_{k \to +\infty} \int_{B_r} |J u_k| \, dx = \liminf_{k \to +\infty} \int_{B_{\rho_k}} |J v_{k,\rho_k}| \, dx = \overline{P}(\gamma), \tag{4.13}$$

the convergence being a consequence of (2.30), (4.10), and (4.9). $\quad\square$

As a consequence of Theorem 4.3 we deduce:

Theorem 4.4 (*Relaxation of $\mathcal{A}$ on Piecewise Constant Maps*). *Let γ and u be as in Theorem 4.3. Then, for any $r > 0$, we have*

$$\overline{\mathcal{A}}_{BV}(u, B_r) = \pi r^2 + r L(\gamma) + \overline{P}(\gamma). \tag{4.14}$$

Proof. Lower bound: Suppose that $v_k \in C^1(B_r; \mathbb{R}^2)$ is such that

$$v_k \to u \quad \text{strictly } BV(B_r; \mathbb{R}^2) \quad \text{and} \quad \lim_{k \to +\infty} \mathcal{A}(v_k, B_r) = \liminf_{k \to +\infty} \mathcal{A}(v_k, B_r).$$
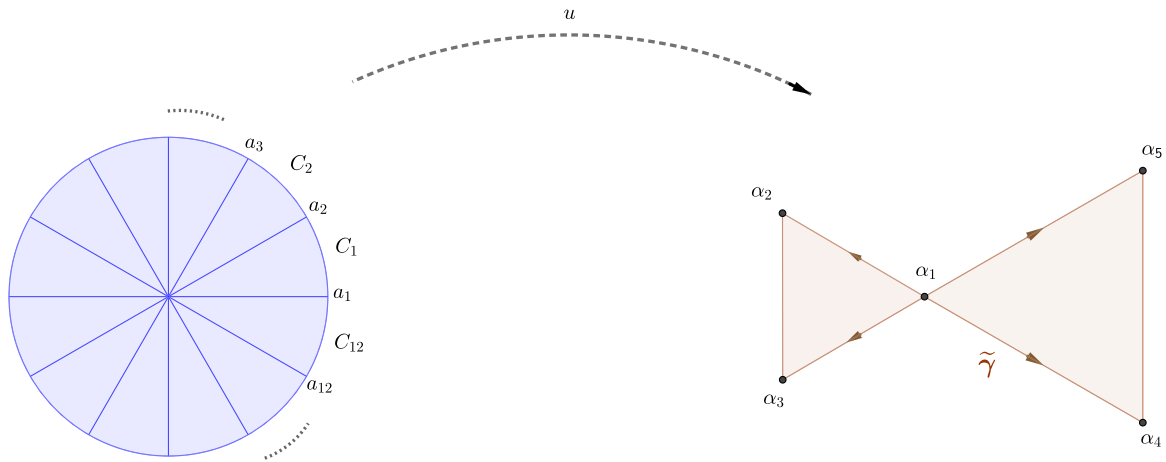
Fig. 4. The map u and the broken line curve $\widetilde{\gamma}$ of Example 4.5.

Now, let $\varepsilon \in (0, r)$ and write $\mathcal{A}(v_k, B_r) = \mathcal{A}(v_k, B_r \setminus B_\varepsilon) + \mathcal{A}(v_k, B_\varepsilon) \geq \mathcal{A}(v_k, B_r \setminus B_\varepsilon) + \int_{B_\varepsilon} |Jv_k|\, dx$, so that, by [1, Theorem 3.7],

$$
\lim_{k \to +\infty} \mathcal{A}(v_k, B_r) \geq \liminf_{k \to +\infty} \mathcal{A}(v_k, B_r \setminus B_\varepsilon) + \liminf_{k \to +\infty} \int_{B_\varepsilon} |Jv_k|\, dx
$$

$$
\geq |B_r \setminus B_\varepsilon| + (r - \varepsilon)L(\gamma) + \liminf_{k \to +\infty} \int_{B_\varepsilon} |Jv_k|\, dx
$$

$$
\geq |B_r \setminus B_\varepsilon| + (r - \varepsilon)L(\gamma) + \overline{P}(\gamma),
$$

where in the last line we have applied Theorem 4.3 with r replaced by ε. We now pass to the limit as $\varepsilon \to 0^+$ to get the lower bound $\overline{\mathcal{A}}_{BV}(u, B_r) \geq \pi r^2 + rL(\gamma) + \overline{P}(\gamma)$ in (4.14).

Upper bound: It is sufficient to consider the sequence $(u_k)_k$ defined in (4.12), for which

$$
\overline{\mathcal{A}}_{BV}(u, B_r) \leq \limsup_{k \to +\infty} \mathcal{A}(u_k, B_1) \leq |B_r| + \lim_{k \to +\infty} \int_{B_r} |\nabla u_k|\, dx + \lim_{k \to +\infty} \int_{B_r} |Ju_k|\, dx
$$

$$
= \pi r^2 + rL(\gamma) + \overline{P}(\gamma). \qquad \square
$$

Now, we are in the position to show an example of a piecewise constant map $u \in BV(B_1; \mathbb{R}^2)$ with infinite relaxed Jacobian total variation but vanishing associated minimal vertical lifting measure $\mu_v[u]$. This map is constructed in Example 4.6, while Example 4.5 is preparatory.

Example 4.5. We want to show here how singular topological phenomena related to the double-eight map [13,15,20,22,24] arise also among piecewise constant maps. In particular, as pointed out in [23], for the homogeneous extension of the double-eight map, a gap phenomenon occurs between the minimal vertical lifting measure and the relaxed Jacobian total variation. We show now that we find such a gap also among piecewise constant maps, by exhibiting a piecewise constant map with vanishing minimal vertical lifting measure but with finite non-zero $\overline{TVJ}$. Namely, we are going to define a map $u : B_1 \to \mathbb{R}^2$ assuming five distinct values, for which the resulting closed curve $\widetilde{\gamma}$ has zero degree, but is homotopically non-trivial, since it is, in fact, homeomorphic to the double-eight curve. Let $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\} \subset \mathbb{R}^2$ be the vertices of two (equilateral for simplicity) triangles with a common vertex, say α_1 (see Fig. 4). Fix a partition of $\mathbb{S}^1$ in twelve disjoint non-empty arcs $C_1, \dots, C_{12}$ (not necessarily of the same length), with extremal points $a_1, \dots, a_{12}$ in counter-clockwise order. Then, define $\gamma : \mathbb{S}^1 \to \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$ to be constant on the arcs $C_1, \dots, C_{12}$, precisely equal to, in the order, $\alpha_1, \alpha_2, \alpha_3, \alpha_1, \alpha_4, \alpha_5, \alpha_1, \alpha_3, \alpha_2, \alpha_1, \alpha_5, \alpha_4$. Then, the broken line curve $\widetilde{\gamma}$ runs consecutively the triangles $T_{123} := T_{\alpha_1 \alpha_2 \alpha_3}$ and $T_{145} := T_{\alpha_1 \alpha_4 \alpha_5}$ twice, and every time with different orientation. Define u as in (4.4), obtaining a 12-point map. Now, by applying Theorem 4.3 and computing the minimum of the Plateau problem (1.5) for $\widetilde{\gamma}$ as in [24, Theorem 5], we obtain

$$
\overline{TVJ}_{BV}(u, B_1) = \overline{P}(\gamma) = \overline{P}(\widetilde{\gamma}) = 2\min\{|T_{123}|, |T_{145}|\}. \tag{4.15}
$$

