Received: 7 June 2020 | Revised: 16 January 2022 | Accepted: 17 January 2022

DOI: 10.1111/meca.12382

METROECONOMICA
INTERNATIONAL REVIEW OF ECONOMICS

WILEY

ORIGINAL ARTICLE

Bertrand–Edgeworth oligopoly: Characterization of mixed strategy equilibria when some firms are large and the others are small

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Funding Information
We thank the University of Pisa for financial support (PRA 2018–2019 Institutions, Markets and Policy Issues).

Abstract

This paper studies Bertrand–Edgeworth competition among firms producing a homogeneous commodity under efficient rationing and constant (and identical across firms) marginal cost until full capacity utilization is reached. Our focus is on a subset of the no pure-strategy equilibrium region of the capacity space in which, in a well-defined sense, some firms are large and the others are small. We characterize equilibria for such subset. For each firm, the payoffs are the same at any equilibrium and, for each type of firm, they are proportional to capacity. While there is a single profile of equilibrium distributions for the large firms, there is a continuum of equilibrium distributions for the small firms: what is uniquely determined, for the latter, is the capacity-weighted sum of their equilibrium distributions and hence the union of the supports of their equilibrium strategies.

KEYWORDS

Bertrand competition, Bertrand–Edgeworth oligopoly, large and small firms, mixed strategy equilibrium

JEL CLASSIFICATION

C72; D43; L13

1 | INTRODUCTION

Bertrand–Edgeworth competition among capacity-constrained sellers of a homogeneous product has been an active field of research since Levitan and Shubik's (1972) reappraisal of such theoretical framework. Assume a given number of firms producing on demand a homogeneous good at constant and identical unit variable cost up to some fixed capacity. Furthermore, assume that rationing takes place according to the surplus-maximizing rule and that demand is a continuous, non-increasing, and non-negative function defined on the set of non-negative prices and is positive, strictly decreasing, twice differentiable and such that the monopolist's profit function is strictly concave when positive. Then there are a few well-established facts about the equilibrium of this price game. First, at any pure strategy equilibrium, the firms earn competitive profit. However, a pure strategy equilibrium need not exist. In this case, existence of a mixed strategy equilibrium is guaranteed by the sufficient conditions of Theorem 5 of Dasgupta and Maskin (1986). Under similar assumptions on demand and cost, the set of mixed strategy equilibria was characterized by Kreps and Scheinkman (1983) for the duopoly within a two-stage capacity and price game. This model was subsequently extended to allow significant convexities in the demand function (by Osborne & Pitchik, 1986) or differences in unit cost among the duopolists (by Deneckere & Kovenock, 1996). This led to the discovery of new phenomena, such as the possibility of the supports of the equilibrium strategies being disconnected and non-identical for the duopolists.

The characterization of equilibria of the price game among capacity-constrained sellers of a homogeneous product under general oligopoly is far from complete in the literature. An important result is that the equilibrium payoff of the largest firm (or any of the largest firms, if more than one firm has the largest size) is equal to the payoff of the Stackelberg follower when the rivals supply their entire capacity (Boccard & Wauthy, 2000; De Francesco 2003).[1] Based on this property, Ubeda (2007) showed, among other things, that the maximum and minimum over all the supports of equilibrium strategies belong to the support of the equilibrium strategies of any firm with the largest capacity.[2] Other results were provided by De Francesco and Salvadori (2010).

Progress on the characterization of equilibria of the price game under given capacities has been made along several directions. One direction was to restrict the number of competing firms. Hirata (2009) and De Francesco and Salvadori (2010, 2015, 2016) have analyzed the triopoly price game with a decreasing and concave demand function, establishing independently a number of features of equilibria. In a recent study on price strategic interaction among capacity-unconstrained sellers facing "captive customers" and price-rigidity of market demand, Mark Armstrong and John Vickers (2018) have also compared the resulting equilibria with equilibria in the more standard Bertrand–Edgeworth framework; such a task has been accomplished for the triopoly, providing a complete characterization of the equilibria arising in the Bertrand–Edgeworth price game with rigid demand.

A second direction of research focused on portions of the whole region of an oligopoly capacity space where no pure strategy equilibria exist (hereafter, the no-pure strategy equilibrium region, for brevity). Vives (1986), among others, characterized the (symmetric) mixed

[1]The proof provided by Boccard and Wauthy (2000) is carried out along the lines followed by Kreps and Sheinkman (1983) for the analogous result under duopoly. After pointing out a mistake in the proof, De Francesco (2003) established the result correctly along the same lines.

[2]In a still unpublished paper, Ubeda (2007) compares discriminatory and uniform auctions among capacity-constrained producers and obtains a number of novel results on discriminatory auctions: a discriminatory auction could be designed in such a way as to be equivalent to Bertrand–Edgeworth competition under the efficient rationing rule.

strategy equilibrium of the price game for the subset in which all firms have the same capacity. De Francesco and Salvadori (2011) generalized Vives' result: they established uniqueness of equilibrium in Vives' symmetric capacity case and, more generally, whenever the capacities of the largest and smallest firm are, in a precise sense, sufficiently close to each other. Furthermore, they characterized the equilibrium in this "quasi-symmetric" oligopoly, showing that the supports of the equilibrium strategies of all firms are intervals, each with the same minimum price whereas the higher a firm's capacity, the higher the maximum price. Within an analysis concerning horizontal merging of firms, Davidson and Deneckere (1984) characterized, for the case of linear demand, equilibria for the subset in which all firms but one have an identical capacity and one firm, the largest, has a capacity that is a multiple of the other firms. Again, the attention was restricted to equilibria in which the strategies of equally-sized firms are symmetrical.

There is one result in Hirata[3] (2009) that extends straightforwardly to the oligopoly. Hirata (2009) showed, not only for the triopoly but also for the oligopoly, that a continuum of equilibria exists in the subset of the no-pure strategy equilibrium region in which the largest firm can meet the highest level of total demand possibly arising at an equilibrium. In fact, while there is one equilibrium strategy for the largest firm, there is a continuum of equilibrium strategies for smaller firms, in that there is a single equation determining the capacity-weighted sum of their cumulative distributions throughout the lowest price and the highest price. The present paper shows constructively that the subset of the no-pure strategy equilibrium region in which a continuum of equilibria exists is much wider.

We specifically analyze a subset of the no-pure strategy equilibrium region in which there are two groups of firms, firms that are "large" and firms that are "small" in the following technical sense: the total capacity of the large firms can meet the highest level of demand that can arise at an equilibrium of the price game, whereas the total capacity of the small firms is so small that total industry capacity minus the capacity of any of the large firms does not exceed the smallest level of total demand that can arise at an equilibrium.

Such a bipolarized industry structure has two interesting and intertwined implications. On the one hand, and similarly as in the mentioned case studied by Hirata (2009), there is no "direct" strategic interaction among the small firms: more specifically, regardless of the prices being charged by the other small firms, each small firm either sells its entire capacity, if at least one of the large firms is more expensive, or sells nothing, if all the large firms are cheaper. On the other hand, each large firm sells its entire capacity if, and only if, at least one of the other large firms is more expensive. In the event of all the other large firms selling cheaper, the expected value of its residual demand falls short of total demand by an amount equal to the total capacity of the other large firms (as it would be in De Francesco & Salvadori, 2011) plus the capacity-weighted sum of the probabilities of all the small firms charging a lower price. We will characterize the equilibria for such a bipolarized industry structure. It will be shown that the above implications are ultimately responsible for the existence of a continuum of equilibrium distributions for the small firms. What is uniquely determined, instead, are the equilibrium payoffs of all firms, the equilibrium distributions of the large firms and hence the supports of their equilibrium strategies, the union of the supports of the equilibrium strategies of the small firms, and the capacity-weighted sum of the equilibrium distributions of the small firms. Most importantly, characterizing the continuum of equilibria

[3]The same result was independently reached by De Francesco and Salvadori (2008).

for any such bipolarized industry structure involves determining the lowest and highest price that small firms can ever charge in equilibrium: the former is generally higher than (in a limit case, equal to) the (uniform) minimum price each large firm will ever charge in equilibrium and the latter is always less than the maximum price any large firm will ever charge. This property of the equilibrium is similar to a property arising in the triopoly under some industry configurations which has been observed both by De Francesco and Salvadori (2008) and Hirata (2009): it implies that the smaller firms have a higher profit per unit of capacity than the larger firms have.

Although our interest here is purely theoretical, as mentioned above, the present study is potentially relevant to a wide array of empiricists. First, the parameter region it covers appears fairly natural: casual observation seems to provide some evidence of industries where a number of relatively few firms of a comparable size coexist with considerably smaller firms.[4] Second, the unique results in terms of each firm's equilibrium payoff, the supports of the equilibrium strategies of the large firms, and the minimum and maximum of the union of the supports of the small firms' equilibrium strategies provide a set of empirically testable predictions. Quite interestingly, carrying out such a test need not require detailed information on the individual capacities of each small firm, which might be more difficult to obtain than an approximate estimate of their total capacity, which is what actually matters for the equilibrium features, since a redistribution of total capacity among the small firms would not affect the total of their equilibrium payoffs. Third, the fraction of industry capacity pertaining to the small-firm segment of the industry is proven to be relevant for the equilibrium payoffs of the remaining firms and therefore the equilibria under such industry structures are worth examining: indeed, that fraction and even the capacity of each small firm need not be negligible compared to the industry size.[5]

The remainder of the paper is organized as follows. Section 2 presents basic properties of the equilibrium of the price game in the no-pure strategy equilibrium region of the capacity space. Section 3 defines an industry containing "large" firms as well as "small" firms and then characterizes the continuum of equilibria arising under such circumstances. These are the main results of the paper. Other two sections concern the motivation of the paper and makes use of numerical examples. Section 4 shows that the role of small firms is not negligible: if there is a change in the sum of their sizes which does not change their role of small firms, the effect on the profits of the other firms may be relevant. Section 5 shows that the part of the region of no pure strategy equilibria investigated in this paper can be quite large indeed. Section 6 briefly concludes. All proofs are in the Mathematical Appendix, which includes also some further results.

2 | PRELIMINARIES

Denote by $\mathcal{Z} = \{1, \ldots, z\}$ the set of firms.[6] Each firm i produces to order a homogeneous commodity with the same constant marginal cost (with no loss of generality normalized to zero) up to its fixed

[4]For instance, the market share held by the first corporate group (the first two corporate groups, the first three corporate groups) for the residential-customer segment of the retail electricity (free) market in Italy was 45,7% (58,2%. 63,3%) in 2020 and was 50,2% (62%, 72,8%) in 2014. Data provided by ARERA, the Italian Regulatory Authority for Energy, Networks and the Environment: https://www.arera.it/it/dati/mr/mre_concentra.htm#domestici.

[5]See the simulations in Section 4.

