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Set Membership State Estimation with Quantized Measurements and Optimal Threshold Selection

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Abstract: The problem of state estimation with quantized measurements is addressed in the set membership estimation setting. The main contribution concerns the optimal selection of the quantizer thresholds in order to minimize the worst-case radius of the feasible state set. This allows one to design adaptive quantizers reducing the uncertainty associated to the state estimates. The proposed solution is applied to several outer approximations of the feasible sets, based on parallelotopes, zonotopes and constrained zonotopes. The benefits of the threshold selection mechanism are assessed on a numerical example, highlighting the trade off between computational burden and uncertainty reduction.

Keywords: State estimation, quantized measurements, adaptive quantizer, set membership estimation.

1. INTRODUCTION

System identification and state estimation based on quantized measurements have been the subject of an intense research activity in recent years. See, e.g., Wang et al. (2010); Godoy et al. (2011); Casini et al. (2012); Cerone et al. (2013); Bottegal et al. (2017) for system identification; Wong and Brockett (1997); Sviestins and Wigren (2000); Fu and de Souza (2009); Zhang et al. (2016) for state estimation. In control systems involving quantized measurements, it is well known that the adaptation of the quantizer thresholds can lead to a dramatic improvement of the control performance (Brockett and Liberzon (2000); Elia and Mitter (2001); Azuma and Sugie (2008)). The possibility of suitably tuning the quantizer thresholds in wireless sensor networks and industrial control networks has motivated the investigation of adaptive quantizers also for system identification (Aguero et al. (2007); Wang et al. (2008); Tsumura (2009); Marelli et al. (2013); You (2015)) and distributed sensing (Papadopoulos et al. (2001); Fang and Li (2008)).

State estimation with quantized measurements has been mainly studied in a stochastic framework. In the seminal work by Wong and Brockett (1997), the quantizer is designed in order to bound the covariance of the estimation error. Sviestins and Wigren (2000) derive the propagation of the state probability density function for a system of cascaded integrators. The joint design of the quantizer and the state observer is addressed by Fu and de Souza (2009). A variance-constrained state estimator with quantized measurements and probabilistic sensor failures is presented in Zhang et al. (2016).

An alternative approach can be envisaged by observing that the information provided by quantized sensor measurements is inherently set theoretic. This motivates the formulation of the state estimation problem in a deterministic setting. For example, Battistelli et al. (2017) propose a state observer based on binary measurements, for systems affected by unknown-but-bounded (UBB) disturbances. Set membership state estimators for linear systems with quantized measurements have been presented in Haimovich et al. (2004) and Zanma et al. (2018). It is worth recalling that the idea of tackling the problem in a set membership setting was already mentioned in Sviestins and Wigren (2000). However, in all these works the quantizer parameters are fixed a priori. Recently, the problem of suitably adapting the threshold of a binary sensor in set membership state estimation has been studied in Casini et al. (2024).

This paper addresses the problem of designing an adaptive quantizer for linear discrete-time systems affected by UBB disturbances. The problem is cast in the set membership estimation framework. The main contribution concerns the optimal design of the quantizer parameters (range and resolution) in order to minimize the uncertainty associated to the state estimates. To the best of our knowledge, this problem has never been studied in a set-theoretic framework. First, a complete solution is provided for first-order systems. Then, such a result is exploited to design adaptive quantizers for general n-th order systems. The threshold adaptation mechanism is also applied to outer approximations of the feasible state sets, provided by recursive approximation algorithms based on parallelotopes, zonotopes and constrained zonotopes. The effectiveness of the proposed techniques for adapting the quantizer parameters is demonstrated on a numerical example, analyzing

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the trade off between the resulting estimation uncertainty and the required computational load.

The rest of the paper is organized as follows. Section 2 introduces some basic set definitions and operations. The quantizer design problem is formulated in Section 3; its solution is given in Section 4 for first-order systems, and in Section 5 for general systems. Quantizer adaptations based on outer approximations of the feasible state set are devised in Section 6. A numerical example is presented in Section 7 and some final remarks are given in Section 8. Due to space limitations, the proofs of the technical results are reported in Casini et al. (2023).