Moreover, it is not difficult to see that

$$
\mu_u^J = (|T_{123}| + |T_{145}| - |T_{123}| - |T_{145}|)\delta_0 = 0.
$$

In this case, we have also $\mu_v[u] = 0$, indeed we can prove that the unique current T_u with minimal completely vertical lifting associated to u is given by

$$
T_u = G_u + S = \sum_{l=1}^{12} [\![\widehat{C}_l]\!] \times [\![c_l]\!] + \sum_{l=1}^{12} [\![0, a_l]\!] \times [\![c_{l-1}, c_l]\!], \tag{4.16}
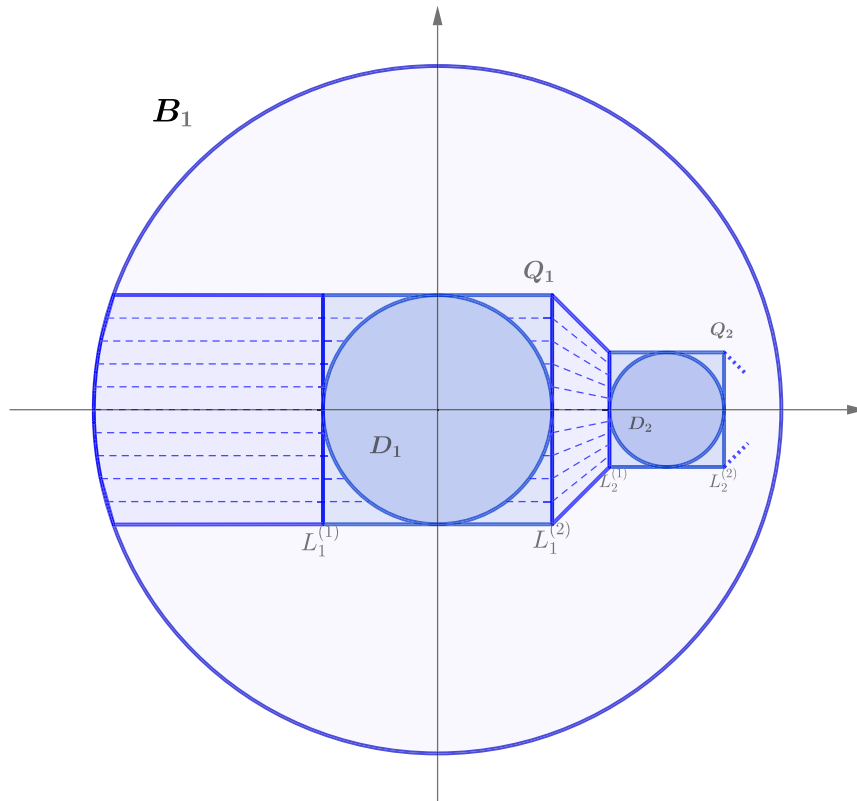$$

Fig. 5. The sequence $\{D_i\} \subset B_1$ of disks of Example 4.6.

where $\hat{C}_l$ is the circular sector corresponding to C_l and c_l is the assigned value of γ on C_l for $l = 1, \ldots, 12$ (we used the convention $c_0 = c_{12}$). Let us show (4.16). One checks that $\mu_i^j[T_u] = \mu_i^j[u]$ for $i, j = 1, 2$ by proceeding as in Remark 3.6. So, it remains to prove that $T_u \in \text{cart}(B_1; \mathbb{R}^2)$: it is enough to check that $(\partial T_u) \llcorner B_1 \times \mathbb{R}^2 = 0$. Compute

$$\partial S = \sum_{l=1}^{12} \partial \left(\llbracket 0, a_l \rrbracket \times \llbracket c_{l-1}, c_l \rrbracket \right) = \sum_{l=1}^{12} \left(-\llbracket 0 \rrbracket \times \llbracket c_{l-1}, c_l \rrbracket + \llbracket 0, a_l \rrbracket \times \llbracket c_l \rrbracket - \llbracket 0, a_l \rrbracket \times \llbracket c_{l-1} \rrbracket \right).$$

Now, since by convention $a_{13} = a_1$,

$$\partial G_u = \sum_{l=1}^{12} \left(\llbracket 0, a_{l+1} \rrbracket \times \llbracket c_l \rrbracket - \llbracket 0, a_l \rrbracket \times \llbracket c_l \rrbracket \right) = -\sum_{l=1}^{12} \left(\llbracket 0, a_l \rrbracket \times \llbracket c_l \rrbracket - \llbracket 0, a_l \rrbracket \times \llbracket c_{l-1} \rrbracket \right).$$

Moreover, by the choice of $\{c_l\}$,

$$\sum_{l=1}^{12} \llbracket 0 \rrbracket \times \llbracket c_{l-1}, c_l \rrbracket = \llbracket 0 \rrbracket \times \llbracket \alpha_1, \alpha_2 \rrbracket + \llbracket 0 \rrbracket \times \llbracket \alpha_2, \alpha_3 \rrbracket + \cdots + \llbracket 0 \rrbracket \times \llbracket \alpha_4, \alpha_1 \rrbracket = 0.$$

Therefore, $\partial G_u = -\partial S$.

Notice that the action of T_u against 2-forms with only vertical differentials is 0, which means that T_u does not have completely vertical part and so $\mu_v[u] = 0$. Roughly, due to cancellations in the part of the boundary of T_u in correspondence to the origin, the current T_u is not able to detect the hole upon the origin in the graph of u, generated by the presence of the multiple junction.

Example 4.6. This example is an adaptation of [22, Theorem 1.3] to the case of piecewise constant maps. Indeed, we construct a piecewise constant map u, taking only five values of $\mathbb{R}^2$, such that

$$\mu_v[u] = 0 \qquad \text{and} \qquad \overline{TVJ}_{BV}(u, B_1) = +\infty.$$

The idea is to replicate the map of Example 4.5 infinitely many times on a sequence $\{D_i\}_{i \in \mathbb{N}} \subset B_1$ of disjoint balls, whose measures form an infinitesimal sequence (see Fig. 5). So, for $i \in \mathbb{N}$, set

$$D_i := B_{r_i}(x_i), \quad \text{with } x_i := \left(-1 + \sum_{j=0}^{i-1} 2^{-j}, 0 \right), \quad r_i := 2^{-i-1}.$$

Let $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\} \subset \mathbb{R}^2$ and $\gamma : \mathbb{S}^1 \to \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$ be as in Example 4.5. Now, define the map $\hat{\gamma} : \mathbb{S}^1 \to \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$ in the same way as γ, but with different order of the values, in a symmetric way with respect to the vertical axis through α_1, namely, in the same arcs $C_1, \ldots, C_{12}$, $\hat{\gamma}$ is equal to $\alpha_1, \alpha_5, \alpha_4, \alpha_1, \alpha_3, \alpha_2, \alpha_1, \alpha_4, \alpha_5, \alpha_1, \alpha_2, \alpha_3$. Then, for $i \in \mathbb{N}$, define $u_{|D_i} := u^{(i)}$ as

$$u^{(i)}(x) = \begin{cases} \gamma\left(\dfrac{x - x_i}{|x - x_i|}\right) & \text{if } i \text{ is odd,} \\[4mm] \hat{\gamma}\left(\dfrac{x - x_i}{|x - x_i|}\right) & \text{if } i \text{ is even.} \end{cases}$$

It remains to define u in $B_1 \setminus \cup_{i \in \mathbb{N}} D_i$. Start by considering, for every $i \in \mathbb{N}$, the square Q_i that circumscribes D_i and extend $u^{(i)}$ to Q_i to be constant along horizontal lines. Now, denote by $L_i^{(1)}$ and $L_i^{(2)}$ the vertical left and right sides of ∂Q_i, then extend u to the convex hull of $L_i^{(2)}$ and $L_{i+1}^{(1)}$ to be constant along straight lines which interpolate pointwise the two sides. Finally, extend u in the strip that connects $L_1^{(1)}$ to ∂B_1 to be constant along horizontal lines and set $u = \alpha_1$ in the rest of B_1 (see Fig. 5). It is not difficult to see that $u \in BV(B_1; \mathbb{R}^2)$, by the choice of the infinitesimal sequence (r_i). Thus, assuming by contradiction that $\overline{\mathcal{A}}_{BV}(u, B_1)$ be finite, one can define the current $T_u = G_u + S$ in a similar way as in Example 4.5, that is to say, by setting S to be the trivial affine interpolation surface on the jump segments of u. One can prove in the same way that T_u is the current with minimal completely vertical lifting associated to u and $\mu_v[u] = 0$. In particular, $T_u \in \text{cart}(B_1; \mathbb{R}^2)$ and has finite mass. On the other hand,

$$\overline{TVJ}_{BV}(u, B_1) \geq \sum_{i=1}^{+\infty} \overline{TVJ}_{BV}(u, D_i) = \sum_{i=1}^{+\infty} 2\min\{|T_{\alpha_1 \alpha_2 \alpha_3}|, |T_{\alpha_1 \alpha_4 \alpha_5}|\} = +\infty.$$

In particular $\overline{\mathcal{A}}_{BV}(u, B_1) = +\infty$ as well.

