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Samuelson's last macroeconomic model: Secular stagnation and endogenous cyclical growth

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ABSTRACT

On the occasion of the centennial of his mentor Alvin Hansen, Paul Samuelson published in 1988 a modified version of his seminal 1939 multiplier-accelerator model to address aspects of Hansen's secular stagnation hypothesis. The "Keynes-Hansen-Samuelson" model (or KHS, as he called it) was built to analyse the effects of population growth on the economy's trajectory. Several changes were then made. Instead of difference equations and a tight accelerator, as in his 1939 model, Samuelson deployed differential equations and a flexible accelerator to produce a nonlinear limit cycle. Despite Samuelson's strong claims for the analytical contributions of his 1988 paper, it has – in contrast with the 1939 model – received only scant attention by macroeconomists and historians of economics alike. Samuelson's 1988 paper was his last published macroeconomic model, based on his long-established tradition of non-optimising macro-dynamics. Our paper provides a close reading of that article and some analytical results that shed new light on the formal aspects of Samuelson's 1988 model. We also discuss how it historically links up with business cycle models advanced by John Hicks, Nicholas Kaldor, Roy Harrod and Richard Goodwin and examine how far Samuelson's use of the term secular stagnation differs from Larry Summers's recent reconstruction of it.

1. Introduction

On the centennial of his mentor Alvin Hansen, Paul Samuelson published in 1988 a modified version of his seminal 1939 multiplier-accelerator model with the specific aim of addressing aspects of Hansen's secular stagnation hypothesis. On that occasion, he introduced a specific propensity-to-save function, giving the fraction of income saved as an invariant function of the ratio of actual income to full employment and managed to build what he called the "Keynes-Hansen-Samuelson" model (or KHS). The model differed from the so-called Harrod-Domar and the Solow neoclassical growth model in the way it accounts for economic stability and the factors responsible for secular growth.

Old business cycle literature, previous to Samuelson (1939a, 1939b) and early macrodynamic models, had been groping toward nonlinear endogenous cycle verbal models, where prosperity created conditions for economic depression as the economy hit its full employment « ceiling » and vice-versa when it reached the « floor ». Samuelson (1939b: 788) challenged the hitherto prevailing notion – which he associated with J. M. Clark, Gottfried Haberler, Hansen and Roy Harrod, among others – that mechanisms akin to the multiplier-accelerator interaction could only bring about a cyclical downturn due to the full-employment ceiling or perverse price-cost movements caused by bottlenecks. Instead, Samuelson (1939b: 792) claimed that his mathematical model was more general than the nonlinear verbal approach since, even without any bottlenecks, the expansion could end for certain values of the marginal propensity to consume and the acceleration coefficient. In addition, as Samuelson (1955: 313, n. 3) would recall, the great merit of a fully determinate linear model was to provide a possibility to account for all phases of the business cycles.

However, what Samuelson saw as the strength of his 1939 cycle model came to be perceived by some as its weakness. Upon describing that model in some detail and calling it a "brilliant" achievement (Haberler 1946: 473–77), Haberler (1949: 85) would complain: give any "sophomore a couple of lags and initial conditions and he will construct systems which display regular, damped or explosive oscillation … as desired." Several decades later, the dependence of the qualitative behaviour of Samuelson's 1939 model upon the values of structural coefficients deployed in the equations became increasingly perceived as

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problematic. It indicated to Kydland and Prescott (1991: 165), for instance, that "pure theory was not providing sufficient discipline" and was lacking proper microfoundations. In an other vein, a Keynesian economists such as Tobin (1983: 195) regretted that Samuelson did not fully recognize at the time the defect of linear models of the business cycle, in the sense that they generate self-sustained cycles for singular values of the parameters only.

Samuelson's 1939 multiplier-accelerator model – shared by macroeconomics in general until the early 1970s – provides a typical example of a "dynamic system that can in no useful sense be related to a maximum problem," as Samuelson (1972: 258) pointed out in his Nobel Lecture. For that very reason, the 1939 model was solved through a mathematical analysis of stability conditions pointing out the various regions corresponding to the possible roots of the quadratic equation that formed the dynamic system's characteristic equation (see Samuelson, 1972: 258). That distinction – between optimisation problems in microeconomics on the one hand and the study of the dynamical properties of aggregative systems under the assumption of stability on the other hand – became the hallmark of Samuelson's (1947) Foundations encapsulated by the "Correspondence Principle" between statics and dynamics.

Samuelson (1947: 284) regarded the Correspondence Principle as a continuation and further elaboration of the "revolution" from static to dynamic modes started by Frisch (1933) (see Boianovsky 2020). He shared with Frisch the view that the economy is a naturally stable system, which, unless disturbed from the outside, always remains around an equilibrium state.[1] That stability postulate was part of Frisch's view of damped propagation mechanisms with cyclical oscillations caused by exogenous shocks. Linear mathematics suited the stability postulate well, as distinct from nonlinear mathematics, later applied to self-sustained fluctuations by Samuelson (1988) under Goodwin's influence.

Our paper provides a close reading of Samuelson (1988), together with a discussion of how it historically links up with Samuelson's early multiplier-accelerator model as well as with business cycle models advanced by John Hicks, Nicholas Kaldor, Roy Harrod, Robert Solow and especially Goodwin. In this way, we hope to highlight how Samuelson's view changed on the issue of secular stagnation and the way to account for cycles. The paper includes simulations highlighting the properties of Samuelson's model and its behaviour in response to various shocks. On that basis, we examine how far Samuelson's use of the term secular stagnation differs from Larry Summers's recent reconstruction of it. Finally, we make a final remark on the possible reasons why Samuelson's 1988 failure to attract a large readership had to do with the fact that mainstream macroeconomists' modelling strategy of endogenous business cycles changed sharply in the 1980s and after.

2. The persistence and many lives of the multiplier-accelerator model

Samuelson often argued that his 1939 multiplier-accelerator model – sometimes regarded as one of the first mathematical endogenous business cycle models – had its origins in Hansen's attempt to explain the sudden and deep 1937–38 American recession on the basis of dynamized elementary Keynesian model.[2] To do so, Hansen developed a determinate numerical example - assuming a propensity to consume equal to 0,5 and a coefficient of acceleration equal to 2 - and concluded that following a temporary rise in autonomous demand (public spending or private investment), national income would not reach a new equilibrium but would eventually slide into recession. This moment is when Samuelson (1939a) came in. Reducing Hansen's analysis to a second-order difference equation, his role consisted of showing that Hansen's example would generate self–sustained cycles. As he would recollect,

> At once I made the inference that the drop in income which had so struck Hansen was not the end of the story. Quite by chance, he had picked numerical values which were on the razor's edge that yielded perpetual oscillations, with no damping and no exploding. In other words, if he had continued his numerical example far enough, his downturn too would have come to an end; and he would have been able to generate a succession of never-ending expansions and contractions (Samuelson 1959: 183).