[6]The assumptions and notation laid down in this section largely draw on De Francesco and Salvadori (2011).

capacity k_i. Denote by K total capacity and, with no loss of generality, let $k_1 \geqslant k_2 \geqslant \cdots \geqslant k_z$. A continuous demand function $D(p)$ which is strictly decreasing and such that $pD(p)$ is strictly concave over the price range in which $D(p) > 0$ is assumed to exist. Firm i's profit at strategy profile (p_i, p_{-i}) is $\Pi_i(p_i, p_{-i}) = p_i\, min\, \{d_i(p_i, p_{-i}), k_i\}$, where $d_i(p_i, p_{-i})$ is the demand forthcoming to firm i at (p_i, p_{-i}), p_i is the price charged by firm i and p_{-i} is the vector of prices charged by all firms except firm i. Under efficient rationing and assuming that such demand is proportional to capacity for equally priced firms, we have that $d_i(p_i, p_{-i}) = max\,\{0, D(p_i) - \sum_{j:p_j<p_i} k_j\} \times \frac{k_i}{\sum_{r:p_r=p_i} k_r}$.

Denote by p^c the competitive price: $D(p^c) = K$ if $D(0) \geqslant K$ and $p^c = 0$ if $D(0) \leqslant K$. As is well known (see, e.g., De Francesco & Salvadori, 2010), $(p_1, \ldots, p_z) = (p^c, \ldots, p^c)$ is an equilibrium of the price game if, and only if, either

$$K - k_1 \geqslant D(0) \text{ when } D(0) \leqslant K, \tag{1}$$

or

$$k_1 \leqslant -p^c \left[D'(p)\right]_{p=p^c} \text{ when } D(0) > K. \tag{2}$$

Holding (2), $(p^c, \ldots, p^c)$ is the unique equilibrium; holding (1), the competitive payoff is earned by each firm at any equilibrium. It is also known that there are no pure strategy equilibria if neither inequality (1) nor inequality (2) holds or, equivalently, if

$$\frac{k_1}{K} > max\left\{1 - \frac{D(0)}{K}, |\varepsilon|_{p=p^c}\right\}. \tag{3}$$

where ε is the price elasticity of demand.

In the remainder, inequality (3) is assumed to hold. It follows from the strict concavity of $pD(p)$ that there is a single solution to $max_p p(D(p) - \sum_{j\neq 1} k_j)$, call it p_M:

$$p_M := \text{arg}max_p p(D(p) - \sum_{j\neq 1} k_j). \tag{4}$$

Furthermore, we call p_m the lower solution of equation $p\, min\,\{D(p), k_1\} = p_M(D(p_M) - \sum_{j\neq 1} k_j)$.

Denote by $\sigma_i: (0, \infty) \rightarrow [0, 1]$ a mixed strategy of firm i, where $\sigma_i(p) = \Pr_{\sigma_i}(p_i < p)$ is the probability that firm i charges a price lower than p under strategy σ_i. Note that $\sigma_i(p)$ is continuous except at any $p°$ such that $\Pr_{\sigma_i}(p_i = p°) > 0$. A mixed strategy equilibrium is denoted by $\phi = (\phi_1, \ldots, \phi_z): (0, \infty)^z \rightarrow [0, 1]^z$, where $\phi_i(p) = \Pr_{\phi_i}(p_i < p)$. We denote by $\Pi_i(\sigma_i, \phi_{-i})$ firm i's expected profit when it follows strategy σ_i and the rivals are playing their equilibrium strategy profile ϕ_{-i}; in particular $\Pi_i(p, \phi_{-i})$ is firm i's expected profit when it charges p with certainty and the rivals are playing their equilibrium strategy profile ϕ_{-i}. We denote by Π_i^* firm i's expected profit at equilibrium ϕ, by S_i the support of ϕ_i and by $p_M^{(i)}$ and $p_m^{(i)}$ the maximum and the minimum of S_i, respectively. Note that $p \in S_i$ when there is $\lambda > 0$ such that $\phi_i(p + h) > \phi_i(p - h)$ for each $h \in (0, \lambda)$. Clearly, $\Pi_i^* \geqslant \Pi_i(\sigma_i, \phi_{-i})$ (each i). For any $p \in S_i$, $\Pi_i^* = \Pi_i(p, \phi_{-i})$ almost everywhere, namely, whenever $\Pr_{\phi_j}(p_j = p) = 0$ (any $j \neq i$). In fact, $\Pi_i^* = lim_{p_i \longrightarrow p-} \Pi_i(p_i, \phi_{-i})$ everywhere for $p \in S_i$ since, quite obviously, $\Pi_i^* \geq lim_{p_i \longrightarrow p-} \Pi_i(p, \phi_{-i})$ (any p) and, furthermore, Π_i^* cannot be greater than $lim_{p_i \longrightarrow p-} \Pi_i(p, \phi_{-i})$ for some $p \in S_i$: since $lim_{p_i \longrightarrow p+} \Pi_i(p, \phi_{-i}) \leqslant \Pi_i(p, \phi_{-i}) \leqslant lim_{p_i \longrightarrow p-} \Pi_i(p, \phi_{-i})$, that event would imply that $\Pi_i(p, \phi_{-i}) < \Pi_i^*$ on a neighborhood of p, contrary to the fact that $p \in S_i$.

We now present some basic properties of mixed strategy equilibria.

Proposition 1 *Let inequality (3) hold. Then, in any equilibrium:*

(i) $max_j\{p_M^{(j)}\} = p_M, min_j\{p_m^{(j)}\} = p_m, \#\mathcal{L} > 1$ and $\#\mathcal{M} > 1$, where $\mathcal{L} := \left\{ i \in \mathcal{Z} \,\middle|\, p_m^{(i)} = p_m \right\}$ and $\mathcal{M} := \left\{ i \in \mathcal{Z} \,\middle|\, p_M^{(i)} = p_M \right\}$; there exists some i such that $k_i = k_1$, $p_M^{(i)} = p_M$, $p_m^{(i)} = p_m$, $\Pi_i^* = p_M(D(p_M) - \sum_{j \neq 1} k_j)$ and $\Pr_{\phi_j}(p_j = p_M) = 0$ for any $j \neq i$.

(ii) $D(p_m) < \sum_{i \in \mathcal{L}} k_i$, $\Pr_{\phi_i}(p_i = p_m) = 0$ for each $i \in \mathcal{Z}$, and $\Pi_i^* = p_m k_i$ for each $i \in \mathcal{L} - \{1\}$.

(iii) If $k_2 = k_1$, then, for any i such that $k_i = k_1$, $\Pr_{\phi_j}(p_j = p_M) = 0$ for any $j \neq i$, $\Pi_i^* = p_M(D(p_M) - \sum_{j \neq 1} k_j) = p_m k_i$, $p_M^{(i)} = p_M$, and $p_m^{(i)} = p_m$.

(iv) $\Pr_{\phi_1}(p_1 = p_M) > 0$ if $k_1 > k_2$.

3 | SOME FIRMS ARE LARGE AND THE OTHERS ARE SMALL

We will focus on the subset of the region of no pure strategy equilibria in which[7]

$$k_1 + \cdots + k_n \geqslant D(p_m) \tag{5}$$

$$D(p_M) \geqslant K - k_n. \tag{6}$$

The sets $\mathcal{N} = \{1, \ldots, n\}$ and $\mathcal{Z} - \mathcal{N}$ will be referred to as the set of "large" firms and the set of "small" firms, respectively. Let us look more deeply at these inequalities in order to grasp the rationale for this terminology. According to inequality (5), large firms as a whole can meet the highest demand that can arise at an equilibrium of the price game, $D(p_m)$. If $n = 1$, inequality (5) coincides with the inequality that defines the subset of the no-pure strategy equilibrium region mentioned in the introduction as explored by Hirata (2009) (and De Francesco & Salvadori, 2008, 2010). According to inequality (6), total industry capacity minus the capacity of any of the large firms does not exceed the smallest level of demand possibly arising at an equilibrium of the price game, $D(p_M)$. If $n = 1$, inequality (6) coincides with inequality $D(p_M) \geqslant K - k_1$, which certainly holds as a strict inequality. Most importantly, since $K > D(p_m) > D(p_M)$, inequalities (5) and (6) imply that

$$k_n > k_{n+1} + \cdots + k_z, \tag{7}$$

$$k_1 - k_n \leqslant D(p_M) - \sum_{j \neq 1} k_j \tag{8}$$

consistent with the "small" labeling of firms from $n + 1$ to z and with the "large" labeling of firms from 1 to n. In the following, we will assume, without further mentioning, that inequalities (5) and (6) hold with $n > 1$. However, footnotes will give some details concerning the case in which $n = 1$.

[7]A simple example can easily show that such subset may be quite large with respect to the the region of no pure strategy equilibria: see Section 5.

Because of inequalities (5) and (6), almost everywhere in the range $[p_m, p_M]$ the payoff function of firm $i \in \mathcal{N}$ in the face of rivals' equilibrium strategies is equal to[8]

$$\Pi_i(p,\phi_{-i}) = p \prod_{j\in\mathcal{N}-\{i\}} \phi_j(p) \left[D(p) - \sum_{j\in\mathcal{N}-\{i\}} k_j - \sum_{r\in\mathcal{Z}-\mathcal{N}} \phi_r(p)k_r \right] + \\ + \left[1 - \prod_{j\in\mathcal{N}-\{i\}} \phi_j(p) \right] pk_i,$$

that is

$$\Pi_i(p,\phi_{-i}) = pk_i - p \prod_{j\in\mathcal{N}-\{i\}} \phi_j(p) \left[\sum_{j\in\mathcal{N}} k_j + \sum_{r\in\mathcal{Z}-\mathcal{N}} \phi_r(p)k_r - D(p) \right], \tag{9}$$

whereas almost everywhere in the same range the payoff function of firm $r \in \mathcal{Z} - \mathcal{N}$ in the face of rivals' equilibrium strategies is equal to[9]

$$\Pi_r(p,\phi_{-r}) = F(p)k_r \tag{10}$$

where

$$F(p) = \left[1 - \prod_{j\in\mathcal{N}} \phi_j(p) \right] p. \tag{11}$$

We can now determine the equilibrium payoff of each large firm (and each small firm in a special case) and prove properties concerning the supports of the strategies, the payoffs and the equilibrium distributions of the large firms.