2. SET DEFINITIONS AND OPERATIONS

Let us denote by $\|v\|_p$ the p-norm of $v \in \mathbb{R}^n$ (we omit the subscript when $p = 2$). $\mathcal{B}_\infty$ is the unit ball in the infinity norm, i.e. $\mathcal{B}_\infty = \{x : \|x\|_\infty \le 1\}$. For a generic set $\mathcal{V}$, its *radius* is defined as

$$\mathrm{rad}(\mathcal{V}) = \inf_z \sup_{v\in\mathcal{V}} \|z - v\|. \tag{1}$$

The argument $\hat{z}$ of the infimum in (1) is the *center* of $\mathcal{V}$. Given $\rho \in \mathbb{R}^n$ and $\gamma \in \mathbb{R}$, a *strip* is defined as

$$\mathcal{S}(\rho, \gamma) = \{x \in \mathbb{R}^n : |\rho' x - \gamma| \le 1\}. \tag{2}$$

A non-degenerate *parallelotope* is the intersection of n linearly independent strips. Given $c \in \mathbb{R}^n$ and a nonsingular matrix $P \in \mathbb{R}^{n\times n}$, it is defined as

$$\begin{aligned}\mathcal{P}(P, c) &= \{x \in \mathbb{R}^n : \|Px - c\|_\infty \le 1\} \\ &= \{x \in \mathbb{R}^n : x = x_c + T\alpha,\ \|\alpha\|_\infty \le 1\}\end{aligned} \tag{3}$$

where $T = P^{-1}$ and $x_c = P^{-1}c$. Its radius is given by

$$\mathrm{rad}(\mathcal{P}(P, c)) = \max_{w\in\mathcal{B}_\infty} \|P^{-1}w\|. \tag{4}$$

A *zonotope* is an extension of a parallelotope defined as

$$\mathcal{Z}(T, x_c) = \{x \in \mathbb{R}^n : x = x_c + T\alpha,\ \|\alpha\|_\infty \le 1\} \tag{5}$$

where $T \in \mathbb{R}^{n\times m}$ and $\alpha \in \mathbb{R}^m$. The columns of T are the generators of the zonotope and its order is $o = \frac{m}{n}$.
A *constrained zonotope* is an extension of a zonotope defined as

$$\begin{aligned}\mathcal{CZ}(T, x_c, A, b) = \{x \in \mathbb{R}^n : x = x_c + T\alpha,& \\ \|\alpha\|_\infty \le 1,\ A\alpha = b\}&\end{aligned} \tag{6}$$

where $A \in \mathbb{R}^{n_c\times m}$ and $b \in \mathbb{R}^{n_c}$. The order of a constrained zonotope is defined as $o = \frac{m-n_c}{n}$, where n_c is the number of equality constraints in (6).
Given sets $\mathcal{V}, \mathcal{W}$ and a matrix A, we define the set operations

$$\mathcal{V} + \mathcal{W} = \{z : z = v + w,\ v \in \mathcal{V},\ w \in \mathcal{W}\}, \tag{7}$$

$$A\mathcal{V} = \{z : z = Av,\ v \in \mathcal{V}\}. \tag{8}$$

3. PROBLEM FORMULATION

Consider the discrete-time linear system

$$x(k + 1) = Ax(k) + Gw(k) \tag{9}$$

$$z(k) = Cx(k) + v(k) \tag{10}$$

where $x(k) \in \mathbb{R}^n$ is the state vector, $z(k) \in \mathbb{R}^p$ is the (inaccessible) output, $w(k) \in \mathbb{R}^m$ is the process disturbance and $v(k) \in \mathbb{R}^p$ is the output noise. The process disturbance $w(k)$ and the output noise $v(k)$ are unknown-but-bounded (UBB) sequences, i.e. they satisfy

$$\|w(k)\|_\infty \le \delta_w \tag{11}$$

$$\|v(k)\| \le \delta_v \tag{12}$$

for all $k \ge 0$. To simplify the presentation, in the following a single output will be considered (i.e., $p = 1$). All the results presented in the paper can be applied to multi-output systems, by processing the quantized measurements sequentially.

Observations of the output signal are obtained from a quantized sensor with d thresholds $\tau_1, \dots, \tau_d$, such that

$$y(k) = \sigma(z(k)) \triangleq \begin{cases} 0 & \text{if } z(k) \le \tau_1 \\ 1 & \text{if } \tau_1 < z(k) \le \tau_2 \\ \vdots & \\ d & \text{if } z(k) > \tau_d \end{cases} \tag{13}$$

The quantization is assumed to be uniform in the range $[\tau_1, \tau_d]$, i.e., the sensor thresholds satisfy $\tau_{i+1} - \tau_i = \Delta$, $i = 1, \dots, d - 1$, for some $\Delta > 0$. For simplicity of exposition, we assume d is odd (hence, the number of possible outcomes of $y(k)$ is even). Therefore, τ_c with $c = (d + 1)/2$ is the center of the quantization range and the thresholds can be expressed as

$$\tau_i = \tau_c - \left(\frac{d+1}{2} - i\right)\Delta, \quad i = 1, \dots, d. \tag{14}$$

Hence, the quantizer is completely defined by the parameters $\theta = [\tau_c\ \Delta]'$.

