# Complete Stability of Neural Networks With Extended Memristors

Mauro Di Marco, Mauro Forti, Riccardo Moretti, Luca Pancioni, and Alberto Tesi

***Abstract*— The article considers a large class of delayed neural networks (NNs) with extended memristors obeying the Stanford model. This is a widely used and popular model that accurately describes the switching dynamics of real nonvolatile memristor devices implemented in nanotechnology. The article studies via the Lyapunov method complete stability (CS), i.e., convergence of trajectories in the presence of multiple equilibrium points (EPs), for delayed NNs with Stanford memristors. The obtained conditions for CS are robust with respect to variations of the interconnections and they hold for any value of the concentrated delay. Moreover, they can be checked either numerically, via a linear matrix inequality (LMI), or analytically, via the concept of Lyapunov diagonally stable (LDS) matrices. The conditions ensure that at the end of the transient capacitor voltages and NN power vanish. In turn, this leads to advantages in terms of power consumption. This notwithstanding, the nonvolatile memristors can retain the result of computation in accordance with the in-memory computing principle. The results are verified and illustrated via numerical simulations. From a methodological viewpoint, the article faces new challenges to prove CS since due to the presence of nonvolatile memristors the NNs possess a continuum of nonisolated EPs. Also, for physical reasons, the memristor state variables are constrained to lie in some given intervals so that the dynamics of the NNs need to be modeled via a class of differential inclusions named differential variational inequalities.**

***Index Terms*— Complete stability (CS), differential variational inequalities, in-memory computing, linear matrix inequality (LMI), Lyapunov diagonally stable (LDS) matrices, Lyapunov method, memristor, neural networks (NNs).**

## I. Introduction

TRADITIONAL computers based on Von Neumann architectures are currently facing severe challenges to process big amount of data in the Internet of Things and edge computing systems [1], [2], [3], [4]. These are due to the Moore's law slowdown and the memory wall bottleneck, i.e., the difficulties to continuously transmit huge amount of data between the central processing unit (CPU) and the memory (e.g., the RAM) located at different sites. The use of emerging nanoscale devices as memristors is a long-term vision aimed at overcoming the limitations of Von Neumann architectures via the implementation of innovative analog and parallel neuromorphic computing paradigms [5], [6], [7]. Especially, nonvolatile memristors can handle their state for a long time and thus they enable to circumvent the memory bottleneck according to the in-memory computing principle [8], [9], [10]. Namely, the same memristor device has a key role in the computation but also in memorizing the result of computation, which is the same mechanism at the core of a biological brain.

The memristor was envisioned by Chua in the seminal 1971 article [11] as the fourth basic passive circuit element together with the resistor, capacitor, and inductor. An *ideal* voltage-controlled memristor is a circuit element defined by a nonlinear relation between flux (the integral of voltage) and charge (the integral of current). In the voltage-current domain it satisfies a state-dependent Ohm's law where the flux has the role of the state variable. The main limitation of ideal memristors is that they are unable to satisfactorily model real memristor devices in nanotechnology [12], [13], [14], [15], [16], [17]. Actually, the most effective way to accurately describe real devices is via more complex models named *extended* memristors [18],[1] which involve more general forms of the quasi-static Ohm's law and also additional internal state variables such as geometric features of the device or the temperature. One of the most popular and widely used real memristor models is the Stanford one [16], [20], [21], [22]. Its importance is due to the fact that it is able to accurately describe the hysteresis loops and nonlinear dynamics displayed by a broad class of filamentary resistive random access memory (RRAM) devices exhibiting bipolar switching characteristics (more details in Section II).

Following the definition in the seminal papers [23], [24], [25], and [26], by *complete stability* (CS) of a neural network (NN) it is meant that any trajectory converges toward an equilibrium point (EP) as $t \to \infty$. In view of the applications, an extremely important case of CS is that where the NN has *multiple EPs* and each solution tends to an EP depending upon the initial condition. In fact, in this case a NN is tailor made to implement content addressable memories (CAMs), where the stable EPs correspond to the memorized patterns, to solve combinatorial optimization problems, where the stable states are the local minima of the cost function, and to solve in real time several other tasks in the fields of image processing and adaptive signal processing [26], [27], [28]. The physical

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[1]A detailed discussion on the nomenclature, genealogy, and classification of memristors is available in [18] and [19].

mechanism of static pattern formation, i.e., the emergence in the long run of some stationary voltage distribution for the neurons, is also inherently related to CS [29]. Adding delays in the NN model further extends the application fields, since a completely stable delayed NN is well suited also to solve motion-related problems [30]. We refer the reader to [31] for a thorough survey of the main recent results in the literature on CS of NNs. It is worth mentioning that in the special case where there exists a *unique EP*, CS reduces to global asymptotic stability (GAS) [32]. However, GAS NNs will not be considered in this article.

CS of NNs without memristors has been widely investigated in the literature using the Lyapunov method, the dichotomy of omega limit sets for cooperative systems, the global consistency of decision schemes for competitive systems and the Łojasiewicz inequality and the concept of trajectories with finite length, see, e.g., [23], [24], [31], [33], [34], [35], [36], [37], and references therein. Quite on the contrary, the study of CS of memristor NNs (MNNs) is an important topic that is still in its infancy, as confirmed by the limited number of available studies. In [38], [39], and [40], CS of MNNs with ideal memristors has been investigated in the case of symmetric interconnections and in that of cooperative (positive) interconnections between neurons, using a technique referred to as flux-charge analysis method [19]. This method is used also in [41] and [42] for multistability of delayed cellular NNs with ideal memristors. Moreover, there are contributions on CS of NNs where memristors are modeled as elements switching between two values of the resistance displayed by the pinched hysteresis loop [43], [44]. We also mention the related relevant papers [45], [46] and [47] dealing with techniques for synchronization of various classes of NNs with delay and switching memristors. However, how to employ such switching models to describe real memristor devices has not been studied so far. Finally, [48] deals with CS of a class of MNNs where memristors obey the ThrEshold Adaptive Memristor (TEAM) model and [49] studies CS of a class of cellular MNNs with bistable neurons.