5. Piecewise Lipschitz maps

In this last section we combine the results of the previous sections and compute the BV-relaxed area for an interesting class of maps that we call *piecewise Lipschitz maps*, quickly mentioned in the Introduction. As stated in our main result (Theorem 1.1), the relaxed area turns out to be composed by a regular term and a singular one, that interestingly further splits into two non-trivial pieces, respectively related to the 1-dimensional and 0-dimensional singularities.

Let $\Omega \subset \mathbb{R}^2$ be a connected bounded open set with boundary of class C^1. We say that a collection $\{\Omega_1, \ldots, \Omega_N\}$ of disjoint nonempty open sets is a Lipschitz partition of Ω if $\overline{\Omega} = \cup_{k=1}^N \overline{\Omega}_k$ and for each $k = 1, \ldots, N$, Ω_k is connected and Lipschitz.

For a given Lipschitz partition of Ω we can consider its interface $\Sigma := \cup_{k=1}^N \partial \Omega_k$. Also, we can define the (possibly empty) set of interior junction points $\{p_i\}_{i=1}^m$, i.e. points $p_i \in \Omega$ such that there exist $r > 0$ and an integer N_i with $3 \leq N_i \leq N$, such that $B_r(p_i) \subset \Omega$ and $B_s(p_i)$ has nonempty intersection with exactly N_i connected components of Ω, for every $s \in (0, r]$.

We shall consider Lipschitz partitions whose interface is a *network* in the following sense:

Definition 5.1 (*Network*). The interface Σ of a Lipschitz partition of Ω is a network if

$$\Sigma := \bigcup_{\ell=1}^n \overline{J}_\ell, \qquad J_\ell = \alpha_\ell(I_\ell), \quad I_\ell = (a_\ell, b_\ell), \tag{5.1}$$

where the curves $\alpha_\ell : \overline{I}_\ell := [a_\ell, b_\ell] \to \overline{\Omega}$, $\ell = 1, \ldots, n$, satisfy the following properties:

- α_ℓ is of class C^2, injective with $|\dot{\alpha}_\ell| \equiv 1$ on I_ℓ, and $J_\ell \subset \Omega$;
- $\ell_1 \neq \ell_2 \Rightarrow J_{\ell_1} \cap J_{\ell_2} = \varnothing$;
- $\alpha_\ell(\{a_\ell, b_\ell\}) \subset \{p_1, \ldots, p_m\} \cup \partial \Omega$ for all $\ell = 1, \ldots, n$ such that $\alpha_\ell(a_\ell) \neq \alpha_\ell(b_\ell)$;
- if $x \in \overline{J}_\ell \cap \partial \Omega$, α_ℓ is transversal to $\partial \Omega$ at x;
- $\ell_1 \neq \ell_2 \Rightarrow \overline{J}_{\ell_1} \cap \overline{J}_{\ell_2} \subset \{p_1, \ldots, p_m\}$.

From the last condition it follows that if two curves have endpoints on $\partial \Omega$, then these points are distinct.

Definition 5.2 (*Piecewise Lipschitz Map*). Let $\{\Omega_k\}_{k=1}^N$ be a Lipschitz partition of Ω whose interface Σ is a network. We say that $u \in BV(\Omega; \mathbb{R}^2)$ is a piecewise Lipschitz map if its jump set S_u coincides with Σ and $u \llcorner \Omega_k \in \text{Lip}(\Omega_k; \mathbb{R}^2)$ for any $k = 1, \ldots, N$.

Since $u \llcorner \Omega_k \in \text{Lip}(\Omega_k; \mathbb{R}^2)$, the trace of u on $\partial \Omega_k$ is also Lipschitz. In particular, for any $i \in \{1, \ldots, m\}$ such that $p_i \in \partial \Omega_k$,

$$\exists \lim_{\substack{x \to p_i \\ x \in \Omega_k}} u(x) =: \beta_i^k \in \mathbb{R}^2.$$

Let $\rho > 0$ be sufficiently small so that $B_\rho(p_i) \subset \Omega$ for $i \in \{1, \ldots, m\}$. Let $\ell \in \{1, \ldots, n\}$ be such that p_i is an endpoint of $\overline{J}_\ell$; since α_ℓ is of class C^2, for ρ small enough the intersection $\overline{J}_\ell \cap \partial B_\rho(p_i)$ consists either of a single point, or of two points if $\alpha_\ell(a_\ell) = \alpha_\ell(b_\ell) = p_i$. Hence, the map $u \llcorner \partial B_\rho(p_i)$ is piecewise Lipschitz and jumps at any point of $\Sigma \cap \partial B_\rho(p_i)$. In particular, the number of these jump points is, by definition of junction point,

$$N_i = \sharp\left(\Sigma \cap \partial B_\rho(p_i)\right) \geq 3, \qquad i = 1, \ldots, m.$$

For $i = 1, \ldots, m$, we denote by $\Omega_1^i, \ldots, \Omega_{N_i}^i$ the connected components of $\Omega \setminus \Sigma$ whose closure contains p_i, chosen in counterclockwise order around p_i. Since Ω_k is Lipschitz for every $k = 1, \ldots, N$, any Ω_k^i has a corner at p_i whose aperture is a positive angle $\theta_k^i \in (0, 2\pi)$.

Lemma 5.3 (*Circular Slices*). *Let $i \in \{1, \ldots, m\}$ be fixed and let $\rho > 0$ be as above. Then the maps $\gamma_\rho^i \in BV(\mathbb{S}^1; \mathbb{R}^2)$ defined by $\gamma_\rho^i(v) := u(p_i + \rho v)$ converge strictly $BV(\mathbb{S}^1; \mathbb{R}^2)$, as $\rho \to 0^+$, to a piecewise constant map $\gamma^i : \mathbb{S}^1 \to \mathbb{R}^2$ taking, in counterclockwise order, the values $\beta_i^1, \beta_i^2, \ldots, \beta_i^{N_i}$ on arcs of size $\theta_i^1, \theta_i^2, \ldots, \theta_i^{N_i}$, respectively.*

The map γ^i has N_i jumps on $\mathbb{S}^1$ whose angular coordinates are denoted by $a_i^1, a_i^2, \ldots, a_i^{N_i}$ (where[7] $a_i^j - a_i^{j-1} = \theta_i^j$, for $j = 1, \ldots, N_i + 1$).

Proof. It is easy to see that (γ_ρ^i) converges to γ^i almost everywhere on $\mathbb{S}^1$ as $\rho \to 0^+$. Moreover, γ_ρ^i, for ρ small enough, has exactly N_i jumps at points $a_{i,\rho}^j$ of amplitude $|u^+(p_i + \rho a_{i,\rho}^j) - u^-(p_i + \rho a_{i,\rho}^j)|$ which tend, by continuity of u in $B_\rho(p_i) \setminus \Sigma$, to $|\beta_i^j - \beta_i^{j+1}|$. Also, on the arcs between $a_{i,\rho}^j$ and $a_{i,\rho}^{j+1}$, $|\dot\gamma_\rho^i| \le L\rho$, where L is the maximum of the Lipschitz constants of u on the sectors Ω_k^i. Hence $|\dot\gamma_\rho^i|(\mathbb{S}^1) \to |\dot\gamma^i|(\mathbb{S}^1)$ and the thesis follows straightforwardly. $\quad\square$

For $\ell = 1, \ldots, n$, we denote by $u_{(\ell)}^\pm$ the two traces of u on J_ℓ, and consider the affine interpolation surface $X_{(\ell)}^{\mathrm{aff}} : [a_\ell, b_\ell] \times I \to \mathbb{R}^3$ spanning the graphs of $u_{(\ell)}^-$ and $u_{(\ell+)}$, given by (1.7):

$$X_{(\ell)}^{\mathrm{aff}}(t, s) = (t, su_{(\ell)}^+(t) + (1-s)u_{(\ell)}^-(t)), \qquad (t, s) \in [a_\ell, b_\ell] \times I. \tag{5.2}$$