Proposition 2 *In any equilibrium*

(i) $\mathcal{L} \supseteq \mathcal{N}$, $\Pi_i^* = p_m k_i$ (each $i \in \mathcal{N}$) and $\phi_i(p)k_i = \phi_j(p)k_j$ everywhere for $p \in S_i \cap S_j$ (any $i,j \in \mathcal{N}$); moreover, $k_j\Pi_i(p,\phi_{-i}) = k_i\Pi_j(p,\phi_{-j})$ almost everywhere for $p \in S_i \cap S_j$ (any $i,j \in \mathcal{N}$);[10]

(ii) $\Pi_r^*/k_r = \Pi_s^*/k_s$ (each $r, s \in \mathcal{Z} - \mathcal{N}$);

(iii) if $k_1 + \dots + k_n > D(p_m)$, then $\mathcal{L} = \mathcal{N}$ and $\Pi_r^* > p_m k_r$ (each $r \in \mathcal{Z} - \mathcal{N}$);

(iv) if $k_1 + \dots + k_n = D(p_m)$, then $\mathcal{L} \supset \mathcal{N}$ and $\Pi_i^* = p_m k_i$ (each $i \in \mathcal{Z}$);

[8] If $n = 1$, then $\prod_{j\in\mathcal{N}-\{i\}} \phi_j(p)$ is the empty product and equality (9) becomes $\Pi_1(p,\phi_{-1}) = p[D(p) - \sum_{r\in\mathcal{Z}-\mathcal{N}} \phi_r(p)k_r]$.

[9] There are two reasons for the "almost everywhere" qualification. First, thus far we have not ruled out yet the event that, for some $p^\circ \in (p_m, p_M)$, $\phi_j(p^{\circ+}) > \phi_j(p^\circ)$ (some $j \in \mathcal{N}$): under that event, for instance, $\Pi_i(p^\circ,\phi_{-i}) < lim_{p\to p^\circ-}\Pi_i(p,\phi_{-i})$. Second, because of Proposition 1(iv), if $k_1 > k_2$ then $lim_{p\to p_M-}\Pi_i(p,\phi_{-i}) > \Pi_i(p_M,\phi_{-i})$ (any $i \in \mathcal{Z} - \{1\}$).

[10] Because of part (vi) $k_j\Pi_i(p,\phi_{-i}) = k_i\Pi_j(p,\phi_{-j})$*everywhere* for $p \in S_i \cap S_j - \{p_M\}$ (any $i,j \in \mathcal{N}$). But the proof of part (vi) requires that $k_j\Pi_i(p,\phi_{-i}) = k_i\Pi_j(p,\phi_{-j})$ almost everywhere for $p \in S_i \cap S_j$ (any $i,j \in \mathcal{N}$).

(v) $S_i = [p_m, p_M^{(i)}]$ (each $i \in \mathcal{N}$); $S_1 = S_2 \supseteq S_3 \supseteq \cdots \supseteq S_n$; moreover, $S_i \supset S_{i+1}$ (each $i \in \mathcal{N} - \{1, n\}$) if and only if $k_i > k_{i+1}$;
(vi) $\Pr_{\phi_i}(p_i = p) = 0$ (any $p \in [p_m, p_M)$ and any $i \in \mathcal{Z}$);
(vii) $max \bigcup_{r \in \mathcal{Z}-\mathcal{N}} S_r \leqslant p_M^{(n)}$;
(viii) if either any of inequalities (5) and (6) is satisfied as a strict inequality or $k_2 > k_n$, then $max \bigcup_{r \in \mathcal{Z}-\mathcal{N}} S_r < p_M^{(n)}$ and $\Pi_r(p_M^{(n)}, \phi_{-r}) < \Pi_r^*$.

Proposition 2 allows segment $[p_m, p_M]$ to be partitioned into three parts: $[p_m, \overline{p})$, $[\overline{p}, \overline{\overline{p}}]$, $(\overline{\overline{p}}, p_M]$, where $\overline{p} = min \bigcup_{r \in \mathcal{Z}-\mathcal{N}} S_r$ and $\overline{\overline{p}} = max \bigcup_{r \in \mathcal{Z}-\mathcal{N}} S_r$. The first part is empty only if $k_1 + \cdots + k_n = D(p_m)$;[11] the second part contains $\bigcup_{r \in \mathcal{Z}-\mathcal{N}} S_r$. In the first and third parts the equilibrium distributions are easily determined.

3.1 | The equilibrium distributions in $[p_m, \overline{p})$

In this subsection, we assume that $k_1 + \cdots + k_n > D(p_m)$. In the range $[p_m, \overline{p})$ the equilibrium distributions are: $\phi_r(p) = 0$ for each $r \in \mathcal{Z} - \mathcal{N}$ and

$$\phi_l(p) = \frac{1}{k_l} \left(\frac{p - p_m}{p} \frac{\prod_{j \in \mathcal{N}} k_j}{\sum_{j \in \mathcal{N}} k_j - D(p)} \right)^{\frac{1}{n-1}} \tag{12}$$

for each $l \in \mathcal{N}$, because of Equation (9) and Proposition 2. It is easily recognized that the RHS of Equation (12) is quasi-concave throughout $[p_m, p_M]$.[12] Moreover, it is larger than 1 for $p = p_M$ and $l = n$ since $p_M \left[D(p_M) - \sum_{j \in \mathcal{N}-\{1\}} k_j \right] > p_m k_1$. Hence there is $\widetilde{p}_M^{(n)} \in (p_m, p_M)$ such that in the range $[p_m, \widetilde{p}_M^{(n)}]$ the RHS of Equation (12) is increasing and no larger than 1 for each $i \in \mathcal{N}$. Hence the functions $F(p)$ and $\Pi_r(p, \phi_{-r}) = F(p)k_r$ are well-defined in the range $[p_m, \overline{p}]$ if and only if $\overline{p} \leqslant \widetilde{p}_M^{(n)}$ and this inequality can easily be proved (by following the same procedure used to prove Proposition 2(vii) & (viii)).

In order to determine $\overline{p}$ and the equilibrium payoff of each small firm, the functions $\phi_l(p)$ (each $l \in \mathcal{N}$) and $F(p)$, as calculated in the range $[p_m, \overline{p}]$—that is, *by keeping* $\phi_r(p) = 0$ (each $r \in \mathcal{Z} - \mathcal{N}$)—need to be extended somewhat beyond $\overline{p}$. Let us call these extended functions $\phi_l^g(p)$ and $G(p)$, respectively. In the range $[p_m, \widetilde{p}_M^{(n)}]$, $\phi_l^g(p)$ consists of the RHS of Equation (12) and $G(p) = \left[1 - \prod_{j \in \mathcal{N}} \phi_l^g(p) \right] p$. The functions $\phi_l^g(p)$ and $G(p)$ are well-defined in the mentioned range. As we will see, $\overline{p}$ equals the argument of a maximum of $G(p)$ in the range $(p_m, \widetilde{p}_M^{(n)})$. We will show that such a maximum exists, but we were not able to prove that it is unique, even if all our simulations suggest that it is so. That said, we prove that $\overline{p}$ coincides with the largest argument in which such a maximum is obtained.

Proposition 3 *Let* $k_1 + \cdots + k_n > D(p_m)$*. Then* $\overline{p} = max\, \mathrm{arg}max_{p \in (p_m, \widetilde{p}_M^{(n)})} G(p)$ *and*

[11] Obviously the first part is empty also in the case in which $n = 1$.

[12] The sign of its first derivative coincides with the sign of function $p_m \left[\sum_{j \in \mathcal{N}} k_j - D(p) \right] + (p - p_m)\, pD'(p)$ which is decreasing in the mentioned range, is positive for $p = p_m$ and negative for $p = p_M$.

$$\Pi_r^* = \left[1 - \left(\frac{\left(\overline{p} - p_m\right) \left(\prod_{j \in \mathcal{N}} k_j \right)^{\frac{1}{n}}}{\overline{p} \left[\sum_{j \in \mathcal{N}} k_j - D(\overline{p}) \right]} \right)^{\frac{n}{n-1}} \right] \overline{p} k_r. \tag{13}$$

A simple intuition can be gained if we spell out the procedure whereby we have determined $\overline{p}$: $\overline{p}$ is the price that maximizes firm r's payoff function when the strategy profile of the large firms is such as to yield them their equilibrium payoffs when the small firms charge a higher price.

3.2 | The equilibrium distributions in $(\overline{\overline{p}}, p_M]$

In this range, $\phi_r(p) = 1$ for each $r \in \mathcal{Z} - \mathcal{N}$ and Equation (9) can thus be written

$$\Pi_i(p, \phi_{-i}) = pk_i - \prod_{j \in \mathcal{N} - \{i\}} \phi_j(p) p \left[K - D(p) \right]. \tag{14}$$

Taking into account Proposition 2(i) & (iv), these equations are enough to determine all the ϕ_i's in the range $(\overline{\overline{p}}, p_M]$. This is done straightforwardly if $k_2 = k_n$. In this case $[\overline{\overline{p}}, p_M] \subset (\cap_{j \in \mathcal{N}} S_j)$: then it follows from Equation (14) that, for each $i \in \mathcal{N}$,

$$\phi_i(p) = \frac{1}{k_i} \left(\frac{p - p_m}{p} \frac{\prod_{j \in \mathcal{N}} k_j}{K - D(p)} \right)^{\frac{1}{n-1}} \tag{15}$$

throughout $(\overline{\overline{p}}, p_M]$. If, instead, $k_2 > k_n$, then $(\overline{\overline{p}}, p_M]$ can be partitioned in a number of non-empty intervals $(p_M^{(i+1)}, p_M^{(i)}]$, where each $i < n$ is such that $k_i > k_{i+1}$ and, by definition, $p_M^{(n+1)} = \overline{\overline{p}}$. In each range $(p_M^{(i+1)}, p_M^{(i)}]$, $\phi_l(p) = 1$ for $l = i + 1, \ldots, n$; then Equation (14) lead to

$$\phi_l(p) = \frac{1}{k_l} \left(\frac{p - p_m}{p} \frac{\prod_{j \leqslant i} k_j}{K - D(p)} \right)^{\frac{1}{i-1}} \tag{16}$$

for each $l = 1, \ldots, i$. The RHS of Equation (16) (each $l = 1, \ldots, i$) is in fact strictly increasing over the range $(p_M^{(i+1)}, p_M]$, its derivative being strictly decreasing over that range and equal to zero at $p = p_M$: hence, $p_M^{(i)}$ is the unique solution of the equation $\left(p - p_m\right) \prod_{j \leqslant i} k_j = p \left[K - D(p)\right] k_i^{i-1}$ over the range $(p_M^{(i+1)}, p_M]$. Thus $p_M^{(i)} = p_M$ if $k_i = k_2$, since $p_m k_1 = p_M \left[D(p_M) - \sum_j k_{j \neq 1}\right]$; if $k_i < k_2$, then $p_M^{(i)} < p_M$ and $\phi_l(p_M^{(i)}) = \frac{k_i}{k_l} < 1$ for any $l < i$ such that $k_l > k_i$.

Next we prove that any large firm l with $k_l < k_2$ would earn strictly less than Π_l^* by charging any price higher than $p_M^{(l)}$. In the next subsection, we prove that any small firm r would earn strictly less than Π_r^* by charging more than $\overline{\overline{p}}$. This will complete the analysis of the range $(\overline{\overline{p}}, p_M]$.