The previous discussion shows that there is a substantial lack of results on CS for NNs with general and reliable real memristor models. In turn, this represents a serious limitation in view of the practical implementation in nanotechnology of MNNs. Goal of this manuscript is to make a first step to fill this relevant gap. Indeed, in this article, we study for the first time CS of a large class of MNNs with nonvolatile extended memristors obeying the widely used and popular Stanford model. For generality, we also allow for the presence of concentrated delays in neuron interconnections. We will refer to these NNs as Stanford MNNs (SMNNs).

Due to the peculiar mathematical structure of the SMNNs equations, there are a number of challenges and mathematical difficulties to analyze CS as follows.

1) Each neuron is described by a third-order nonlinear dynamical system since, in addition to the capacitor voltage, there are two more state variables given by the length $g$ of the gap of the memristor insulating material and the temperature $T$ of the filament tip. Therefore, for $n$ neurons, an SMNN is described by a set of $3n$ coupled differential equations. This is remarkably more complex than in a traditional memristor-less NN, or in MNNs with ideal memristors, which can be described by a set of $n$ differential equations.
2) Due to the nonvolatility of the memristors, it follows that for structural reasons an SMNN has a continuum of nonisolated EPs, a new scenario calling for peculiar methods to address CS.
3) Due to physical limitations, one state variable of each neuron, i.e., the gap $g$, is constrained to lie in a given interval. Such hard constraints are mathematically described by a class of differential inclusions termed differential variational inequalities (DVIs) [50]. Their study thus needs nonstandard tools from nonsmooth analysis.

The main results in the article can be summarized as follows.

1) We provide sets of conditions ensuring CS of the considered class of MNNs. These hold for any concentrated delay and they are robust with respect to variations of the neuron interconnections.
2) The conditions for CS can be effectively tested numerically since they are expressed in the form of linear matrix inequalities (LMIs) [51]. For several relevant classes of interconnection matrices they can also be checked analytically via the concept of Lyapunov diagonally stable (LDS) matrices [52].
3) The conditions for CS guarantee that at the end of the transient the capacitor voltages vanish, hence power consumption of the SMNNs drops off at a steady state. This is a desirable feature for NNs used in the Internet of Things or edge computing systems [3], [4], where power efficiency is crucial. It is stressed that, although capacitor voltages vanish, the nonvolatile memristors can retain in memory the result of the SMNN computation, i.e., the asymptotic values of the memristors gaps, in accordance with the in-memory computing principle.

The article is organized as follows. In the remaining part of this section we give the notation and some preliminary mathematical results. Then, Section II discusses the Stanford memristor and SMNN models considered in the article. The main results on CS of SMNNs are given in Section III, while Section IV provides numerical simulations illustrating the applicability of the results on CS. Finally, Section V draws the main conclusions of the article.

*Notation:* If $x, y \in \mathbb{R}^n$ are column vectors, $\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i$ is the scalar product, while $\|x\| = \langle x, x\rangle^{1/2} = (\sum_{i=1}^{n} x_i^2)^{1/2}$ denotes the Euclidean norm of $x$ and $\|x\|_\infty = \max_{i=1,\dots,n} |x_i|$ is the infinity norm. Given $x \in \mathbb{R}^n$ and set $Q \subset \mathbb{R}^n$, we let $\text{dist}(x, Q) = \inf_{y\in Q} \|x - y\|$. If $A = A^\top \in \mathbb{R}^{n\times n}$, where superscript $\top$ means the transpose, is a symmetric square matrix, $\Lambda_m(A)$ (resp., $\Lambda_M(A)$) is the minimum eigenvalue (resp., maximum eigenvalue) of $A$. Moreover, $A > 0$ (resp. $A \geq 0$) means that $A$ is positive definite (resp. positive semidefinite). If $A \in \mathbb{R}^{n\times n}$ is (in general) nonsymmetric, $\|A\|_2 = [\Lambda_M(A^\top A)]^{1/2}$ is the induced two-norm of $A$. We denote by $C([-\tau, 0], \mathbb{R}^n)$ the space of continuous functions from $[-\tau, 0]$ to $\mathbb{R}^n$, where $\tau > 0$.

### A. Preliminaries

*1) Tangent and Normal Cones:* We recall some properties of tangent and normal cones that are useful in the article. The reader is referred to [50] for more details.

Let $Q \subset \mathbb{R}^n$ be a nonempty closed convex set. The tangent cone to $Q$ at $x \in Q$ is defined as

$$T_Q(x) = \left\{ z \in \mathbb{R}^n : \liminf_{\zeta \to 0^+} \frac{\text{dist}(x + \zeta z, Q)}{\zeta} = 0 \right\}$$

while the normal cone to $Q$ at $x \in Q$ is given by

$$N_Q(x) = \left\{ w \in \mathbb{R}^n : \langle w, z \rangle \le 0 \quad \forall z \in T_Q(x) \right\}.$$

It can be shown that, for any $x \in Q$, $T_Q(x)$ and $N_Q(x)$ are nonempty closed convex cones in $\mathbb{R}^n$. Furthermore, $N_Q(\cdot)$ is a monotone operator, i.e., for any $x, y \in Q$ and any $n_x \in N_Q(x)$, $n_y \in N_Q(y)$, we have $\langle x - y, n_x - n_y \rangle \ge 0$.