According to the set membership estimation paradigm, the information provided by the quantized measurement $y(k)$ is captured by the *feasible measurement set*, i.e., the set of states $x(k)$ which are compatible with $y(k)$. By using (10) and (12), this is given by [1]

$$\begin{aligned}\mathcal{M}(k) = \{x \in \mathbb{R}^n\ :\ & Cx \le \tau_1 + \delta_v && \text{if } y(k) = 0; \\ & \tau_1 - \delta_v \le Cx \le \tau_2 + \delta_v && \text{if } y(k) = 1; \\ & \vdots && \\ & Cx \ge \tau_d - \delta_v && \text{if } y(k) = d\}.\end{aligned} \tag{15}$$

The set $\mathcal{M}(k)$ is a strip in the state space when $1 \le y(k) \le d - 1$, and a half-space when $y(k) = 0$ or $y(k) = d$ (notice that this occurs even in the noiseless case $\delta_v = 0$).

The *feasible state set* $\Xi(k|k)$, i.e. the set of all the state vectors $x(k)$ which are compatible with the quantized measurements collected up to time k, is defined by the recursion

$$\Xi(k|k) = \Xi(k|k - 1) \bigcap \mathcal{M}(k) \tag{16}$$

$$\Xi(k + 1|k) = A\Xi(k|k) + \delta_w G\mathcal{B}_\infty \tag{17}$$

which is initialized by the set $\Xi(0| - 1)$, containing all the feasible initial states $x(0)$. Hereafter, we will denote $\rho(k) = \mathrm{rad}(\Xi(k|k))$. By choosing the center of $\Xi(k|k)$ as a pointwise estimate of the state $x(k)$, $\rho(k)$ represents the maximum uncertainty associated to the estimate.

If the parameter vector θ of the quantizer is constant, the resulting state estimation uncertainty may be large, or even grow unbounded as in the following example.

[1] With a slight abuse of notation, strict inequalities are transformed in nonstrict ones, in order to deal only with closed feasible sets.

Example 1. Let $n = 1$, $A = G = C = 1$, $\delta_w = 1$, $\delta_v = 0$. Assume $x(0) = \frac{1}{2}$ and $w(k) = 1$, $\forall k \geq 0$, so that $z(k) = x(k) = \frac{1}{2} + k$. Consider the quantizer in (13) with $d = 3$, $\tau_1 = -1$, $\tau_2 = 0$, $\tau_3 = 1$. Then, $y(k) = 3$, $\forall k \geq 1$, and hence $\mathcal{M}(k) = \{x : x \geq 1\}$. By choosing $\Xi(0|-1) = [0, 1]$ and applying the recursion (16)-(17), one gets the sequence of feasible state sets $\Xi(k|k) = [1, k+1]$ which is asymptotically unbounded.

Example 1 suggests that, in order to reduce the estimation uncertainty (namely, the size of the feasible state set), it may be useful to adapt online the quantizer parameters. Let us assume that at each time k it is possible to select the parameters of the quantizer $\theta(k) = [\tau_c(k)\ \Delta(k)]'$. The aim is to choose $\theta(k)$ in order to minimize the size of the resulting feasible set $\Xi(k|k)$. Since $\mathcal{M}(k)$ depends on the actual value taken by the output $y(k)$, which in turn depends on the feasible predicted states $\Xi(k|k-1)$ and the realization of the measurement noise $v(k)$, a meaningful objective is to minimize the radius of $\Xi(k|k)$ with respect to the worst-case value taken by $y(k)$, i.e., to select

$$\theta^*(k) = \arg\inf_{\theta} \sup_{y(k)} \rho(k; \theta, y(k)) \tag{18}$$

where $\rho(k; \theta, y(k)) = \text{rad}\,(\Xi(k|k-1) \bigcap \mathcal{M}(k))$, in which the dependence of the radius on both θ and $y(k)$, through $\mathcal{M}(k)$ in (15), has been made explicit. Notice that in the information-based complexity framework, this corresponds to trigger the quantizer in order to minimize the so-called *radius of information* (see Milanese and Vicino (1991)). In the next section, the solution of problem (18) is derived for first-order systems.

4. FIRST-ORDER SYSTEMS

Let $n = m = 1$. Without loss of generality, consider the system

$$x(k+1) = ax(k) + w(k) \tag{19}$$

$$z(k) = x(k) + v(k) \tag{20}$$

with $|w(k)| \leq \delta_w$, $|v(k)| \leq \delta_v$, $\forall k \geq 0$. Moreover, let us assume $a > 0$. If $\Xi(0|-1) \subset \mathbb{R}$ is an interval, then all feasible sets $\Xi(k|k)$ and $\Xi(k+1|k)$ are also intervals. Hence, let us adopt the notation $\Xi(k|k) = [l(k), r(k)]$, whose radius is $\rho(k) = (r(k) - l(k))/2$.