In undertaking a comprehensive analysis of the implicit algebraic structure of Hansen's model, Samuelson highlighted how various dynamic behaviors could be generated for different values of the marginal propensity to consume and the coefficient of acceleration.[3] Four movements, reflecting the stability of the economy, were eventually pointed out: stable movements comprising monotonic and cyclical convergence towards stationary equilibrium and unstable movements comprising monotonic and cyclical divergence from stationary equilibrium.[4]

In the case of a stable movement, Samuelson (1939a) concluded that a single impulse – like a temporary rise in public spending – would have a transitory effect, reinforcing Hansen's doubt that the government just had to "prime the pump" before balancing its budget to induce private investment. In addition, if the accelerator proved critical for the trajectory of the system and its stability, it did not influence the final level attained (Samuelson 1939b), a result which seemed to vindicate Keynes's lack of interest in the acceleration principle: "From the long-run point of view Keynes was partially justified in ignoring the acceleration principle completely. The average level of the system is independent of its operation, depending rather upon the level of investment outlets" (Samuelson, 1939b: 795).[5]

In the end, only in unstable cases could an initial change in government expenditures lead to ever-increasing consumption levels, induced investment and income. Only then "[a] constant level of governmental expenditure will result in an ever-increasing national income, eventually approaching a compound interest rate of growth" (Samuelson, 1939a: 77 and 1940: 502). That corresponded to Region D in Samuelson's (1939a) stability diagram. But Samuelson viewed this possibility as an extreme case, characterized by a particularly high sensitivity of expectations to income – it was more likely that private enthusiasm would soon peter out and a downturn would occur once the acceleration coefficient had been reduced (Samuelson, 1940: 502–503).

This motion was even more likely to happen if, in accordance with Keynes's idea (shared by Harrod), the marginal propensity to consume tended to fall, causing an expanding economy to move away from

[1] In the meantime, Samuelson thought that a system in which money wages would respond to unemployment was highly likely to be unstable. See Assous and Carret (2020) on Samuelson's early take on instability of full employment equilibria.

[2] Oskar Lange's 1938 Keynesian model was a common reference to Hansen and Samuelson, who both met him after his move to the US. See Assous and Lampa (2014) and Backhouse (2017, chapter 18) on Samuelson's and Lange's correspondence on the properties of that model.

[3] In that respect, Samuelson's 1939 multiplier-accelerator model was tightly connected to the works of his fellow econometricians. He viewed it as "a useful introduction to the mathematical theory of [Jan Tinbergen's] work" (Samuelson, 1939a: 78) as well as an example of the importance, repeatedly underlined by Frisch in the early 1930s, of having a unified argument accounting for the turning points of the phases of the business cycle (Samuelson, 1939b: 785, 789). See Assous and Carret (2022, chapters 7 and 8) on the connection between Samuelson and early econometricians.

[4] In each case, stability referred to movement with respect to the stationary state, as the dynamic equation was derived under the assumption that the macroeconomic equilibrium condition is verified at each point of time.

[5] See Assous and Carret (2022 sections 7.4 and 8.3) for a detailed account of the various properties of Samuelson's multiplier-accelerator model.

regions of instability – another blow to the possibility of "pure pump priming." Of course, changes in the propensity to consume, while preventing cumulative upward movement, would intensify downward movements. Nevertheless, Samuelson thought that such movements would be avoided because of the existence of a lower bound on net investment. In his discussion, he also raised the possibility that for certain values of the propensity to consume and the coefficient of acceleration - but without working out the mathematics behind the analysis - there would probably exist a "periodic motion of definite amplitude" which would be approached regardless of initial conditions (Samuelson, 1939b: 795), or, in modern terms, a limit cycle. Such a case was thought quite like a stable scenario that excluded the possibility of endogenous cyclical growth and highlighted the major risk of stagnation.

The so-called Harrod-Domar model – particularly in Harrod's version – shared with Samuelson (1939a, 1939b) the notion that the interaction between the multiplier and the accelerator determines the dynamic path. Samuelson did not develop or anticipate that growth model. Looking back on this from the vantage point of four decades later, he regretted that he had suppressed the "development of the Harrod-Domar exponential growth aspects that kept thrusting themselves on anyone who worked with accelerator–multiplier systems" (Samuelson, 1974: 10). But some did not join him on that path. Eager to show how one could make the transition from Samuelson's cyclical model to Evsey Domar's and Roy Harrod's growth analyses, economists like Thomas Schelling, Sydney Alexander and Goodwin (all of them with links to Harvard University then) came up with new insights about economic growth and business cycles that Samuelson eventually tackled upfront in his 1988 Hansen anniversary article.

As a member of that cohort of Harvard-trained economists and a PhD student of Hansen, Schelling partially overlapped with Domar. At the time, Domar worked at the Federal Reserve Board in Washington. Schelling most likely attended the seminars Domar organized (together with Hansen) on macroeconomic policy from 1943 to 1946 and interacted with him about his growth models. Schelling (1947) was attracted to Domar's guaranteed growth rate, which he saw as shedding new light on Harrod's (1939) involved notion of the "warranted growth rate" (see Boianovsky 2021).

In order to indicate the closeness of the "multiplier-accelerator" approach to Domar's and Harrod's approaches (Schelling 1947 was the first to refer to the «Harrod-Domar model»), Schelling examined a first model in which consumption was assumed to operate with a one-period lag, and the investment demand function was of the form $I_t = \beta(Y_t - Y_{t-1})$ (with β being the acceleration coefficient, with a difference regarding Samuelson's model that investment does not depend on changes in consumption but on changes in income). As long as β is greater than one, he argued that the economy would grow at a compound rate (Schelling 1947: 872). The problem, Schelling argued, was that the steady growth rate thus defined would prove to be unstable. This result is because any chance discrepancy between aggregate output and aggregate demand which may occur will cause either income to increase indefinitely (in case of excess aggregate demand) or to decrease indefinitely (in case of excess aggregate supply), all within one time period. Drawing a parallel with the Keynesian cross diagram, Schelling concluded: "In fact, a value of β greater than unity will necessarily yield a 'negative multiplier,' indicating the same kind of instability as would result from a marginal propensity to consume greater than unity" (Schelling 1947: 873).

The issue then arose: "May not there be more resiliency in the system than we have supposed, so that disturbances will be cushioned?" (Schelling 1947: 875). To examine it, Schelling built a second model in which consumption operates with no time lag, but investment was now geared to past investment only, with $I_t = \beta(Y_{t-1} - Y_{t-2})$ so that investment no longer depends on current income. As long as the propensity to consume remained lower than 1, stability at any point in time was then shown to be ensured. But, as in Samuelson's original model, steady growth will now be possible only if the two roots of the characteristic equations are real and greater than one, which is the case only if $\beta > 4(1 - a)$ where a is the propensity to consume. So, apart from the fact that the growth rate thus generated by any combination of coefficients a and β is "rather frightening," any demand disturbance will have a permanent effect on the growth rate, generating thus a new form of instability (Schelling 1948: 876), as already pointed out by Samuelson. Schelling eventually sided with Samuelson and dismissed the growth solutions, thus highlighting the likelihood that the economy will stagnate in absence of shocks.