*2) Differential Variational Inequalities:* A DVI is a particular class of differential inclusion that is especially useful to model the dynamics of systems evolving in a closed convex subset of $\mathbb{R}^n$ defined for instance by some hard constraints. Next, we recall the definition and some basic properties of DVIs. For a more thorough treatment the reader is referred to [50, Ch. 5].

*Definition 1 [50, pp. 265]:* Let $Q \subset \mathbb{R}^n$ be a nonempty closed convex set and $F : Q \to \mathbb{R}^n$. A DVI is a problem of the form: find an absolutely continuous function $x(t)$, $t \in [t_1, t_2]$, such that $x(t) \in Q$ for all $t \in [t_1, t_2]$ and

$$\dot{x}(t) \in F(x(t)) - N_Q(x(t)) \tag{1}$$

for almost all (a.a.) $t \in [t_1, t_2]$.

*Property 1 [50, Th. 1, pp. 267]:* Let $Q \subset \mathbb{R}^n$ be a nonempty compact convex set. Also, suppose that $F : Q \to \mathbb{R}^n$ is continuous in $Q$. Then, for any initial condition (IC) $x_0 \in Q$, the DVI (1) has at least a solution $x(t)$, such that $x(0) = x_0$, which is defined for $t \ge 0$.

*3) Relevant Classes of Matrices:* In the article we will consider these classes of matrices $A \in \mathbb{R}^{n \times n}$ [53].

Matrix $A \in \mathcal{P}$ if all its principal minors are positive.

Matrix $A$ is said to be (positive) LDS if there exists a diagonal matrix $D \in \mathbb{R}^{n \times n}$, $D > 0$, such that $DA + A^\top D > 0$. If $A$ is LDS then $A \in \mathcal{P}$. The converse, however does not hold in general.

Matrix $A$ is an $M$-matrix if $a_{ij} \le 0$ for any $i \ne j$ and $A \in \mathcal{P}$, while $A$ is an $H$-matrix if the comparison matrix of $A$, i.e.,

$$M(A) = \begin{cases} |a_{ii}|, & i = j \\ -|a_{ij}|, & i \ne j \end{cases}$$

for $i, j = 1, \ldots, n$, is an $M$-matrix.

More details about $\mathcal{P}$ and LDS matrices and their relationships, and a discussion of their importance in the stability analysis of NNs and interconnected dynamical systems, can be found in [52].

## II. SMNN Model

### A. Stanford Memristor Model

Let $v(t)$ [resp., $i(t)$] be the voltage (resp., the current) in a memristor. Moreover, define the flux $\phi(t) = \int_{-\infty}^{t} v(\sigma) d\sigma$ and charge $q(t) = \int_{-\infty}^{t} i(\sigma) d\sigma$. An ideal flux-controlled memristor is defined by the constitutive relation $q = \hat{q}(\phi)$ [11], where $\hat{q} : \mathbb{R} \to \mathbb{R}$ is a nonlinear $C^1$ function. By differentiating in time, the memristor satisfies the quasi-static Ohm's law

$$i = \dot{q} = \hat{q}'(\phi) \frac{d\phi}{dt} = G_{\text{id}}(\phi) v \tag{2}$$

$$\dot{\phi} = v \tag{3}$$

where $G_{\text{id}}(\phi) \doteq \hat{q}'(\phi)$ has dimension of $\text{Ohm}^{-1}$ and is named (state dependent) memductance. The state variable is the flux $\phi$. When subject to a zero-mean sinusoidal voltage, an ideal memristor displays its main fingerprint consisting in a pinched hysteresis loop in the $v - i$ plane.

An extended memristor satisfies [18], [19]

$$i = G_{\text{ext}}(x, v) v \tag{4}$$

$$\dot{x} = F_{\text{ext}}(x, v) \tag{5}$$

where $x \in \mathbb{R}^n$ is a vector of state variables, $G_{\text{ext}}(\cdot, \cdot) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is the memductance and $F_{\text{ext}}(\cdot, \cdot) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$. It is further assumed that $G_{\text{ext}}(\cdot, \cdot)$ is bounded in a neighborhood of $(x, 0)$ for any $x \in \mathbb{R}^n$.

Clearly, an extended memristor has a more general form of quasi-static Ohm's law with respect to an ideal memristor. Moreover, it may have more than one internal state variable as defined by the state vector $x$. Due to (4), an extended memristor still displays a pinched hysteresis loop in the $v - i$ plane when subject to a zero-mean sinusoidal voltage.

We consider the popular and widely used Stanford memristor model [16], [20], [21], [22]

$$i = I_0 \exp\left(-\frac{g}{\tilde{g}}\right) \sinh\left(\frac{v}{V_0}\right) \tag{6}$$

$$\dot{g} = -v_0 \exp\left(-\frac{q_e E}{KT}\right) \sinh\left(\frac{q_e a_0 \psi}{\ell KT} v\right) \tag{7}$$

$$\dot{T} = \frac{vi}{C_{\text{th}}} - \frac{T - T_{\text{amb}}}{\Theta_{\text{th}}} \tag{8}$$

which accurately describes the switching mechanism and dynamics due to the formation/rupture of conductive filaments in a broad class of filamentary RRAM devices exhibiting bipolar switching characteristics.