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Lower bound: Consider a sequence $(v_k) \subset C^1(\Omega; \mathbb{R}^2)$ converging to u strictly $BV(\Omega; \mathbb{R}^2)$. For any $\rho > 0$ small enough, we take a family of mutually disjoint balls $B_\rho(p_i) \subset \Omega$, $i = 1, \ldots, m$. By Lemma 2.5, there exists a subsequence $(v_{k_h}) \subset (v_k)$ depending on ρ such that for $i = 1, \ldots, m$

$$v_{k_h} \llcorner \partial B_\rho(p_i) \to u \llcorner \partial B_\rho(p_i) \quad \text{strictly } BV(\partial B_\rho(p_i); \mathbb{R}^2). \tag{5.3}$$

We may also assume that for $i = 1, \ldots, m$

$$\liminf_{k \to +\infty} \int_{B_\rho(p_i)} |Jv_k| \, dx = \lim_{h \to +\infty} \int_{B_\rho(p_i)} |Jv_{k_h}| \, dx.$$

Then

$$\mathcal{A}(v_{k_h}, \Omega) = \mathcal{A}(v_{k_h}, \Omega \setminus \cup_{i=1}^m \overline{B}_\rho(p_i)) + \sum_{i=1}^m \mathcal{A}(v_{k_h}, \overline{B}_\rho(p_i)) \ge \mathcal{A}(v_{k_h}, \Omega \setminus \cup_{i=1}^m \overline{B}_\rho(p_i)) + \sum_{i=1}^m \int_{B_\rho(p_i)} |Jv_{k_h}| dx.$$

By Corollary 3.12, we get

$$\liminf_{h \to +\infty} \mathcal{A}(v_{k_h}, \Omega \setminus \cup_{i=1}^m \overline{B}_\rho(p_i)) \ge \overline{\mathcal{A}}_{BV}(u, \Omega \setminus \cup_{i=1}^m \overline{B}_\rho(p_i))$$

$$= \int_{\Omega \setminus \cup_{i=1}^m B_\rho(p_i)} |\mathcal{M}(\nabla u)| dx + \sum_{\ell=1}^n \int_{[a_\ell^\rho, b_\ell^\rho] \times I} |\partial_t X_{(\ell)}^{\mathrm{aff}} \wedge \partial_s X_{(\ell)}^{\mathrm{aff}}| \, dtds$$

$$\longrightarrow \int_\Omega |\mathcal{M}(\nabla u)| dx + \sum_{\ell=1}^n \int_{[a_\ell, b_\ell] \times I} |\partial_t X_{(\ell)}^{\mathrm{aff}} \wedge \partial_s X_{(\ell)}^{\mathrm{aff}}| dtds \quad \text{as } \rho \to 0^+,$$

where $(a_\ell^\rho), (b_\ell^\rho) \subset [a_\ell, b_\ell]$ are respectively a decreasing and increasing sequence of numbers satisfying $a_\ell^\rho \to a_\ell$ and $b_\ell^\rho \to b_\ell$ as $\rho \to 0^+$ and $\alpha_\ell([a_\ell^\rho, b_\ell^\rho]) = \alpha_\ell([a_\ell, b_\ell]) \setminus \cup_{i=1}^m B_\rho(p_i)$.

Let us recall that, by Lemma 2.14, $P(\tilde\gamma^i) = \overline{P}(\gamma^i)$, with γ^i as in Lemma 5.3. So, it remains to show that

$$\liminf_{\rho \to 0^+} \lim_{h \to +\infty} \int_{B_\rho(p_i)} |Jv_{k_h}| \, dx \ge \overline{P}(\gamma^i) \qquad \forall i = 1, \ldots, m. \tag{5.4}$$

By definition (2.40), using (2.31) and (5.3), we readily conclude that

$$\lim_{h \to +\infty} \int_{B_\rho(p_i)} |Jv_{k_h}| \, dx \ge \overline{P}(\gamma_\rho^i),$$

where γ_ρ^i is defined in Lemma 5.3. Then, since γ_ρ^i converge to γ^i strictly $BV(\mathbb{S}^1; \mathbb{R}^2)$ as $\rho \to 0^+$, (5.4) follows, thanks to Lemma 5.3 and Corollary 2.16.

Upper bound: Fix $r > 0$ small enough and consider mutually disjoint balls $B_r(p_i) \subset \Omega$, $i = 1, \ldots, m$, such that, for every $\ell \in \{1, \ldots, n\}$, $J_\ell \cap \partial B_s(p_i)$, if nonempty, consists either of a single point, or of two points if $\alpha_\ell(a_\ell) = \alpha_\ell(b_\ell) = p_i$, for every $s \in (0, r)$.

Clearly, the difficulty of the proof is concentrated around the junction points p_i. The idea is to modify u on $\cup_{i=1}^m B_r(p_i)$ by constructing a new map u_r (see (5.8) and (5.20)), which coincides with u out of $\cup_{i=1}^m B_r(p_i)$ and converges to u strictly $BV(\Omega; \mathbb{R}^2)$ as r tends to 0^+. The map u_r will be again a piecewise Lipschitz map with the same set $\{p_i\}$ of junction points, but different jump set

---

[7] With the convention $N_i + 1 = 1$.

Σ_r, with $\Sigma_r \cap B_{r/2}(p_i)$ made of segments, i.e. u_r is of the form (4.4) in $B_{r/2}(p_i)$. The difficult point will be to provide that Σ_r is still a union of (pairwise disjoint up to the endpoints) C^2-curves $\hat{\alpha}_\ell$, in particular that each on e hits $\partial B_{r/2}(p_i)$ with vanishing second derivative. At the end, we will apply Theorem 4.4 to u_r in $\cup_{i=1}^m B_{r/2}(p_i)$ and Corollary 3.12 to u_r in $\Omega \setminus (\cup_{i=1}^m B_{r/2}(p_i))$, and conclude by lower semicontinuity of $\overline{\mathcal{A}}_{BV}(\cdot, \Omega)$.

We start by considering a smooth strictly increasing surjective function $\psi_r : [\frac{r}{2}, +\infty) \to [0, +\infty)$ with[8]

$$\psi_r(\rho) = \rho \quad \forall \rho \geq r, \quad \psi_r(\rho) = \left(\rho - \frac{r}{2}\right)^3 \text{ in a right neighborhood of } \frac{r}{2}, \quad |\psi_r'| \leq C \quad \text{in } \left(\frac{r}{2}, r\right) \tag{5.5}$$

with $C > 0$ independent of r. We define the radial map $\Phi_r : \mathbb{R}^2 \setminus B_{\frac{r}{2}}(0) \to \mathbb{R}^2 \setminus \{0\}$ as

$$\Phi_r(x) = \psi_r(|x|)\frac{x}{|x|},$$

whose inverse is $\Phi_r^{-1}(y) = f_r(|y|)\frac{y}{|y|}$, where $f_r := \psi_r^{-1}$, and set

$$\hat{u}_r(x) := u(p_i + \Phi_r(x - p_i)) \qquad \text{for } x \in B_r(p_i) \setminus \overline{B}_{\frac{r}{2}}(p_i), \ i = 1, \dots, m. \tag{5.6}$$

The jump set of $\hat{u}_r$ in $B_r(p_i) \setminus B_{r/2}(p_i)$ is parametrized by the curves

$$\hat{\alpha}_\ell := p_i + \Phi_r^{-1}(\alpha_\ell - p_i) \qquad \forall \ell = 1, \dots, n. \tag{5.7}$$

Notice carefully that $\hat{\alpha}_\ell$ is parametrized on the same parameter interval of α_ℓ, but this is not an arc length parametrization for $\hat{\alpha}_\ell$. Moreover, thanks to the regularity of Φ_r, the map

$$u_r := \begin{cases} u & \text{in } \Omega \setminus (\cup_{i=1}^m B_r(p_i)) \\ \hat{u}_r & \text{in } B_r(p_i) \setminus B_{\frac{r}{2}}(p_i), \quad i = 1, \dots, m, \end{cases} \tag{5.8}$$

has jump set Σ_r which is parametrized by the curves $\hat{\alpha}_\ell$, whose supports $\hat{J}_\ell$ are pairwise disjoint and in turn coincide with the ones of α_ℓ in $\Omega \setminus (\cup_{i=1}^m B_r(p_i))$.