The effort, by Schelling and others,[6] to solve Harrod's and Domar's growth instability problems returning to Samuelson's 1939 multiplier-accelerator model attracted significant attention from the late 1940s to mid 1950s, as illustrated by Hamberg's (1956) contemporary survey. Hamberg agreed with Schelling and Alexander (1949) about the accelerator as an unreliable generator of steady growth and on the notion that the source of long-period trends in investment and income – and the reason for their oscillations – should be sought in autonomous investment. Solow (1957) charged Hamberg (and the rest of the contemporary growth-instability literature) for overlooking the role of the substitution between capital and labour in reaction to relative price changes. Indeed, capital deepening is conspicuously absent from Schelling and Alexander, as it was from Samuelson (1939a, 1939b; 1940) and even Samuelson (1988) later on.[7]

Like Schelling and Alexander, Goodwin was concerned with accounting for growth oscillations within the multiplier-accelerator interaction, with limited success. Since the publication of Frisch's 1933 Cassel paper, as Goodwin argued in the introduction to his 1951 Econometrica article, economists had faced the following "unpleasant dilemma": either assuming that the economy is unstable and likely to explode or collapse or is stable but kept alive by outside forces. The only way to avoid the dilemma was to build a new class of determinate models displaying nonlinearities. The first reason is clear. Only such models - except for Hansen's razor-edge solution - made it possible to account for self-sustained oscillations. The second reason is less obvious. Goodwin argued that the great merit of such models was their ability to be "frequency converters", revealing that any steady force acting on the system (like the steady change in technical progress) would change the movement period but not its amplitude. Similarly, Goodwin (1951: 8) claimed that the approach might more easily account for the impact of "historical events" whose effects would prolong or shorten the boom or depression without changing the economy's trajectory.

Goodwin's work on nonlinear business cycles, together with contributions by Kaldor (1940) and Hicks (1950), eventually indicated to Samuelson the limits of his own model. Donald Gordon (1955) challenged the empirical and theoretical validity of Samuelson's (1947) key assumption that the real world is dynamically stable. Gordon (1955: 308) pointed out that "recent theories of the business cycle … suggest that actual economic variables may possess no stable equilibrium values over the observable range, yet the values observed may all be points on stable functions." As Samuelson (1955: 313) remarked, Gordon was referring to auto-relaxation business cycle models of the kind proposed by Kaldor, Hicks and especially Goodwin, based on local instability at their stationary levels and featuring limited oscillations due to nonlinearities. That was distinct from Samuelson's linear multiplier-accelerator model. Gordon's point was that – instead of

[6] See for instance Alexander (1949).

[7] As Samuelson put it in a letter of 1 July 2002 to Craufurd Goodwin, regarding some refereeing work he was then doing for *History of Political Economy*, "in this important *trend* area, there is à la Cassel and Solow a distinctly *neoclassical* role for investment (as both cause and effect) that is 180 degrees from Keynes-Harrod-Samuelson 1930s investment role as merely a way of generating purchasing-power stimulus – to the neglect of its role as capital formation (capital deepening) to raise full-employment productivity within a Say's Law world where pump-priming is not 'needed.'"

Samuelson's (1947, p. 5) claim that actual observations are either points of dynamically stable or unstable equilibrium, which makes the latter very unlikely to be observed – what we may observe, as implied by the mentioned business cycle models, are neither. As Samuelson acknowledged,

> Well, maybe the system is unstable. That is one possibility, and as Gordon is cogently pointing out, many of the cobweb cycles and auto-relaxation trade cycle theories of such moderns as Kaldor, Goodwin, Hicks, and others are squarely based on the notion of a system that is locally unstable at its stationary levels so that it oscillates — but because of such nonlinear elements as full-employment ceilings, capacity limitations, impossibility of dis-investing faster than at certain limiting rates, the system oscillates with a preferred finite amplitude (Samuelson 1955: 313).

Later on, Samuelson introduced in the 1964 edition of Economics a new section titled "Interactions of accelerator and multiplier" and a new chapter on growth economics. In his presentation of the Harrod-Domar growth models, he (1964a: 743, n. 3) remarked that the multiplier-accelerator interaction was then applied to the economic growth trend rather than to the business cycle as deviations from that trend.[8] That was quite distinct from his 1939 approach. And so was his discussion of the cyclical effects resulting from the collision of Harrod's warranted growth rate with the ceiling represented by the natural growth rate. The bouncing back of the system – which he associated with Hicks (1950) – became from then on an essential element of Samuelson's 1964 statement of the multiplier-accelerator business cycle model:

> But how can a system grow forever at 5 or 6 per cent if its labour force grows only at 1 or 2 per cent? It can't. The self-warranting expansion … must ultimately bump into the full-employment ceiling. Like a tennis ball … it is likely to bounce back from the full-employment ceiling into a recession. Why? Because the minute the system stops its fast growth, the accelerator dictates the end of the high investment supporting the boom. [Similarly] when output plummets downward rapidly, the acceleration principle calls for negative investment … greater than the rate at which machines can wear out. This wear-out rate puts a floor on how far disinvestment can push the economy below its break-even point. Bumping along such a basement floor means that eventually firms will work down their capital stock to the level called for by that low level of income; and now the acceleration principle calls for a termination of disinvestment! (Samuelson 1964a: 263).

This long quotation forcefully illustrates how, by the early 1960s, Samuelson had shifted toward a nonlinear explanation of the multiplier-acceleration interaction. Throughout the several editions of his Economics, including those co-authored with William Nordhaus from 1985 to 2010, Samuelson claimed that the multiplier-accelerator model (primarily, but not only, in its nonlinear version) provided a suitable account of endogenous macroeconomic fluctuations from a Keynesian perspective. It is however only in the late 1980s that he eventually managed to formulate a nonlinear business cycle model as he attempted to make sense of Hansen's notion of secular stagnation while paying a last tribute at his old mentor's centennial.

[8] Investment appears in Domar's aggregate supply and demand functions, with asymmetrical effects: the former is a function of the (net) *level* of investment, while the latter depends on its *rate of growth*. A higher investment level has a permanent effect on capacity, but a temporary one on income as the multiplier mechanism peters out (Domar [1947] 1957, p. 98). Hence, in order that sufficient demand is generated, and capacity remains fully utilized it is necessary for investment (and income) to grow at a certain rate, determined by the equality between aggregate demand and supply.

3. The 1988 keynes-hansen-samuelson (KHS) model

On Hansen's centennial, Samuelson (1988) produced his last contribution to macroeconomic dynamics, an extended reformulation of his first 1939 multiplier-accelerator model. At a time when macroeconomics had moved toward optimization (with frequent references to his 1958 overlapping generation model), he chose to go against the trend, stick to macro-dynamics and start from the model which marked his early career.

