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QUASI-OPTIMAL ERROR ESTIMATES FOR THE MEAN CURVATURE FLOW WITH A FORCING TERM*

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Abstract. We study a singularly perturbed reaction-diffusion equation with a small parameter $\varepsilon > 0$. This problem is viewed as an approximation of the evolution of an interface by its mean curvature with a forcing term. We derive a quasi-optimal error estimate of order $\mathcal{O}(\varepsilon^2|\log\varepsilon|^2)$ for the interfaces, which is valid before the onset of singularities, by constructing suitable sub and super solutions. The proof relies on the behavior at infinity of functions appearing in the truncated asymptotic expansion, and by using a modified distance function combined with a vertical shift.

1. Introduction. It is known that the singularly perturbed reaction-diffusion equation, with the quartic double equal well potential $\Psi(s) = (1-s^2)^2$ and forcing term g

$$\partial_t u_\varepsilon - \Delta u_\varepsilon + \tfrac{1}{2\varepsilon^2}\Psi'(u_\varepsilon) = \tfrac{c_0}{2\varepsilon} g \qquad \text{in } \mathbf{R}^n \times (0,T),$$

where $c_0 = \int_{-1}^{1}\sqrt{\Psi(s)}ds$, provides an approximation for an interface $\Sigma(t)$ evolving by the law

$$V = \kappa + g,$$

where V is the normal velocity of the interface $\Sigma(t)$ and κ the sum of its principal curvatures [1–4, 6–9, 12–13]. Such an equation was introduced by Allen and Cahn [1] in order to describe the motion of antiphase boundaries in crystalline solids, thus showing its relevance in phase transitions. It was independently suggested by De Giorgi [6] as a variational approach to the mean curvature flow. Such a connection has been rigorously established by Evans, Soner, and Souganidis [9] and Barles, Soner, and Souganidis [2], who proved convergence of the zero level set of u_ε to the generalized motion by mean curvature [10], even beyond the onset of singularities, provided the limit interface does not develop interior; see also [4, 12]. Asymptotic analyses were carried out prior to those convergence results, but apply only to smooth evolutions [3, 7, 8, 13].

The goal of this paper is to prove a quasi-optimal error estimate, valid before the onset of singularities, for the Hausdorff distance between the flow $\Sigma(t)$ and the approximate interface $\Sigma_\varepsilon(t) = \{\mathbf{x} \in \Omega : u_\varepsilon(\mathbf{x},t) = 0\}$. In fact, assuming that

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$\Sigma = \{(\mathbf{x}, t) \in \Omega \times [0, T] : \mathbf{x} \in \Sigma(t)\}$ is sufficiently regular (see (2.4)) and $u_\varepsilon(\cdot, 0)$ has the correct shape (see (6.6)), we prove that

$$\Sigma_\varepsilon(t) \subseteq \{\mathbf{x} \in \Omega : \mathrm{dist}(\mathbf{x}, \Sigma(t))) \leq C\varepsilon^2 |\log \varepsilon|^2\} \qquad \forall t \in [0, T],$$
$$\Sigma(t) \subseteq \{\mathbf{x} \in \Omega : \mathrm{dist}(\mathbf{x}, \Sigma_\varepsilon(t))) \leq C\varepsilon^2 |\log \varepsilon|^2\} \qquad \forall t \in [0, T],$$

where C is a constant depending on T but independent of ε. We cannot expect estimates of order higher than two in view of the formal asymptotics and the results of [15]. Our estimate improves the results obtained by Chen [4], who shows a first order error estimate via comparison arguments. If Ψ is replaced by a non smooth potential giving rise to a double obstacle problem, similar results have been established in [5, 14–16].

The order of the interface error estimate is a consequence of the Maximum Principle and the explicit construction of sub and supersolutions, which in turn are inspired by, and indeed rely on, the formal asymptotics developed in [17]. The use of the asymptotic expansion, as in [7, 8], would also lead to a rate of convergence for interfaces via nondegeneracy properties of both the continuous and truncated solutions. In this light, a nontrivial by-product of the rigorous asymptotic analysis of De Mottoni and Schatzman [7, 8] would give an $\mathcal{O}(\varepsilon^2)$ interface error estimate for the special case $g = 0$, but under higher regularity restrictions on Σ than in the present discussion. Regardless of convergence, the formal asymptotics of §4 still provides valuable information on the shape of u_ε, which, together with a modified distance function combined with a vertical shift and the nondegeneracy property of sub and supersolutions, plays a fundamental role in our subsequent rigorous analysis; see §6. Furthermore, the functions appearing in the definition of the sub and super solutions need to be constant far from the interface, thus requiring proper shape corrections.

The outline of the paper is as follows. In §2 we introduce some notation. §3 is devoted to the analysis of suitable limit problems (suggested by the formal asymptotics of §4) and accurate decay estimates at infinity of its solutions. These solutions, properly combined, provide the sub and supersolutions of §6. For the sake of completeness, a simple but crucial comparison result is proved in §5. The sub and supersolutions are fully examined in §6, and then used to derive interface error estimates.

2. Notations. In what follows $\Omega \subseteq \mathbf{R}^n$ will be a bounded open set with smooth boundary, and ν will denote the outward unit normal vector to $\partial\Omega$.

Given $T > 0$ and a function $v \in H^2(\Omega \times (0, T))$, we denote by ∇v and Δv the gradient and the Laplacian of v with respect to the spatial variable $\mathbf{x} \in \Omega$, and by $\mathcal{H}v$ the heat operator of v, i.e., $\mathcal{H}v = \partial_t v - \Delta v$.

Let

$$g(\cdot, t) \in W^{3,\infty}(\Omega), \qquad \partial_t g \in W^{1,\infty}(\Omega \times (0, T)); \tag{2.1}$$

for any $t \in [0, T]$ we indicate by $\Sigma(t)$ a mean curvature flow with forcing term g (see (2.2)). We shall assume that, for any $t \in [0, T]$, $\Sigma(t)$ is a smooth closed manifold of dimension $n - 1$, oriented by the inward unit normal vector $\mathbf{n}(\mathbf{x}, t)$ to $\Sigma(t)$ at $\mathbf{x} \in \Sigma(t)$. Furthermore, we suppose that $\Sigma(t) \subset\subset \Omega$ for any $t \in [0, T]$. The precise regularity requirements on $\Sigma(t)$ are listed in (2.4).

We set $O(t) =$ outside of $\Sigma(t)$, $I(t) =$ inside of $\Sigma(t)$, and $\Sigma = \bigcup_{t\in[0,T]} \Sigma(t) \times \{t\}$. We denote by $\kappa_1(\mathbf{x}, t), \ldots, \kappa_{n-1}(\mathbf{x}, t)$ the principal curvatures of $\Sigma(t)$ at $\mathbf{x} \in \Sigma(t)$, and we set

$$\kappa(\mathbf{x},t)=\sum_{i=1}^{n-1}\kappa_i(\mathbf{x},t),\qquad h(\mathbf{x},t)=\sum_{i=1}^{n-1}\kappa_i^2(\mathbf{x},t).$$

Note that $\kappa(\mathbf{x},t)$ equals $(n-1)$-times the mean curvature of $\Sigma(t)$ at $\mathbf{x}\in\Sigma(t)$. The evolution of Σ is then defined by

$$V(\mathbf{x},t)=\kappa(\mathbf{x},t)+g(\mathbf{x},t)\qquad\forall(\mathbf{x},t)\in\Sigma,\tag{2.2}$$

where, for any $t\in[0,T]$, $V(\cdot,t)$ denotes the normal velocity of $\Sigma(t)$, with the positive sign in the direction of $\mathbf{n}(\cdot,t)$.

The signed distance function $d:\mathbf{R}^n\times[0,T]\to\mathbf{R}$ from $\Sigma(t)$ is defined by

$$d(\mathbf{x},t)=\begin{cases}\operatorname{dist}\big(\mathbf{x},\Sigma(t)\big) & \text{if } t\in[0,T] \text{ and } \mathbf{x}\in O(t)\\ 0 & \text{if } t\in[0,T] \text{ and } \mathbf{x}\in\Sigma(t)\\ -\operatorname{dist}\big(\mathbf{x},\Sigma(t)\big) & \text{if } t\in[0,T] \text{ and } \mathbf{x}\in I(t).\end{cases}$$

We then have

$$\nabla d(\mathbf{x},t)=-\mathbf{n}(\mathbf{x},t)\qquad\forall t\in[0,T],\ \forall\mathbf{x}\in\Sigma(t).$$

We fix a positive number D in such a way that, for any $t\in[0,T]$, the tubular neighborhood $\mathcal{T}(t)$ of $\Sigma(t)$ defined by

$$\mathcal{T}(t)=\{\mathbf{x}\in\Omega:|d(\mathbf{x},t)|\le D\}\tag{2.3}$$

is relatively compact in Ω. We set $\mathcal{T}=\bigcup_{t\in[0,T]}\mathcal{T}(t)\times\{t\}$.

The interface error estimate in §6 will be proved under the assumptions

$$d,\partial_t d,\partial_t\partial_{\mathbf{xx}}d,\partial_{\mathbf{x}}^i d\in C^0(\overline{\mathcal{T}})\qquad\forall i=1,2,3,4.\tag{2.4}$$

If D is sufficiently small, from (2.4) it follows that any point $(\mathbf{x},t)\in\mathcal{T}$ has a unique projection $\mathbf{s}(\mathbf{x},t)\in\Sigma(t)$ such that

$$\operatorname{dist}\big(\mathbf{s}(\mathbf{x},t),\mathbf{x}\big)=|d(\mathbf{x},t)|.$$

Given a scalar or vector function f defined on Σ, we indicate with $\overline{f}(\mathbf{x},t)$ the composite function $f\big(\mathbf{s}(\mathbf{x},t),t\big)$, which is defined on $\mathcal{T}$. Hence, if f is scalar, we have $\nabla d\cdot\nabla\overline{f}=0$ on $\mathcal{T}$. We point out that (2.4) yields

$$\|\overline{h}\|_{L^\infty(\mathcal{T})},\|\partial_t\overline{h}\|_{L^\infty(\mathcal{T})},\|\nabla\overline{h}\|_{L^\infty(\mathcal{T})},\|\Delta\overline{h}\|_{L^\infty(\mathcal{T})}<+\infty.\tag{2.5}$$

The following property holds for all $(\mathbf{x},t)\in\mathcal{T}$ [11, 14.6]:

$$\Delta d(\mathbf{x},t)=\sum_{i=1}^{n-1}\frac{\overline{\kappa}_i(\mathbf{x},t)}{1-d(\mathbf{x},t)\overline{\kappa}_i(\mathbf{x},t)}=\overline{\kappa}(\mathbf{x},t)+d(\mathbf{x},t)\overline{h}(\mathbf{x},t)+\mathcal{O}(d^2(\mathbf{x},t)).$$