Step 1: Let us first check that the length of $\hat{\alpha}_\ell$ in $\cup_{i=1}^m (B_r(p_i) \setminus B_{r/2}(p_i))$ is controlled, more precisely, we will show that for each i and ℓ, the length of $\hat{\alpha}_\ell$ in $B_r(p_i) \setminus B_{r/2}(p_i)$ goes to 0 as $r \to 0^+$. We suppose that $J_\ell \cap \partial B_s(p_i)$, for every $s \leq r$, consists of a single point, because the argument adapts also if α_ℓ has two arcs exiting from p_i, simply by considering them separately. To this aim, fix i and ℓ and denote $\alpha_\ell = \alpha$, $J_\ell = J$. Without loss of generality, assume $p_i = 0$, $B_r(0) = B_r$, and suppose that $J \cap B_r$ is parametrized by arc length on $[0, R]$, with $\alpha(0) = 0$ and $\alpha(R) \in \partial B_r$, where $R(r) = R = \mathcal{H}^1(J \cap B_r)$. We can express the gradient of Φ_r^{-1} as follows:

$$\nabla \Phi_r^{-1}(y) = f_r'(|y|)\frac{y}{|y|} \otimes \frac{y}{|y|} + f_r(|y|)\nabla\left(\frac{y}{|y|}\right) = f_r'(|y|)\frac{y}{|y|} \otimes \frac{y}{|y|} + \frac{f_r(|y|)}{|y|}\Pi(y), \tag{5.9}$$

where

$$\Pi(y) := \text{Id} - \frac{y \otimes y}{|y|^2},$$

and we used that

$$\nabla\left(\frac{y}{|y|}\right) = \frac{1}{|y|}\Pi(y). \tag{5.10}$$

From (5.7), we have $\dot{\hat{\alpha}} = \nabla \Phi_r^{-1}(\alpha)\dot{\alpha}$, and using (5.9) and $|\dot{\alpha}| = 1$,

$$|\dot{\hat{\alpha}}| \leq f_r'(|\alpha|) + \frac{f_r(|\alpha|)}{|\alpha|}|\Pi(\alpha)\dot{\alpha}|. \tag{5.11}$$

Notice that if r is small, the function $t \mapsto |\alpha(t)| =: \sigma(t)$ is C^1 and invertible from $[0, R]$ to $[0, r]$. Moreover, $\sigma'(t) = \frac{\alpha(t)}{|\alpha(t)|} \cdot \dot{\alpha}(t) \to \frac{\dot{\alpha}(0)}{|\dot{\alpha}(0)|} \cdot \dot{\alpha}(0) = |\dot{\alpha}(0)| = 1$ as $t \to 0^+$. Let us integrate on $[0, R]$ the term $f_r'(|\alpha|)$: performing the change of variable $\sigma(t) = \rho$, we get

$$\int_0^R f_r'(|\alpha(t)|)dt = \int_0^R f_r'(\sigma(t))\,dt = \int_0^r f_r'(\rho)\frac{d\rho}{\sigma'(\sigma^{-1}(\rho))} \leq 2\int_0^r f_r'(\rho)d\rho,$$

where in the last inequality we used that, for small r, $\sigma'(\sigma^{-1}(\rho)) \geq \frac{1}{2}$ for every $\rho \in [0, r]$. Sending r to 0^+, we have that $\int_0^R f_r'(|\alpha(t)|)dt \to 0$ by integrability of f' near to the origin.

In order to estimate the second term on the right hand side of (5.11), we can use a Taylor expansion of α around 0, writing $\alpha(t) = vt + wt^2 + o(t^2)$, with $v = \dot{\alpha}(0)$, $w = \frac{\ddot{\alpha}(0)}{2}$, and $\lim_{t \to 0^+} o(t^p)/t^p = 0$. We have

$$\Pi(\alpha)\dot{\alpha} = \Pi(vt + wt^2 + o(t^2))(v + 2wt + o_2(t)) = \Pi(v + wt + o_1(t))(v + 2wt + o_2(t)),$$

where $o_1(t) = o(t^2)/t$ and $o_2(t) = o(t)$. Writing $v + 2wt + o_2(t) = v + wt + o_1(t) + wt + o_2(t) - o_1(t)$, we get

$$\Pi(\alpha)\dot{\alpha} = \Pi(v + wt + o_1(t))(v + wt + o_1(t)) + \Pi(v + wt + o_1(t))(wt + o_2(t) - o_1(t)).$$

---

[8] The exponent must be chosen greater than 2 in order to ensure (5.19).

The first term on the right hand side is 0 and the norm of the second term can be estimated from above by $|w|t + o(t)$. Now, by definition of arc length parameter, $R = \mathcal{H}^1(\mathrm{spt}\,\alpha \cap B_r(0)) \to 0$ as $r \to 0^+$. Moreover, by Taylor expansion, $|\alpha(t)| > \frac{t}{2}$ for t small enough. Therefore, since $f_r(0) = \frac{r}{2}$, for r small enough we have $\frac{f_r(|\alpha(t)|)}{|\alpha(t)|} \leq \frac{2r}{t}$ on $[0, R]$. So, integrating on $[0, R]$ the second term on the right hand side of (5.11),

$$\int_0^R \frac{f_r(|\alpha(t)|)}{|\alpha(t)|}\,|\Pi(\alpha(t))\dot{\alpha}(t)|\,dt \leq \int_0^R \frac{2r}{t}(|w|t + o(t))dt \to 0 \qquad \text{as } r \to 0^+.$$

Step 2: Let $\hat{J} = \hat{J}_\ell$ be the support of $\hat{\alpha}$; let us show that there is a parametrization of $\hat{J} \cap (B_r \setminus B_{r/2})$ on an interval $[0, L]$, which is of class C^2 up to 0 and with vanishing second derivative at 0. Indeed, set $L := \mathcal{H}^1(\hat{J} \cap (B_r \setminus B_{r/2}))$ and consider the arc-length parameter $s \in [0, L]$ given by

$$s(t) = \int_0^t |V_r(\alpha(\tau))|\,d\tau,$$

where

$$V_r(\alpha) := \nabla \Phi_r^{-1}(\alpha)\dot{\alpha}.$$

We compute

$$\frac{d^2}{ds^2}\hat{\alpha}(t) = \frac{d}{ds}\left(\frac{V_r(\alpha)}{|V_r(\alpha)|}\right) = \Pi(V_r(\alpha))\left(\frac{\nabla^2\Phi_r^{-1}(\alpha) : (\dot{\alpha}\otimes\dot{\alpha}) + \nabla\Phi_r^{-1}(\alpha)\ddot{\alpha}}{|V_r(\alpha)|^2}\right). \tag{5.12}$$

Here and in what follows, α is evaluated at $t = t(s)$ and $\dot{\alpha}$ and $\ddot{\alpha}$ denote the first and second derivative of α with respect to t. The operation $:$ between a tensor $T = (T_{ijk}) \in \mathbb{R}^{2\times2\times2}$ and a matrix $M = (M_{ij}) \in \mathbb{R}^{2\times2}$ is defined as the vector $T : M \in \mathbb{R}^2$ with components $(T : M)_k = T_{ijk}M_{ij}$ for $k = 1, 2$.

We get

$$\left|\frac{d^2}{ds^2}\hat{\alpha}(t)\right| \leq \left|\Pi(V_r(\alpha))\left(\frac{\nabla^2\Phi_r^{-1}(\alpha) : (\dot{\alpha}\otimes\dot{\alpha})}{|V_r(\alpha)|^2}\right)\right| + \frac{|\nabla\Phi_r^{-1}(\alpha)\ddot{\alpha}|}{|V_r(\alpha)|^2}$$

$$\leq \left|\Pi(V_r(\alpha))\left(\frac{\nabla^2\Phi_r^{-1}(\alpha) : (\dot{\alpha}\otimes\dot{\alpha})}{|V_r(\alpha)|^2}\right)\right| + C\frac{f_r'(|\alpha|) + \frac{f_r(|\alpha|)}{|\alpha|}}{|V_r(\alpha)|^2}. \tag{5.13}$$

where we have used (5.9) and that $\ddot{\alpha}$ is bounded.