Proposition 4 For any $l \in \mathcal{N} - \{1,2\}$ such that $k_l < k_2$, $\Pi_l(p, \phi_{-l}) < \Pi_l^*$ over the range $(p_M^{(l)}, p_M]$.

3.3 | The equilibrium distributions in $[\overline{p}, \overline{\overline{p}}]$

Let $p \in \bigcup_{r \in \mathcal{Z}-\mathcal{N}} S_r$. Then we obtain from Equation (10) and Proposition 3 that

$$\prod_{j \in \mathcal{N}} \phi_j(p) = \frac{p - F(\overline{p})}{p}. \tag{17}$$

and, afterwords, from equations (9) and Proposition 2 that

$$\phi_l(p) = \frac{1}{k_l} \frac{p - F(\overline{p})}{p - p_m} \left[\sum_{j \in \mathcal{N}} k_j + \sum_{r \in \mathcal{Z}-\mathcal{N}} \phi_r(p) k_r - D(p) \right] \quad l \in \mathcal{N}. \tag{18}$$

As a consequence, also by using Equation (17) again,

$$\sum_{r \in \mathcal{Z}-\mathcal{N}} \phi_r(p) k_r = \left[\frac{p}{p - F(\overline{p})} \right]^{\frac{n-1}{n}} \frac{p - p_m}{p} \left(\prod_{j \in \mathcal{N}} k_j \right)^{\frac{1}{n}} - \left[\sum_{j \in \mathcal{N}} k_j - D(p) \right]. \tag{19}$$

Finally, from equations (18) and (19) we obtain

$$\phi_l(p) = \frac{1}{k_l} \left[\frac{p - F(\overline{p})}{p} \right]^{\frac{1}{n}} \left(\prod_{j \in \mathcal{N}} k_j \right)^{\frac{1}{n}} \quad l \in \mathcal{N}. \tag{20}$$

Remark 1 By construction the RHS of Equation (12) equals the RHS of Equation (20) for $p = \overline{p}$, whereas it is larger than the latter for $p > \overline{p}$. As a consequence, the RHS of Equation (19) equals zero for $p = \overline{p}$ and is positive for $p > \overline{p}$.

Another remark concerns a constant finding of our simulations, according to which the RHS of Equation (19) is strictly increasing over the relevant subset. Whenever this is the case, Equations (19) and (20) hold throughout $[\overline{p}, \overline{\overline{p}}]$ and $[\overline{p}, \overline{\overline{p}}] = \bigcup_{r \in \mathcal{Z}-\mathcal{N}} S_r$. On the other hand, we have not been able to establish theoretically the generality of the above finding, except for the special case in which $k_1 + \cdots + k_n = D(p_m)$. Nevertheless, a general characterization of equilibria is possible. This will be done in the Mathematical Appendix. The following proposition is stated in the assumption that the RHS of Equation (19) is strictly increasing over the relevant subset. Let us clarify that if the RHS of Equation (19) is *not* so, then there is in $[\overline{p}, \overline{\overline{p}}]$ some interval which is not in the support of any of the small firms and therefore the union of the supports of the small firms is not connected.

Proposition 5 *Let the RHS of Equation (19) be strictly increasing over the range* $(\overline{p}, p_M^{(n)})$. *Then,*

(i) $\overline{\overline{p}}$ is the unique solution of the equation

$$\left[\frac{p - F(\overline{p})}{p}\right]^{\frac{n-1}{n}} = \frac{p - p_m}{p\left[K - D(p)\right]}\left(\prod_{j\in\mathcal{N}} k_j\right)^{\frac{1}{n}} \tag{21}$$

over the range $(\overline{p}, p_M^{(n)})$.

(ii) $\Pi_r(p, \phi_{-r}) < \Pi_r^*$ over the range $(\overline{\overline{p}}, p_M]$, each $r \in \mathcal{Z} - \mathcal{N}$.

Remark 2 There is a continuum of profiles of equilibrium distributions for the small firms, and this is so whether or not the RHS of Equation (19) is strictly increasing over the relevant subset. The continuum of equilibria includes one in which the equilibrium distributions are the same for each small firm: at the "symmetric" equilibrium,

$$\phi_r(p) = \frac{\left[\frac{p}{p-F(\overline{p})}\right]^{\frac{n-1}{n}} \frac{p-p_m}{p}\left(\prod_{j\in\mathcal{N}} k_j\right)^{\frac{1}{n}} - \left[\sum_{j\in\mathcal{N}} k_j - D(p)\right]}{\sum_{r\in\mathcal{Z}-\mathcal{N}} k_r} \tag{22}$$

for any $p \in \bigcup_{r\in\mathcal{Z}-\mathcal{N}} S_r$ (each $r \in \mathcal{Z} - \mathcal{N}$).

Some considerations are in order about the role played by firms $r \in \mathcal{Z} - \mathcal{N}$. Although the total capacity of these firms is fairly small, their impact on the equilibrium may well be sizeable. Simple comparative statics will help to see this point. This will be shown in next section.

4 | THE ROLE OF SMALL FIRMS IS NOT NEGLIGIBLE

In this section, we show that the impact of total capacity of small firms on the equilibrium may well be sizeable. Take the number and capacities of the small firms as an independent variable while keeping fixed the number and capacities of the large firms. Of course, mere reshuffling of capacities among the small firms would not affect S_i, Π_i^* (each $i \in \mathcal{N}$), $\sum_{r\in\mathcal{Z}-\mathcal{N}} \Pi_r^*$, and $\cup S_{r\in\mathcal{Z}-\mathcal{N}}$. On the other hand, there is room for a significant (upward or downward) change in $\xi := \sum_{r\in\mathcal{Z}-\mathcal{N}} k_r$ that does not violate inequalities (5) and (6): any such change would have a considerable impact on the equilibria. The resulting change of the equilibrium payoff is $\Delta\Pi_i^* \approx -\frac{k_i}{k_1} p_M \Delta\xi$ for any large firm; thus, for each large firm, the proportional change in the equilibrium payoff is $\frac{\Delta\Pi_i^*}{\Pi_i^*} \approx -\frac{\Delta\xi}{D(p_M) - \sum_{j\neq 1} k_j}$, which may be far from negligible, as the following example illustrates.

Let $n = 5$, $D(p) = 22 - p$, $k_1 = 9.2$, $k_2 = 8.5$, $k_3 = 6$, $k_4 = 0.4$, $k_5 = 0.2$. Then $p_M = 3.45$, $p_m = 1.29375$, $\Pi_1^* = p_m k_1 = 11.9025$, and $\Pi_2^* = p_m k_2 = 10.996875$. Since $k_1 + k_2 + k_3 = 23.7 > D(p_m) = 20.70625$ and $D(p_M) = 18.55 > K - k_3 = 18.3$, then firms 1, 2, and 3 are "large" firms, consequently $\Pi_3^* = p_m k_3 = 7.7625$, and firms 4 and 5 are "small" firms. Inequality (5) is strict and hence $L = \{1, 2, 3\}$. According to Equation (12), over the range $[p_m, \overline{p}]$, $\phi_1(p) = \frac{1}{9.2}\sqrt{469.2\frac{p - 1.29375}{p(1.7 + p)}}$, $\phi_2(p) = \frac{1}{8.5}\sqrt{469.2\frac{p - 1.29375}{p(1.7 + p)}}$ and $\phi_3(p) = \frac{1}{6}\sqrt{469.2\frac{p - 1.29375}{p(1.7 + p)}}$; hence, $G(p)k_r = p\left[1 - 21.66102489\left(\frac{p-1.29375}{p(1.7+p)}\right)^{\frac{3}{2}}\right]k_r$

$(r = 3, 4)$ over the range $[p_m, \tilde{p}_M^{(3)}] = [1.29375; 1.761639635]$. Then it is found that $\text{arg}max_{p \in (p_m, \tilde{p}_M^{(3)})} G(p) = 1.330357324$, implying that $F(\overline{p}) = 1.305422514$ and hence $\Pi_4^* = F(\overline{p})k_4 = 0.5221690056$ and $\Pi_5^* = F(\overline{p})k_5 = 0.2610845028$. Over the range $\cup_{r \in \{4,5\}} S_r$, $(\phi_4(p), \phi_5(p))$ is any pair of continuous and non-decreasing functions such that Equation (19) holds, namely:

$$0.4\phi_4(p) + 0.2\phi_5(p) = \frac{p - 1.29375}{p} \left(\frac{p}{p - 1.305422514} \right)^{\frac{2}{3}} 469.\,20^{\frac{1}{3}} - 1.7 - p. \tag{23}$$

The RHS of Equation (22) is strictly increasing throughout $[\overline{p}, \tilde{p}_M^{(3)}]$, implying that $\cup_{r \in \{4,5\}} S_r = [\overline{p}, \overline{\overline{p}}]$, where $\overline{\overline{p}} = 1.423433842$, the single value of $p \in [\overline{p}, \tilde{p}_M^{(3)}]$ such that the RHS of (22) is equal to 0.6. According to Equation (20), over the range $[\overline{p}, \overline{\overline{p}}] = [1.330357324; 1.423433842]$, $\phi_1(p) = \frac{1}{9.2}\left(469.2\frac{p-1.305422514}{p}\right)^{\frac{1}{3}}$, $\phi_2(p) = \frac{1}{8.5}\left(469.2\frac{p-1.305422514}{p}\right)^{\frac{1}{3}}$, and $\phi_3(p) = \frac{1}{6}\left(469.2\frac{p-1.305422514}{p}\right)^{\frac{1}{3}}$.

Over the range $(\overline{\overline{p}}, p_M^{(3)}] = (1.423433842; 1.911346695]$, $\phi_1(p) = \frac{1}{9.2}\sqrt{\frac{p-1.29375}{p(2.3+p)}469.2}$, $\phi_2(p) = \frac{1}{8.5}\sqrt{\frac{p-1.29375}{p(2.3+p)}469.2}$ and $\phi_3(p) = \frac{1}{6}\sqrt{\frac{p-1.29375}{p(2.3+p)}469.2}$; $p_M^{(3)} = 1.911346695$. Over the remaining range $(p_M^{(3)}, p_M] = (1.911346695; 3.45]$, $\phi_1(p) = 8.5\frac{p-1.29375}{p(2.3+p)}$ and $\phi_2(p) = 9.2\frac{p-1.29375}{p(2.3+p)}$: of course, $\phi_2(p_M) = 1$ while $\phi_1(p_M) = \frac{k_2}{k_1} = \frac{8.5}{9.2} = 0.9239130437$. Figure 1 provides a graphical representation of one of the equilibria. More specifically, the dashed curve represents the uniform cumulative distribution of the small firms in the "symmetric" equilibrium, in which $\phi_r(p) = \frac{1}{0.6}\left(\frac{p-1.29375}{p}\left(\frac{p}{p-1.305422514}\right)^{\frac{2}{3}} 469.\,20^{\frac{1}{3}} - 1.7 - p\right)$ for $r \in \{4, 5\}$ throughout $[\overline{p}, \overline{\overline{p}}]$; the other curves represent the unique cumulative distributions of the large firms in any equilibrium.