The Stanford memristor state variables are $x = (g, T)^\top \in \mathbb{R}^2$, where the gap $g$ is the distance between the top electrode and the tip of the conductive filament and $T$ is the temperature of the filament tip. The model is characterized by a number of positive parameters. We refer to Table I for the nomenclature and for specific parameter values of some real memristor devices implemented in nanotechnology [54]. One first set of parameters $a_0, E, \ell, T_{\text{amb}}, \Theta_{\text{th}}$ and $C_{\text{th}}$ are named 'process parameters' since they depend upon the fabrication aspects as device structure and material properties and measurement setup. The second set $I_0, \tilde{g}, V_0, v_0$ an $\psi$ are named 'switching parameters' since they mainly describe the filament evolution (gap dynamics) and the hysteresis loops in response to periodic

TABLE I

STANFORD MEMRISTOR PARAMETERS AND TYPICAL VALUES FOR SOME REAL DEVICES

| Symbol | Unit | Value | Description |
|---|---|---|---|
| $a_0$ | m | $0.25 \times 10^{-9}$ | Atomic hopping distance |
| $E$ | eV | 1.25 | Activation energy for vacancy generation/recombination |
| $\ell$ | m | $5.0 \times 10^{-9}$ | Oxide thickness |
| $g_{\mathrm{m}}$ | m | $0.6 \times 10^{-9}$ | Minimum gap distance |
| $g_{\mathrm{M}}$ | m | $1.1 \times 10^{-9}$ | Maximum gap distance |
| $I_0$ | A | $2.0 \times 10^{-3}$ | $i - v$ fitting parameter |
| $\tilde{g}$ | m | $4.0 \times 10^{-10}$ | = |
| $V_0$ | V | 0.45 | = |
| $v_0$ | m/s | 500 | Gap dynamics fitting parameter |
| $\psi_0$ | – | 16.5 | = |
| $T_{\mathrm{amb}}$ | K | 298 | Ambient temperature |
| $C_{\mathrm{th}}$ | J/K | $3.18 \times 10^{-16}$ | Effective thermal capacitance |
| $\Theta_{\mathrm{th}}$ | s | $2.3 \times 10^{-10}$ | Effective thermal time constant |

inputs. Additionally, $K = 1.38 \times 10^{-23}$ J/K is the Boltzmann's constant and $q_e = 1.6 \times 10^{-19}$ C is the elementary unit charge. We refer the reader to [20], [21], [22], and [16] for a detailed discussion on the physical mechanisms described by (6)–(8).

Due to physical constraints, the filament gap $g$ has a lower and an upper bound [16]

$$g \in [g_m, g_M].$$

From a mathematical viewpoint, the best way to impose such hard constraint is to replace (7) with the following DVI [50]

$$\dot{g} \in -v_0 \exp\left(-\frac{q_e E}{KT}\right) \sinh\left(\frac{q_e a_0 \psi}{\ell K T} v\right) - N_{[g_m, g_M]}(g) \quad (9)$$

where $N_{[g_m, g_M]}(g)$, the normal cone to $[g_m, g_M]$ at $g$, ensures that $g$ evolves within $[g_m, g_M]$ for all times.

Henceforth, we suppose without loss of generality that $T(0) \geq T_{\mathrm{amb}}$. Note that in (8) the term $v(t)i(t) \geq 0$ for all $t$. It easily follows that $T(0) \geq T_{\mathrm{amb}}$ implies $T(t) \geq T_{\mathrm{amb}}$ for $t \geq 0$. Note that if power is turned off, i.e., we let $v = 0$ and $T = T_{\mathrm{amb}}$ (the ambient temperature), then we have $\dot{g} = 0$ and $\dot{T} = 0$, i.e., any point of the form

$$(\bar{g}, T_{\mathrm{amb}}), \bar{g} \in [g_m, g_M] \quad (10)$$

is an EP of (6)–(8). This implies that Stanford model is nonvolatile since it is able to retain in memory a continuum of nonisolated EPs as given in (10). Finally, note that Stanford model can be put in the form of an extended memristor by defining its memductance as $G_{\mathrm{ext}}(v, g) = I_0 e^{-g/\tilde{g}} \sinh(v/V_0)/v$ for $v \neq 0$ and $G_{\mathrm{ext}}(v, g) = (I_0/V_0)e^{-g/\tilde{g}}$ if $v = 0$.

### B. SMNN Equations

We consider a class of delayed MNNs with an additive interconnection structure where each neuron has a capacitor $C$ and a memristor satisfying Stanford model. The SMNNs obey the system of delayed DVIs

$$C\dot{v}_i(t) = -I_0 \exp\left(-\frac{g_i(t)}{\tilde{g}}\right) \sinh\left(\frac{v_i(t)}{V_0}\right) + \sum_{j=1}^{n} a_{ij} f(v_j(t)) + \sum_{j=1}^{n} a_{ij}^{\tau} f(v_j(t-\tau)) \quad (11)$$

$$\dot{g}_i(t) \in -v_0 \exp\left(-\frac{q_e E}{K T_i(t)}\right) \sinh\left(\frac{q_e a_0 \psi}{\ell K T_i(t)} v_i(t)\right) - N_{[g_m, g_M]}(g_i(t)) \quad (12)$$

$$\dot{T}_i(t) = \frac{I_0 \exp\left(-\frac{g_i(t)}{\tilde{g}}\right) v_i(t) \sinh\left(\frac{v_i(t)}{V_0}\right)}{C\mathrm{th}} - \frac{T_i(t) - T_{\mathrm{amb}}}{\Theta_{\mathrm{th}}} \quad (13)$$

for $i = 1, \ldots, n$, where $a_{ij}$ (resp., $a_{ij}^{\tau}$), $i, j = 1, \ldots, n$, are the neuron interconnections (resp., delayed neuron interconnections) and $0 < \tau < +\infty$ is a constant concentrated delay.