The Hessian of Φ_r^{-1} can be computed as

$$\nabla^2\Phi_r^{-1}(y) = f_r''(|y|)\frac{y}{|y|}\otimes\frac{y}{|y|}\otimes\frac{y}{|y|} + f_r'(|y|)\nabla\left(\frac{y}{|y|}\otimes\frac{y}{|y|}\right) +$$

$$+ f_r'(|y|)\frac{y}{|y|}\otimes\nabla\left(\frac{y}{|y|}\right) + f_r(|y|)\nabla^2\left(\frac{y}{|y|}\right)$$

$$= f_r''(|y|)\frac{y}{|y|}\otimes\frac{y}{|y|}\otimes\frac{y}{|y|} + f_r'(|y|)\nabla\left(\frac{y}{|y|}\right)\otimes\frac{y}{|y|} +$$

$$+ 2f_r'(|y|)\frac{y}{|y|}\otimes\nabla\left(\frac{y}{|y|}\right) + f_r(|y|)\nabla\left(\nabla\left(\frac{y}{|y|}\right)\right).$$

Then, by (5.10), we have

$$\nabla^2\Phi_r^{-1}(\alpha) = f_r''(|\alpha|)\frac{\alpha}{|\alpha|}\otimes\frac{\alpha}{|\alpha|}\otimes\frac{\alpha}{|\alpha|} + \left(\frac{f_r'(|\alpha|)}{|\alpha|} - 2\frac{f_r(|\alpha|)}{|\alpha|^2}\right)\Pi(\alpha)\otimes\frac{\alpha}{|\alpha|}$$

$$+ \left(2\frac{f_r'(|\alpha|)}{|\alpha|} - \frac{f_r(|\alpha|)}{|\alpha|^2}\right)\frac{\alpha}{|\alpha|}\otimes\Pi(\alpha).$$

So, for $k = 1, 2$, we have

$$\left(\nabla^2\Phi_r^{-1}(\alpha) : (\dot{\alpha}\otimes\dot{\alpha})\right)_k$$

$$= f_r''(|\alpha|)\left(\left(\frac{\alpha}{|\alpha|}\otimes\frac{\alpha}{|\alpha|}\otimes\frac{\alpha}{|\alpha|}\right) : (\dot{\alpha}\otimes\dot{\alpha})\right)_k$$

$$+ \left(\frac{f_r'(|\alpha|)}{|\alpha|} - 2\frac{f_r(|\alpha|)}{|\alpha|^2}\right)\left(\left(\Pi(\alpha)\otimes\frac{\alpha}{|\alpha|}\right) : (\dot{\alpha}\otimes\dot{\alpha})\right)_k \tag{5.14}$$

$$+ \left(2\frac{f_r'(|\alpha|)}{|\alpha|} - \frac{f_r(|\alpha|)}{|\alpha|^2}\right)\left(\left(\frac{\alpha}{|\alpha|}\otimes\Pi(\alpha)\right) : (\dot{\alpha}\otimes\dot{\alpha})\right)_k. \tag{5.15}$$

Notice that, since $\Pi(\alpha)$ is symmetric,

$$\Pi(\alpha)_{ij}\alpha_j = 0, \quad \Pi(\alpha)_{ij}\alpha_i = 0, \tag{5.16}$$

where we sum on repeated indices. So, using (5.16) and that, from Taylor expansion, $\dot{\alpha}(t) = v + 2wt + o(t) = \frac{\alpha(t)}{t} + wt + o(t)$, we have

$$\left(\left(\Pi(\alpha) \otimes \frac{\alpha}{|\alpha|}\right) : (\dot{\alpha} \otimes \dot{\alpha})\right)_k = \Pi(\alpha)_{ij}\dot{\alpha}_i\dot{\alpha}_j\frac{\alpha_k}{|\alpha|} = \Pi(\alpha)_{ij}\left(\frac{\alpha_i}{t} + w_i t + o(t)\right)\dot{\alpha}_j\frac{\alpha_k}{|\alpha|} =$$
$$= \Pi(\alpha)_{ij}\left(w_i t + o(t)\right)\dot{\alpha}_j\frac{\alpha_k}{|\alpha|};$$

$$\left(\left(\frac{\alpha}{|\alpha|} \otimes \Pi(\alpha)\right) : (\dot{\alpha} \otimes \dot{\alpha})\right)_k = \frac{\alpha_i}{|\alpha|}\Pi(\alpha)_{jk}\dot{\alpha}_i\dot{\alpha}_j = \frac{\alpha_i}{|\alpha|}\Pi(\alpha)_{jk}\left(\frac{\alpha_j}{t} + w_j t + o(t)\right)\dot{\alpha}_i$$
$$= \frac{\alpha_i}{|\alpha|}\Pi(\alpha)_{jk}\left(w_j t + o(t)\right)\dot{\alpha}_i.$$

So, the norm of the sum of (5.14) and (5.15) can be easily estimated by

$$3\left(\frac{f_r'(|\alpha|)}{|\alpha|} + \frac{f_r(|\alpha|)}{|\alpha|^2}\right)(|w|t + o(t)) \leq C\left(f_r'(|\alpha|) + \frac{f_r(|\alpha|)}{|\alpha|}\right),$$

where we used that, for t small, $|\alpha(t)| \geq \frac{t}{2}$.

Therefore, (5.13) becomes

$$\left|\frac{d^2}{ds^2}\hat{\alpha}(t)\right| \leq |f_r''(|\alpha|)|\left|\Pi(V_r(\alpha))\left(\frac{\frac{\alpha}{|\alpha|} \otimes \frac{\alpha}{|\alpha|} \otimes \frac{\alpha}{|\alpha|} : (\dot{\alpha} \otimes \dot{\alpha})}{|V_r(\alpha)|^2}\right)\right| + C\frac{f_r'(|\alpha|) + \frac{f_r(|\alpha|)}{|\alpha|}}{|V_r(\alpha)|^2}. \tag{5.17}$$

Now we treat the first term of the right hand side of (5.17). For $j = 1, 2$, by definition of $V_r(\alpha)$, using Taylor expansion and (5.16), we have

$$(V_r)_j(\alpha) = f_r'(|\alpha|)\frac{\alpha_i\alpha_j}{|\alpha|^2}\dot{\alpha}_i + f_r(|\alpha|)\Pi(\alpha)_{ij}\dot{\alpha}_i$$
$$= f_r'(|\alpha|)\frac{\alpha_i\alpha_j}{|\alpha|^2}\left(\frac{\alpha_i}{t} + w_i t + o(t)\right) + f_r(|\alpha|)\Pi(\alpha)_{ij}\left(\frac{\alpha_i}{t} + w_i t + o(t)\right)$$
$$= f_r'(|\alpha|)\left(\frac{\alpha_j}{t} + \frac{\alpha_i\alpha_j}{|\alpha|^2}w_i t + o(t)\right) + f_r(|\alpha|)\Pi(\alpha)_{ij}\left(w_i t + o(t)\right) \tag{5.18}$$
$$= f_r'(|\alpha|)\left(\frac{\alpha_j}{t} + o(t)\right) + f_r(|\alpha|)O_j(t),$$

where in the last equality we used that $\alpha_i w_i = o(t)$, since $v_i w_i = 0$ because $|\dot{\alpha}| = 1$, and we setted $O_j(t) := \Pi(\alpha)_{ij}(w_i t + o(t))$, meaning that $\lim_{t\to 0^+}|O_j(t)|/t < +\infty$. Then, we get

$$\alpha = t\left(\frac{V_r(\alpha) - O(t)}{f_r'(|\alpha|)} + o(t)\right).$$

So,

$$\Pi(V_r(\alpha))\frac{\frac{\alpha}{|\alpha|} \otimes \frac{\alpha}{|\alpha|} \otimes \frac{\alpha}{|\alpha|} : (\dot{\alpha} \otimes \dot{\alpha})}{|V_r(\alpha)|^2} = \frac{\alpha_i\alpha_j}{|\alpha|^2}\dot{\alpha}_i\dot{\alpha}_j\Pi(V_r(\alpha))\frac{\frac{\alpha}{|\alpha|}}{|V_r(\alpha)|^2}$$
$$= \frac{\alpha_i\alpha_j}{|\alpha|^2}\dot{\alpha}_i\dot{\alpha}_j\frac{t}{|\alpha|}\Pi(V_r(\alpha))\frac{\left(\frac{V_r(\alpha) - O(t)}{f_r'(|\alpha|)} + o(t)\right)}{|V_r(\alpha)|^2}$$
$$= \frac{\alpha_i\alpha_j}{|\alpha|^2}\dot{\alpha}_i\dot{\alpha}_j\frac{t}{|\alpha|}\Pi(V_r(\alpha))\frac{\left(\frac{O(t)}{f_r'(|\alpha|)} + o(t)\right)}{|V_r(\alpha)|^2},$$

where we used that $\Pi(V_r(\alpha))V_r(\alpha) = 0$. For t small, we get

$$\left|\Pi(V_r(\alpha))\frac{\frac{\alpha}{|\alpha|} \otimes \frac{\alpha}{|\alpha|} \otimes \frac{\alpha}{|\alpha|} : (\dot{\alpha} \otimes \dot{\alpha})}{|V_r(\alpha)|^2}\right| \leq 2\frac{\frac{O(t)}{f_r'(|\alpha|)} + o(t)}{|V_r(\alpha)|^2}.$$