Hence, as $\partial_t d(\mathbf{x},t)=\overline{V}(\mathbf{x},t)=\overline{\kappa}(\mathbf{x},t)+\overline{g}(\mathbf{x},t)$ for any $\mathbf{x}\in\Sigma(t)$ (recall (2.2)), we have

$$\mathcal{H}d(\mathbf{x},t)=\overline{g}(\mathbf{x},t)+d(\mathbf{x},t)\overline{h}(\mathbf{x},t)+\mathcal{O}(d^2(\mathbf{x},t))\qquad\forall(\mathbf{x},t)\in\mathcal{T}.\tag{2.6}$$

We denote by $\Psi : \mathbf{R} \to [0, +\infty[$ the double equal well potential $\Psi(s) = (1 - s^2)^2$, and we set

$$\psi = \tfrac{1}{2}\Psi', \qquad c_0 = \int_{-1}^{1} \sqrt{\Psi(s)}ds = \frac{4}{3}, \tag{2.7}$$

$$\alpha = \psi'(1) = \psi'(-1) = 4, \qquad \beta = \psi''(1) = -\psi''(-1) = 12. \tag{2.8}$$

Finally, for any $\varepsilon > 0$ we denote by u_ε the classical solution of the problem

$$\begin{aligned} &\mathcal{H}u_\varepsilon + \varepsilon^{-2}\psi(u_\varepsilon) - \varepsilon^{-1}\frac{c_0}{2}g = 0 \qquad \text{in } \Omega \times (0,T), \\ &u_\varepsilon(\cdot, 0) = u_\varepsilon^0(\cdot) \in \mathcal{C}^2(\Omega) \cap L^\infty(\Omega) \quad \text{on } \Omega, \\ &\frac{\partial u_\varepsilon}{\partial \nu} = 0 \qquad \text{on } \partial\Omega \times (0,T). \end{aligned} \tag{2.9}$$

for a given initial datum u_ε^0 which will be specified later on (see (6.6)). Existence of such a solution u_ε can be proved by classical methods (see, for instance, [18, p. 98]).

3. Decay estimates. Any absolute minimizer γ of the functional

$$\int_{\mathbf{R}} \left(|\zeta'(x)|^2 + \Psi(\zeta(x))\right)dx \tag{3.1}$$

defined on $\{\zeta \in H^1_{\text{loc}}(\mathbf{R}) : \lim_{x\to\pm\infty} \zeta(x) = \pm 1\}$ is a solution of the problem

$$\gamma''(x) - \psi(\gamma(x)) = 0 \qquad \forall x \in \mathbf{R}. \tag{3.2}$$

One can show that such a minimizer γ must be nondecreasing. Since the functional in (3.1) is greater than or equal to $2\int_{\mathbf{R}} \zeta'(x)\sqrt{\Psi(\zeta(x))}\,dx = 2c_0$, it follows that γ satisfies

$$\gamma'(x) = \sqrt{\Psi(\gamma(x))} \qquad \forall x \in \mathbf{R}. \tag{3.3}$$

It turns out that imposing the condition $\gamma(0) = 0$, the unique nondecreasing solution γ of (3.2) is given by $\gamma(x) = \text{tgh}(x)$, $x \in \mathbf{R}$.

Note that there exists a positive constant c such that

$$|1 - \gamma(x)| \le c\gamma'(x) \qquad \forall x \in]0, +\infty[. \tag{3.4}$$

3.1. Some remarks on the operator $\mathcal{A}$. Let $\mathcal{A} : H^1(\mathbf{R}) \to H^{-1}(\mathbf{R})$ be the linear operator defined by $\mathcal{A}\zeta = \zeta'' - \psi'(\gamma)\zeta$. One can verify that $\mathcal{A}$ is selfadjoint and, by (3.2), that $\gamma' \in \text{Ker}(\mathcal{A}) \subseteq H^2_{\text{loc}}(\mathbf{R})$. Let us show that $\text{Ker}(\mathcal{A}) = \text{span}\{\gamma'\}$. Denote by $\widetilde{\mathcal{A}} : H^2_{\text{loc}}(\mathbf{R}) \to L^2_{\text{loc}}(\mathbf{R})$ the operator $\widetilde{\mathcal{A}}\zeta = \zeta'' - \psi'(\gamma)\zeta$. Take $\zeta_0, \zeta_1 \in]0, +\infty[$, and let $x_0 > 0$ be such that $\psi'(\gamma(x_0)) > 0$. Let $\zeta \in H^2_{\text{loc}}(\mathbf{R})$ be the unique solution of the backward and forward Cauchy problem

$$\zeta'' - \psi'(\gamma)\zeta = 0, \qquad \zeta(x_0) = \zeta_0, \ \zeta'(x_0) = \zeta_1.$$

It is not difficult to see that $\lim_{x\to+\infty} \zeta(x) = +\infty$, so that $\zeta \in \text{Ker}(\widetilde{\mathcal{A}}) \setminus \text{Ker}(\mathcal{A})$. Since obviously $\gamma' \in \text{Ker}(\widetilde{\mathcal{A}})$ and $\text{Ker}(\widetilde{\mathcal{A}})$ is a two dimensional linear space, it follows that $\text{Ker}(\mathcal{A})$ must be one dimensional, which implies that $\text{Ker}(\mathcal{A}) = \text{span}\{\gamma'\}$.

Let $\mathcal{R} : H^{-1}(\mathbf{R}) \to H^1(\mathbf{R})$ be the isometric linear operator given by the Riesz Representation Theorem on $H^1(\mathbf{R})$ endowed with the scalar product

$$\langle f, g\rangle = \int_{\mathbf{R}} f'g'\ dx + \alpha \int_{\mathbf{R}} fg\ dx,$$

where α is defined in (2.8). Let $\mathcal{B} : H^1(\mathbf{R}) \to H^{-1}(\mathbf{R})$ be the linear operator defined by $\mathcal{B}\zeta = -\zeta'' + \alpha\zeta$ for any $\zeta \in H^1(\mathbf{R})$. Then $\mathcal{R}\mathcal{A} : H^1(\mathbf{R}) \to H^1(\mathbf{R})$, and, as $\mathcal{R}\mathcal{B} = \mathrm{Id}$ on $H^1(\mathbf{R})$, we have $\mathcal{R}\mathcal{A} = -\mathrm{Id}+\mathcal{R}(\mathcal{A}+\mathcal{B})$. Now $(\mathcal{A}+\mathcal{B})\zeta = (\alpha-\psi'(\gamma))\zeta$ for any $\zeta \in H^1(\mathbf{R})$; hence, as $\alpha-\psi'(\gamma)$ decreases exponentially to zero as $|x| \to +\infty$, the operator $\mathcal{A}+\mathcal{B} : H^1(\mathbf{R}) \to H^{-1}(\mathbf{R})$ is compact. As $\mathcal{R}$ is an isometry, $\mathcal{R}(\mathcal{A}+\mathcal{B})$ is also compact. It follows that the composite operator $\mathcal{R}\mathcal{A}$ is a Fredholm operator, so that $\mathcal{A}\zeta = f$ has a solution $\zeta \in H^2(\mathbf{R})$ for $f \in L^2(\mathbf{R})$ if and only if $\int_{\mathbf{R}} f\gamma'\ dx = 0$.

For our purposes, we need an extension of this result. Denote by $\mathbf{P}^-$ (respectively by $\mathbf{P}^+$) the ring of all polynomials with real coefficients defined on $]-\infty,-1[$ (respectively on $]1,+\infty[$).

Lemma 3.1. *Let $f \in L^2_{\mathrm{loc}}(\mathbf{R})$ be such that there exist $p_- \in \mathbf{P}^-$, $p_+ \in \mathbf{P}^+$ with*

$$f_{|]-\infty,-1[} - p_- \in L^2(-\infty,-1), \qquad f_{|]1,+\infty[} - p_+ \in L^2(1,+\infty). \tag{3.5}$$

If $\int_{\mathbf{R}} f\gamma'\ dx = 0$ then there exist a function $\zeta \in H^2_{\mathrm{loc}}(\mathbf{R})$ and two polynomials $q_- \in \mathbf{P}^-$, $q_+ \in \mathbf{P}^+$ such that

$$\zeta - q_- \in H^2(-\infty,-1), \quad \zeta - q_+ \in H^2(1,+\infty), \qquad \widetilde{\mathcal{A}}\zeta = f. \tag{3.6}$$

Moreover, such a function ζ is unique up to an addition of a real multiple of γ'.

Proof. Define $q_- \in \mathbf{P}^-$ (respectively $q_+ \in \mathbf{P}^+$) as the unique polynomial solution of $v'' - \psi'(-1)v = p_-$ on $]-\infty,-1[$ (respectively of $v'' - \psi'(1)v = p_+$ on $]1,+\infty[$). Let $q \in H^2_{\mathrm{loc}}(\mathbf{R})$ be an arbitrary function such that $q = q_-$ on $]-\infty,-1[$ and $q = q_+$ on $]1,+\infty[$. We claim that $f - \widetilde{\mathcal{A}}q \in L^2(\mathbf{R})$. Indeed on $]1,+\infty[$ we have $\widetilde{\mathcal{A}}q = q_+'' - \psi'(\gamma)q_+ = p_+ + (\psi'(1) - \psi'(\gamma))q_+$, so that, using (3.5),

$$f - \widetilde{\mathcal{A}}q = f - p_+ - (\psi'(1) - \psi'(\gamma))q_+ \in L^2(1,+\infty)$$

(recall that $\psi'(\gamma)$ tends exponentially to $\psi'(1)$ at $+\infty$). The claim then follows by a similar argument on $]-\infty,-1[$.

Observe now that, by the exponential decrease of γ' at infinity and recalling that $\gamma' \in \mathrm{Ker}(\mathcal{A})$, we get

$$\int_{\mathbf{R}}(f - \widetilde{\mathcal{A}}q)\gamma'\,dx = \int_{\mathbf{R}} f\gamma'\,dx - \int_{\mathbf{R}}\widetilde{\mathcal{A}}q\gamma'\,dx = \int_{\mathbf{R}} f\gamma'\,dx - \int_{\mathbf{R}} q\mathcal{A}\gamma'\,dx = \int_{\mathbf{R}} f\gamma'\,dx.$$

Therefore, if $\int_{\mathbf{R}} f\gamma'\ dx = 0$, by the Fredholm Alternative there exists a function $v \in H^2(\mathbf{R})$ (which is unique up to an addition of a real multiple of γ') such that $\mathcal{A}v = f - \widetilde{\mathcal{A}}q$, so that the function $\zeta = v + q$ satisfies properties (3.6).