The next theorem provides a solution of problem (18).

Theorem 1. Consider system (19)-(20) with the quantized sensor (13). If $a\rho(k-1) + \delta_w > \delta_v$, a solution of problem (18) is given by

$$\tau_c(k) = a\,\frac{l(k-1) + r(k-1)}{2}, \tag{21}$$

$$\Delta(k) = \frac{2}{d+1}\left(a\rho(k-1) + \delta_w - \delta_v\right). \tag{22}$$

When $a\rho(k-1) + \delta_w \leq \delta_v$, any $\theta(k)$ with $\tau_c(k)$ given by (21) and $\Delta(k) > 0$ is an optimal solution of (18).

Proof. See (Casini et al., 2023, Theorem 1). □

The rational behind the quantizer adaptation (21)-(22) is clarified by the following example.

Example 2. Let $a = 2$, $\delta_w = 1$, $\delta_v = 2$. Assume that at time $k-1$ one has $\Xi(k-1|k-1) = [-5,\ 5]$. Then, one gets $\Xi(k|k-1) = [-11,\ 11]$. By applying (21)-(22) with $d = 5$, one has $\tau_c(k) = 0$ and $\Delta(k) = 3$. The resulting thresholds $\tau_i(k)$ in (14) are equal to $-6, -3, 0, 3, 6$. It is worth stressing that such thresholds do not partition the predicted feasible set $\Xi(k|k-1)$ in equal parts. Rather, they are selected in such a way that the corrected feasible set $\Xi(k|k) = \Xi(k|k-1) \bigcap \mathcal{M}(k)$ has always the same radius, equal to $\frac{1}{2}\Delta(k) + \delta_v = \frac{7}{2}$, irrespectively of the sensor output $y(k)$, thus achieving the optimal solution of problem (18).

By using the result in Theorem 1 it is possible to establish conditions for the asymptotic boundedness of the feasible set $\Xi(k|k)$.

Theorem 2. By choosing the quantizer parameters as in (21)-(22), the radius of information $\bar{\rho}(k) = \sup_{y(k)} \rho(k)$ evolves according to

$$\bar{\rho}(k) = \begin{cases} a\bar{\rho}(k-1) + \delta_w & \text{if } \bar{\rho}(k-1) \leq \dfrac{\delta_v - \delta_w}{a}, \\ \dfrac{1}{d+1}\left(a\bar{\rho}(k-1) + \delta_w + d\delta_v\right) & \\ & \text{if } \bar{\rho}(k-1) \geq \dfrac{\delta_v - \delta_w}{a}. \end{cases} \tag{23}$$

Moreover, if $d > a - 1$, one has

$$\lim_{k\to+\infty} \bar{\rho}(k) = \begin{cases} \dfrac{1}{1-a}\delta_w & \text{if } \delta_w \leq (1-a)\delta_v, \\ \dfrac{1}{d+1-a}\left(\delta_w + d\delta_v\right) & \text{if } \delta_w > (1-a)\delta_v \end{cases} \tag{24}$$

and if $\delta_w > (1-a)\delta_v$,

$$\lim_{k\to+\infty} \sup_{y(k)} \Delta(k) = \frac{2}{d+1-a}\left(\delta_w + (a-1)\delta_v\right). \tag{25}$$

Proof. See (Casini et al., 2023, Theorem 2). □

Theorem 2 states that in order to guarantee asymptotic boundedness of the feasible state set, the number of thresholds of a uniform quantizer must exceed $a - 1$. Notice that if system (19) is stable, this condition is always satisfied. Conversely, in the special case of a binary sensor ($d = 1$), asymptotic boundedness holds if and only if $a < 2$.

Remark 1. If $a < 0$, Theorems 1 and 2 still hold by replacing a with $|a|$ in (22) and (23)-(25). If d is even, the results do not change, except that there will be no threshold coinciding with the center of $\Xi(k|k-1)$. In particular, the optimal Δ in (22) remains the same.

Remark 2. The result (24) in Theorem 2 can be seen as the set-theoretic counterpart of that in (Wong and Brockett, 1997, Th. 3) for a stochastic estimation framework. In fact, the cited result provides the minimum number n of bits for a coder-estimator guaranteeing asymptotic boundedness of the covariance of the state estimation error. Specifically, under the assumption that all the involved stochastic disturbances have a finite-support density function, it is shown that $2^n > a$ implies asymptotic stability of a coder estimator for a first-order system with pole a. By observing that an n-bit coder corresponds to a quantizer with $d = 2^n - 1$ thresholds, the condition $d > a - 1$ is recovered. In addition, Theorem 2 provides exact asymptotic expressions for the estimation uncertainty $\rho(k)$ and the quantizer resolution $\Delta(k)$, in the considered set-theoretic framework.