3.1. Chronic unemployment and growth

The 1988 article begins by examining the working of a "Model T Keynesian system" in which income is the sole determinant of saving so that if capital is assumed to widen to match population growth, equilibrium is shown to be set at any time at the intersection of a rising *SS* saving schedule with a horizontal *II* investment curve (Samuelson 1988: 6–7). As indicated in Fig. 1, at point *E*, equilibrium happens at level Y^E below full employment equilibrium Y^F. That was reminiscent of the famous Keynesian cross diagram Samuelson had introduced in 1939.

A characteristic of that model, Samuelson (1988: 7) argued, is to account for Simon Kuznets's 1941 finding that the saving/income ratio is an invariant function of the ratio Y^E to Y^F or of actual income Y (as long as $I = S$) to full employment income Y^*. Assume that following a significant rise in population, the *II* curve has shifted upwards. As long as full employment income and investment rise by the same amount, no change in the saving/income ratio will occur while the economy will keep growing smoothly with unchanged chronic unemployment. This can be formally expressed in the following way:

$$\frac{S}{Y} = f\left(\frac{Y}{Y^*}\right) \tag{1}$$

Defining $y = Y/Y^*$ as the ratio of actual to full employment income, we have that $0 < [Yf(y)]' < 1$ is the marginal propensity to save.

Along those lines, the system is dynamized in accordance with a "tight accelerator mechanism", which indicates that, in the absence of "feasible deepening" of capital and no autonomous investment, the rate of net capital formation is proportional to the rate of change of actual income (Samuelson 1988: 8):

$$\dot{K} = \beta\dot{Y} \tag{2}$$

where the numerical value of β, i.e. Harrod's 'relation', depends on the time unit used to measure income and saving rates. To keep notation as simple as possible, in the remainder of this paper, for any generic variable x, its time derivative dx/dt will be represented as $\dot{x}$.

As long as equilibrium between saving and investment is maintained, we have that:

$$f(y) = \beta\frac{\dot{Y}}{Y} \tag{3}$$

Hence, it turns out that the rate of growth of actual income is equal to

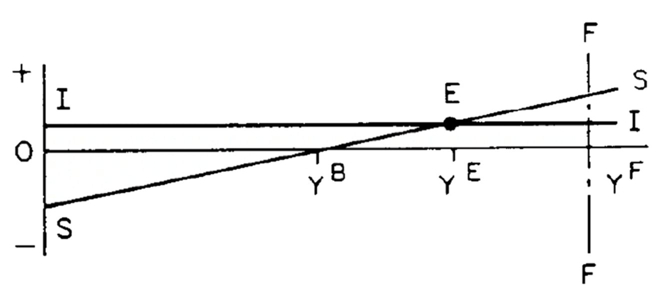

Fig. 1. Model T Keynesian system and the invariance of the saving propensity (Source: Samuelson 1988: 7).

$f(y)/\beta$.

On the assumption that full-employment income grows exponentially at the 'natural' rate n, which reflects mere population growth and neutral labour-saving technical change, Samuelson (1988: 9) derived the equilibrium level $\bar{y}$ for which actual income keeps pace with population, that is $\bar{y} = f^{-1}(n\beta)$. Therefore, if the rate of population slackens off permanently, the long-run rate level of underemployment eventually increases by the same proportion. On that basis, Samuelson argued that Hansen had figured out a way to dynamize Keynes's 1936 General Theory analysis to a situation in which the level of full employment income is not constant but is changing over time, an idea that Keynes mentioned in his 1937 Galton lectures published the same year in the Eugenics Review. So, he chose to name it the Keynes-Hansen-Samuelson (KHS), which, he argued, is different from most contemporaries' macroeconomic models in several ways.

For example, in Harrod's (alleged) model in which the propensity to save is constant, there is no possibility to "have a determination solution for equilibrium Y/Y^*" and for relative unemployment" (Samuelson 1988: 9). This is because the equality between saving and investment is possible for any level of unemployment and thus any value of $\bar{y}$. "A determinate theory of relative employment" would hence be lacking in Harrod's model (Samuelson 1988: 10). The KHS system would also crucially differ from the Solow neoclassical growth model in which the whole population is assumed to be fully employed and y (equal to Y/Y^*) is assumed to be permanently equal to 1. To see these differences more clearly, suppose, Samuelson argues, a production function that takes the neoclassical form $Y^* = Q(K,L) = LQ(K/L)$. When the investment-saving condition is permanently met at full employment, we have:

$$\dot{K} = f(1)LQ(K/L) \tag{4}$$

Defining $Z = K/L$ as an auxiliary variable and dividing both sides of Eq. (4) by K, we have $\dot{K}/K = f(1)(L/K)Q(K/L)$ or $\dot{K}/K = f(1)[Q(Z)/Z]$. Log-differentiating Z with respect to time and substituting the previous expression in it:

$$\dot{Z}/Z = \dot{K}/K - \dot{L}/L = f(1)[Q(Z)/Z] - n \tag{5}$$

Such a relation implies that in this Solow-Meade framework, Z converges to its steady state value $\overline{Z}$ defined and given by:

$$n = \frac{f(1)}{[\overline{Z}/Q(\overline{Z})]} \tag{6}$$

That is distinct from the equations for both the KHS, in which $n = f(\bar{y})/\beta$, and Harrod's warranted-growth-rate versions, $g_w = s/\beta$.

In accordance with the notion of the neoclassical synthesis he put forward in the 1950s and early 1960s Samuelson did not fail to emphasize that both Solow-Meade and KHS models complement each other: the Solow model simply showing how the economy behaves when factors of production are substitutable and Keynesian forces of effective demand are managed by economic policy to ensure full employment so that y remains permanently equal to 1. Precisely because of this complementarity, Samuelson could state that "Keynesianism as a tool of analysis" has for a time superseded early Keynesianism "as a depression ideology": "The old King is dead; long live the new King of the neoclassical synthesis'" (Samuelson 1988: 18). The KHS system would also differ significantly from Kaldor's [1955–56] full-employment model or the "'Cambridge' long-term generalization" (Samuelson, undated). His-point was that in those formalizations, changes in income distribution are assumed to stabilize the full employment growth path automatically – thereby ultimately and dangerously reducing the case for Keynesian government interventions.

What the KHS model has in common with Harrod is to display an unstable growth rate resulting from "positive feedback" effects (Samuelson 1988: 11). Assume for instance that the economy is on the right of the break-even point (intersection between the SS and II curves in Fig. 1). This means that y has increased, which will cause a rise in the propensity to save, but due to the accelerator, the economy will keep expanding the gap between the rate of growth of actual income and the rate of growth of full employment income until full employment is reached. Inversely, if the economy is on the left of the break-even point, the departure from $\bar{y}$ will be self-aggravating until "the unemployment rate soars toward 100 percent!" (Samuelson 1988: 11). So, Samuelson concludes, "the KHS model is seen, transiently, to possess some of Harrod's razor's-edge pathology" (Samuelson 1988: 10). But, since y cannot exceed 1 and the saving propensity is flexible, such instability proves to be only local.