As $\gamma'(0) \neq 0$, by adding to ζ a suitable real multiple of γ', we can always obtain $\zeta(0) = 0$, so that ζ is uniquely determined. □

It can be checked directly that if f is odd, then the solution ζ through the origin is itself odd. Also, if f is even, ζ is itself even, as it follows by considering the function $\frac{1}{2}(\zeta(x) + \zeta(-x)))$, which is still a solution of the problem.

Lemma 3.2. *Let $\zeta \in H^2(\mathbf{R})$ and assume*

$$|\mathcal{A}\zeta| \le c(1+|x|^m)\gamma' \qquad \forall x \in \mathbf{R} \tag{3.7}$$

for some constant $c > 0$ and some $m \in \mathbf{N}$. Then there exists a positive constant C such that

$$|\zeta| \le C(1+|x|^{m+1})\gamma' \qquad \forall x \in \mathbf{R}. \tag{3.8}$$

If, in addition, $\zeta \in H^3(\mathbf{R})$ and

$$|(\mathcal{A}\zeta)'| \le c(1+|x|^m)\gamma' \qquad \forall x \in \mathbf{R}, \tag{3.9}$$

then

$$|\zeta'| \le C(1+|x|^{m+1})\gamma' \qquad \forall x \in \mathbf{R}. \tag{3.10}$$

Proof. As $\gamma' = \sqrt{\Psi(\gamma)}$ and $\gamma'' = \psi(\gamma)$ (see (3.3) and (3.2)), we have $(\frac{\gamma''}{\gamma'})^2 = \frac{\psi(\gamma)^2}{\Psi(\gamma)}$, so that, with the notation introduced in (2.7),

$$\lim_{x\to+\infty} \frac{\gamma''}{\gamma'} = \lim_{x\to+\infty} -\sqrt{\frac{\psi(\gamma)\psi'(\gamma)}{\psi(\gamma)}} = -\sqrt{\alpha} = -2. \tag{3.11}$$

Therefore we can choose $x_0 \ge m+1$ large enough in such a way that

$$\gamma''(x) < -\gamma'(x) \qquad \forall x \ge x_0, \tag{3.12}$$

and

$$\psi'(\gamma(x)) \ge \delta > 0 \qquad \forall x \ge x_0. \tag{3.13}$$

Let $k \ge 1$ be such that $|\zeta(x_0)| \le k x_0^{m+1}\gamma'(x_0)$, and let $z(x) = kx^{m+1}\gamma'(x)$ for any $x \ge x_0$. Recalling that $\mathcal{A}\gamma' = 0$ and using (3.12), we have

$$\begin{aligned}\mathcal{A}z &= kx^{m+1}\mathcal{A}\gamma' + 2(m+1)kx^m\gamma'' + m(m+1)kx^{m-1}\gamma' \\ &= 2(m+1)kx^m\gamma'' + m(m+1)kx^{m-1}\gamma' \le -k\gamma'(m+1)(2x^m - mx^{m-1}).\end{aligned}$$

Then, using the assumption (3.7),

$$-\mathcal{A}(-z-\zeta) = \mathcal{A}z + \mathcal{A}\zeta \le -k\gamma'(m+1)(2x^m - mx^{m-1}) + |\mathcal{A}\zeta| \le 0,$$

for any $x \ge x_0$, provided that k is large enough (depending on c). Since $-z(x_0) - \zeta(x_0) \le 0$, by the Maximum Principle (recall (3.13)) we have $-z(x) - \zeta(x) \le 0$ for any $x \ge x_0$, i.e.,

$$-kx^{m+1}\gamma'(x) \le \zeta(x) \qquad \forall x \ge x_0. \tag{3.14}$$

In a similar way, one can prove that $-\mathcal{A}(\zeta - z) \le 0$, so that

$$\zeta(x) \le kx^{m+1}\gamma'(x) \qquad \forall x \ge x_0. \tag{3.15}$$

Since analogous results are valid on $]-\infty, -x_0]$, assertion (3.8) follows from (3.14), (3.15), and the compactness of $[0, x_0]$, provided that C is large enough.

Let us prove (3.10). Recalling the definition of $\mathcal{A}$, it follows that

$$|\mathcal{A}(\zeta')| \leq |(\mathcal{A}\zeta)'| + |\psi''(\gamma)\gamma'\zeta|.$$

Since $\psi''(\gamma) = 12\gamma \in L^\infty(\mathbf{R})$ and $\zeta \in L^\infty(\mathbf{R})$ by (3.8), using assumption (3.9) we get

$$|\mathcal{A}(\zeta')| \leq c(1 + |x|^m)\gamma' \qquad \forall x \in \mathbf{R}.$$

The result then follows from the first part of the Lemma. $\square$

3.2. Estimates on the functions ξ, η, ω, π. We denote by $\xi \in H^2(\mathbf{R})$ the solution of the problem

$$\mathcal{A}\xi = x\gamma', \qquad \xi(0) = 0,$$

which exists by the Fredholm Alternative, since $x\gamma'$ is odd. Furthermore ξ is odd. By Lemma 3.2 we have

$$|\xi|, |\xi'| \leq c(1 + |x|^2)\gamma' \qquad \forall x \in \mathbf{R}. \tag{3.16}$$

We denote by $\eta \in H^2_{\text{loc}}(\mathbf{R})$ the polynomially increasing solution of the problem

$$\widetilde{\mathcal{A}}\eta = \gamma' - \frac{c_0}{2}, \qquad \eta(0) = 0, \tag{3.17}$$

where c_0 is defined in (2.7). Such a solution exists by Lemma 3.1 applied with $f = \gamma' - \frac{c_0}{2}$, $p_- = p_+ = -\frac{c_0}{2}$ (an easy calculation yields $\int_{\mathbf{R}}(\gamma' - \frac{c_0}{2})\gamma'\, dx = 0$). Furthermore η is even, since $\gamma' - \frac{c_0}{2}$ is even. With the notation of Lemma 3.1, we find $q_- = q_+ = \frac{c_0}{2\alpha}$, where α is defined in (2.8). Hence

$$\lim_{x \to \pm\infty} \eta(x) = \frac{c_0}{2\alpha} = \eta_\infty.$$

Now

$$\mathcal{A}(\eta - \frac{c_0}{2\alpha}) = \gamma' - \frac{c_0}{2\alpha}(\alpha - \psi'(\gamma)),$$

and

$$\lim_{x \to \pm\infty} \frac{\alpha - \psi'(\gamma)}{\gamma'} = \lim_{x \to \pm\infty} -\frac{\psi''(\gamma)\gamma'}{\gamma''} = \frac{1}{2}\psi''(\pm 1) \tag{3.18}$$

(see (3.11)). It follows that there exists a constant $c > 0$ such that

$$|\mathcal{A}(\eta - \frac{c_0}{2\alpha})| \leq c\gamma' \qquad \forall x \in \mathbf{R}, \tag{3.19}$$

and hence, by Lemma 3.2,

$$|\eta - \frac{c_0}{2\alpha}| \leq c(1 + |x|)\gamma' \qquad \forall x \in \mathbf{R}. \tag{3.20}$$

We denote by $\omega \in H^2(\mathbf{R})$ the solution of the problem

$$\mathcal{A}\omega = \eta', \qquad \omega(0) = 0. \tag{3.21}$$

Such a solution exists by the Fredholm Alternative, since η is even and hence η' is odd. Furthermore ω is odd. Using (3.17) we have

$$|(\mathcal{A}(\eta - \frac{c_0}{2\alpha}))'| = |(\gamma' - \frac{c_0}{2} + \frac{c_0}{2\alpha}\psi'(\gamma))'| = |\gamma'' + \frac{c_0}{2\alpha}\psi''(\gamma)\gamma'| \le c\gamma'$$

for a suitable positive constant c, so that, by (3.19), using Lemma 3.2, we have

$$|\mathcal{A}\omega| = |\eta'| \le c(1+|x|)\gamma' \qquad \forall x \in \mathbf{R}. \tag{3.22}$$

Using again Lemma 3.2 and (3.21), it follows that

$$|\omega| \le c(1+|x|^2)\gamma' \qquad \forall x \in \mathbf{R}. \tag{3.23}$$

In addition

$$\begin{aligned}(\mathcal{A}\omega)' = \eta'' = \widetilde{\mathcal{A}}\eta + \psi'(\gamma)\eta &= \gamma' - \frac{c_0}{2} + \psi'(\gamma)\eta \\ &= \gamma' + \psi'(\gamma)(\eta - \frac{c_0}{2\alpha}) + \frac{c_0}{2\alpha}(\psi'(\gamma) - \alpha),\end{aligned}$$

so that, by (3.18) and (3.20), we have $|(\mathcal{A}\omega)'| \le c(1+|x|)\gamma'$ on $\mathbf{R}$, which, together with (3.22) and Lemma 3.2, gives

$$|\omega'| \le c(1+|x|^2)\gamma' \qquad \forall x \in \mathbf{R}. \tag{3.24}$$

Finally, we denote by $\pi \in H^2_{\mathrm{loc}}(\mathbf{R})$ the polynomially increasing solution of the problem

$$\widetilde{\mathcal{A}}\pi = \frac{1}{2}\eta^2\psi''(\gamma), \qquad \pi(0) = 0. \tag{3.25}$$

Such a solution exists by Lemma 3.1 applied with $f = \frac{1}{2}\eta^2\psi''(\gamma)$, $p_- = -p_+ = -\frac{1}{2}(\frac{c_0}{2\alpha})^2\beta$, (recall that $\frac{1}{2}\eta^2\psi''(\gamma)$ is odd), where β is defined in (2.8). Furthermore π is odd, since $\frac{1}{2}\eta^2\psi''(\gamma)$ is odd. With the notation of Lemma 3.1, we find $q_- = -q_+ = \frac{c_0^2\beta}{8\alpha^3}$. Hence,

$$\lim_{x\to\pm\infty} \pi(x) = \mp\frac{c_0^2\beta}{8\alpha^3} = \pm\pi_\infty.$$

Now, setting q as in the proof of Lemma 3.1, we have

$$\mathcal{A}(\pi - q) = \begin{cases} \frac{1}{2}\eta^2\psi''(\gamma) - \frac{c_0^2\beta}{8\alpha^3}\psi'(\gamma)\gamma + \frac{c_0^2\beta}{8\alpha^3}\gamma'' & \text{on }]-\infty,-1], \\ \frac{1}{2}\eta^2\psi''(\gamma) + \frac{c_0^2\beta}{8\alpha^3}\psi'(\gamma)\gamma - \frac{c_0^2\beta}{8\alpha^3}\gamma'' & \text{on } [1,+\infty[, \end{cases}$$

and

$$\begin{aligned}&\lim_{x\to\pm\infty} \frac{\eta^2\psi''(\gamma) \mp \frac{c_0^2\beta}{4\alpha^3}\psi'(\gamma)\gamma}{x\gamma'} \\ &\le \lim_{x\to\pm\infty} \frac{\eta^2\psi'''(\gamma)\gamma' \mp \frac{c_0^2\beta}{4\alpha^3}\psi''(\gamma)\gamma'\gamma \mp \frac{c_0^2\beta}{4\alpha^3}\psi'(\gamma)\gamma''}{\gamma' + x\gamma''} + \limsup_{x\to\pm\infty} \frac{2\eta\eta'\psi''(\gamma)}{\gamma' + x\gamma''}.\end{aligned}$$