5. GENERAL CASE

The solution of problem (18) in the general case of n-th order systems is intractable, because it involves a nonconvex min-max optimization over n-dimensional sets. Nevertheless, we can leverage the solution for first-order systems, in order to solve an alternative formulation of the problem, which is both meaningful and computationally feasible.

Instead of minimizing the radius of the feasible state set at time k, the aim is to minimize the radius of the *feasible output signal set*, i.e., the set of feasible values that the nominal output $\bar{z}(k) = Cx(k)$ can take at time k. This is given by

$$\begin{aligned} C\Xi(k|k) &= C\left(\Xi(k|k-1)\bigcap\mathcal{M}(k)\right) \\ &= C\Xi(k|k-1)\bigcap C\mathcal{M}(k) \end{aligned} \tag{26}$$

where the last equality follows from the definition of $\mathcal{M}(k)$ in (15). Then, one can design the quantizer parameters according to

$$\begin{aligned} \theta^*(k) &= \arg\inf_{\theta}\sup_{y(k)} \mathrm{rad}(C\Xi(k|k)) \\ &= \arg\inf_{\theta}\sup_{y(k)} \mathrm{rad}\left(C\Xi(k|k-1)\bigcap C\mathcal{M}(k)\right). \end{aligned} \tag{27}$$

Since $C\Xi(k|k)$ in (26) is a one-dimensional interval, one can solve problem (27) by adopting the same strategy as in the case of first-order systems. In particular, the optimal quantizer parameters in Theorem 1 can be rewritten as

$$\tau_c(k) = \frac{l(k|k-1) + r(k|k-1)}{2}, \tag{28}$$

$$\Delta(k) = \frac{r(k|k-1) - l(k|k-1) - 2\delta_v}{d+1}, \tag{29}$$

where $l(k|k-1)$ and $r(k|k-1)$ are, respectively, the infimum and the supremum of the interval $\Xi(k|k-1)$. Similarly to (28)-(29), for an n-th oder system one can choose the quantizer parameters as

$$\tau_c(k) = \frac{\bar{l}(k) + \bar{r}(k)}{2}, \tag{30}$$

$$\Delta(k) = \frac{\bar{r}(k) - \bar{l}(k) - 2\delta_v}{d+1}, \tag{31}$$

where

$$\bar{l}(k) = \inf_{x\in\Xi(k|k-1)} Cx \ , \quad \bar{r}(k) = \sup_{x\in\Xi(k|k-1)} Cx. \tag{32}$$

In words, this choice corresponds to selecting the sensor thresholds in such a way that the feasible output signal set in (26) turns out to have the same radius, irrespectively of the value taken by the output $y(k)$, thus leading to the optimal solution of problem (27).

6. THRESHOLD SELECTION BASED ON SET APPROXIMATIONS

The main drawback of the threshold selection mechanism (30)-(32) is that it requires the propagation of the true feasible state set $\Xi(k|k)$ according to the recursion (16)-(17). Unfortunately, this is not computationally feasible, even for low state dimensions n. This is a well-known problem in the set membership estimation literature, which led to the development of a number of set approximation techniques. A popular approach is based on the choice of a class of approximating sets Π, from which an outer approximation of the true feasible set is selected at each time k through the recursion

$$\Pi(k|k) \supseteq \Pi(k|k-1)\bigcap\mathcal{M}(k) \tag{33}$$

$$\Pi(k+1|k) \supseteq A\Pi(k|k) + \delta_w G\mathcal{B}_\infty \tag{34}$$

which is initialized by picking a $\Pi(0|-1) \supseteq \Xi(0|-1)$. Many different set classes have been considered in the literature, including ellipsoids (Gollamudi et al. (1996); Durieu et al. (2001); El Ghaoui and Calafiore (2001)), parallelotopes (Chisci et al. (1996)), zonotopes (Alamo et al. (2005); Combastel (2015); Wang et al. (2019); Althoff and Rath (2021)) and constrained zonotopes (Scott et al. (2016)).