Moreover, the neuron activation $f : \mathbb{R} \to \mathbb{R}$ is such that $f(0) = 0$, it is Lipschitz continuous, i.e., $|f(\zeta_1) - f(\zeta_2)| \leq L_f|\zeta_1 - \zeta_2|$ for any $\zeta_1, \zeta_2 \in \mathbb{R}$, for some $0 < L_f < +\infty$ and it satisfies the following condition.

*Assumption 1:* For any $V > 0$ there exists $\gamma(V)$ such that

$$0 < \gamma(V) \leq \frac{f(v_i)}{v_i} \leq 1 \quad (14)$$

for all $v_i \in [-V, V]$, $v_i \neq 0$, and $i = 1, \ldots, n$.

Several widely used neuron activations satisfy Assumption 1. For example, the piecewise linear activation $f(\zeta) = (1/2)(|\zeta + 1| - |\zeta - 1|)$ of standard cellular NNs [26] satisfies this assumption with $\gamma(V) = \max\{1, 1/V\}$, while the sigmoidal activation $f(\zeta) = (2/\pi)\arctan(\pi\zeta/2)$ of Hopfield NNs [55] satisfies the same assumption with $\gamma(V) = 1/V$. Moreover, note that there are unbounded activations that satisfy Assumption 1, as for instance $f(\zeta) = \zeta$ if $\zeta \geq 0$, $f(\zeta) = (1/2)\zeta$ if $\zeta < 0$.

Note that (11)–(13) are $3n$ coupled differential equations in $3n$ state variables, i.e., the $n$ capacitor voltages $v_i$, the $n$ memristor gaps $g_i$ and the $n$ memristor temperatures $T_i$.

### C. Existence and Uniqueness of the Solution

Since a SMNN obeys a class of differential inclusions defined by a discontinuous and multivalued vector field, first of all, we have to discuss what is meant by a solution of an initial value problem (IVP) associated with the SMNN.

Let $v = (v_1, \ldots, v_n)^{\top}, g = (g_1, \ldots, g_n)^{\top}, T = (T_1, \ldots, T_n)^{\top}, \tilde{T}_{\mathrm{amb}} = (T_{\mathrm{amb}}, \ldots, T_{\mathrm{amb}})^{\top} \in \mathbb{R}^n$, $A =$

$[a_{ij}]$, $A^\tau = [a^\tau_{ij}] \in \mathbb{R}^{n\times n}$.[2] Moreover, introduce the following diagonal mappings:

1) $\mathfrak{F}(v) = (f(v_1), \dots, f(v_n))^\top : \mathbb{R}^n \to \mathbb{R}^n$
2) $\mathfrak{D}(v, g) = (\mathfrak{d}(v_1, g_1), \dots, \mathfrak{d}(v_n, g_n))^\top : \mathbb{R}^{2n} \to \mathbb{R}^n$, where for $i = 1, \dots, n$

$$\mathfrak{d}(v_i, g_i) = I_0 \exp\left(-\frac{g_i}{\tilde g}\right) \sinh\left(\frac{v_i}{V_0}\right). \tag{15}$$

3) $\mathfrak{G}(v, T) = (\mathfrak{g}(v_1, T_1), \dots, \mathfrak{g}(v_n, T_n))^\top : \mathbb{R}^{2n} \to \mathbb{R}^n$, where for $i = 1, \dots, n$

$$\mathfrak{g}(v_i, T_i) = -v_0 \exp\left(-\frac{q_e E}{K T_i}\right) \sinh\left(\frac{q_e a_0 \psi}{\ell K T_i} v_i\right). \tag{16}$$

4) $\mathfrak{T}(v, g) = (\mathfrak{t}(v_1, g_1), \dots, \mathfrak{t}(v_n, g_n))^\top : \mathbb{R}^{2n} \to \mathbb{R}^n$, where for $i = 1, \dots, n$

$$\mathfrak{t}(v_i, g_i) = \frac{I_0 \exp\left(-\frac{g_i}{\tilde g}\right) v_i \sinh\left(\frac{v_i}{V_0}\right)}{C\text{th}}. \tag{17}$$

Finally, let

$$D_m \doteq \frac{I_0}{V_0} \text{diag}\left(\exp\left(-\frac{g_M}{\tilde g}\right), \dots, \exp\left(-\frac{g_M}{\tilde g}\right)\right) \in \mathbb{R}^{n\times n}. \tag{18}$$

Note that the diagonal elements of $D_m$ are equal to the minimum value attained by the small signal memductance $(I_0/V_0)e^{-g_i/\tilde g}$ when $g_i \in [g_m, g_M]$, $i = 1, \dots, n$.

Then, we can rewrite the SMNN equations as

$$C\dot v(t) = -\mathfrak{D}(v(t), g(t)) + A\mathfrak{F}(v(t)) + A^\tau \mathfrak{F}(v(t-\tau)) \tag{19}$$
$$\dot g(t) \in \mathfrak{G}(v(t), T(t)) - N_\Gamma(g(t)) \tag{20}$$
$$\dot T(t) = \mathfrak{T}(v(t), g(t)) - \frac{T(t) - \tilde T_{\text{amb}}}{\Theta_{\text{th}}} \tag{21}$$

where

$$\Gamma \doteq [g_m, g_M]^n \subset \mathbb{R}^n.$$

*Definition 2 (IVP):* Consider function $\nu \in C^0([-\tau, 0], \mathbb{R}^n)$ and vectors $g_0, T_0 \in \mathbb{R}^n$ such that $g_{0,i} \in [g_m, g_M]$ and $T_{0,i} \geq T_{\text{amb}}$, $i = 1, \dots, n$. By a solution in $[-\tau, \tilde t]$ of the SMNN with ICs $\nu, g_0, T_0$, we mean a function $(v, g, T) : [-\tau, \tilde t] \to \mathbb{R}^{3n}$, such that:

1) $(v, g, T)$ is continuous in $[-\tau, \tilde t]$ and absolutely continuous in $[0, \tilde t]$;
2) we have $v(t) = \nu(t)$, $g(t) = g_0$, $T(t) = T_0$, $t \in [-\tau, 0]$;
3) $(v, g, T)$ satisfies (19)–(21) for a.a. $t \in [0, \tilde t]$; and
4) we have $g(t) \in \Gamma$ for all $t \in [0, \tilde t]$.