It is easy to see that the first limit is zero, and, by (3.22),

$$\limsup_{x\to\pm\infty} \frac{2\eta\eta'\psi''(\gamma)}{\gamma' + x\gamma''} \le \frac{c(1+|x|)\gamma'\eta\psi''(\gamma)}{\gamma' + x\gamma''} \le c,$$

for a suitable positive constant c. It follows that there exists a constant $c > 0$ such that

$$|\mathcal{A}(\pi - q)| \le c(1 + |x|)\gamma' \qquad \forall x \in \mathbf{R},$$

and hence, by Lemma 3.2,

$$|\pi - q| \le c(1 + |x|^2)\gamma' \qquad \forall x \in \mathbf{R}. \tag{3.26}$$

Since by (3.25) and (3.22) we have

$$|(\widetilde{\mathcal{A}}\pi)'| \le |\eta\eta'\psi''(\gamma)| + \frac{1}{2}\eta^2\psi'''(\gamma)\gamma' \le c(1 + |x|)\gamma',$$

using Lemma 3.2 we conclude that

$$|\pi'| \le c(1 + |x|^2)\gamma' \qquad \forall x \in \mathbf{R}. \tag{3.27}$$

4. Formal asymptotics. The shape of the subsolution is suggested by a formal asymptotic expansion that can be obtained following the ideas in [17]. Since the way such an expansion is obtained is not important in our context, we simply give the final result.

Let u_ε be a classical solution of problem (2.9). Introduce the vector $\widetilde{\mathbf{n}}$, the stretched variable $\widetilde{y} = \frac{\widetilde{d}(\mathbf{x},t)}{\varepsilon}$, and the projection $\widetilde{\mathbf{s}}$, as in §2 with $\Sigma(t)$ replaced by the set $\{\mathbf{x} \in \Omega : u_\varepsilon(\mathbf{x}, t) = 0\}$, and define

$$U_\varepsilon(\widetilde{y}, t) = u_\varepsilon\big(\widetilde{\mathbf{s}}(\mathbf{x}, t) - \varepsilon\widetilde{y}\ \widetilde{\mathbf{n}}(\mathbf{x}, t), t\big).$$

Then U_ε can be expressed formally in terms of ε (inner expansion) as follows:

$$U_\varepsilon(\widetilde{y}, t) = \sum_{i=0}^{2} \varepsilon^i U_i(\widetilde{y}, t) + \mathcal{O}(\varepsilon^3),$$

where

$$\begin{aligned}
U_0(\widetilde{y}, t) =& \gamma(\widetilde{y}),\\
U_1(\widetilde{y}, t) =& g(\mathbf{x}, t)\eta(\widetilde{y}),\\
U_2(\widetilde{y}, t) =& \big(\overline{h}(\mathbf{x}, t) - 2\nabla\widetilde{d}(\mathbf{x}, t) \cdot \nabla g(\mathbf{x}, t)\big)\xi(\widetilde{y})\\
&+ \big(g^2(\mathbf{x}, t) - 2\nabla\widetilde{d}(\mathbf{x}, t) \cdot \nabla g(\mathbf{x}, t)\big)\omega(\widetilde{y}) + g^2(\mathbf{x}, t)\pi(\widetilde{y}).
\end{aligned}$$

5. Comparison Lemma. In this section we prove an elementary Comparison Lemma for subsolutions similar to that in [5] (see also [18, p. 98]).

Lemma 5.1. *Fix $\varepsilon > 0$, let $u, v \in L^2(0, T; H^2(\Omega)) \cap H^1(0, T; L^2(\Omega))$, and consider the following three conditions:*

$$\begin{gathered}
\mathcal{H}u + \varepsilon^{-2}\psi(u) - \varepsilon^{-1}\frac{c_0}{2}g \ge \mathcal{H}v + \varepsilon^{-2}\psi(v) - \varepsilon^{-1}\frac{c_0}{2}g \qquad \textit{a.e. in } \Omega \times (0, T),\\
u(\mathbf{x}, 0) \ge v(\mathbf{x}, 0) \qquad \textit{for a.e. } \mathbf{x} \in \Omega,\\
\frac{\partial u}{\partial \nu} \ge \frac{\partial v}{\partial \nu} \qquad \textit{a.e. on } \partial\Omega \times (0, T).
\end{gathered} \tag{5.1}$$

Then

$$u \geq v \qquad a.e.\ in\ \Omega\times(0,T). \tag{5.2}$$

Proof. Set $e = \max(v-u,0)$. Multiply (5.1) by e, integrate over Ω and subtract. We get

$$\frac{d}{dt}\|e(\cdot,t)\|^2_{L^2(\Omega)} \leq 2\varepsilon^{-2}\langle \psi(u)-\psi(v), e\rangle_{L^2(\Omega)} \qquad \text{for a.e. } t\in[0,T]. \tag{5.3}$$

Let $t \in (0,T)$ be a point where (5.3) holds, and write $\psi = \psi_l + \psi_i$, where ψ_l is Lipschitz continuous on $\mathbf{R}$, and ψ_i non decreasing on $\mathbf{R}$. Integrating (5.3) on $(0,t)$ and recalling that by assumption $e(\mathbf{x},0)=0$ for almost every $\mathbf{x}\in\Omega$, we get

$$\begin{aligned}\|e(\cdot,t)\|^2_{L^2(\Omega)} &\leq 2\varepsilon^{-2}\int_0^t |\langle \psi_l(u)-\psi_l(v), e(\cdot,\tau)\rangle_{L^2(\Omega)}|d\tau \\ &\leq 2\mathrm{lip}(\psi_l)\varepsilon^{-2}\int_0^t \|e(\cdot,\tau)\|^2_{L^2(\Omega)}d\tau.\end{aligned}$$

Then for almost every $t\in[0,T]$ Gronwall's Lemma implies that $\|e(\cdot,t)\|_{L^2(\Omega)}=0$ and thus, by Fubini's Theorem, $e=0$ almost everywhere in $\Omega\times(0,T)$, i.e., (5.2). ☐

6. Subsolution and supersolution. In this section we construct a sub and supersolution for problem (2.9). Such functions will be used to derive the desired interface error estimate.

Let $\delta\geq 3$ be a fixed natural number; for any $\varepsilon>0$ let $x_\varepsilon = \delta|\log\varepsilon|$. Note that $\gamma(x_\varepsilon) = 1-2\varepsilon^{2\delta}(1+\varepsilon^{2\delta})^{-1} = 1-\mathcal{O}(\varepsilon^{2\delta})$, and $\gamma'(x_\varepsilon) = 1-\gamma^2(x_\varepsilon) = \mathcal{O}(\varepsilon^{2\delta})$. As a consequence, using (3.20), (3.22), (3.16), (3.23), (3.24), (3.26) and (3.27), we get

$$\begin{aligned}&|\gamma(x_\varepsilon)-1|, |\gamma'(x_\varepsilon)| = \mathcal{O}(\varepsilon^{2\delta}),\\ &|\eta(x_\varepsilon)-\eta_\infty|, |\eta'(x_\varepsilon)| = |\log\varepsilon|\mathcal{O}(\varepsilon^{2\delta}),\\ &|\xi(x_\varepsilon)|, |\xi'(x_\varepsilon)|, |\omega(x_\varepsilon)|, |\omega'(x_\varepsilon)|, |\pi(x_\varepsilon)-\pi_\infty|, |\pi'(x_\varepsilon)| = |\log\varepsilon|^2\mathcal{O}(\varepsilon^{2\delta}).\end{aligned}$$

We construct five functions

$$\gamma_\varepsilon, \xi_\varepsilon, \eta_\varepsilon, \omega_\varepsilon, \pi_\varepsilon \in \mathcal{C}^{1,1}(\mathbf{R})\cap\mathcal{C}^\infty(\mathbf{R}\setminus\{\pm x_\varepsilon, \pm 2x_\varepsilon\}),$$

which coincide, respectively, with $\gamma, \xi, \eta, \omega, \pi$ on $[-x_\varepsilon, x_\varepsilon]$ and are constant outside the interval $[-2x_\varepsilon, 2x_\varepsilon]$, as follows:

$$\gamma_\varepsilon(x) = \begin{cases}\gamma(x) & 0\leq x< x_\varepsilon\\ P_\gamma(x) & x_\varepsilon\leq x\leq 2x_\varepsilon\\ 1 & x>2x_\varepsilon\\ -\gamma_\varepsilon(-x) & x<0,\end{cases} \qquad \xi_\varepsilon(x) = \begin{cases}\xi(x) & 0\leq x< x_\varepsilon\\ P_\xi(x) & x_\varepsilon\leq x\leq 2x_\varepsilon\\ 0 & x>2x_\varepsilon\\ -\xi_\varepsilon(-x) & x<0,\end{cases}$$

$$\eta_\varepsilon(x) = \begin{cases}\eta(x) & 0\leq x< x_\varepsilon\\ P_\eta(x) & x_\varepsilon\leq x\leq 2x_\varepsilon\\ \frac{c_0}{2\alpha} & x>2x_\varepsilon\\ \eta_\varepsilon(-x) & x<0,\end{cases} \qquad \omega_\varepsilon(x) = \begin{cases}\omega(x) & 0\leq x< x_\varepsilon\\ P_\omega(x) & x_\varepsilon\leq x\leq 2x_\varepsilon\\ 0 & x>2x_\varepsilon\\ -\omega_\varepsilon(-x) & x<0,\end{cases}$$

$$\pi_\varepsilon(x) = \begin{cases} \pi(x) & 0 \le x < x_\varepsilon \\ P_\pi(x) & x_\varepsilon \le x \le 2x_\varepsilon \\ -\frac{c_0^2\beta}{8\alpha^3} & x > 2x_\varepsilon \\ -\pi_\varepsilon(-x) & x < 0. \end{cases}$$

Here $P_\gamma, P_\xi, P_\eta, P_\omega, P_\pi : [x_\varepsilon, 2x_\varepsilon] \to \mathbf{R}$ are the five polynomials of degree 3 such that $\gamma_\varepsilon, \xi_\varepsilon, \eta_\varepsilon, \omega_\varepsilon, \pi_\varepsilon \in \mathcal{C}^{1,1}(\mathbf{R})$. For example, for any $x \in [x_\varepsilon, 2x_\varepsilon]$

$$\begin{aligned} P_\gamma(x) =& \Big(\frac{2x_\varepsilon - x}{x_\varepsilon}\Big)^3 [2(1-\gamma(x_\varepsilon)) - x_\varepsilon\gamma'(x_\varepsilon)] \\ &+ \Big(\frac{2x_\varepsilon - x}{x_\varepsilon}\Big)^2 [x_\varepsilon\gamma'(x_\varepsilon) - 3(1-\gamma(x_\varepsilon))] + 1, \\ P_\xi(x) =& \Big(\frac{2x_\varepsilon - x}{x_\varepsilon}\Big)^3 [-2\xi(x_\varepsilon) - x_\varepsilon\xi'(x_\varepsilon)] + \Big(\frac{2x_\varepsilon - x}{x_\varepsilon}\Big)^2 [x_\varepsilon\xi'(x_\varepsilon) + 3\xi(x_\varepsilon)]. \end{aligned}$$

Note that these polynomials are not necessarily monotone (for instance P_γ is decreasing at the point $2x_\varepsilon$).