A few variants of this numerical example also allow us to assess the role played by the small firms. Suppose that, other things being equal, the total capacity of the small firms decreases from 0.6 to zero. This would result in a sizeable increase in p_M, p_m, and Π_i^* (each $i \in \mathcal{N}$): Π_1^* would rise to 14.0625, meaning that Π_i^* (each $i \in \mathcal{N}$) would increase approximately by 18.15%. Alternatively, let total capacity of the small firms increase from 0.6 to 1.1. Note that firms 1, 2, and 3 are still "large" firms while the remaining firms are still "small" firms in that inequalities (5) and (6) still hold. By straightforward computation it is found that the equilibrium payoff of firm 1 would now fall to 10.24, meaning a fall by approximately 13.97% for each large firm, compared to the initial industry structure.

Quite interestingly, what is, according to our criterion, the small-firm segment of a bipolarized industry might account for a remarkable share of industry capacity. As before, let $D(p) = 22 - p$, $n = 5$, and $k_1 + k_2 + k_3 = 23.7$ but now let $k_1 = k_2 = k_3 = 7.9$. Now, firms 4 and 5 would be "small" firms even with capacities much closer to k_1 than in previous simulations. Let, for instance, $k_4 = k_5 = 3$. Then $p_M = 0.1$, $p_m = 0.0012658$, $\Pi_1^* = p_m k_1 = 11.9025$, and hence $k_1 + k_2 + k_3 = 23.7 > D(p_m) = 21.9987$ and $D(p_M) = 21.9 > K - k_3 = 21.8$: as before, inequalities (5) and (6) both hold: firms 1, 2, and 3 are large firms while firms 4 and 5 are small firms. But now small firms are more than one third as large as the largest firms; and the small-firm segment accounts for 20% of industry capacity. Thus, a "bipolarized" industry structure need not involve that the small-firm segment is a "fringe" or that each small firm is negligible.

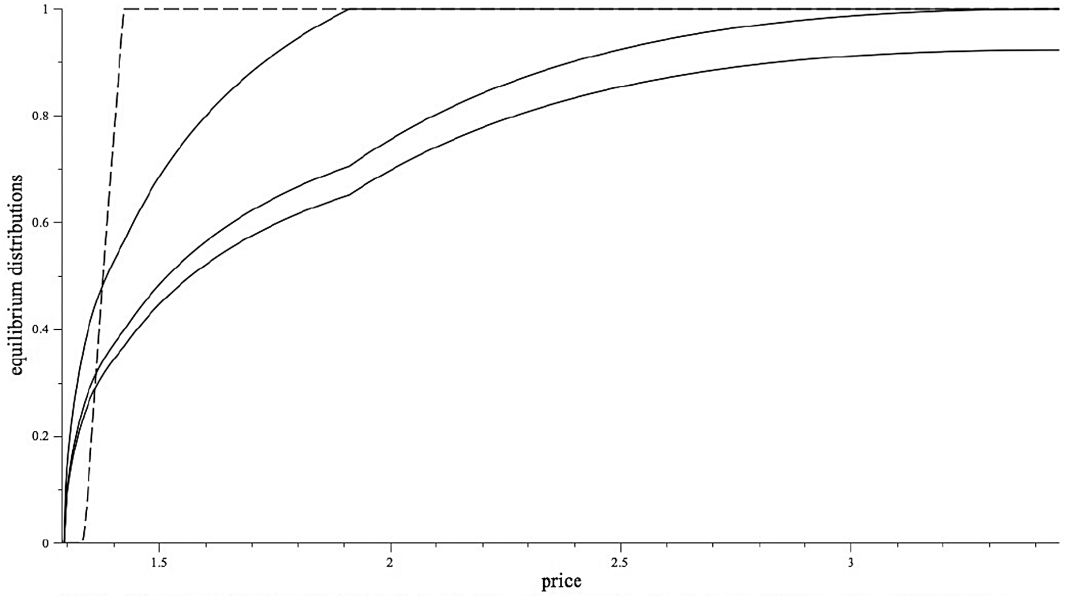

FIGURE 1 The cumulative distributions in the symmetric equilibrium

5 | THE REGION OF NO PURE STRATEGY EQUILIBRIA INVESTIGATED IN THIS PAPER IS NOT SMALL

In this section, we show that the part of the region of no pure strategy equilibria in which inequalities (5) (6) hold can be quite large indeed. In order to do so and to represent our data in a plane, we consider a subset of the part of the region of no pure strategy equilibria in which inequalities (5) (6) hold, and precisely the subset in which $k_{n+1} + \cdots + k_z = \frac{1}{10}K$ and $n = 2$, so that $k_n = k_2 = \frac{9}{10}K - k_1$. As a consequence, inequalities (5) (6) can be represented in terms of K and k_1 only:

$$\frac{9}{10}K \geqslant D(p_m) \tag{24}$$

$$D(p_M) \geqslant \frac{1}{10}K + k_1. \tag{25}$$

since p_M and p_m are determined by K and k_1 (and the demand function) only.

Figure 2 represents a partition of the space in the case in which the demand is $D(p) = 1 - p$ (and therefore $p_M = \frac{1-K+k_1}{2}$ and $p_m = \frac{(1-K+k_1)^2}{4k_1}$) and $z = 25$. K is on the horizontal axis and k_1 is on the vertical axis. Of course the whole space is below the 45° line and either above or along the straight line $k_1 = \frac{1}{z}K$. The portions of space A and B are the regions in which pure strategy equilibria exist. Portion C_1, above the curve $k_1 = \frac{1}{5}[1 + K + 2\sqrt{3K - K^2 - 1}$, is the subset of the region of no pure strategy equilibria in which $K_1 > D(p_m)$. Portion C_2, above or along the curve $k_1 = 1 - \frac{4}{5}K + 2\sqrt{\frac{1}{10}K - \frac{9}{100}K^2}$ and below or along the straight line $k_1 = \frac{1}{3} + \frac{4}{15}K$, is the subset of

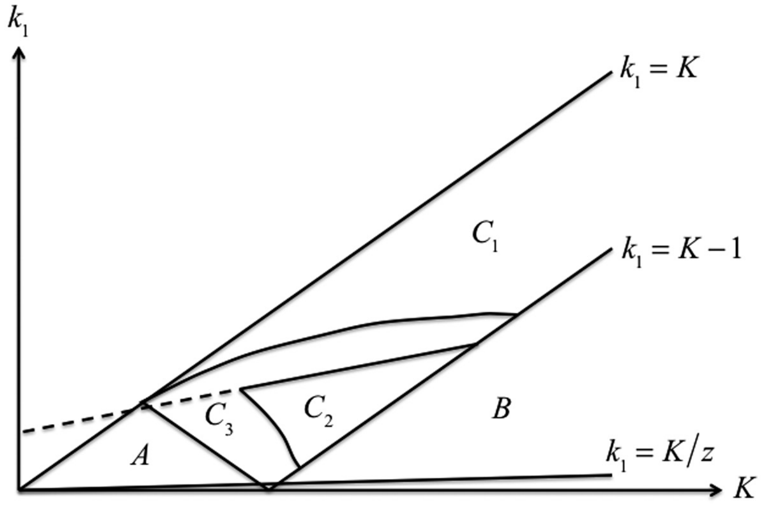

FIGURE 2 A partition of the space

the region of no pure strategy equilibria in which inequalities (23) (24) hold. Portion C_3 is the remaining part of the region of no pure strategy equilibria.

6 | CONCLUDING REMARKS

This paper is a further contribution to the analysis of equilibria of the price game in a setting of given capacities. We in fact characterized the equilibria in a specific subset of the no-pure strategy equilibrium region of the capacity space, the subset where, according to a well-defined distinction, there are "large" firms along with "small" firms. It was found that, with an industry structure like this, the interval between the minimum price p_m and maximum price p_M being quoted in equilibria can be partitioned into three intervals, $[p_m, \overline{p})$, $[\overline{p}, \overline{\overline{p}}]$, and $(\overline{\overline{p}}, p_M]$, where $\overline{p}$ and $\overline{\overline{p}}$ are, respectively, the minimum and the maximum of the union of the supports of the small firms. The first part is empty in a limit case, whereas the other two are never so. We determined the equilibrium payoffs for all firms and we saw that, for firms of the same type, the equilibrium payoffs are proportional to capacities. Except in the limit case in which $\overline{p} = p_m$, the equilibrium payoff per unit of capacity is larger for the small firms and we saw that $\overline{p}$, and correspondingly the equilibrium payoff of each small firm, is the solution of a maximization problem facing any small firm. Finally, although a continuum of equilibrium distributions exists for the small firms, the capacity-weighted sum of these distributions is the same at each equilibrium and hence the union of the supports of their equilibrium strategies is also the same.

To conclude, there is undoubtedly still a long way to go before the equilibria of the price game among capacity-constrained sellers across the whole region of no-pure strategy equilibria are characterized. Yet it is encouraging that such a task could be performed for the bipolarized distribution of total capacity assumed in the present paper. It seems reasonable to expect that the findings obtained—most notably, the procedure to determine the equilibrium payoff and the minimum price for the relatively small firms—may also be helpful to characterize equilibria in parts of that region that lie somewhere in between the "quasi-symmetric" case (De Francesco & Salvadori, 2011) and the bipolarized industry structure of this paper.

ACKNOWLEDGEMENTS

We are grateful to two anonymous referees for their suggestions to improve the paper. Special thanks go to Elmar Wolfstetter, who supervised this submission and did it off the platform generally used to ensure the anonymity of the referees. Open Access Funding provided by Universita degli Studi di Pisa.

CONFLICT OF INTEREST

No conflict of interest nor competing interests were detected.