An EP of the SMNN is a constant solution $(\bar v, \bar g, \bar T)$, $t \geq -\tau$. It can be easily checked that any point in $\mathbb{R}^{3n}$ such that $\bar v_i = 0$, $\bar g_i \in [g_m, g_M]$ and $\bar T_i = T_{\text{amb}}$, $i = 1, \dots, n$, is an EP. Then, a SMNN has a continuum of nonisolated EPs.

Let

$$v_M \doteq \max\left\{1, V_0 \sqrt{6\left(\frac{V_0}{I_0} \exp\left(\frac{g_M}{\tilde g}\right)(M_F + \varepsilon) - 1\right)}\right\} \tag{22}$$

[2]Note that, with some abuse of notation, henceforth we use $v, g$ and $T$ to denote the vectors of voltages, gaps and temperatures, while in (6)–(8) the same notations are used for the scalar quantities given by the voltage, gap, and temperature of a single memristor.

$$T_M \doteq T_{\text{amb}} + \Theta_{\text{th}}\left(I_0 v_M \exp\left(-\frac{g_m}{\tilde g}\right) \sinh\left(\frac{v_M}{V_0}\right) + \varepsilon\right) \tag{23}$$

where $\varepsilon > 0$

$$M_F = \max_{i=1,\dots,n} \left(\sum_{j=1}^n |a_{ij}| + \sum_{k=1}^n |a^\tau_{ik}| V_F\right)$$
$$V_F = \max_{t\in[-\tau,0]} \|\nu(t)\|_\infty.$$

Consider sets

$$\Xi \doteq [-v_M, v_M]^n \subset \mathbb{R}^n, \quad \Upsilon \doteq [T_{\text{amb}}, T_M]^n \subset \mathbb{R}^n.$$

*Lemma 1:* Let $(v(t), g(t), T(t))$, $t \geq -\tau$, be a solution of an IVP associated with (19)–(21). Then, there exists $t_f < \infty$ such that $v(t) \in \Xi$ and $T(t) \in \Upsilon$ for $t \geq t_f$. Moreover, $\Xi \times \Gamma \times \Upsilon \subset \mathbb{R}^{3n}$ is positively invariant for the dynamics of (19)–(21).

*Proof:* See Appendix A. ■

Due to Lemma 1, there is no loss in generality if we study CS of SMNNs obeying the following delayed DVI

$$\dot v(t) \in -\mathfrak{D}(v(t), g(t)) + A\mathfrak{F}(v(t)) + A^\tau \mathfrak{F}(v(t-\tau)) - N_\Xi(v(t)) \tag{24}$$
$$\dot g(t) \in \mathfrak{G}(v(t), T(t)) - N_\Gamma(g(t)) \tag{25}$$
$$\dot T(t) \in \mathfrak{T}(v(t), g(t)) - \frac{T(t) - \tilde T_{\text{amb}}}{\Theta_{\text{th}}} - N_\Upsilon(T(t)) \tag{26}$$

where the ICs $\nu \in C^0([-\tau, 0], \Xi)$, $g_0 \in \Gamma$ and $T_0 \in \Upsilon$.

*Proposition 1:* There exists a unique solution of any IVP associated with a SMNN which is defined and bounded for $t \geq -\tau$.

*Proof:* See Appendix B. ■

Note that, although a SMNN obeys a DVI defined by a discontinuous and multivalued vector field, yet it enjoys the important property of uniqueness of the solution of any IVP.

## III. CS of SMNNs

In Section III-A and III-B we establish the main results on CS of a SMNN. Then, in Section III-C we provide a number of remarks to discuss the significance of the obtained results.

### A. Main Result

Next, we recall the classical definition of CS.

*Definition 3 ([23], [24], [25], [26]):* A SMNN is said to be completely stable if and only if the solution of any IVP converges to an EP as $t \to \infty$.

To study CS we enforce the following assumption on the interconnection and delayed interconnection matrices $A$ and $A^\tau$, and the diagonal matrix $D_m$ in (18).

*Assumption 2:* There exist a diagonal matrix $P \in \mathbb{R}^{n\times n}$, $P > 0$, and a matrix $Z = Z^\top \in \mathbb{R}^{n\times n}$, $Z > 0$, such that the following condition

$$S = \begin{pmatrix} 2PD_m - PA - A^\top P - Z & -PA^\tau \\ -A^{\tau\top} P & Z \end{pmatrix} > 0 \tag{27}$$

holds.

It can be readily verified that condition (27) amounts to solve an LMI feasibility problem with respect to the $n$ positive diagonal entries of the matrix $P$ and the $n(n+1)/2$ entries of the positive definite matrix $Z$. It is well known that there are effective numerical tools based on interior point polynomial methods for solving such LMI problems [31], [56]. Moreover, the particular structure of the matrix $S$ makes it possible to derive some necessary and simplified sufficient conditions for (27), as discussed in Section III-B.

The next fundamental result holds.