When an outer approximation of the feasible set is available, one can design the quantizer thresholds by applying the ideas developed in Section 5 to the approximating set $\Pi(k|k-1)$. In particular, $\tau_c(k)$ and $\Delta(k)$ can still be chosen according to (30)-(31), where $\bar{l}(k)$ and $\bar{r}(k)$ are computed as

$$\bar{l}(k) = \inf_{x\in\Pi(k|k-1)} Cx \ , \quad \bar{r}(k) = \sup_{x\in\Pi(k|k-1)} Cx. \tag{35}$$

Notice that $\bar{l}(k)$ and $\bar{r}(k)$ are easy to compute when the approximating sets Π are parallelotopes or zonotopes. If $x_c(k|k-1)$ and $T(k|k-1)$ denote respectively the center and the generator matrix associated to $\Pi(k|k-1)$, according to the notation in (3) and (5), one gets

$$\bar{l}(k) = Cx_c(k|k-1) - \|CT(k|k-1)\|_1 \tag{36}$$

$$\bar{r}(k) = Cx_c(k|k-1) + \|CT(k|k-1)\|_1 \tag{37}$$

and hence (30)-(31) boil down to

$$\tau_c(k) = Cx_c(k|k-1), \tag{38}$$

$$\Delta(k) = \frac{2}{d+1}(\|CT(k|k-1)\|_1 - \delta_v), \tag{39}$$

which can be easily computed at each time step. When constrained zonotopes (6) are used as approximating sets, the computation of $\bar{l}(k)$ and $\bar{r}(k)$ in (35) requires to solve two linear programs.

7. NUMERICAL EXAMPLE

The proposed threshold selection strategy technique is validated on a numerical example presented in Althoff (2021) as a benchmark for set membership state estimation. It is a six-tank system, whose dynamics have been linearized and discretized. A constant input flow is considered in the first tank and water levels are measured in the third and sixth tank. The resulting system matrices are

$$A = \begin{bmatrix} 0.939 & 0 & 0 & 0 & 0 & 0 \\ 0.061 & 0.98 & 0 & 0 & 0 & 0 \\ 0 & 0.02 & 0.955 & 0 & 0 & 0 \\ 0 & 0 & 0.045 & 0.875 & 0 & 0 \\ 0 & 0 & 0 & 0.125 & 0.92 & 0 \\ 0 & 0 & 0 & 0 & 0.08 & 0.969 \end{bmatrix},$$

$$G = [\,1\ 0\ 0\ 0\ 0\ 0\,]',$$

$$C = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The input flow is equal to 0.2 $[m^3/s]$ and is corrupted by a process disturbance satisfying (11) with $\delta_w = 0.01$. The output measurements are affected by a UBB noise satisfying (12) with $\delta_v = 0.2$. Both the process disturbance and the output noise are generated as uniformly distributed white processes, within the given bounds.

The threshold selection mechanism presented in Section 6 has been tested on different families of approximating sets Π. The following recursive approximation algorithms have been considered:

(a) the parallelotopic algorithm proposed in Chisci et al. (1996) (hereafter referred to as Par), with minimum volume intersection step (33) performed according to Vicino and Zappa (1996);
(b) the zonotopic algorithm presented in Alamo et al. (2005), denoted as $Zon(o_z)$ (where o_z is the maximum order of the zonotope), with minimum volume intersection step (33) performed according to Bravo et al. (2006);
(c) the technique based on constrained zonotopes proposed in Scott et al. (2016), denoted as $CZ(o_c, n_c)$ (where o_c is the maximum order of the constrained zonotope and n_c its maximum number of constraints), with intersection step (33) based on the minimization of the Frobenius norm of the zonotope generator matrix.

For both zonotopes and constrained zonotopes, the order reduction procedure is the one proposed in Scott et al. (2016). The $Zon(o_z)$ and $CZ(o_c, n_c)$ have been implemented by using the CORA toolbox for Matlab (Althoff (2021)). In all algorithms, at each time k the two output measurements are processed sequentially, assuming that it is possible to adapt the quantizer parameters independently for each measurement channel.

In order to compare the performance of the different approximating techniques, the radius of the resulting uncertainty interval associated to each state variable has been considered, namely $r_i(k) = \text{rad}(e_i'\Pi(k|k))$, $i = 1, \ldots, 6$, where e_i denotes the i-th column of the identity matrix. The values of $r_i(k)$ have been collected for 10 runs of 200 samples each, with different realizations of $w(k)$ and $v(k)$. The average values of $r_i(k)$ with respect to variables i, time k and multiple runs are provided in Table 1, for different number of thresholds d. In order to assess the benefit of employing time-varying thresholds, Table 2 reports the same values in the case of fixed thresholds, equally distributed in the range of the output signals $z(k)$. Table 3 shows the average relative computation times for one iteration of the algorithms, with respect to one iteration of Par. It can be seen that the threshold selection mechanism leads to a remarkable reduction of the uncertainty affecting the state estimates (on average, by approximately a factor 2.5). Another nice feature of the adaptive quantization is that the minimum uncertainty level compatible with each set approximation technique is achieved with a very small number of thresholds. In fact, in Table 1 uncertainty values do not significantly change for $d > 5$, while they keep reducing even for $d > 20$ when a fixed quantizer is employed. The comparison between set approximation techniques shows that constrained zonotopes provide smaller uncertainties, as expected, especially in the case of few fixed thresholds. On the other hand, their much higher computational complexity may prevent their application in online estimation schemes. Conversely, parallelotopic and zonotopic algorithms provide a good compromise between quality of the approximation and required computational burden, this being more evident in the case of an adaptive quantizer.