3.2. Autorelaxation cycles

It is for this very reason that Samuelson thought that the KHS model provided a way to account for Goodwin's (1951) and Hicks' (1950) as well as Kaldor's (1940) nonlinear analyses. Let us replace Eq. (2) with a flexible accelerator and write the dynamic equation for change in current income as:

$$\lambda\frac{\dot{Y}}{Y} = \left(\frac{\dot{K}}{Y} - f(y)\right) \tag{7}$$

where $0 < \lambda \ll 1$ is a positive constant which denotes the inverse of the speed of adjustment, $\dot{K}/Y - f(y)$ represents the deviation between aggregate demand and aggregate output. The floor to capital accumulation is given by the depreciation rate δ or the rate at which machines wear out. Hence, he writes:

$$\dot{K} = max\left[\frac{\beta Y - K}{\epsilon}; -\delta K\right] \tag{8}$$

where $0 < \epsilon \ll 1$ is a rate-of-adjustment parameter, which makes K changes with respect to desired capital. Once Eqs. (7) and (8) are rewritten with ratio-to-trend variables $y = Y/Y^*$ and $k = K/Y^*$, one eventually gets a timeless two-dimensional piece-wise system in which the state variables are y and k.

$$\dot{k}\Big/k = max\left[\frac{\beta y k^{-1} - 1}{\epsilon}, -\delta\right] - n = H(k, y; n) \tag{9}$$

$$\lambda\dot{y}\Big/y = \begin{cases} [H(k, y; n) + n]\left[\frac{k}{y}\right] - f(y) - \lambda n \equiv G(k, y; n), & 0 < y < 1 \\ min[0, G(k, y; n)], & y = 1 \end{cases}$$

Samuelson did not present a detailed analysis of the properties of this system. He just provided a figure illustrating the working of the process over a complete limit cycle that is reproduced as Fig. 2. As population grows exponentially at the given rate n, the system features a moving equilibrium of $\bar{y}$ determined by n. The horizontal straight line $F'F''$

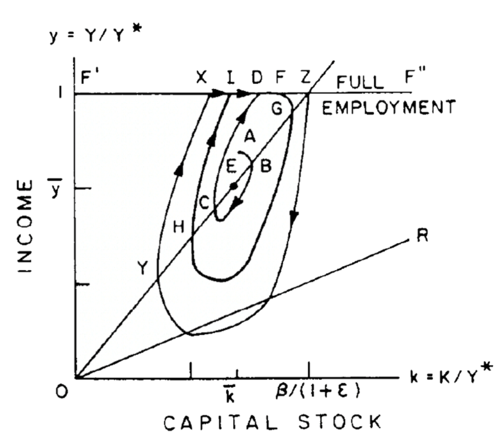

Fig. 2. Hansen Limit-Cycle (Source: Samuelson 1988: 13).

describes the ceiling due to resource limitations, i.e., the maximal growth path. An actual trajectory starting at a point where y is off its stationary state (with y above $\bar{y}$ and k/y below $\bar{k}/\bar{y}$) is described in the following way: as the population is rising, the system, in accordance with the acceleration principle, is exploding. Actual income therefore rises with a growth rate higher than full employment output. Once the ceiling is reached, the growth rate of income is limited by that of the resource capacity. As Y grows that slowly, K/Y becomes excessive until $\dot{K}$ turns negative and the system is sent in a self-aggravating way below its break-even point. Eventually, k/y is so low that $f(y)$ becomes large relative to β, which generates a positive $\dot{k}$. Following this, the system features again a positive growth rate of y until reaching the full employment ceiling.

Samuelson's KHS model has the advantage that it is no longer necessary to specify the parameter values that lead to self-sustained oscillations. In addition, it allows accounting simultaneously for growth and cycles. However, because it relies on the assumption of an exogenous population growth rate, as long as that rate is no longer changing, the economy starts cycling around a static position.

On that basis, Samuelson concluded that "all that Kaldor (1951) found lacking in Hicks (1950) is achieved by KHS" (Samuelson 1988:14). First, the model solves the limitations of linear systems by providing a way to account for self-sustained oscillations. Second, it is derived from a flexible accelerator, the value of which varies in the cycles, falling off during the downswing and becoming null once full employment has been reached. However, unlike Hicks, Samuelson did not need to assume the existence of autonomous investment to account for the low turning points. In the KHS model, as y adjusts downwards, $f(y)$ becomes infinitely low. The bottom line is that the KHS model does not rely on introducing any arbitrarily assumed bounds. Third, as in Hicks' model, cyclical oscillations are treated as deviations around a rising long-term trend of output, but while Hicks explained the trend by growing autonomous investment, Samuelson explained it by the rise in population.

For those reasons, Samuelson claimed that the KHS model "adds Harrodian trends to the effective-demand system of Keynes, but in a less *ad hoc* fashion than the Hicks [1950] bald axiomatizing of autonomous exponential investment approach" (Samuelson undated). Of course, Samuelson added, one can complain that the KHS model is not derived from rigorous microeconomics foundations. "But even worse defects must be acknowledged; sometimes [systems derived from proper microeconomic foundations] lack proper macroeconomic foundations also" (Samuelson 1988: 12).

To provide a more concrete view of the properties of system (9), we present a brief numerical exercise that further illustrates some of the main dynamics of the model. First, we need to choose a functional form of $f(\cdot)$. Samuelson refers to a Kuznets-like function such that:

$$f(y) = \alpha\left(Y - \frac{1}{2}Y^*\right)\frac{Y}{Y^*} = \alpha\left(1 - \frac{1}{2y}\right) \tag{10}$$

where $0 < \alpha \ll 1$ is the marginal propensity to save. It has the nice property that $\lim_{y\to 0} f(y) = -\infty$. In this way, we avoid y going below the zero of no employment.

In his original article, Samuelson assumed $\alpha = 0.2$. Furthermore, the Harrod's relation was obtained from the literature on capital-output ratios. Annual data suggests $\beta = 3$. These are the only two parameters for which his 1988 paper provides specific values. Here, we adopted a depreciation rate, $\delta = 0.05$, and the natural rate of growth, $n = 0.02$, in line with the Solovian growth literature. Finally, we fixed the speed of adjustment in the two dynamic relations as $\epsilon = \lambda = 0.99$. Analogous results can be obtained for different combinations of ϵ and λ as long as they lie between zero and one. Still, given the model is highly stylised, its interpretation is inevitable more qualitative rather than quantitative.

Fig. 3 plots, in black, our baseline scenario using the calibration described before. Cycles are clockwise oriented in the $y - k$ space, and the small squares indicate the nontrivial equilibrium point. In dashed-grey, the diagram on the left depicts the model's response to an increase in the propensity to save from 0.2 to 0.35. We observe a reduction in the size of the orbit. Recall that output adjusts to the difference between investment and saving rates. As the latter is a function of y, a higher α is associated with a faster adjustment, leading to a smaller cycle. Furthermore, despite the economy never being in a state of rest, the values of $(\bar{k}, \bar{y})$ are also reduced. This "Keynesian" result follows from the fact that Y is the adjustment variable in the investment-saving equilibrium equality. For a given investment level compatible with full employment, the higher the marginal propensity to save, the lower the required $\bar{y}$ to bring the goods market to equilibrium.