*Theorem 1:* Suppose that Assumptions 1 and 2 are satisfied. Let

$$k \doteq \min\left\{ \frac{\Lambda_m(S)\gamma^2(v_M)}{4\Lambda_M(P)}, \frac{1}{2\tau}\ln\left(1+\frac{\Lambda_m(S)}{2\Lambda_M(PA^\tau Z^{-1}(PA^\tau)^\top)}\right)\right\} > 0 \tag{28}$$

where $v_M$ is given in (22) and $\gamma(v_M)$ is defined in Assumption 1. Then, the solution of any IVP associated with the SMNN (24)–(26) converges exponentially to an EP $(\bar{v}, \bar{g}, \bar{T})$ depending upon the ICs and such that $\bar{v}=0$, $\bar{g}\in\Gamma$ and $\bar{T}=\tilde{T}_{\text{amb}}$. More precisely, the convergence rate can be estimated as follows. There exist $M_v, M_g, M_T > 0$ such that

$$\|v(t)\| \le M_v e^{-kt}$$
$$\|g(t)-\bar{g}\| \le M_g e^{-kt}$$
$$\|T(t)-\tilde{T}_{\text{amb}}\| \le M_T e^{-k_T t}$$

for $t \ge 0$, where $k_T = \min\{k, 1/\Theta_{\text{th}}\}$.

*Proof:* See Appendix C. ■

*Theorem 2:* A SMNN is completely stable if Assumptions 1 and 2 are satisfied.

*Proof:* Immediately follows from Theorem 1. ■

### B. Checking Conditions for CS

Since Assumption 2 plays a key role to assess CS via Theorem 2, in this section we investigate if it is possible to explicitly characterize classes of matrices $A$, $A^\tau$, and $D_m$ for which condition (27) holds. We first show that (27) admits an equivalent condition.

*Lemma 2:* The diagonal matrix $P\in\mathbb{R}^{n\times n}$, $P>0$, and the matrix $Z=Z^\top\in\mathbb{R}^{n\times n}$, $Z>0$, solve (27) if and only if the matrix

$$\Omega \doteq 2PD_m - PA - A^\top P - Z - PA^\tau Z^{-1}(PA^\tau)^\top \tag{29}$$

is positive definite.

*Proof:* It follows directly by applying Schur's lemma [51] to the matrix $S$. ■

Next, we investigate how both necessary and sufficient conditions ensuring that $\Omega>0$ can be derived using the class of LDS matrices (see Section I-A3). Assessing whether a given matrix is LDS in general amounts to solving a simplified LMI feasibility problem. Moreover, as shown later, there are relevant classes of matrices for which checking the LDS condition can be performed analytically via a finite number of inequalities.

*1) Necessary Conditions:* First of all, we observe that $\Omega>0$ if and only if

$$2PD_m - PA - A^\top P > Z + PA^\tau Z^{-1}(PA^\tau)^\top \tag{30}$$

where the right-side term is positive definite for all $Z>0$. Hence, inequality (30) can be solved for $P$ and $Z$ only if $P(D_m-A)+(D_m-A^\top)P>0$, which implies that a necessary condition for Assumption 2 is that $D_m-A$ is LDS.

Indeed, a sharpened necessary condition is derived next.

*Proposition 2:* Assumption 2 holds only if both matrices $D_m-A-A^\tau$ and $D_m-A+A^\tau$ are LDS.

*Proof:* We first observe that the condition

$$(PA^\tau \pm Z)Z^{-1}(PA^\tau \pm Z)^\top \ge 0$$

is satisfied for all $P$ and $Z>0$. Hence, we have

$$Z + PA^\tau Z^{-1}(PA^\tau)^\top \ge \mp\left(PA^\tau + (PA^\tau)^\top\right)$$

which implies that $\Omega>0$ only if

$$2PD_m - PA - A^\top P \pm \left(PA^\tau + (PA^\tau)^\top\right) > 0$$

thus completing the proof. ■

*2) Sufficient Conditions:* A sufficient condition ensuring that $\Omega>0$ can be derived by finding a diagonal matrix $P$, $P>0$, such that $D_m-A$ is LDS and condition (30) is satisfied for some choice of the matrix $Z$, $Z>0$. Suppose that $A^\tau$ is nonsingular and assume that $Z$ enjoys the following structure

$$Z = Z_0 \doteq \alpha(A^\tau)^\top PA^\tau \tag{31}$$

where $\alpha$ is a positive real parameter. Clearly, $Z_0>0$ for all $\alpha>0$. By replacing $Z$ with $Z_0$ in (30) we have

$$2PD_m - PA - A^\top P > \alpha(A^\tau)^\top PA^\tau + \frac{1}{\alpha}P. \tag{32}$$

Clearly, if $P$ is a solution of (32) for a given $\alpha>0$, then $\Omega>0$ once $Z=Z_0$. We observe that for a given $P>0$ condition (32) holds for some $\alpha>0$ only if it is satisfied for the value of $\alpha$ which makes its right-side as less positive definite as possible. This can be pursued by noting that the following condition

$$\alpha\|A^\tau\|_2^2\Lambda_M(P) + \frac{1}{\alpha}\Lambda_M(P) \ge \alpha(A^\tau)^\top PA^\tau + \frac{1}{\alpha}P \tag{33}$$

holds for all $\alpha>0$. Hence, a suitable choice for minimizing the right-side term of (32) consists in selecting for $\alpha$ the value which minimizes the left-side of (33), which amounts to

$$\alpha = \frac{1}{\|A^\tau\|_2}. \tag{34}$$

This leads to the next result.