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How to cite this article: De Francesco, M. A. & Salvadori, N. (2022). Bertrand–Edgeworth oligopoly: Characterization of mixed strategy equilibria when some firms are large and the others are small. *Metroeconomica*, 73, 803–824. https://doi.org/10.1111/meca.12382

APPENDIX A

MATHEMATICAL APPENDIX

Proof of Proposition 1

(i) The contents of this part has already been established in the recent literature: see, for example, Claim 1 in Hirata (2009).[13]

(ii) If $D(p_m) > \sum_{i\in\mathcal{L}} k_i$, then $\Pi_i(p,\phi_{-i}) = pk_i > p_m k_i = \Pi_i^*$ (each $i \in \mathcal{L}$) for p larger than and close enough to p_m. If $D(p_m) = \sum_{i\in\mathcal{L}} k_i$, then $p_m k_1 \geqslant \prod_{i\in\mathcal{L}-\{1\}} \phi_i(p)[D(p) - \sum_{i\in\mathcal{L}-\{1\}} k_j] + (1 - \prod_{i\in\mathcal{L}-\{1\}} \phi_i(p))pk_1$ for p larger than and close enough to p_m. This implies a contradiction since $\prod_{i\in\mathcal{L}} \phi_i(p) \geqslant \frac{(p-p_m)k_j}{p[\sum_{i\in\mathcal{L}} k_i - D(p)]} > 1$ for p larger than and close enough to p_m, since $lim_{p\to p_m+} \frac{(p-p_m)k_1}{p[\sum_{i\in\mathcal{L}} k_j - D(p)]} = \frac{k_1}{-p_m D'(p_m)} > 1$: indeed, $p_m k_1 > -p^2 D'(p)$ over the range $[p_m, p_M)$, since $-p^2 D'(p)$ is strictly increasing over that range, because of strict concavity of $pD(p)$, and $[-p^2 D'(p)]_{p=p_M} = p_m k_1$. The last equality derives from equalities $[D(p) - \sum_{j\neq 1} k_j + pD'(p)]_{p=p_M} = 0$ and $p_m k_1 = p_M[D(p_M) - \sum_{i\neq 1} k_i]$ because of part (i). If $\Pr_{\phi_i}(p_i = p_m) > 0$ for some $i \in \mathcal{L}$, then $\Pi_j(p,\phi_{-j}) < \Pi_j(p_m,\phi_{-j})$ (any $j \in \mathcal{L} - \{i\}$) for p larger than and close enough to p_m, and hence $p \notin (\cup S_{j\in\mathcal{L}-\{i\}})$. This means that $\Pr_{\phi_j}(p_j = p_m) > 0$ (each $j \in \mathcal{L} - \{i\}$), in its turn implying that $\Pi_j^* = \Pi_j(p_m,\phi_{-j}) < p_m k_j = lim_{p\to p_m^-} \Pi_j(p,\phi_{-j})$. Finally, since $D(p_m) > D(p_M) > \sum_{j\neq 1} k_j \geq \sum_{i:i\in\mathcal{L},k_i<k_1} k_i$ and given that $\Pr_{\phi_i}(p_i = p_m) = 0$ (each $i \in \mathcal{L}$), it follows that $\Pi_i^* = p_m k_i$ for each $i \in \mathcal{L} - \{1\}$.

(iii) Without loss of generality, let $p_m^{(1)} = p_m$, $p_M^{(1)} = p_M$, $\Pi_1^* = p_M(D(p_M) - \sum_{j\neq 1} k_j)$ and, by way of contradiction, let $p_m^{(i)} > p_m$ for some i such that $k_i = k_1$. Then, since $D(p_M) - \sum_{j\neq 1} k_j > 0$, it would be $D(p_m) > \sum_{j\in\mathcal{L}} k_j$ contrary to part (i). Thus, $p_m^{(i)} = p_m$ and hence $\Pi_i^* = \Pi_1^* = p_m k_1$. This in its turn implies that $p_M^{(i)} = p_M$: if not, then, $\phi_i(p) = 1 > \phi_1(p)$ for any $p \in S_1 \cap [p_M^{(i)}, p_M)$, but then from $\Pi_1(p,\phi_{-i}) = \Pi_1^*$ it would follow $\Pi_i(p,\phi_{-i}) > \Pi_i^*$, an obvious contradiction. And, of course, $\Pr_{\phi_i}(p_i = p_M) = 0$ for any i such that $k_i = k_1$, for otherwise $\Pi_i^* > p_M(D(p_M) - \sum_{j\neq 1} k_j)$.

(iv) Let $k_1 > k_2$. If $\Pr_{\phi_1}(p_1 = p_M) = 0$, then $\Pi_i^* = \Pi_i(p_M,\phi_{-i}) = p_M\, max\{D(p_M) - \sum_{j\neq i} k_j, 0\}$ for $i \in \mathcal{M} - \{1\}$. We are already done if $D(p_M) - \sum_{j\neq i} k_j \leq 0$. If $D(p_M) - \sum_{j\neq i} k_j > 0$, then firm i has failed to make a best response since $p[D(p) - \sum_{j\neq i} k_j]$ is a decreasing function for p less than and close enough to p_M.

[13] In De Francesco (2003) this statement was proved under the assumption that $D''(p) \leq 0$ over the range $[0,\overline{p}]$. The proof in Hirata (2009) relies upon the weaker assumption that $pD(p)$ is strictly concave over the range in which $D(p)$ is positive $[0, p]$, which assures that $\arg max_p p(D(p) - \sum_{j\neq 1} k_j)$ is a singleton.

Proof of Proposition 2

(i) Since $\sum_{i\in\mathcal{L}} k_i > D(p_m) > D(p_M) \geqslant K - k_n$ because of Proposition 1(ii), inequality $p_m < p_M$, and inequality (6), if $\mathcal{L} \not\supseteq \mathcal{N}$, and therefore $K - k_n \geqslant \sum_{i\in\mathcal{L}} k_i$, then a contradiction is obtained. $\mathcal{L} \supseteq \mathcal{N}$ implies $\Pi_i^* = p_m k_i$ (each $i \in \mathcal{N}$), because of Proposition 1(ii) and inequalities $D(p_m) > D(p_M) \geqslant K - k_n > k_1$; the last inequality being a consequence of $n > 1$.[14] Hence

$$\prod_{j\in\mathcal{N}-\{i\}} \phi_j(p) = \frac{(p-p_m)k_i}{p\left[\sum_{j\in\mathcal{N}} k_j + \sum_{r\in\mathcal{Z}-\mathcal{N}} \phi_r(p)k_r - D(p)\right]}, \forall p \in S_i \tag{A1}$$

because of Equation (9). As a consequence, for any $p \in S_i \cap S_j (i, j \in \mathcal{N})$, $\phi_i(p)k_i = \phi_j(p)k_j$. Moreover, from equations (9), we obtain that, almost everywhere throughout $[p_m, p_M]$,

$$\frac{\Pi_j(p, \phi_{-j})}{k_j} - \frac{\Pi_i(p, \phi_{-i})}{k_i} = \left[\frac{1}{\phi_i(p)k_i} - \frac{1}{\phi_j(p)k_j}\right] A(p)$$

where

$$A(p) = p\prod_{l\in\mathcal{N}} \phi_l(p)\left[\sum_{l\in\mathcal{N}} k_l + \sum_{r\in\mathcal{Z}-\mathcal{N}} \phi_r(p)k_r - D(p)\right].$$

(ii) Equations (10) imply that $\Pi_r(p, \phi_{-r})k_s = \Pi_s(p, \phi_{-s})k_r$ (any $r, s \in \mathcal{Z} - \mathcal{N}$), almost everywhere throughout $[p_m, p_M]$.[15] Then the claim follows straightforwardly. Indeed, if $\Pi_r^*/k_r < \Pi_s^*/k_s$, then $S_r \cap S_s = \emptyset$ since, at any $p \in S_r \cap S_s$, $\Pi_r(p, \phi_{-r}(p)) = \Pi_r^*$ and $\Pi_s(p, \phi_{-s}(p)) = \Pi_s^*$; but then firm r's strategy would not be a best response to ϕ_{-r}, since a payoff of $\Pi_r(p, \phi_{-r}) = (k_r/k_s)\Pi_s^* > \Pi_r^*$ is obtained by quoting any $p \in S_s$.

(iii) If $\mathcal{L} \supset \mathcal{N}$, then, by Proposition 1(ii) and part (ii), $\Pi_r^* = p_m k_r$ (each $r \in \mathcal{Z} - \mathcal{N}$). Then, again by Proposition 1(ii), $p \in S_r$ (some $r \in \mathcal{L} - \mathcal{N}$) for p larger than and close enough to p_m. Hence Equation (10) implies $\prod_{i\in\mathcal{N}} \phi_i(p) = \frac{p-p_m}{p}$. Thus, on a right neighborhood of p_m, $\sum_{j\in\mathcal{N}} k_j + \sum_{r\in\mathcal{Z}-\mathcal{N}} \phi_r(p)k_r - D(p) = \phi_i(p)k_i$, because of Equation (25) (each $i \in \mathcal{N}$); but then it follows from $lim_{p\to p_m+}\phi_i(p) = 0$ that $lim_{p\to p_m+}\sum_{r\in\mathcal{Z}-\mathcal{N}} \phi_r(p)k_r < 0$. Thus $\mathcal{L} = \mathcal{N}$.[16] Further, since $lim_{p\to p_m+} F'(p) = 1$, $F(p)$ is increasing on a right neighborhood of p_m. As a consequence, $\Pi_r^* > p_m k_r$ (any $r \in \mathcal{Z} - \mathcal{N}$). Otherwise firm r would have failed to make a best response given that $\Pi_r(p, \phi_{-r}) > p_m k_r$ for p close enough to p_m.

(iv) If $\mathcal{L} = \mathcal{N}$, then Proposition 1(ii) is contradicted. Therefore, by Proposition 1(ii) and part (ii), $\Pi_r^* = p_m k_r$ (each $r \in \mathcal{Z} - \mathcal{N}$).

(v) The first of the two claims is obviously equivalent to:

$$S_{n-u} = \left[p_m, p_M^{(n-u)}\right] = \cap_{h\in[1,\ldots,n-u]} S_h \qquad u = 0, 1, \ldots, n-2 \tag{A2}$$

[14]If $n = 1$, $\Pi_1^* \leqslant p_m k_1$; the equality holds only when inequality (5) is satisfied as an equality; the whole part (i) collapses to $\mathcal{L} \supseteq \mathcal{N}$.

[15]The argument in the text would work even if, contrary to part (vi), not yet proved, $\Pr_{\phi_i}(p_i = p^\circ) > 0$ (some $p^\circ \in S_i$ and some $i \in \mathcal{N}$), except for $\Pi_r(p^\circ, \phi_{-r})$ and $\Pi_s(p^\circ, \phi_{-s})$ being replaced by $lim_{p\to p^\circ-}\Pi_r(p, \phi_{-r})$ and $lim_{p\to p^\circ-}\Pi_s(p, \phi_{-r})$, respectively.

[16]If $n = 1$ Equation (25) does not hold (unless $D(p_m) = k_1$), and indeed $\mathcal{L} \supset \mathcal{N} = \{1\}$ because of Proposition 1(i).