d	Par	Zon(2)	Zon(4)	CZ(2,5)	CZ(4,5)
3	0.542	0.483	0.42	0.361	0.305
5	0.485	0.441	0.381	0.361	0.304
10	0.464	0.420	0.364	0.358	0.302
15	0.468	0.417	0.361	0.366	0.300
20	0.475	0.415	0.357	0.363	0.302

Table 1. Average uncertainty intervals in the case of adaptive thresholds.

d	Par	Zon(2)	Zon(4)	CZ(2,5)	CZ(4,5)
3	8.854	2.951	1.502	0.684	0.501
5	2.137	1.243	0.860	0.534	0.427
10	0.895	0.807	0.652	0.422	0.345
15	0.664	0.687	0.596	0.373	0.321
20	0.570	0.658	0.542	0.339	0.289

Table 2. Average uncertainty intervals in the case of fixed thresholds.

Par	Zon(2)	Zon(4)	CZ(2,5)	CZ(4,5)
1	3.2	5.3	1587.9	2127.1

Table 3. Average relative computation times.

8. CONCLUSIONS

The design of adaptive quantizers for set membership state estimation has been addressed. The quantizer parameters are selected in such a way to minimize the size of the worst-case feasible state set. As it is common in set membership estimation, the problem presents a trade off between the computational load associated to the approximation of the feasible sets and the uncertainty associated to the state estimates. The adaptation of the quantizer parameters based on parallelotopic and zonotopic approximations turns out to be computationally feasible for online implementation and provides a remarkable uncertainty reduction with respect to quantizers with fixed thresholds. Ongoing research concerns the asymptotic behavior of the set theoretic observer and the extension of the proposed approach to nonlinear systems.

REFERENCES

Aguero, J.C., Goodwin, G.C., and Yuz, J.I. (2007). System identification using quantized data. In *46th IEEE Conference on Decision and Control*, 4263 –4268.

Alamo, T., Bravo, J.M., and Camacho, E.F. (2005). Guaranteed state estimation by zonotopes. *Automatica*, 41(6), 1035–1043.

Althoff, M. and Rath, J.J. (2021). Comparison of guaranteed state estimators for linear time-invariant systems. *Automatica*, 130, 109662.

Althoff, M. (2021). Guaranteed state estimation in CORA 2021. In *Proc. of the 8th International Workshop on Applied Verification of Continuous and Hybrid Systems.*

Azuma, S. and Sugie, T. (2008). Optimal dynamic quantizers for discrete-valued input control. *Automatica*, 44(2), 396–406.

Battistelli, G., Chisci, L., and Gherardini, S. (2017). Moving horizon estimation for discrete-time linear systems with binary sensors: algorithms and stability results. *Automatica*, 85, 374–385.

Bottegal, G., Hjalmarsson, H., and Pillonetto, G. (2017). A new kernel-based approach to system identification with quantized output data. *Automatica*, 85, 145–152.

Bravo, J.M., Alamo, T., and Camacho, E.F. (2006). Bounded error identification of systems with time-varying parameters. *IEEE Transactions on Automatic Control*, 51(7), 1144–1150.

Brockett, R.W. and Liberzon, D. (2000). Quantized feedback stabilization of linear systems. *IEEE transactions on Automatic Control*, 45(7), 1279–1289.

Casini, M., Garulli, A., and Vicino, A. (2012). Input design in worst-case system identification with quantized measurements. *Automatica*, 48(12), 2997–3007.

Casini, M., Garulli, A., and Vicino, A. (2023). Adaptive threshold selection for set membership state estimation with quantized measurements. *arXiv.org*, 2311.00611.

Casini, M., Garulli, A., and Vicino, A. (2024). Set membership state estimation for discrete-time linear systems with binary sensor measurements. *Automatica*, 159, 111396.

Cerone, V., Piga, D., and Regruto, D. (2013). Fixed-order FIR approximation of linear systems from quantized input and output data. *Systems & Control Letters*, 62(12), 1136–1142.