One can check by direct computation that

$$\begin{aligned} \|P_\zeta - \zeta_\infty\|_{L^\infty(x_\varepsilon,2x_\varepsilon)} &\le C(|\zeta(x_\varepsilon) - \zeta_\infty| + x_\varepsilon|\zeta'(x_\varepsilon)|), \\ \|P'_\zeta\|_{L^\infty(x_\varepsilon,2x_\varepsilon)} &\le \frac{C}{x_\varepsilon}(|\zeta(x_\varepsilon) - \zeta_\infty| + x_\varepsilon|\zeta'(x_\varepsilon)|), \\ \|P''_\zeta\|_{L^\infty(x_\varepsilon,2x_\varepsilon)} &\le \frac{C}{x_\varepsilon^2}(|\zeta(x_\varepsilon) - \zeta_\infty| + x_\varepsilon|\zeta'(x_\varepsilon)|), \end{aligned}$$

where ζ stands for any one of the functions $\gamma, \xi, \eta, \omega, \pi$, $\zeta_\infty = \lim_{x\to+\infty}\zeta(x)$, and C is a suitable positive constant. As a consequence, we get the following estimates:

$$\begin{aligned} &\|P_\gamma - 1\|_{L^\infty(x_\varepsilon,2x_\varepsilon)} = |\log\varepsilon|\mathcal{O}(\varepsilon^{2\delta}) = o(\varepsilon^{2\delta-1}), \\ &\|P'_\gamma\|_{L^\infty(x_\varepsilon,2x_\varepsilon)} = \mathcal{O}(\varepsilon^{2\delta}), \quad \|P''_\gamma\|_{L^\infty(x_\varepsilon,2x_\varepsilon)} = |\log\varepsilon|^{-1}\mathcal{O}(\varepsilon^{2\delta}) = o(\varepsilon^{2\delta}), \\ &\|P_\eta - \eta_\infty\|_{L^\infty(x_\varepsilon,2x_\varepsilon)} = |\log\varepsilon|^2\mathcal{O}(\varepsilon^{2\delta}) = o(\varepsilon^{2\delta-1}), \\ &\|P'_\eta\|_{L^\infty(x_\varepsilon,2x_\varepsilon)} = |\log\varepsilon|\mathcal{O}(\varepsilon^{2\delta}) = o(\varepsilon^{2\delta-1}), \quad \|P''_\eta\|_{L^\infty(x_\varepsilon,2x_\varepsilon)} = \mathcal{O}(\varepsilon^{2\delta}), \\ &\|P_\xi\|_{L^\infty(x_\varepsilon,2x_\varepsilon)}, \|P_\omega\|_{L^\infty(x_\varepsilon,2x_\varepsilon)}, \|P_\pi - \pi_\infty\|_{L^\infty(x_\varepsilon,2x_\varepsilon)} = |\log\varepsilon|^3\mathcal{O}(\varepsilon^{2\delta}), \\ &\|P'_\xi\|_{L^\infty(x_\varepsilon,2x_\varepsilon)}, \|P'_\omega\|_{L^\infty(x_\varepsilon,2x_\varepsilon)}, \|P'_\pi\|_{L^\infty(x_\varepsilon,2x_\varepsilon)} = |\log\varepsilon|^2\mathcal{O}(\varepsilon^{2\delta}), \\ &\|P''_\xi\|_{L^\infty(x_\varepsilon,2x_\varepsilon)}, \|P''_\omega\|_{L^\infty(x_\varepsilon,2x_\varepsilon)}, \|P''_\pi\|_{L^\infty(x_\varepsilon,2x_\varepsilon)} = |\log\varepsilon|\mathcal{O}(\varepsilon^{2\delta}). \end{aligned} \tag{6.1}$$

Finally, from (6.1), it follows that

$$\|\gamma_\varepsilon - \gamma\|_{L^\infty(\mathbf{R})} = \|\gamma_\varepsilon - \gamma\|_{L^\infty(x_\varepsilon,2x_\varepsilon)} = o(\varepsilon^{2\delta-1}), \tag{6.2}$$

and similarly

$$\|\xi_\varepsilon - \xi\|_{L^\infty(\mathbf{R})}, \|\eta_\varepsilon - \eta\|_{L^\infty(\mathbf{R})}, \|\omega_\varepsilon - \omega\|_{L^\infty(\mathbf{R})}, \|\pi_\varepsilon - \pi\|_{L^\infty(\mathbf{R})} = o(\varepsilon^{2\delta-1}). \tag{6.3}$$

6.1. Subsolution. For any $\varepsilon > 0$ and any $(\mathbf{x}, t) \in \Omega \times [0, T]$ we define the modified distance function $d_\varepsilon^-(\mathbf{x}, t)$ as

$$d_\varepsilon^-(\mathbf{x}, t) = d(\mathbf{x}, t) - c_1(t)\varepsilon^2|\log\varepsilon|^2,$$

where $c_1 : [0,T] \to]0,+\infty[$ is a continuous (exponentially increasing) function to be determined later on independently of ε (see (6.16)). For any $t \in [0,T]$, let

$$\mathcal{T}_\varepsilon^-(t) = \{\mathbf{x} \in \Omega : |d_\varepsilon^-(\mathbf{x},t)| < 2\delta\varepsilon|\log\varepsilon|\}, \qquad \mathcal{T}_\varepsilon^- = \bigcup_{t\in[0,T]} \mathcal{T}_\varepsilon^-(t) \times \{t\}.$$

For convenience, we remove the superscript $^-$, thus denoting $d_\varepsilon = d_\varepsilon^-$, $\mathcal{T}_\varepsilon^-(t) = \mathcal{T}_\varepsilon(t)$, and $\mathcal{T}_\varepsilon^- = \mathcal{T}_\varepsilon$.

Observe that there exists $\varepsilon_0 > 0$ (depending on $\delta, c_1(\cdot), D$) such that $\mathcal{T}_\varepsilon(t) \subseteq \mathcal{T}(t)$ for any $0 < \varepsilon \le \varepsilon_0$ and any $t \in [0,T]$, so that $\mathcal{T}_\varepsilon \subseteq \mathcal{T}$ for any $0 < \varepsilon \le \varepsilon_0$. Moreover

$$d(\mathbf{x},t) = \mathcal{O}(\varepsilon|\log\varepsilon|) \qquad \forall(\mathbf{x},t) \in \mathcal{T}_\varepsilon. \tag{6.4}$$

Note that $\nabla d_\varepsilon = \nabla d$ and $(\nabla d_\varepsilon, \nabla\overline{h}) = 0$ on $\mathcal{T}_\varepsilon$ (see §2). In addition, on $\mathcal{T}_\varepsilon$, using (2.6), (2.5), and (6.4), we have

$$\mathcal{H}d_\varepsilon = \mathcal{H}d - c_1'\varepsilon^2|\log\varepsilon|^2 = \overline{g} + d_\varepsilon\overline{h} + \varepsilon^2|\log\varepsilon|^2\big(c_1\overline{h} - c_1'\big) + \mathcal{O}(\varepsilon^2|\log\varepsilon|^2). \tag{6.5}$$

Our aim is to introduce a subsolution v_ε^- for problem (2.9). As we shall see, the definition of v_ε^- is suggested by the formal asymptotics of §4. Define the real stretched variable $y = y(\mathbf{x},t) = \varepsilon^{-1}d_\varepsilon(\mathbf{x},t)$, set

$$\mathbf{b} = (\overline{h} - \nabla d\cdot\nabla g, g^2 - 2\nabla d\cdot\nabla g, g^2), \qquad \mathbf{p}_\varepsilon = (\xi_\varepsilon, \omega_\varepsilon, \pi_\varepsilon),$$

and define on $\Omega \times [0,T]$

$$v_\varepsilon^-(\mathbf{x},t) = \begin{cases} \gamma_\varepsilon(y) + \varepsilon g(\mathbf{x},t)\eta_\varepsilon(y) + \varepsilon^2\mathbf{b}(\mathbf{x},t)\cdot\mathbf{p}_\varepsilon(y) - c_2\varepsilon^3|\log\varepsilon|^2 \text{ on } \mathcal{T}_\varepsilon, \\ 1 + \varepsilon\eta_\infty g(\mathbf{x},t) + \varepsilon^2\pi_\infty g^2(\mathbf{x},t) - c_2\varepsilon^3|\log\varepsilon|^2 \text{ on } \{d_\varepsilon \ge 2\delta\varepsilon|\log\varepsilon|\}, \\ -1 + \varepsilon\eta_\infty g(\mathbf{x},t) - \varepsilon^2\pi_\infty g^2(\mathbf{x},t) - c_2\varepsilon^3|\log\varepsilon|^2 \text{ on } \{d_\varepsilon \le -2\delta\varepsilon|\log\varepsilon|\}, \end{cases}$$

where $c_2 > 0$ is a constant to be determined later on independently of ε. For simplicity, we will use the notation $v_\varepsilon = v_\varepsilon^-$. It is clear, from the regularity of d_ε and the properties of the functions $\gamma_\varepsilon, \xi_\varepsilon, \eta_\varepsilon, \omega_\varepsilon, \pi_\varepsilon$, that v_ε belongs to $L^2(0,T;H^2(\Omega)) \cap H^1(0,T;L^2(\Omega))$.

Note that, in view of the definition of d_ε, the term $\gamma_\varepsilon(y)$ corresponds to a right shift of the function γ_ε of order ε, which is natural when looking for a subsolution. On the other hand, the term $\varepsilon g\eta_\varepsilon + \varepsilon^2\mathbf{b}\cdot\mathbf{p}_\varepsilon$ is a shape correction suggested by the formal asymptotic expansion given in §4; one can prove that this correction is of higher order with respect to the previous right translation. Finally, the term $c_2\varepsilon^3|\log\varepsilon|^2$ provides a further downward translation, and it is necessary both for the comparison with the initial datum and to provide control of "bad" terms far from the interface.