Finally, from (5.17), we obtain

$$\left| \frac{d^2}{ds^2} \widehat{\alpha}(t) \right| \leq |f_r''(|\alpha|)| \, \frac{\frac{O(t)}{f_r'(|\alpha|)} + o(t)}{|V_r(\alpha)|^2} + C \frac{f_r'(|\alpha|) + \frac{f_r(|\alpha|)}{|\alpha|}}{|V_r(\alpha)|^2}.$$

From the definition of f_r, we have that $f_r(|\alpha(t)|) = \frac{r}{2} + t^{\frac{1}{3}} + o(t^{\frac{1}{3}})$ for t near to 0. So, by (5.18), we have $|V_r(\alpha(t))| \geq Cf_r'(|\alpha(t)|) = Ct^{-\frac{2}{3}} + o(t^{-\frac{2}{3}})$. Then, since $|f_r''(|\alpha(t)|)| = Ct^{-\frac{5}{3}} + o(t^{-\frac{5}{3}})$, a straightforward check shows that

$$\frac{d^2}{ds^2} \widehat{\alpha}(t) \to 0 \quad \text{as } t \to 0^+. \tag{5.19}$$

We conclude that the curve $\widehat{\alpha}$ is C^2 up to 0 with vanishing second derivative, and hence can be extended on the interval $(-\frac{r}{2}, 0)$ to a (not relabeled) curve $\widehat{\alpha}$ whose support is a straight segment connecting $\widehat{\alpha}(0)$ to 0 (namely a radius of $B_{r/2}(0)$). Going back to the curves $\widehat{\alpha}_\ell$, we have just proved that we can extend them in $B_{r/2}(p_i)$ with C^2-regularity using a segment along a radius, reaching p_i. In particular, the new supports of $\widehat{\alpha}_\ell$'s form a N^i-junction point around p_i in $B_{r/2}(p_i)$, whose circular sectors $\widehat{C}_j^i$ ($j = 1, \ldots, N_i$) have amplitudes $\theta_i^1, \ldots, \theta_i^{N_i}$ (according to Lemma 5.3). Up to a reparametrization by arc-length of $\widehat{\alpha}_\ell$, we will suppose that $\widehat{\alpha}_\ell : [\widehat{a}_\ell, \widehat{b}_\ell] \to \mathbb{R}^2$ have always derivative of modulus 1.

Step 3: We are ready to extend the map u_r in $B_{r/2}(p_i)$. We observe that, from (5.8), $u_r(x) = \gamma^i \left(\frac{2}{r}(x - p_i) \right)$ on $\partial B_{r/2}(p_i)$ (see Lemma 5.3), and hence it is constant on any arc with angular coordinate in (a_i^{j-1}, a_i^j). Hence we define

$$u_r(x) := \gamma^i \left(\frac{x - p_i}{|x - p_i|} \right) \qquad x \in B_{\frac{r}{2}}(p_i). \tag{5.20}$$

Now, u_r satisfies the hypotheses of Corollary 3.12 in $\Omega_r := \Omega \setminus (\cup_{i=1}^m \overline{B}_{r/4}(p_i))$, where all the curves $\widehat{\alpha}_j$ satisfy hypotheses (H3), and they run on a straight segment (along a radius of $B_{r/2}(p_i)$) inside $B_{r/2}(p_i) \setminus B_{r/4}(p_i)$. Then we introduce a sequence of Lipschitz maps $\widetilde{v}_k : \Omega_r \to \mathbb{R}^2$ which are defined as in 3.10, where, we recall, $\varepsilon = \frac{1}{k}$, with u_r in place of u and $\Lambda = \text{id}$; in particular, for k large enough, the trace of $\widetilde{v}_k$ on $\partial B_{r/3}(p_i)$ is a piecewise affine map coinciding with γ_k in (4.8), with β_i in place of α_i. Thus, if we introduce also the sequence of Lipschitz maps $\widehat{v}_k : B_{r/2}(p_i) \to \mathbb{R}^2$ as in (4.12) (with B_r replaced by $B_{r/2}(p_i)$) we see that $\widetilde{v}_k = \widehat{v}_k$ on $\partial B_{r/3}(p_i)$. Therefore we define

$$v_k^r := \begin{cases} \widetilde{v}_k & \text{in } \Omega \setminus (\cup_{i=1}^m B_{r/3}(p_i)) \\ \widehat{v}_k & \text{in } \cup_{i=1}^m B_{r/3}(p_i), \end{cases} \tag{5.21}$$

and we readily see that $v_k^r \to u_r$ strictly $BV(\Omega; \mathbb{R}^2)$.

Since the supports of α_ℓ and $\widehat{\alpha}_\ell$ coincide out of $\cup_i B_r(p_i)$, there exist $\widehat{a}_\ell^r, \widehat{b}_\ell^r \in [\widehat{a}_\ell, \widehat{b}_\ell]$, $\widehat{a}_\ell^r < \widehat{b}_\ell^r$, and $a_\ell^r, b_\ell^r \in [a_\ell, b_\ell]$, $a_\ell^r < b_\ell^r$, such that

$$\widehat{\alpha}_\ell([\widehat{a}_\ell^r, \widehat{b}_\ell^r]) = \alpha_\ell([a_\ell^r, b_\ell^r]), \quad \widehat{\alpha}_\ell(\widehat{a}_\ell^r) = \alpha_\ell(a_\ell^r), \quad \widehat{\alpha}_\ell(\widehat{b}_\ell^r) = \alpha_\ell(b_\ell^r).$$

In particular, $\widehat{b}_\ell^r - \widehat{a}_\ell^r = b_\ell^r - a_\ell^r$, so up to a translation of the parameter interval of $[\widehat{a}_\ell, \widehat{b}_\ell]$, we can suppose $\widehat{a}_\ell^r = a_\ell^r$ and $\widehat{b}_\ell^r = b_\ell^r$. Clearly, $a_\ell^r \to a_\ell$ non increasingly and $b_\ell^r \to b_\ell$ non decreasingly as $r \to 0^+$.

In view of Corollary 3.12 and Theorem 4.4 we conclude

$$\begin{aligned}
\overline{\mathcal{A}}_{BV}(u_r, \Omega) &\leq \lim_{k \to +\infty} \mathcal{A}(v_k^r, \Omega) = \int_{\Omega \setminus (\cup_{i=1}^m B_r(p_i))} |\mathcal{M}(\nabla u)| \, dx + \sum_{\ell=1}^n \int_{[\widehat{a}_\ell, \widehat{b}_\ell] \times I} |\partial_t X_{\ell,r}^{\text{aff}} \wedge \partial_s X_{\ell,r}^{\text{aff}}| \, dt \, ds \\
&\quad + \int_{\cup_{i=1}^m (B_r(p_i) \setminus B_{r/3}(p_i))} |\mathcal{M}(\nabla u_r)| \, dx + m \frac{\pi r^2}{9} + \sum_{i=1}^m \overline{P}(\gamma^i) \\
&= \int_{\Omega \setminus (\cup_{i=1}^m B_r(p_i))} |\mathcal{M}(\nabla u)| \, dx + \sum_{\ell=1}^n \int_{[a_\ell^r, b_\ell^r] \times I} |\partial_t X_\ell^{\text{aff}} \wedge \partial_s X_\ell^{\text{aff}}| \, dt \, ds + \sum_{i=1}^m \overline{P}(\gamma^i) \\
&\quad + \int_{\cup_{i=1}^m (B_r(p_i) \setminus B_{r/3}(p_i))} |\mathcal{M}(\nabla u_r)| \, dx + \sum_{\ell=1}^n \int_{([\widehat{a}_\ell^{r/3}, a_\ell^r] \cup [b_\ell^r, \widehat{b}_\ell^{r/3}]) \times I} |\partial_t X_{\ell,r}^{\text{aff}} \wedge \partial_s X_{\ell,r}^{\text{aff}}| \, dt \, ds \\
&\quad + \frac{r}{3} \sum_{i=1}^m \sum_{j=1}^{N_i} |\beta_i^j - \beta_i^{j+1}| + m \frac{\pi r^2}{9},
\end{aligned} \tag{5.22}$$