We also study the case in which there is an increase in the natural growth rate. In a scenario with no technical progress, n is equal to

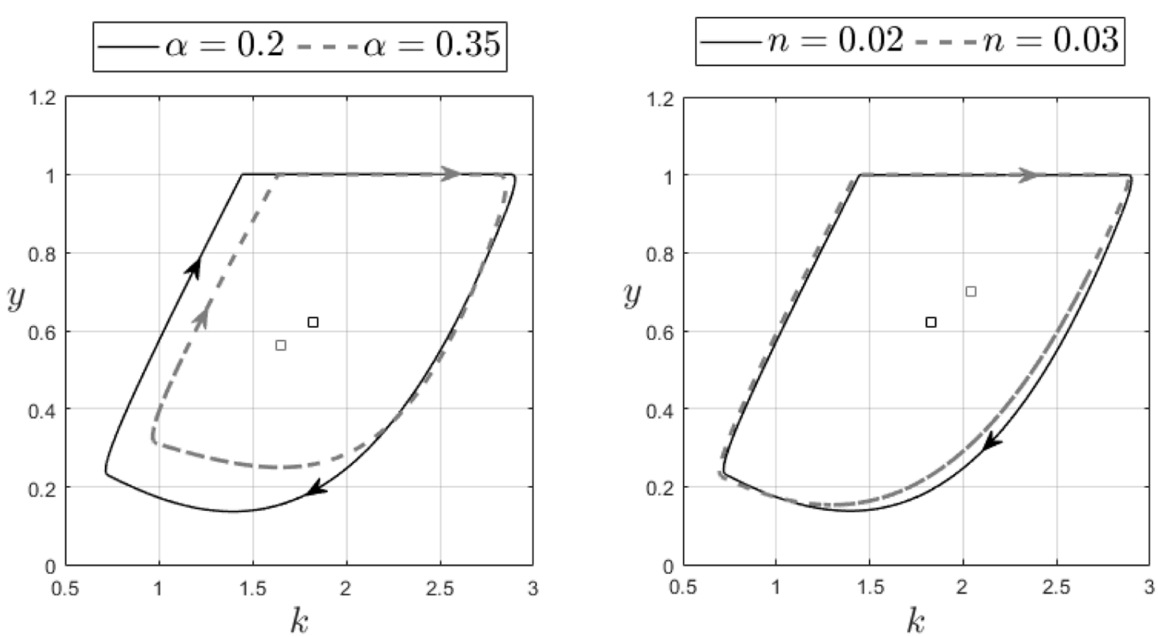

Fig. 3. Robustness of the KHS business cycles (Source: Authors elaboration).

population growth. Fig. 3, on the right, shows that raising n from 0.02 to 0.03 has somehow a neglectable impact on the amplitude of the business cycle. Still, in such a scenario, $(\bar{k},\bar{y})$ is significantly higher. The rationale of this finding is quite simple. Facing higher warranted growth, firms increase investment to guarantee productive capacity matches sales. For a given propensity to save, the output level necessary to bring the goods market to equilibrium will be higher, moving the grey square in the diagram closer to the upper-right corner. Such a change is barely perceived over the business cycle as the orbits are similar. However, in the long run, full-employment income will be lower in the black case relative to the grey one because the economy is growing comparatively slower.

It is worth evaluating in more detail the response of $(\bar{k},\bar{y})$ to α and n. This step allows us to address how the "centre of gravity" of the economy moves for different combinations of these structural parameters. Fig. 4, on the left, provides a clearer view of the negative relationship suggested in Fig. 3. The black dashed-dotted line marks full-employment output. Thus, its difference with the grey dashed line indicates the amount of unemployment in the economy. As the marginal propensity to save is increased to 0.5, the ratio between current and full employment output deepens and gets closer to 0.5. Still, notice that given that the equilibrium point is unstable – due to the Harrodian feature of the KHS model – the economy continues to hit full employment for a significantly long period, and the cycle's amplitude is reduced. On the other hand, the right diagram in Fig. 4, shows that a reduction of the natural growth rate is related to lower $\bar{k}$ and $\bar{y}$.

3.3. iA khs model with life-cycle savings

Modigliani's life-cycle hypothesis stressed the crucial role of growth in affecting savings. With no growth, the dissaving of retirees would cancel out all people's savings during their active working years. Samuelson argued that life cycle effects reduced the stagnation effects of population decline. Still, the main qualitative intuition from the KHS model should be preserved. It must be noted that he did not provide a numerical analysis of the main implications of this augmented version of his model. From a modelling point of view, Samuelson's (1988: 15) tactic consisted in rewriting $f(\cdot)$ as:

$$f(y;n) = \alpha\left(1-\frac{1}{2y}\right)[1/2+M(n)/2] \tag{11}$$

where $M(n)$ is the Modigliani life-cycle factor.

He argued that if population grows like $exp(nt)$, the saving-income ratio is given by considering the $exp(-na)$ profile of people of different ages a, such that:

$$M(n) = 1-\left[\frac{W}{W+R}\right]\left[\frac{\int_0^{W+R} exp(-na)\,da}{\int_0^{W} exp(-na)\,da}\right] \tag{12}$$

where W is the years of work and R stands for those of retirement. Solving the integrals in Eq. (12) and substituting the resulting expression into Eq. (11), we have that:

$$f(y;n) = \alpha\left(1-\frac{1}{2y}\right)\left\{1-\left(\frac{W}{W+R}\right)\left[\frac{1-exp(-n(W+R))}{1-exp(-nW)}\right]\bigg/2\right\} \tag{13}$$

The new dynamic system is equivalent to (9), with the main novelty in the savings function determined now by Eq. (13).

Fig. 5 provides some insights on the robustness of the persistent and endogenous fluctuations generated by this augmented version. We use similar parameters as in our previous benchmark calibration with the following novelties. It is assumed people live 85 years; they start to work at 20 and retire at 65. Thus, we have W=45 and R=20, as indicated by the black line in the two diagrams. What happens when life expectancy increases, let us say, by 5 years? The panel on the left depicts in grey-dashed the scenario in which people will work during that extra-life period, i.e., W=50. On the right, we have the other extreme: there is no change in W and the retirement period goes from 20 to 25 years. In both cases, the magnitude of the orbit is similar, which basically result from the properties of Eq. (12). More specifically, the ratio $W/(W+R)$ changes very little between cases being ≈ 0.7. Still, comparing Figs. 4 and 5, two main results emerge. First, the magnitude of business cycles in the augmented KHS increases. Both models deliver full employment as the ceiling of the economic system. However, introducing life-cycle savings makes y to touch the floor. Such a result follows from $\partial f/\partial n < 0$, making the economy more responsive to the accelerator. Second, $(\bar{k},\bar{y})$ is higher as, ceteris paribus, a reduction in average savings requires a higher level of income to match investment, which in turn adjusts to the warranted growth rate.