6.2. Comparison with the initial datum. Fix $\varepsilon > 0$; inspired by the asymptotics in §4, we assume that

$$u_\varepsilon(\mathbf{x},0) = \gamma(\varepsilon^{-1}d(\mathbf{x},0)) + \varepsilon\frac{c_0}{2\alpha}g(\mathbf{x},0) - \varepsilon^2\frac{c_0^2\beta}{8\alpha^3}g^2(\mathbf{x},0)\gamma(\varepsilon^{-1}d(\mathbf{x},0)). \tag{6.6}$$

Let $z = \varepsilon^{-1}d(\mathbf{x},0)$, and, as before, let $y = \varepsilon^{-1}d_\varepsilon(\mathbf{x},0)$. Note that $z = y + c_1(0)\varepsilon|\log\varepsilon|^2 > y$. We have to check

$$v_\varepsilon(\mathbf{x},0) \le u_\varepsilon(\mathbf{x},0) \qquad \forall\mathbf{x} \in \Omega. \tag{6.7}$$

Define $\mathbf{p} = (\xi, \omega, \pi)$, and

$$w_\varepsilon(\mathbf{x}, t) = \gamma(y(\mathbf{x}, t)) + \varepsilon g(\mathbf{x}, t)\eta(y(\mathbf{x}, t)) + \varepsilon^2 \mathbf{b}(\mathbf{x}, t) \cdot \mathbf{p}(y(\mathbf{x}, t)) - \frac{c_2}{2}\varepsilon^3 |\log \varepsilon|^2.$$

To prove (6.7) it will be enough to show

$$w_\varepsilon(\mathbf{x}, 0) \le u_\varepsilon(\mathbf{x}, 0) \qquad \forall \mathbf{x} \in \Omega. \tag{6.8}$$

Indeed, by (2.1), (6.2) and (6.3) we have

$$|v_\varepsilon - w_\varepsilon + \frac{c_2}{2}\varepsilon^3 |\log \varepsilon|^2| \le \mathcal{O}(\varepsilon^{2\delta - 1}),$$

so that, if (6.8) is true,

$$\begin{aligned} v_\varepsilon =&(v_\varepsilon - w_\varepsilon) + w_\varepsilon \le w_\varepsilon - \frac{c_2}{2}\varepsilon^3 |\log \varepsilon|^2 + \mathcal{O}(\varepsilon^{2\delta-1}) \\ \le& u_\varepsilon(\mathbf{x}, 0) - \frac{c_2}{2}\varepsilon^3 |\log \varepsilon|^2 + \mathcal{O}(\varepsilon^{2\delta-1}) \le u_\varepsilon(\mathbf{x}, 0), \end{aligned}$$

as $\varepsilon \to 0$.

Let us show (6.8). We have

$$\begin{aligned} w_\varepsilon(\mathbf{x}, 0) - u_\varepsilon(\mathbf{x}, 0) = (\gamma(y) - \gamma(z)) + \varepsilon g(\mathbf{x}, 0)(\eta(y) - \frac{c_0}{2\alpha}) + \varepsilon^2(\mathbf{b}(\mathbf{x}, 0) \cdot \mathbf{p}(y) \\ + \frac{c_0^2 \beta}{8\alpha^3} g^2(\mathbf{x}, 0)\gamma(z)) - \frac{c_2}{2}\varepsilon^3 |\log \varepsilon|^2 =: \mathrm{I} + \mathrm{II} + \mathrm{III} - \frac{c_2}{2}\varepsilon^3 |\log \varepsilon|^2. \end{aligned}$$

We need the following result.

Lemma 6.1. *Let $z = \varepsilon^{-1} d(\mathbf{x}, 0)$, $y = \varepsilon^{-1} d_\varepsilon(\mathbf{x}, 0) = z - c_1(0)\varepsilon |\log \varepsilon|^2$. We have*

$$\frac{1}{2}\gamma'(y) \le \gamma'(t) \le 2\gamma'(y) \tag{6.9}$$

for all $t \in [y, z]$ and ε sufficiently small.

Proof. Observe that $(\log(\gamma'))' = \frac{\gamma''}{\gamma'} = -2\gamma \in L^\infty(\mathbf{R})$, so that $\log(\gamma')$ is Lipschitz continuous with Lipschitz constant 2. Therefore,

$$|\log(\frac{\gamma'(t_1)}{\gamma'(t_2)})| = |\log(\gamma'(t_1)) - \log(\gamma'(t_2))| \le 2|t_2 - t_1|.$$

Hence, if $|t_2 - t_1| \le c_1(0)\varepsilon |\log \varepsilon|^2$, we have

$$\frac{\gamma'(t_1)}{\gamma'(t_2)} \ge \exp(-2|t_2 - t_1|) \ge \exp(-2c_1(0)\varepsilon |\log \varepsilon|^2) \to 1$$

as $\varepsilon \to 0$, and the left inequality of (6.9) follows. Similarly,

$$\frac{\gamma'(t_1)}{\gamma'(t_2)} \le \exp(2|t_2 - t_1|) \le \exp(2c_1(0)\varepsilon |\log \varepsilon|^2) \to 1$$

as $\varepsilon \to 0$, and the Lemma follows. $\square$

Using Lemma 6.1, we find

$$\mathrm{I} \le -\frac{1}{2}\gamma'(y)c_1(0)\varepsilon|\log\varepsilon|^2; \tag{6.10}$$

moreover, by (2.1) and (3.20), there exists a positive constant c such that

$$|\mathrm{II}| \le c\varepsilon(1+|y|)\gamma'(y). \tag{6.11}$$

In addition, by (2.1), the fact that $\mathbf{b}(\cdot,0) \in L^\infty(\Omega)$, and (3.16), (3.23), (3.26), (3.4), and Lemma 6.1, we have

$$\begin{aligned}|\mathrm{III}| \le&\varepsilon^2|\mathbf{b}(\mathbf{x},0)\cdot\mathbf{p}(y) + \frac{c_0^2\beta}{8\alpha^3}g^2(\mathbf{x},0)\gamma(y)| + \varepsilon^2\frac{c_0^2\beta}{8\alpha^3}g^2(\mathbf{x},0)|\gamma(z)-\gamma(y)|\\ \le&\varepsilon^2|\mathbf{b}(\mathbf{x},0)\cdot\mathbf{p}(y) - g^2(\mathbf{x},0)q| + \varepsilon^2 g^2(\mathbf{x},0)|\frac{c_0^2\beta}{8\alpha^3}\gamma(y) - q|\\ &+\varepsilon^2\frac{c_0^2\beta}{8\alpha^3}g^2(\mathbf{x},0)|\gamma(z)-\gamma(y)| \le c\varepsilon^2(1+|y|^2)\gamma'(y),\end{aligned}$$

where q is as in the proof of Lemma 3.1 with $q_- = q_+ = \frac{c_0^2\beta}{8\alpha^3}$.

Let us distinguish two different cases. If $|y| > 2|\log\varepsilon|$, by direct computation we have $\gamma'(y) < c\varepsilon^4$ for some positive constant c, hence the terms II and III can be controlled by the negative term $-\frac{c_2}{2}\varepsilon^3|\log\varepsilon|^2$. On the contrary, if $|y| \le 2|\log\varepsilon|$, they are easily controlled by the term I. This proves (6.8), and so the proof of (6.7) is concluded.

6.3. v_ε **is a subsolution.** In order to apply the Comparison Lemma 5.1 we must show

$$\mathcal{H}v_\varepsilon + \varepsilon^{-2}\psi(v_\varepsilon) - \varepsilon^{-1}\frac{c_0}{2}g \le 0 \qquad \text{for a.e. } (\mathbf{x},t) \in \Omega\times(0,T). \tag{6.12}$$

We first restrict ourselves to points $(\mathbf{x},t) \in \mathcal{T}_\varepsilon$. Direct computations yield

$$\nabla v_\varepsilon = \varepsilon^{-1}\gamma_\varepsilon'(y)\nabla d_\varepsilon + g\eta_\varepsilon'\nabla d_\varepsilon + \varepsilon\mathbf{b}\cdot\mathbf{p}_\varepsilon'\nabla d_\varepsilon + \varepsilon\eta_\varepsilon\nabla g + \varepsilon^2\mathbf{p}_\varepsilon\cdot\nabla\mathbf{b} \qquad \text{in } \mathcal{T}_\varepsilon,$$

where, if $\mathbf{b} = (b_1, b_2, b_3)$, then $\nabla\mathbf{b} = (\nabla b_1, \nabla b_2, \nabla b_3)$. Enforcing the equality $|\nabla d_\varepsilon| = 1$ and (2.1), we get

$$\begin{aligned}\partial_t v_\varepsilon =&\varepsilon^{-1}\gamma_\varepsilon'\partial_t d_\varepsilon + g\eta_\varepsilon'\partial_t d_\varepsilon + \varepsilon\mathbf{b}\cdot\mathbf{p}_\varepsilon'\partial_t d_\varepsilon + \varepsilon\eta_\varepsilon\partial_t g + \varepsilon^2\mathbf{p}_\varepsilon\cdot\partial_t\mathbf{b},\\ \Delta v_\varepsilon =&\varepsilon^{-2}\gamma_\varepsilon'' + \varepsilon^{-1}g\eta_\varepsilon'' + \varepsilon^{-1}\gamma_\varepsilon'\Delta d_\varepsilon + g\eta_\varepsilon'\Delta d_\varepsilon + \mathbf{b}\cdot\mathbf{p}_\varepsilon''\\ &+\varepsilon\mathbf{b}\cdot\mathbf{p}_\varepsilon'\Delta d_\varepsilon + 2\eta_\varepsilon'\nabla g\cdot\nabla d_\varepsilon + \mathcal{O}(\varepsilon).\end{aligned}$$

Hence, using (6.5), in the layer $\mathcal{T}_\varepsilon$ we have

$$\begin{aligned}\mathcal{H}v_\varepsilon =& -\varepsilon^{-2}\gamma_\varepsilon'' - \varepsilon^{-1}g\eta_\varepsilon'' - \mathbf{b}\cdot\mathbf{p}_\varepsilon'' + (\varepsilon^{-1}\gamma_\varepsilon' + g\eta_\varepsilon')\mathcal{H}d_\varepsilon - 2\eta_\varepsilon'\nabla g\cdot\nabla d_\varepsilon + \mathcal{O}(\varepsilon)\\ =& -\varepsilon^{-2}\gamma_\varepsilon'' - \varepsilon^{-1}g\eta_\varepsilon'' + \varepsilon^{-1}\gamma_\varepsilon'\overline{g} - \mathbf{b}\cdot\mathbf{p}_\varepsilon'' - 2\eta_\varepsilon'\nabla g\cdot\nabla d_\varepsilon + g\overline{g}\eta_\varepsilon' + y\gamma_\varepsilon'\overline{h}\\ &+ c_1\varepsilon|\log\varepsilon|^2\gamma_\varepsilon'\overline{h} - c_1'\varepsilon|\log\varepsilon|^2\gamma_\varepsilon' + \mathcal{O}(\varepsilon|\log\varepsilon|^2).\end{aligned}$$