where for all $\ell = 1, \ldots, n$ we have $\widehat{a}_\ell \leq \widehat{a}_\ell^{r/3} \leq a_\ell^r < b_\ell^r \leq \widehat{b}_\ell^{r/3} \leq \widehat{b}_\ell$, where $\widehat{\alpha}_\ell(\widehat{a}_\ell^{\frac{r}{3}}) \in \partial B_{r/3}(p_i)$, $\widehat{\alpha}_\ell(\widehat{b}_\ell^{\frac{r}{3}}) \in \partial B_{r/3}(p_j)$ for some $i, j \in \{1, \ldots, m\}$, unless one of them belongs to $\partial \Omega$, and where $X_{\ell,r}^{\text{aff}}$ is defined as X_ℓ^{aff} with u_r replacing u.

Now, since by (5.5) $|\psi_r'| \leq C$, u_r is still a piecewise Lipschitz map on Ω, hence, by *Step 1*, the last four terms in (5.22) are negligible as $r \to 0^+$. We then conclude, provided that $u_r \to u$ strictly $BV(\Omega; \mathbb{R}^2)$, that

$$\overline{\mathcal{A}}_{BV}(u, \Omega) \leq \liminf_{r \to 0^+} \overline{\mathcal{A}}_{BV}(u_r, \Omega) \leq \int_\Omega |\mathcal{M}(\nabla u)| \, dx + \sum_{\ell=1}^n \int_{[a_\ell, b_\ell] \times I} |\partial_t X_\ell^{\text{aff}} \wedge \partial_s X_\ell^{\text{aff}}| \, dt \, ds + \sum_{i=1}^m \overline{P}(\gamma^i),$$

that is the thesis. In order to check that $u_r \to u$ strictly $BV(\Omega; \mathbb{R}^2)$ it is sufficient to observe that $u = u_r$ outside $\cup_{i=1}^{m} B_r(p_i)$ and that

$$
\begin{aligned}
& \limsup_{r \to 0^+} |Du_r|(\cup_{i=1}^{m} B_r(p_i)) \\
& \leq \limsup_{r \to 0^+} \limsup_{k \to +\infty} \int_{\cup_{i=1}^{m} B_r(p_i)} \sqrt{1 + |\nabla v_k^r|^2} \, dx \\
& \leq \limsup_{r \to 0^+} \lim_{k \to +\infty} \mathcal{A}(v_k^r, \cup_{i=1}^{m} B_r(p_i)) \\
& = \limsup_{r \to 0^+} \Big(\int_{\cup_{i=1}^{m} (B_r(p_i) \setminus B_{r/3}(p_i))} |\mathcal{M}(\nabla u_r)| \, dx + m \frac{\pi r^2}{9} \\
& \quad + \sum_{\ell=1}^{n} \int_{([\hat{a}_\ell^{r/3}, \hat{a}_\ell^r] \cup [\hat{b}_\ell^r, \hat{b}_\ell^{r/3}]) \times I} |\partial_t X_{\ell,r}^{\mathrm{aff}} \wedge \partial_s X_{\ell,r}^{\mathrm{aff}}| \, dt ds + \frac{r}{3} \sum_{i=1}^{m} \sum_{j=1}^{N_i} |\alpha_j^i - \alpha_{j+1}^i| \Big) = 0.
\end{aligned}
$$

The proof is complete. $\quad\square$

Acknowledgments

References

[1] E. Acerbi, G. Dal Maso, New lower semicontinuity results for polyconvex integrals, Calc. Var. Partial Differential Equations 2 (1994) 329–371.

[2] L. Ambrosio, N. Fusco, D. Pallara, Functions of Bounded Variation and Free Discontinuity Problems, in: Oxford Mathematical Monographs, Oxford University Press, New York, 2000.

[3] G. Bellettini, S. Carano, R. Scala, The relaxed area of $\mathbb{S}^1$-valued singular maps in the strict BV-convergence, ESAIM Control Optim. Calc. Var. 28 (2022) 1–38.

[4] G. Bellettini, A. Elshorbagy, M. Paolini, R. Scala, On the relaxed area of the graph of discontinuous maps from the plane to the plane taking three values with no symmetry assumptions, Ann. Mat. Pura Appl. 199 (2020) 445–477.

[5] G. Bellettini, A. Elshorbagy, R. Scala, The L^1-relaxed area of the graph of the vortex map, 2021, (submitted for publication). Preprint arXiv 2107.07236, https://arxiv.org/abs/2107.07236.

[6] G. Bellettini, R. Marziani, R. Scala, A non-parametric plateau problem with partial free boundary, 2022, (submitted for publication). Preprint arXiv 2201.06145, https://arxiv.org/abs/2201.06145.

[7] G. Bellettini, M. Paolini, On the area of the graph of a singular map from the plane to the plane taking three values, Adv. Calc. Var. 3 (2010) 371–386.

[8] G. Bellettini, M. Paolini, L. Tealdi, On the area of the graph of a piecewise smooth map from the plane to the plane with a curve discontinuity, ESAIM Control Optim. Calc. Var. 22 (2015) 29–63.

[9] G. Bellettini, M. Paolini, L. Tealdi, Semicartesian surfaces and the relaxed area of maps from the plane to the plane with a line discontinuity, Ann. Mat. Pura Appl. 195 (2016) 2131–2170.

[10] H. Brezis, P. Mironescu, A. Ponce, W^1, 1-maps with values into $\mathbb{S}^1$, geometric analysis of PDE and several complex variables, Contemp. Math. 368 (2005) 69–100.

[11] G. Dal Maso, Integral representation on $BV(\Omega)$ of Γ-limits of variational integrals, Manuscripta Math. 30 (1980) 387–416.

[12] E. De Giorgi, On the relaxation of functionals defined on cartesian manifolds, in: Developments in Partial Differential Equations and Applications in Mathematical Physics, Ferrara 1992, Plenum Press, New York.

[13] G. De Philippis, Weak notions of Jacobian determinant and relaxation, ESAIM Control Optim. Calc. Var. 18 (2012) 181–207.

[14] I. Fonseca, N. Fusco, P. Marcellini, On the total variation of the Jacobian, J. Funct. Anal. 207 (2004) 1–32.

[15] M. Giaquinta, G. Modica, J. Souček, Graphs of finite mass which cannot be approximated in area by smooth graphs, Manuscripta Math. 78 (1993) 259–271.

[16] M. Giaquinta, G. Modica, J. Souček, Cartesian Currents in the Calculus of Variations I, in: Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 37, Springer-Verlag, Berlin-Heidelberg, 1998.

[17] M. Giaquinta, G. Modica, J. Souček, Cartesian Currents in the Calculus of Variations II. Variational Integrals, in: Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 38, Springer-Verlag, Berlin-Heidelberg, 1998.

[18] R.L. Jerrard, N. Jung, Strict convergence and minimal liftings in BV, Proc. Royal Soc. Edinburgh Sect. A 134 (2004) 1163–1176.

[19] S. Krantz, H. Parks, Geometric Integration Theory, Birkhäuser, Boston, 2008.

[20] J. Malý, L^p-approximation of Jacobians, Comment. Math. Univ. Carolin. 32 (1991) 659–666.

[21] J. Milnor, Topology from the Differentiable Viewpoint, Princeton University Press, 1997.

[22] D. Mucci, Remarks on the total variation of the Jacobian, NoDEA 13 (2006) 223–233.

[23] D. Mucci, Strict convergence with equibounded area and minimal completely vertical liftings, Nonlinear Anal. 221 (2022).

[24] E. Paolini, On the relaxed total variation of singular maps, Manuscripta Math. 111 (2003) 499–512.

[25] R. Scala, Optimal estimates for the triple junction function and other surprising aspects of the area functional, Ann. Sc. Norm. Super. Pisa Cl. Sci. 20 (2020) 491–564.