4. The khs model and the modern debate on secular stagnation

The modern debate about secular stagnation, started mainly by Lawrence Summers around 2013, brought back an interest in Hansen's original ideas (e.g., Summers 2015; Backhouse and Boianovsky 2016). With reference to Japanese experience of the 1990s and the poor performance of the United States and Europe after the 2007–2008, Summers casted doubt on whether the rate of interest may be low enough for full employment. With aggregate demand a function of real interest rates which in turn depend on expected inflation or deflation, he argued that even at zero nominal interest rate, full employment might not be reached. Several factors acting on the demand side were pointed out. Amongst them, slow population growth across the developed world acting negatively on investment and rising inequalities acting positively on saving were seen as central. On the political front, the conclusion was obvious. In a context of low nominal interest rate, the way to cope with stagnation should rest with fiscal policy. Only such a policy could reduce savings, raise interest rates, and stimulate growth especially when they are articulated to large public investment programs.

The rate of interest is hardly mentioned by Samuelson (1988), who made investment demand a function of the rate of change of income only. Despite this significant difference, Samuelson, like Summers, saw the problem of stagnation on the demand side. This is especially visible if one considers the effects of a change in the population growth rate in the KHS model. Assume that the population growth rate, n, has fallen. Therefore, the gap between actual output Y and full employment output Y^* is reduced causing y (defined as the ratio of Y to Y^*) to increase. In accordance with the saving function $f(y)$, y will eventually stabilize at its long run value $\bar{y}$ for which actual output keeps pace with the new population growth rate. This is the scenario illustrated on he right panel in Fig. 3.

According to Hansen (1947: 177), there was a tendency for investment to outrun the requirements of technical progress and population growth in boom periods, called temporary "saturation" of investment opportunities. Hansen argued that the "amount of investment needed to maintain full employment has historically far exceeded the amount needed for growth and progress" (1947: 177). Only in boom years had the amount of investment been adequate to provide for full employment, but "this amount of investment could not be maintained continuously without exceeding by far the requirements of growth and progress", claimed Hansen (1947: 178). Such abrupt end of investment, amplified by the acceleration mechanism, was the "essential cause of depressions and unemployment," concluded Hansen (ibid). Suppose the "growth and progress" factors and the multiplier and accelerator coefficient values are weak. In that case, the economy is set for "secular stagnation," with stillborn recoveries and recurring depressions caused by an excess of the secular propensity to save over the long-run maintainable investment

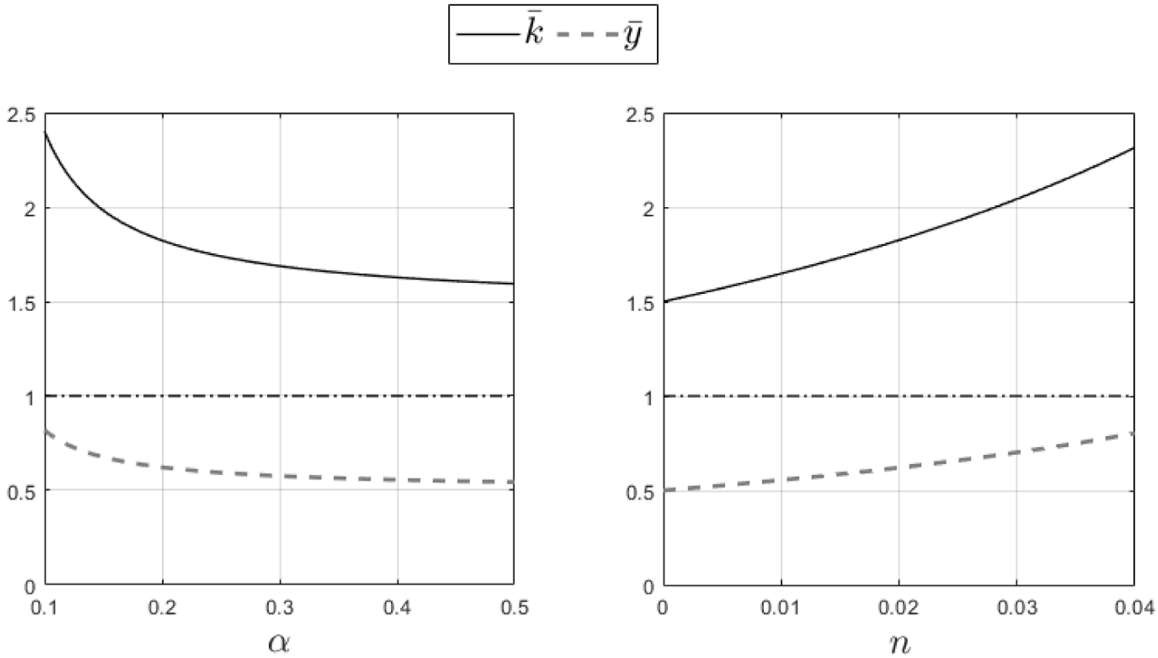

Fig. 4. Sensitivity of the non-trivial equilibrium point to the propensity to save and the natural growth rate (Source: Authors elaboration).

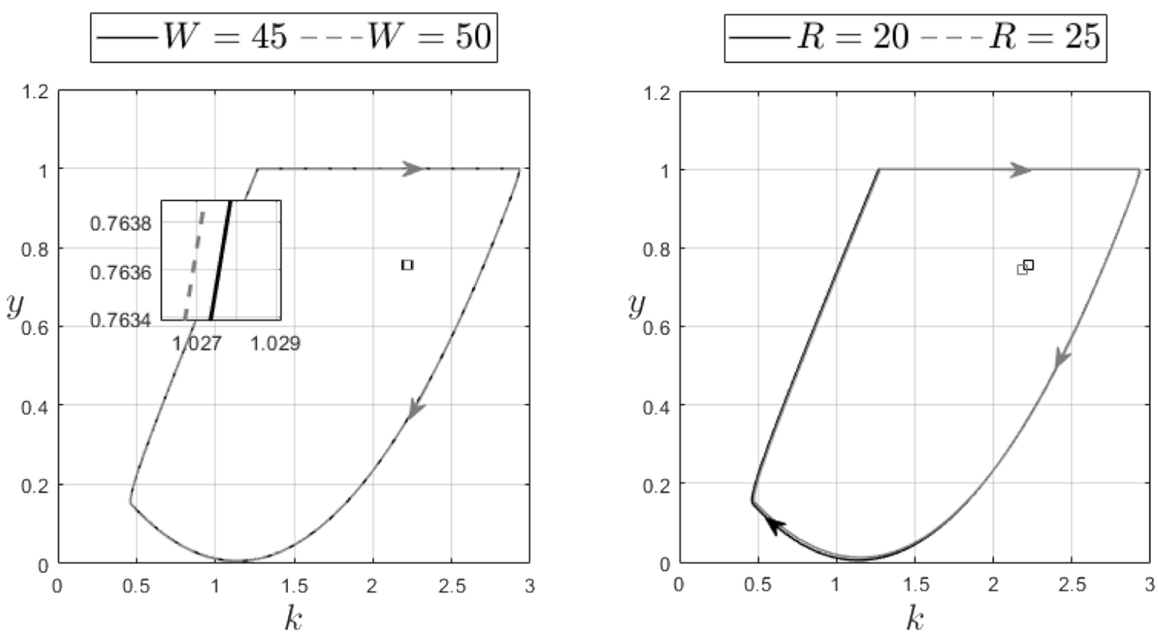

Fig. 5. Robustness of the KHS augmented business cycles (Source: Authors elaboration).