Chisci, L., Garulli, A., and Zappa, G. (1996). Recursive state bounding by parallelotopes. *Automatica*, 32, 1049–1055.

Combastel, C. (2015). Zonotopes and Kalman observers: Gain optimality under distinct uncertainty paradigms and robust convergence. *Automatica*, 55, 265–273.

Durieu, C., Walter, E., and Polyak, B. (2001). Multi-input multi-output ellipsoidal state bounding. *Journal of optimization theory and applications*, 111(2), 273–303.

El Ghaoui, L. and Calafiore, G. (2001). Robust filtering for discrete-time systems with bounded noise and parametric uncertainty. *IEEE Transactions on Automatic Control*, 46(7), 1084–1089.

Elia, N. and Mitter, S.K. (2001). Stabilization of linear systems with limited information. *IEEE transactions on Automatic Control*, 46(9), 1384–1400.

Fang, J. and Li, H. (2008). Distributed adaptive quantization for wireless sensor networks: From delta modulation to maximum likelihood. *IEEE Transactions on Signal Processing*, 56(10), 5246–5257.

Fu, M. and de Souza, C.E. (2009). State estimation for linear discrete-time systems using quantized measurements. *Automatica*, 45(12), 2937–2945.

Godoy, B.I., Goodwin, G.C., Aguero, J.C., Marelli, D., and Wigren, T. (2011). On identification of FIR systems having quantized output data. *Automatica*, 47(9), 1905 – 1915.

Gollamudi, S., Nagaraj, S., Kapoor, S., and Huang, Y. (1996). Set-membership state estimation with optimal bounding ellipsoids. In *Proceedings of the international symposium on information theory and its applications*, 262–265.

Haimovich, H., Goodwin, G.C., and Welsh, J.S. (2004). Set-valued observers for constrained state estimation of discrete-time systems with quantized measurements. In *2004 5th Asian Control Conference (IEEE Cat. No. 04EX904)*, volume 3, 1937–1945. IEEE.

Marelli, D., You, K., and Fu, M. (2013). Identification of arma models using intermittent and quantized output observations. *Automatica*, 49(2), 360–369.

Milanese, M. and Vicino, A. (1991). Estimation theory for nonlinear models and set membership uncertainty. *Automatica*, 27, 403–408.

Papadopoulos, H.C., Wornell, G.W., and Oppenheim, A.V. (2001). Sequential signal encoding from noisy measurements using quantizers with dynamic bias control. *IEEE Transactions on information theory*, 47(3), 978–1002.

Scott, J.K., Raimondo, D.M., Marseglia, G.R., and Braatz, R.D. (2016). Constrained zonotopes: A new tool for set-based estimation and fault detection. *Automatica*, 69, 126–136.

Sviestins, E. and Wigren, T. (2000). Optimal recursive state estimation with quantized measurements. *IEEE Transactions on Automatic Control*, 45(4), 762–767.

Tsumura, K. (2009). Optimal quantization of signals for system identification. *IEEE transactions on automatic control*, 54(12), 2909–2915.

Vicino, A. and Zappa, G. (1996). Sequential approximation of feasible parameter sets for identification with set membership uncertainty. *IEEE Transactions on Automatic Control*, 41, 774–785.

Wang, L.Y., Yin, G.G., Zhang, J.F., and Zhao, Y. (2008). Space and time complexities and sensor threshold selection in quantized identification. *Automatica*, 44(12), 3014 – 3024.

Wang, L.Y., Yin, G.G., Zhang, J.F., and Zhao, Y. (2010). *System Identification with Quantized Observations*. Springer.

Wang, Y., Wang, Z., Puig, V., and Cembrano, G. (2019). Zonotopic set-membership state estimation for discrete-time descriptor LPV systems. *IEEE Transactions on Automatic Control*, 64(5), 2092–2099. doi: 10.1109/TAC.2018.2863659.

Wong, W.S. and Brockett, R.W. (1997). Systems with finite communication bandwidth constraints - Part I: State estimation problems. *IEEE Transactions on Automatic Control*, 42(9), 1294–1299.

You, K. (2015). Recursive algorithms for parameter estimation with adaptive quantizer. *Automatica*, 52, 192–201.

Zanma, T., Ohtsuka, T., and Liu, K.Z. (2018). Set-based state estimation in quantized state feedback control systems with quantized measurements. *IEEE Transactions on Control Systems Technology*, 28(2), 550–557.

Zhang, Y., Wang, Z., and Ma, L. (2016). Variance-constrained state estimation for networked multi-rate systems with measurement quantization and probabilistic sensor failures. *International Journal of Robust and Nonlinear Control*, 26(16), 3507–3523.