Now observe that, on $\mathcal{T}_\varepsilon$,

$$\begin{aligned}g(\mathbf{x},t) =&g(\mathbf{s}(\mathbf{x},t) + d\nabla d, t) = \overline{g} + d\nabla d\cdot\overline{\nabla g} + \mathcal{O}(d^2)\\ =&\overline{g} + \varepsilon y\nabla d_\varepsilon\cdot\overline{\nabla g} + c_1\varepsilon^2|\log\varepsilon|^2\nabla d_\varepsilon\cdot\overline{\nabla g} + \mathcal{O}(\varepsilon^2|\log\varepsilon|^2),\end{aligned}$$

so that

$$g\overline{g}\eta_\varepsilon' = g^2\eta_\varepsilon' + \mathcal{O}(\varepsilon|\log\varepsilon|), \tag{6.13}$$

and

$$\varepsilon^{-1}\gamma_\varepsilon'\overline{g} = \varepsilon^{-1}\gamma_\varepsilon' g - y\gamma_\varepsilon'\nabla d_\varepsilon\cdot\overline{\nabla g} - c_1\varepsilon|\log\varepsilon|^2\gamma_\varepsilon'\nabla d_\varepsilon\cdot\overline{\nabla g} + \mathcal{O}(\varepsilon|\log\varepsilon|^2). \tag{6.14}$$

Inserting (6.13) and (6.14) into the previous expression of $\mathcal{H}v_\varepsilon$, we get

$$\begin{aligned}\mathcal{H}v_\varepsilon = &-\varepsilon^{-2}\gamma_\varepsilon'' - \varepsilon^{-1}g\eta_\varepsilon'' + \varepsilon^{-1}\gamma_\varepsilon' g - y\gamma_\varepsilon'\nabla d_\varepsilon\cdot\overline{\nabla g} - c_1\varepsilon|\log\varepsilon|^2\gamma_\varepsilon'\nabla d_\varepsilon\cdot\overline{\nabla g} - \mathbf{b}\cdot\mathbf{p}_\varepsilon'' \\ &-2\eta_\varepsilon'\nabla g\cdot\nabla d_\varepsilon + g^2\eta_\varepsilon' + y\gamma_\varepsilon'\overline{h} + c_1\varepsilon|\log\varepsilon|^2\gamma_\varepsilon'\overline{h} - c_1'\varepsilon|\log\varepsilon|^2\gamma_\varepsilon' + \mathcal{O}(\varepsilon|\log\varepsilon|^2).\end{aligned}$$

Enforcing the formula

$$\begin{aligned}\varepsilon^{-2}\psi(v_\varepsilon) =&\varepsilon^{-2}\psi(\gamma_\varepsilon) + \varepsilon^{-1}g\eta_\varepsilon\psi'(\gamma_\varepsilon) + \mathbf{b}\cdot\mathbf{p}_\varepsilon\psi'(\gamma_\varepsilon) \\ &+ \frac{1}{2}g^2\eta_\varepsilon^2\psi''(\gamma_\varepsilon) - c_2\varepsilon|\log\varepsilon|^2\psi'(\gamma_\varepsilon) + \mathcal{O}(\varepsilon),\end{aligned}$$

we finally get that, on $\mathcal{T}_\varepsilon$,

$$\mathcal{H}v_\varepsilon + \varepsilon^{-2}\psi(v_\varepsilon) - \varepsilon^{-1}\frac{c_0}{2}g = \mathrm{I}_\varepsilon + \mathrm{II}_\varepsilon + \mathrm{III}_\varepsilon + \mathrm{IV}_\varepsilon + \mathcal{O}(\varepsilon|\log\varepsilon|^2),$$

where, recalling also the expression of $\mathbf{b}$ and $\mathbf{p}_\varepsilon$ and the equality $\nabla d_\varepsilon = \nabla d$,

$$\begin{aligned}\mathrm{I}_\varepsilon =& -\varepsilon^{-2}(\gamma_\varepsilon'' - \psi(\gamma_\varepsilon)),\\ \mathrm{II}_\varepsilon =& -\varepsilon^{-1}g[\eta_\varepsilon'' - \eta_\varepsilon\psi'(\gamma_\varepsilon) + \frac{c_0}{2} - \gamma_\varepsilon'],\\ \mathrm{III}_\varepsilon =& -\mathbf{b}\cdot\mathbf{p}_\varepsilon'' + \mathbf{b}\cdot\mathbf{p}_\varepsilon\psi'(\gamma_\varepsilon) - 2\eta_\varepsilon'\nabla g\cdot\nabla d + g^2\eta_\varepsilon' + y\gamma_\varepsilon'\overline{h} + \frac{1}{2}g^2\eta_\varepsilon^2\psi''(\gamma_\varepsilon) - y\gamma_\varepsilon'\nabla d\cdot\overline{\nabla g}\\ =& -(\overline{h} - \nabla d\cdot\nabla g)(\xi_\varepsilon'' - \psi'(\gamma_\varepsilon)\xi_\varepsilon - y\gamma_\varepsilon') + y\gamma_\varepsilon'\nabla d\cdot(\nabla g - \overline{\nabla g})\\ &-(g^2 - 2\nabla d\cdot\nabla g)(\omega_\varepsilon'' - \psi'(\gamma_\varepsilon)\omega_\varepsilon - \eta_\varepsilon') - g^2(\pi_\varepsilon'' - \psi'(\gamma_\varepsilon)\pi_\varepsilon - \frac{1}{2}\eta_\varepsilon^2\psi''(\gamma_\varepsilon)),\\ \mathrm{IV}_\varepsilon =& c_1\varepsilon|\log\varepsilon|^2\gamma_\varepsilon'\overline{h} - c_1'\varepsilon|\log\varepsilon|^2\gamma_\varepsilon' - c_2\varepsilon|\log\varepsilon|^2\psi'(\gamma_\varepsilon) - c_1\varepsilon|\log\varepsilon|^2\gamma_\varepsilon'\nabla d\cdot\overline{\nabla g}\\ =& \mathrm{IV}_\varepsilon^1 + \mathrm{IV}_\varepsilon^2 + \mathrm{IV}_\varepsilon^3 + \mathrm{IV}_\varepsilon^4.\end{aligned}$$

Observe now that

$$\gamma_\varepsilon'' - \psi(\gamma_\varepsilon) = o(\varepsilon^{2\delta-1}) \qquad \forall y\in\mathbf{R}\setminus\{\pm x_\varepsilon, \pm 2x_\varepsilon\}. \tag{6.15}$$

Indeed $\gamma_\varepsilon'' - \psi(\gamma_\varepsilon) = 0$ on the set $]-\infty, -2x_\varepsilon[\cup]-x_\varepsilon, x_\varepsilon[\cup]2x_\varepsilon, +\infty[$. Let now $y\in]-2x_\varepsilon, -x_\varepsilon[\cup]x_\varepsilon, 2x_\varepsilon[$. By the definition of γ_ε and the second relation in (6.1), one has $\gamma_\varepsilon''(y) = o(\varepsilon^{2\delta})$; in addition, by the first relation in (6.1),

$$\psi(\gamma_\varepsilon(y)) = o(\varepsilon^{2\delta-1}),$$

and this gives (6.15). Similarly, the definitions of $\xi_\varepsilon, \eta_\varepsilon, \omega_\varepsilon, \pi_\varepsilon$ and iterated applications of formulae (6.1) yield

$$\begin{aligned}\eta_\varepsilon'' - \psi'(\gamma_\varepsilon)\eta_\varepsilon + \frac{c_0}{2} - \gamma_\varepsilon' =& o(\varepsilon^{2\delta-1}),\\ \xi_\varepsilon'' - \psi'(\gamma_\varepsilon)\xi_\varepsilon - y\gamma_\varepsilon' =& \omega_\varepsilon'' - \psi'(\gamma_\varepsilon)\omega_\varepsilon - \eta_\varepsilon'\\ =& \pi_\varepsilon'' - \psi'(\gamma_\varepsilon)\pi_\varepsilon - \frac{1}{2}\eta_\varepsilon^2\psi''(\gamma_\varepsilon) = o(\varepsilon^{2\delta-1}),\end{aligned}$$

for any $y \in \mathbf{R} \setminus \{\pm x_\varepsilon, \pm 2x_\varepsilon\}$. Noting that $y\gamma_\varepsilon' \nabla d \cdot (\nabla g - \overline{\nabla g}) = \mathcal{O}(\varepsilon |\log \varepsilon|)$, the previous estimates imply that, on $\mathcal{T}_\varepsilon$,

$$\mathrm{I}_\varepsilon = \mathrm{II}_\varepsilon = \mathrm{III}_\varepsilon = o(\varepsilon^{2\delta-3}) + \mathcal{O}(\varepsilon|\log\varepsilon|).$$

Take

$$c_1(t) = c\exp((1+K)t), \tag{6.16}$$

where c and K are positive constants. Then we have

$$\mathrm{IV}_\varepsilon^1 + \mathrm{IV}_\varepsilon^2 + \mathrm{IV}_\varepsilon^4 \le -c_1 \varepsilon |\log\varepsilon|^2 \gamma_\varepsilon'.$$

Choose now $K = \|h\|_{L^\infty(\Sigma)} + \|\nabla g\|_{L^\infty(\Omega\times(0,T))}$. As $c_3\gamma_\varepsilon' + \psi'(\gamma_\varepsilon)$ is uniformly positive for a proper choice of the positive constant c_3, we realize that, if c and c_2 are large enough (independently of ε), then

$$\mathcal{H}v_\varepsilon + \varepsilon^{-2}\psi(v_\varepsilon) - \varepsilon^{-1}\frac{c_0}{2} g \le 0 \qquad \text{on } \mathcal{T}_\varepsilon,$$

for ε sufficiently small (depending on c_1 and c_2).