Property (A2) will be proved by induction. Let us first prove that it holds for $u = 0$. Because of part (i) and Proposition 1(ii), there is $\tilde{p}$ such that $[p_m, \tilde{p}] \subseteq \cap_{i\in\mathcal{N}} S_i$. Note that $\frac{d\Pi_i(p,\phi_{-i})}{dp} = \frac{\partial\Pi_i(p,\phi_{-i})}{\partial p} + \sum_{j\in\mathcal{Z}-\{i\}} \frac{\partial\Pi_i(p,\phi_{-i})}{\partial\phi_j}\phi'_j(p) = 0$ in the range $[p_m, \tilde{p}]$, and therefore in the same range $\frac{\partial\Pi_i(p,\phi_{-i})}{\partial p} > 0$, since

$$\frac{\partial\Pi_i(p,\phi_{-i})}{\partial\phi_j} = p\prod_{h\in\mathcal{N}-\{i,j\}}\phi_h(p)\left[D(p) - \sum_{h\in\mathcal{N}}k_h - \sum_{r\in\mathcal{Z}-\mathcal{N}}\phi_r(p)k_r\right] < 0,$$

if $j \in \mathcal{N} - \{i\}$ and $\frac{\partial\Pi_i(p,\phi_{-i})}{\partial\phi_j} = -pk_j \prod_{h\in\mathcal{N}-\{i\}}\phi_h(p) < 0$, if $j \in \mathcal{Z} - \mathcal{N}$. If there is $\tilde{\tilde{p}} > \tilde{p}$ such that $(\tilde{p}, \tilde{\tilde{p}}) \cap (\cap_{i\in\mathcal{N}} S_i) = \emptyset$, then either (a) $\tilde{p} = p_M^{(n)}$, or (b) $\Pr_{\phi_i}(p_i = \tilde{p}) > 0$ for some $i \in \mathcal{Z}$, or (c) there is a gap $(\tilde{p}, p^\circ)$ in S_j (some $j \in \mathcal{N}$ and some $p^\circ \geqslant \tilde{\tilde{p}}$): namely, $\phi_j(p^\circ) = \phi_j(\tilde{p})$, while $\phi_j(p)$ is increasing in both $\tilde{p}$ and p°. Obviously property (A2) for $u = 0$ holds in event (a). Let us first exclude event (b). By way of contradiction, let $\Pr_{\phi_r}(p_r = \tilde{p}) > 0$ (some $r \in \mathcal{Z} - \mathcal{N}$). As a consequence, there is $p^\circ \in (\tilde{p}, p_M)$ such that $(\tilde{p}, p^\circ) \cap (\cup_{j\in\mathcal{N}} S_j) = \emptyset$, since $lim_{p\to\tilde{p}+}\Pi_j(p,\phi_{-j}) < lim_{p\to\tilde{p}-}\Pi_j(p,\phi_{-j}) = \Pi_j^*$, each $j \in \mathcal{N}$. But then it follows from Equation (10) that $\Pi_r(p,\phi_{-r}) > \Pi_r(\tilde{p},\phi_{-r}) = \Pi_r^*$ over the range $(\tilde{p}, p^\circ)$. Quite similarly, if $\Pr_{\phi_i}(p_i = \tilde{p}) > 0$ (some $i \in \mathcal{N}$), then $(\tilde{p}, p^\circ) \cap (\cup_{j\in\mathcal{Z}-\{i\}} S_j) = \emptyset$ for some $p^\circ \in (\tilde{p}, p_M)$, but then $\Pi_i(p,\phi_{-i}) > \Pi_i(\tilde{p},\phi_{-i}) = \Pi_i^*$ on the right of $\tilde{p}$. Let us now exclude event (c). If there is $p^{\circ\circ} \in (\tilde{p}, p^\circ]$ such that $(\tilde{p}, p^{\circ\circ}) \cap (\cup_{i\in\mathcal{Z}} S_i) = \emptyset$, the same argument applies. Then there is $h \neq j$ such that $(\tilde{p}, p^\circ) \cap S_h \neq \emptyset$ and $\phi_j(p^\circ)k_j = \phi_j(\tilde{p})k_j = \phi_h(\tilde{p})k_h < \phi_h(p^\circ)k_h$, and therefore $\Pi_j(p^\circ,\phi_{-j}) < \frac{k_j}{k_h}\Pi_h(p^\circ,\phi_{-h}) \leqslant \frac{k_j}{k_h}\Pi_h^* = \Pi_j^*$, contrary to the fact that $p^\circ \in S_j$. Now assume that property (A2) holds for $u = v < n-2$; then there is $\tilde{p} \geqslant p_M^{(n-v)}$ such that $[p_m, \tilde{p}] = \cap_{h\in\mathcal{N};h\leqslant n-v-1} S_h$. Hence the same argument used above proves that property (A2) holds for $u = v + 1$. Note that $p_M^{(n-v)} = p_M^{(n-v-1)}$ if and only if $k_{n-v-1} = k_{n-v}$, because of part (i).

(vi) It is an obvious consequence of part (v).

(vii) By way of contradiction, let $\bar{\bar{p}} > p_M^{(n)}$.[17] Then $\phi_1(\bar{\bar{p}}) > \phi_1(p_M^{(n)}) = \frac{k_n}{k_1}$, the equality being a consequence of part (i). Therefore,

$$\prod_{j\in\mathcal{N}}\phi_j(\bar{\bar{p}}) = \phi_1(\bar{\bar{p}})\frac{\bar{\bar{p}} - p_m}{\bar{\bar{p}}}\frac{k_1}{K - D(\bar{\bar{p}})} > \frac{\bar{\bar{p}} - p_m}{\bar{\bar{p}}}\frac{k_n}{K - D(\bar{\bar{p}})} \geqslant \frac{\bar{\bar{p}} - p_m}{\bar{\bar{p}}}: \tag{A3}$$

the equality is a consequence of Equation (9) and part (i) since $\sum_{r\in\mathcal{Z}-\mathcal{N}}\phi_r(\bar{\bar{p}})k_r = \sum_{r\in\mathcal{Z}-\mathcal{N}}k_r$; the second inequality is a consequence of inequality (6). Thus $\Pi_r(\bar{\bar{p}},\phi_{-r})) < p_m k_r$ because of Equation (10) and the definition of $\bar{\bar{p}}$ is contradicted.

(viii) By way of contradiction, let $\bar{\bar{p}} = p_M^{(n)}$. Then instead of (A3) we have[18]

[17] If $n = 1$, $\bar{\bar{p}} > p_M^{(n)} = p_M^{(1)}$ contradicts Proposition 1(i), and therefore the statement holds, but the proof provided in the text does not apply since part (i) does not hold and therefore the equality (A3) is not satisfied in general; it is, of course, when $D(p_m) = k_1$.

[18] If $n = 1$, the first equality (A4) does not hold, unless $D(p_m) = k_1$ (see previous footnote); this time also the statement is false since $\bar{\bar{p}} = p_M^{(n)} = p_M^{(1)}$ because of Proposition 1(i).

$$\prod_{j\in\mathcal{N}}\phi_j(\bar{\bar{p}})=\phi_1(\bar{\bar{p}})\frac{\bar{\bar{p}}-p_m}{\bar{\bar{p}}}\frac{k_1}{K-D(\bar{\bar{p}})}=\frac{\bar{\bar{p}}-p_m}{\bar{\bar{p}}}\frac{k_n}{K-D(\bar{\bar{p}})}\geqslant\frac{\bar{\bar{p}}-p_m}{\bar{\bar{p}}}. \tag{A4}$$

It follows that $\Pi_r(\bar{\bar{p}},\phi_{-r}))\leqslant p_m k_r$. Hence, if inequality (5) holds as a strict inequality, part (iii) is contradicted; if either inequality (6) holds as a strict inequality or $k_2>k_n$ (or both), then the weak inequality in (A4) is satisfied as a strict inequality and hence $\Pi_r(\bar{\bar{p}},\phi_{-r}))<p_m k_r$. Finally, it follows from $\bar{\bar{p}}<p_M^{(n)}$ that

$$\prod_{j\in\mathcal{N}}\phi_j(p_M^{(n)})=\phi_1(p_M^{(n)})\frac{(p_M^{(n)}-p_m)k_1}{p_M^{(n)}\left[K-D(p_M^{(n)})\right]}=\frac{(p_M^{(n)}-p_m)k_n}{p_M^{(n)}\left[K-D(p_M^{(n)})\right]}\geqslant\frac{p_M^{(n)}-p_m}{p_M^{(n)}},$$

implying that $\Pi_r(p_M^{(n)},\phi_{-r}))\leqslant p_m k_r\leqslant\Pi_r^*$, with at least one strict inequality: the last inequality is strict if inequality (5) is strict and the first inequality is strict if either $k_2>k_n$, or inequality (6) is strict (or both).

Proof of Proposition 3 Since $G(\tilde{p}_M^{(n)})=\left[1-\prod_{j\in\mathcal{N}}\phi_j^g(\%\tilde{p}_M^{(n)})\right]\tilde{p}_M^{(n)}<p_m$, as can easily be checked, $G(p)$ has a maximum at some $p\in(p_m,\tilde{p}_M^{(n)})$. By way of contradiction, let $G(p)\geqslant F(\bar{p})$ for some $p\in(\bar{p},\tilde{p}_M^{(n)})$. Then, $\left[1-\prod_{j\in\mathcal{N}}\phi_j(p)\right]p\leqslant F(\bar{p})=G(\bar{p})\leqslant G(p)=\left[1-\prod_{j\in\mathcal{N}}\phi_j^g(p)\right]p$, where the first weak inequality is certainly an equality for $p\in\cup S_r$. Therefore, $\prod_{j\in\mathcal{N}}\phi_j(p)\geqslant\prod_{j\in\mathcal{N}}\phi_j^g(p)$ and, as a consequence of Equation (A1) and the definition of functions $\phi_j^g(p)$'s, $\sum_{r\in\mathcal{Z}-\mathcal{N}}\phi_r(p)k_r\leqslant 0$, an obvious contradiction. Next, again by way of contradiction, let $G(p)>G(\bar{p})$ for some $p\in(p_m,\bar{p})$. Under such an event, firm r would get $G(p)k_r>\Pi_r^*=G(\bar{p})k_r$ by charging a price somewhat less than $\bar{p}$. Finally, Equation (13) derives straightforwardly from $\Pi_r^*=F(\bar{p})k_r$ (each $r\in\mathcal{Z}-\mathcal{N}$) and Equation (12).

Proof of Proposition 4 It is enough to remark that over any non-empty range $(p_M^{(i+1)},p_M^{(i)}]$, $\Pi_l(p,\phi_{-l})/k_l<\Pi_i(p,\phi_{-i})/k_i=p_m$ for any $l\geqslant i+1$, since $\phi_i(p)>\frac{k_l}{k_i}$.

Proof of Proposition 5

(i) By definition $\bar{\bar{p}}$ is the unique solution to Equation (21) and $\bar{p}<\bar{\bar{p}}<p_M^{(n)}$ because of Proposition 2(v)&(viii).

(ii) Since $\sum_{r\in\mathcal{Z}-\mathcal{N}}\phi_r(p)k_r=\sum_{r\in\mathcal{Z}-\mathcal{N}}k_r$ is lower than the RHS of Equation (19) over the range $(\bar{\bar{p}},p_M^{(n)}]$, over the same range $\phi_l(p)$ is larger than the RHS of Equation (20), each $l\in\mathcal{N}$, and, as a consequence, $F(p)<F(\bar{p})$. If $k_n<k_2$, so that $p_M^{(n)}<p_M$, then $F(p)=p\left[1-\phi_1(p)\frac{p-p_m}{p}\frac{k_1}{K-D(p)}\right]<p_m<F(\bar{p})$ over the range $(p_M^{(n)},p_M)$. The first inequality derives from $\phi_1(p)>\phi_1(p_M^{(n)})=\frac{k_n}{k_1}>\frac{K-D(p)}{k_1}$, whereas the last inequality holds since inequality (5) is strict and Proposition 2(iii) holds.