rate.[9]

Accordingly, Hansen (1939: 4) defined the "essence of secular stagnation" as "sick recoveries which die in their infancy and depressions which feed on themselves and leave a hard and seemingly immovable core of unemployment". This passage is consistent with Samuelson's (1988: 16; italics in the original) remark that the KHS model describes what would be the "*average* level of spontaneous unemployment throughout the business cycle that will occur in economies experiencing different natural rates of growth than in describing truly steady states of equilibrium." Hence, even if full employment is the business cycle ceiling, secular stagnation, as in Summers's approach, should be interpreted from the perspective of the average performance of the economy over time.

Second World War growth not to mention the recent episodes marked by post Covid 19 expansionary fiscal policies are often seen as rebuttals to the secular stagnation hypothesis. In response, Summers repeatedly put the emphasis on the political dimension of secular stagnation and argued that it was precisely because of massive political interventions that economies had grown in the past. On this matter, that is precisely what Samuelson says when he argued that the neoclassical growth model was in fact a KHS model in which full employment was postulated and was for that very reason only part and parcel of the

[9] This may be understood in terms of an excess of Harrod's warranted growth rate over the natural growth rate.

"neoclassical synthesis."

5. Final remark

By closely reading the KHS, we hope our study proves itself helpful in redirecting attention to Samuelson's last contribution to macroeconomics and its usefulness to relevant contemporary phenomena such as the secular stagnation hypothesis.

Several reasons may explain why Samuelson's model received only scant attention from his contemporaries. In a letter dated 11 February 1997, addressed to one of the authors (Boianovsky), Samuelson wrote:

For your interest, I enclose a 1988 reprint few have noticed. This Keynes-Hansen-Samuelson non-linear limit cycle captures the empirical content of the 1936 Harrod, 1930s Kalecki, 1940 Kaldor, 1940s Goodwin, 1950 Hicks cyclical model, avoiding certain infelicities and omissions; and it enabled me to discern (50 years later!) that decelerating population growth, at the same time that it lowered the acceleration-principle investment propensity, also lowered (by virtue of Modigliani's lifecycle theory of saving) the propensity to save. In principle, *Prosperity and Depression* would agree with its spirit.

Hence, nearly ten years after its publication, Samuelson was painfully aware that, unlike his 1939 articles, the 1988 KHS model did not attract a large readership despite his strong claims regarding its analytical achievements. Such claims for what he named the Keynes-Hansen-Samuelson multiplier–accelerator model may be found in the article, where he stated that "among other virtues", it "provides a nice fulfilment of the limit-cycle paradigm sought by Kaldor, Goodwin, and Hicks" (Samuelson 1988: 4). The fact that it came out in the first issue of a not well-known Japanese journal, probably contributed to the negligible impact of that article.[10]

Except for Pagano and Sbracia's (2014: 31, n. 38) brief reference to Samuelson's (1988) formalization of Hansen's secular stagnation, the KHS model has gone unnoticed in the recent revival of interest in secular stagnation. Before that, Rostow (1998: 213) – while discussing the Japanese stagnation of the 1990s in Hansenian terms – mentioned Samuelson's (1988) modelling of Keynes' and Hansen's intuitions about the perverse effects of lower population growth. More recently, Velupillai (2019: 357; 362–63) addressed some aspects of the KHS model as part of a chapter on Samuelson's macroeconomics. Velupillai discussed the assumptions leading to the KHS limit-cycle fluctuations while remarking that the growth path remains exogenous in that model despite Samuelson's intention to provide a mechanism of cyclical growth.

By the late 1980s, when Samuelson's KHS model came out, his 1939 multiplier-accelerator model, while retaining its status as a classic reference, had long lost its role as an influential account of economic fluctuations. Grandmont (1989: 279), for instance, mentioned Samuelson (1939a) as an example of expectations-driven endogenous deterministic cycles in Keynesian economics. According to Grandmont, since the 1980s, in contrast with earlier macroeconomic formulations, economists relied on "explicit modeling of the traders' optimizing behavior", which permitted analysis of "how expectations interact with the internal mechanisms of the economic system" to generate fluctuations. That was related to the fact that the "internal nonlinear dynamics" of the economy could generate complex periodic orbits, as well as to the existence of multiple stochastic equilibria produced by random factors (sunspots) that influence traders' expectations.

It is, therefore, hardly surprising that Samuelson (1988) would fail to attract attention when macroeconomists' nonlinear dynamics at the frontier of business cycle research differed from the Goodwin-like mathematical foundations of that article. Samuelson was not oblivious to such developments. He noticed that "these days it is fashionable to complain that various macroeconomic systems – like the General Theory or the present KHS model – lack rigorous microeconomic foundations. Fair enough" (Samuelson 1988: 12, n. 1). However, his main concern was the lack of proper macro-foundations, in the sense that the assumption of a given price level cannot be sustained when the economy approaches its full employment ceiling. Even so, the KHS model, he claimed, could be still acceptable if the central-bank reaction to accelerating inflation was considered to produce a bounce-back from the full-employment ceiling. Samuelson stuck to his guns while elaborating non-optimizing macroeconomic models along the methodological lines argued back in the Foundations. Hence, it is eventually not so surprising that Samuelson's 1939 multiplier-accelerator model has become a topic of research mostly for mathematical economists outside more mainstream circles (see, e.g. Westerhoff, 2006; Sushko et al., 2010; Matsumoto and Szidarovszk, 2015; Cavalli et al., 2019).

[10] The inaugural issue of *Japan and the World Economy*, edited by Ryuzo Sato, featured articles as well by other prominent economists, such as Paul Krugman and Hal Varian.

6. Author statement

I hereby declare that michaël Assous, Mauro Boianovsky and **Marwil J. Dávila-Fernández** am the sole author of this article and that we have not used any sources other than those listed in the bibliography and identified as references. We further declare that we have not submitted this article at any other journal for publication.

CRediT authorship contribution statement

Michaël Assous: Conceptualization, Supervision, Writing – original draft, Writing – review & editing, Formal analysis. **Mauro Boianovsky:** Conceptualization, Writing – original draft, Writing – review & editing. **Marwil J. Dávila-Fernández:** Conceptualization, Formal analysis, Writing – original draft, Writing – review & editing.

Data availability

No data was used for the research described in the article.

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