On the other hand, outside the transition layer $\mathcal{T}_\varepsilon$, we have

$$v_\varepsilon = \pm 1 + \varepsilon \frac{c_0}{2\alpha} g \mp \varepsilon^2 \frac{c_0^2 \beta}{8\alpha^3} g^2 - c_2 \varepsilon^3 |\log\varepsilon|^2,$$

so that, by the assumption (2.1), one easily gets

$$\partial_t v_\varepsilon = \mathcal{O}(\varepsilon), \qquad \Delta v_\varepsilon = \mathcal{O}(\varepsilon).$$

Moreover,

$$\begin{aligned} \varepsilon^{-2}\psi(v_\varepsilon) &= \varepsilon^{-2}\psi(\pm 1) + \varepsilon^{-2}\alpha(\varepsilon \frac{c_0}{2\alpha} g \mp \varepsilon^2 \frac{c_0^2\beta}{8\alpha^3} g^2 - c_2\varepsilon^3|\log\varepsilon|^2) \\ &\pm \frac{1}{2}\varepsilon^{-2}(\varepsilon\frac{c_0}{2\alpha} g + \mathcal{O}(\varepsilon^2))^2\beta + \mathcal{O}(\varepsilon) = \varepsilon^{-1}\frac{c_0}{2} g - c_2\alpha\varepsilon|\log\varepsilon|^2 + \mathcal{O}(\varepsilon). \end{aligned}$$

Therefore,

$$\mathcal{H}v_\varepsilon + \varepsilon^{-2}\psi(v_\varepsilon) - \varepsilon^{-1}\frac{c_0}{2} g = -c_2\alpha\varepsilon|\log\varepsilon|^2 + \mathcal{O}(\varepsilon),$$

so that $\mathcal{H}v_\varepsilon + \varepsilon^{-2}\psi(v_\varepsilon) - \varepsilon^{-1}\frac{c_0}{2} g \le 0$ as $\varepsilon \to 0$ for any choice of the positive constant c_2.

Hence, applying the Comparison Lemma 5.1 with u replaced by the solution u_ε of problem (2.9) with initial datum (6.6) and v replaced by v_ε, we conclude

$$v_\varepsilon^-(\mathbf{x},t) \le u_\varepsilon(\mathbf{x},t) \qquad \forall\ (\mathbf{x},t) \in \Omega\times[0,T].$$

If $(\mathbf{x},t) \in \Omega\times(0,T)$, the construction of a supersolution

$$v_\varepsilon^+(\mathbf{x},t) = \begin{cases} \gamma_\varepsilon(y) + \varepsilon g(\mathbf{x},t)\eta_\varepsilon(y) + \varepsilon^2 \mathbf{b}(\mathbf{x},t)\cdot\mathbf{p}_\varepsilon(y) + c_2\varepsilon^3|\log\varepsilon|^2 & \text{on } \mathcal{T}_\varepsilon^+, \\ 1 + \varepsilon\eta_\infty g(\mathbf{x},t) + \varepsilon^2\pi_\infty g^2(\mathbf{x},t) + c_2\varepsilon^3|\log\varepsilon|^2 & \text{on } \{d_\varepsilon^+ \ge 2\delta\varepsilon|\log\varepsilon|\}, \\ -1 + \varepsilon\eta_\infty g(\mathbf{x},t) - \varepsilon^2\pi_\infty g^2(\mathbf{x},t) + c_2\varepsilon^3|\log\varepsilon|^2 & \text{on } \{d_\varepsilon^+ \le -2\delta\varepsilon|\log\varepsilon|\}, \end{cases}$$

where $y = \frac{d_\varepsilon^+(\mathbf{x},t)}{\varepsilon}$, and

$$d_\varepsilon^+(\mathbf{x},t) = d(\mathbf{x},t) + c_1(t)\varepsilon^2|\log\varepsilon|^2,$$

is similar, and thus is omitted.

Using the sub-solution v_ε^- and supersolution v_ε^+ defined in §6, it is now possible to deduce the following error estimates, valid before the onset of singularities, for interfaces evolving by mean curvature with a forcing term g satisfying (2.1).

Theorem 6.1. *Let $\Sigma(t)$ be a mean curvature flow with forcing term g which satisfies (2.4). For any $\varepsilon > 0$ let u_ε be a solution of problem (2.9) with initial datum (6.6). Let $\Sigma_\varepsilon(t) = \{\mathbf{x} \in \Omega : u_\varepsilon(\mathbf{x}, t) = 0\}$. Then there exist $0 < \varepsilon_0 < 1$ and a constant C depending on Σ, g, and T such that for all $0 < \varepsilon \le \varepsilon_0$*

$$\Sigma_\varepsilon(t) \subseteq \{\mathbf{x} \in \Omega : \mathrm{dist}(\mathbf{x}, \Sigma(t))) \le C\varepsilon^2 |\log \varepsilon|^2\} \quad \forall t \in [0, T], \tag{6.17}$$

$$\Sigma(t) \subseteq \{\mathbf{x} \in \Omega : \mathrm{dist}(\mathbf{x}, \Sigma_\varepsilon(t))) \le C\varepsilon^2 |\log \varepsilon|^2\} \quad \forall t \in [0, T]. \tag{6.18}$$

Proof. Let us prove (6.17). Fix $t \in [0, T]$, and let $\mathbf{x} \in \Sigma_\varepsilon(t)$. We claim that $\mathbf{x} \in \mathcal{T}(t)$ (see (2.3)). Indeed, if $\mathbf{x} \in \Sigma_\varepsilon(t)$, then $u_\varepsilon(\mathbf{x}, t) = 0$, so that

$$v_\varepsilon^-(\mathbf{x}, t) \le u_\varepsilon(\mathbf{x}, t) = 0 \le v_\varepsilon^+(\mathbf{x}, t). \tag{6.19}$$

If by contradiction $\mathbf{x} \notin \mathcal{T}(t)$, by the inclusions $\mathcal{T}_\varepsilon^\mp(t) \subseteq \mathcal{T}(t)$ we have $\mathbf{x} \notin \mathcal{T}_\varepsilon^-(t)$ and $\mathbf{x} \notin \mathcal{T}_\varepsilon^+(t)$. Recalling the definitions of $v_\varepsilon^\mp$, we deduce that, for ε sufficiently small, $v_\varepsilon^-(\mathbf{x}, t)$ and $v_\varepsilon^+(\mathbf{x}, t)$ have the same sign, a contradiction with (6.19). We have

$$v_\varepsilon^-(\mathbf{x}, t) = \gamma_\varepsilon(\frac{d_\varepsilon^-(\mathbf{x}, t)}{\varepsilon}) + \mathcal{O}(\varepsilon) \le 0,$$

so that

$$\gamma_\varepsilon(\frac{d_\varepsilon^-(\mathbf{x}, t)}{\varepsilon}) \le \mathcal{O}(\varepsilon).$$

As $\gamma'(0) = 1$, we deduce that $\frac{d_\varepsilon^-(\mathbf{x},t)}{\varepsilon} \le \mathcal{O}(\varepsilon)$; hence, recalling the definition of d_ε^-, we get $d(\mathbf{x}, t) \le \mathcal{O}(\varepsilon^2 |\log \varepsilon|^2)$. A similar argument applied with v_ε^- replaced by v_ε^+ gives $d(\mathbf{x}, t) \ge \mathcal{O}(\varepsilon^2 |\log \varepsilon|^2)$. We conclude that $|d(\mathbf{x}, t)| \le \mathcal{O}(\varepsilon^2 |\log \varepsilon|^2)$, which proves (6.17).

Let us prove (6.18). Fix $t \in [0, T]$, and let $\mathbf{x} \in \Sigma(t)$. We indicate by I the connected component of the intersection between $\mathcal{T}(t)$ and the normal line to $\Sigma(t)$ at $\mathbf{x}$ which contains $\mathbf{x}$. Set $\{\mathbf{x}^-, \mathbf{x}^+\} = I \cap \partial\mathcal{T}(t)$, where $v_\varepsilon^-(\mathbf{x}^+) > 0$ (see the definition of v_ε^-).

Set $\Sigma_\varepsilon^\mp(t) = \{\mathbf{x} \in \Omega : v_\varepsilon^\mp(\mathbf{x}, t) = 0\}$. We claim that there exist $\mathbf{x}_\varepsilon^\mp \in I \cap \Sigma_\varepsilon^\mp(t)$ such that $|\mathbf{x}_\varepsilon^\mp - \mathbf{x}| \le C\varepsilon^2 |\log \varepsilon|^2$, for a suitable absolute positive constant C and ε small enough. Indeed, we have $v_\varepsilon^-(\mathbf{x}, t) = \gamma_\varepsilon(-c_1(t)\varepsilon|\log \varepsilon|^2) + \mathcal{O}(\varepsilon) < 0$, and $v_\varepsilon^-(\mathbf{x}^+, t) > 0$, so that there exists a point $\mathbf{x}_\varepsilon \in I$ lying between $\mathbf{x}$ and $\mathbf{x}^+$ such that $v_\varepsilon(\mathbf{x}_\varepsilon^-, t) = 0$, i.e., $\mathbf{x}_\varepsilon^- \in I \cap \Sigma_\varepsilon^-(t)$. Moreover,

$$|v_\varepsilon^-(\mathbf{x}_\varepsilon^-, t) - v_\varepsilon^-(\mathbf{x}, t)| = |v_\varepsilon^-(\mathbf{x}, t)| \le C\varepsilon |\log \varepsilon|^2, \tag{6.20}$$

and, in view of the non degeneracy property of v_ε, namely $|\nabla v_\varepsilon^- \cdot \mathbf{n}| = \frac{1}{\varepsilon}$, we have

$$|v_\varepsilon^-(\mathbf{x}_\varepsilon^-, t) - v_\varepsilon^-(\mathbf{x}, t)| = |\mathbf{x}_\varepsilon^- - \mathbf{x}||\nabla v_\varepsilon^-(\xi, t) \cdot \mathbf{n}| \ge C\frac{|\mathbf{x}_\varepsilon^- - \mathbf{x}|}{\varepsilon}, \tag{6.21}$$

for a suitable absolute positive constant C and a suitable ξ between $\mathbf{x}_\varepsilon^-$ and $\mathbf{x}$. Then (6.20) and (6.21) yield $|\mathbf{x}_\varepsilon^- - \mathbf{x}| \le C\varepsilon^2 |\log \varepsilon|^2$.

A similar argument applied with v_ε^- replaced by v_ε^+ concludes the proof of the claim.

As $\mathbf{x}_\varepsilon^\mp \in \Sigma_\varepsilon^\mp(t)$, we have $u_\varepsilon(\mathbf{x}_\varepsilon^-, t) \geq v_\varepsilon(\mathbf{x}_\varepsilon^-, t) = 0$, and $u_\varepsilon(\mathbf{x}_\varepsilon^+, t) \leq v_\varepsilon(\mathbf{x}_\varepsilon^-, t) = 0$, so that there exists a point $\mathbf{z} \in I$ between $\mathbf{x}_\varepsilon^-$ and $\mathbf{x}_\varepsilon^+$ such that $u_\varepsilon(\mathbf{z}, t) = 0$. Therefore

$$\operatorname{dist}(\mathbf{x}, \Sigma_\varepsilon(t)) \leq |\mathbf{z} - \mathbf{x}| \leq C\varepsilon^2 |\log \varepsilon|^2,$$

and this gives (6.18), and concludes the proof of the theorem. □

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