As mentioned in the main text, we have not proved that the RHS of Equation (19) is strictly increasing over the relevant subset. For this reason we establish here the following results, that complete Proposition 5.

Proposition A1 *If $k_1+\cdots+k_n>D(p_m)$, in any equilibrium*

(i) $\bar{\bar{p}}$ is the largest solution of the Equation (21) over the range $(\bar{p},p_M^{(n)})$;

(ii) the set of equilibrium distributions of the small firms is any set of non-negative, continuous and non-decreasing functions no larger than 1 such that

$$\sum_{r\in\mathcal{Z}-\mathcal{N}}\phi_r(p)k_r = min_{y\in[\underline{p},\overline{\overline{p}}]}\left\{\left[\frac{y}{y-F(\overline{p})}\right]^{\frac{n-1}{n}}\frac{y-p_m}{y}\left(\prod_{j\in\mathcal{N}}k_j\right)^{\frac{1}{n}}-\left[\sum_{j\in\mathcal{N}}k_j-D(y)\right]\right\} \tag{A5}$$

over the range $[\overline{p},\overline{\overline{p}}]$;

(iii) the equilibrium distributions of the large firms are uniquely determined by the equations

$$\phi_i(p)=\frac{1}{k_i}\left(\frac{p-p_m}{p}\frac{\prod_{j\in\mathcal{N}}k_j}{\sum_{j\in\mathcal{N}}k_j+\sum_{r\in\mathcal{Z}-\mathcal{N}}\phi_r(p)k_r-D(p)}\right)^{\frac{1}{n-1}} \tag{A6}$$

over the range $[\overline{p},\overline{\overline{p}}]$;

(iv) $\Pi_r(p,\phi_{-r})<\Pi_r^*$ over the range $(\overline{\overline{p}},p_M]$, each $r\in\mathcal{Z}-\mathcal{N}$.

(v) If $k_1+\cdots+k_n=D(p_m)$, then the RHS of Equation (19) is increasing in the whole range $[\overline{p},\overline{\overline{p}}]$, so that $[\overline{p},\overline{\overline{p}}]=\bigcup_{r\in\mathcal{Z}-\mathcal{N}}S_r$,[19] and $\overline{\overline{p}}<p_M^{(n)}\leqslant p_M$; $\overline{\overline{p}}$ is the single solution of the equation

$$\left(\frac{p-p_m}{p}\prod_{j\in\mathcal{N}}k_j\right)^{\frac{1}{n}}-[K-D(p)]=0 \tag{A7}$$

over the range $(\overline{p},p_M^{(n)})$; the set of equilibrium distributions of the small firms is, over the range $[\overline{p},\overline{\overline{p}}]$, any set of non-negative, continuous and non-decreasing functions no larger than 1 such that

$$\sum_{r\in\mathcal{Z}-\mathcal{N}}\phi_r(p)k_r=\left(\frac{p-p_m}{p}\prod_{j\in\mathcal{N}}k_j\right)^{\frac{1}{n}}-\left[\sum_{j\in\mathcal{N}}k_j-D(p)\right]; \tag{A8}$$

the equilibrium distributions of the large firms are uniquely determined by the Equation (A6) over the range $[\overline{p},\overline{\overline{p}}]$; $\Pi_r(p,\phi_{-r})<\Pi_r^*$ over the range $(\overline{\overline{p}},p_M]$, each $r\in\mathcal{Z}-\mathcal{N}$.

Proof (i) By definition $\overline{\overline{p}}$ is a solution to Equation (21) and $\overline{p}<\overline{\overline{p}}<p_M^{(n)}$ because of Proposition 2(v)&(viii). Note, furthermore, that the RHS of (21) is lower (higher) than the LHS at any p where the RHS of (19) is lower (higher) than $\sum_{j\in Z-\mathcal{N}}k_r$. Over $(\overline{p},p_M^{(n)})$ Equation (21) has an odd number of solutions. Indeed, since the RHS of Equation (19) equals zero at $\overline{p}$ (Remark 1), the RHS of Equation (21) is smaller than the LHS at $\overline{p}$ too. On the other side, the RHS of Equation (21) is larger than the LHS at $p_M^{(n)}$. In order to recognize this fact, we obtain from equations (11) and (16) for $i=n$, that

[19]Equality $\bigcup_{r\in\mathcal{Z}-\mathcal{N}}S_r=[p_m,p_M]$ is easily proved if $n=1$. In such a case, $\sum_{r\in\mathcal{Z}-\mathcal{N}}\phi_r(p)k_r=D(p)-\Pi_1^*/p$ over the range $[p_m,p_M]$, where $\Pi_1^*=p_mD(p_m)$: indeed, by the strict concavity of $pD(p)$, $\frac{pD(p)-p_mD(p_m)}{p}$ is strictly increasing.

$$F(p_M^{(n)}) = \left[1 - \frac{1}{\prod_{j\in\mathcal{N}} k_j}\left(\frac{p_M^{(n)} - p_m}{p_M^{(n)}}\frac{\prod_{j\in\mathcal{N}} k_j}{K - D(p_M^{(n)})}\right)^{\frac{n}{n-1}}\right] p_M^{(n)}$$

and since $F(p_M^{(n)}) < F(\overline{p})$ because of Proposition 2(viii), we obtain that the RHS of Equation (21) is larger than the LHS at $p_M^{(n)}$.

Let us say that a solution is odd if there is a left neighborhood in which the RHS of Equation (21) is smaller than the LHS, whereas a solution is even if there is a left neighborhood in which the RHS of Equation (21) exceeds the LHS. Let p' be an odd solution differing from the largest one and p'' be the lowest even solution larger than p'. Clearly, $\overline{\overline{p}} \neq p''$ since the RHS of Equation (19) is decreasing for p less than and close enough to p'' whereas, because of Proposition 2(vi), $p \in \bigcup_{r\in\mathcal{Z}-\mathcal{N}} S_r$ on some left neighborhood of $\overline{\overline{p}}$. Nor can it be that $\overline{p} = p'$. Under such an event, $\sum_{r\in\mathcal{Z}-\mathcal{N}} \phi_r(p)k_r = \sum_{r\in\mathcal{Z}-\mathcal{N}} k_r$ is larger than the RHS of Equation (19) in a right neighborhood of p'' that is part of $\bigcap_{i\in\mathcal{N}} S_i$ (see Proposition 2(v)-(vi)) and therefore $\phi_j(p)$ is lower than the RHS of Equation (20) (each $j \in \mathcal{N}$), but then $F(p) = F(\overline{p})$, an obvious contradiction.

(ii) and (iii) The RHS of Equation (A5) is a non-decreasing function that equals 0 at $\overline{p}$, also because of Remark 1, and equals $\sum_{r\in\mathcal{Z}-\mathcal{N}} k_r$ at $\overline{\overline{p}}$. Whenever the RHS of Equation (A5) is increasing, it equals the RHS of Equation (19) and the RHS of Equation (A6) equals the RHS of Equation (20). Therefore, $F(p) = F(\overline{p})$ whenever the RHS of Equation (A5) is increasing. Over any range $(p', p'') \subset [\overline{p}, \overline{\overline{p}}]$ in which the RHS of Equation (A5) is constant, it is lower than the RHS of Equation (19) and the RHS of Equation (A6) is higher than the RHS of Equation (20). Therefore $F(p) < F(\overline{p})$, consistent with the fact that $(p', p'') \cap (\cup_{r\in\mathcal{Z}-\mathcal{N}} S_r) = \emptyset$.

(iv) By exploring the proof of part (i) we obtain that $\sum_{r\in\mathcal{Z}-\mathcal{N}} \phi_r(p)k_r = \sum_{r\in\mathcal{Z}-\mathcal{N}} k_r$ is lower than the RHS of Equation (19) over the range $(\overline{\overline{p}}, p_M^{(n)}]$. The the proof follows along the same lines of the proof of Proposition 5(ii).

(v) Because of Proposition 2(iii), $\overline{p} = F(\overline{p}) = p_m$. As a consequence, Equation (19) can be written as Equation (A8) and Equation (21) can be written as Equation (A7). The derivative of the RHS of Equation (A8) is positive if, and only if,

$$p_m\left(\prod_{j\in\mathcal{N}} k_j\right)^{\frac{1}{n}} + n\left(\frac{p - p_m}{p}\right)^{\frac{n-1}{n}} p^2 D'(p) > 0. \tag{A9}$$

The LHS of inequality (A1) is a strictly decreasing function in the range $[p_m, p_M]$ since the second addend is strictly decreasing due to the strict concavity of $pD(p)$. This is enough since the LHS of inequality (A1) is by definition non-negative for $p = \overline{\overline{p}}$. Now we will prove that $\overline{\overline{p}} < p_M^{(n)} = p_M$. Because of Proposition 2(viii) we can concentrate on the case in which $k_2 = k_n$ and $K - k_n = D(p_M) > K - k_1$ (the inequality is a consequence of Proposition 1(i)). In this case the RHS of Equation (A8) equals $\sum_{r\in\mathcal{Z}-\mathcal{N}} k_r$ also at $p_M = p_M^{(n)}$. Nevertheless, at $p = p_M$, the LHS of inequality (A1) is negative since, because of the fact that $p_M^2 D'(p_M) = -p_m k_1$, it equals $p_m k_1^{\frac{1}{n}} k_2^{\frac{n-1}{n}} - n\left(\frac{p_M - p_m}{p_M}\right)^{\frac{n-1}{n}} p_m k_1 = p_m k_1^{\frac{1}{m}}\left[k_2^{\frac{n-1}{n}} - n\left(\frac{p_M - p_m}{p_M} k_1\right)^{\frac{n-1}{n}}\right] = p_m k_1^{\frac{1}{n}}(1 - m)\left[K - D(p_M)\right]^{\frac{n-1}{n}} < 0$, the last equality deriving since $\frac{p_M - p_m}{p_M} = \phi_1(p_M) = \frac{k_2}{k_1}$. Hence, because of quasi-concavity of the RHS, Equation (A7) has two solutions in the range $[\overline{p}, p_M]$: the former is $\overline{\overline{p}}$, the latter is p_M. Clearly, over the range $(\overline{\overline{p}}, p_M)$, the RHS of Equation (A8) is higher than $\sum_{r \in \mathcal{Z} - \mathcal{N}} k_r$. Therefore, $\Pi_r(p, \phi_{-r}) < \Pi_r^*$ over that range and $\Pi_r(p_M, \phi_{-r}) < \Pi_r^*$.