*Proposition 3:* Let $A^\tau$ be nonsingular. Then, Assumption 2 holds if there exists a diagonal matrix $P$, $P>0$, such that

$$2PD_m - PA - A^\top P - \frac{1}{\|A^\tau\|_2}(A^\tau)^\top PA^\tau - \|A^\tau\|_2 P > 0. \tag{35}$$

*Proof:* It is enough to replace $\alpha$ in (32) with the expression in (34). ■

The above sufficient condition requires the solution of an LMI feasibility problem involving the $n$ positive entries of $P$

and the assumption that $A^\tau$ is nonsingular. Indeed, if $A^\tau$ is singular then a similar reasoning can be applied by adding to $Z_0$ an arbitrarily small diagonal positive definite matrix. The analysis requires some technical developments and it is omitted for space limitations.

Now, we derive a sufficient condition which is more simple to be checked and it covers also the case when $A^\tau$ is singular. Indeed, we exploit the fact that if $P > 0$ is such that $D_m - A$ is LDS then there exists $\delta > 0$ ensuring that

$$2PD_m - PA - A^\top P \geq \delta \Lambda_M(P) I_n \tag{36}$$

where $I_n$ is the $n$th-order identity matrix.

*Proposition 4:* Let $P > 0$ satisfy condition (36) for some $\delta > 0$. Then, Assumption 2 holds if

$$\|A^\tau\|_2 < \frac{\delta}{2}. \tag{37}$$

*Proof:* Let

$$Z = \Lambda_M(P)\|A^\tau\|_2 I_n$$

where $P$, $P > 0$, solves (36) for some $\delta$. According to this choice of $Z$, (30) boils down to

$$2PD_m - PA - A^\top P > \Lambda_M(P)\|A^\tau\|_2 I_n + \frac{1}{\Lambda_M(P)\|A^\tau\|_2} PA^\tau (PA^\tau)^\top.$$

To prove that the above condition is satisfied, it is enough to observe that (37) ensures that $P$ is such that

$$2PD_m - PA - A^\top P > 2\Lambda_M(P)\|A^\tau\|_2 I_n$$

and to verify that the following condition

$$\Lambda_M(P)\|A^\tau\|_2 I_n \geq \frac{1}{\Lambda_M(P)\|A^\tau\|_2} PA^\tau (PA^\tau)^\top$$

holds true. ■

Clearly, the larger is $\delta$ the larger is the set of matrices $A^\tau$ to which Proposition 4 applies. In the next theorem, which summarizes and at the same time improves the previous sufficient conditions, we address the problem of choosing the optimal value of $\delta$.

*Theorem 3:* Suppose that Assumption 1 is satisfied and $D_m - A$ is LDS. Let the diagonal matrix $P^*$, $P^* > 0$, be such that $(D_m - A^\top)P^* + P^*(D_m - A) > 0$ and let

$$d^* = \frac{1}{2}\Lambda_{\mathrm{m}}\left[\left((D_m - A^\top)P^* + P^*(D_m - A)P^*\right)(P^*)^{-1}\right] > 0. \tag{38}$$

Then, a SMNN is CS for all the matrices $A^\tau$ satisfying

$$\|A^\tau\|_2 < d^* \frac{\Lambda_m(P^*)}{\Lambda_M(P^*)}. \tag{39}$$

*Proof:* Consider

$$2P^*D_m - P^*A - A^\top P^* \geq 2dP^* \tag{40}$$

where $d$ is a positive scalar. It is seen that $d^*$ in (38) is the maximum value of $d$ such that (40) holds. Now, exploiting the relation $[\Lambda_m(P^*)]^{-1}P^* \geq I_n$ we have

$$2P^*D_m - P^*A - A^\top P^* \geq 2d^*\Lambda_m(P^*)I_n.$$

Hence, by comparing the above condition with (36) it can be concluded that, once $d^*$ and $P^*$ are available, condition (37) boils down to (39). This ensures that Assumption 2 is satisfied and a SMNN is CS due to Theorem 2. ■

We stress that Theorem 3 provides a robustness condition for CS with respect to the magnitude of the induced matrix 2-norm of $A^\tau$. The theorem assumes that $D_m - A$ is LDS, whose verification in general requires to solve an LMI problem with $n$ unknowns. However, there are relevant classes of matrices for which $D_m - A$ is LDS is equivalent to $D_m - A \in \mathcal{P}$ and hence it can be simply checked via a finite number of inequalities (see Section I-A3).

*Theorem 4:* Suppose that Assumption 1 is satisfied. Then, the result on CS in Theorem 3 holds if (39) is met and one of the following conditions is satisfied:

1) $D_m - A$ is symmetric and positive definite, i.e., $D_m - A \in \mathcal{P}$;
2) $D_m - A$ is an $M$-matrix;
3) $D_m - A$ is a nonsingular $H$-matrix with nonnegative diagonal entries;
4) $D_m - A$ is an acyclic $\mathcal{P}$ matrix; and
5) $D_m - A$ is a skew symmetric matrix with positive diagonal entries.

*Proof:* All the stated conditions ensure that $D_m - A$ is LDS [52]. ■

### *C. Discussion*

1) To the authors knowledge, Theorems 2–4 are the first results on CS of MNNs with a class of extended memristors. The memristors obey Stanford model and they can accurately describe the behavior of real memristor devices. These theorems are not only of theoretic interest, but they can also be used to design practical MNNs in nanotechnology enjoying the important property of CS.
2) Previous papers [48] and [49] deal with CS of MNNs with the TEAM memristor model and a memristor model used in ameba learning. Such models are not as general and effective as the Stanford model to describe the dynamics of real memristor devices [16]. Moreover, [49] deals with CS of a class of NNs without delay and with uncoupled second-order cells.
3) According to Theorems 1–4, a SMNN is not only completely stable but in addition each solution is exponentially convergent to an EP with a known convergence rate not depending on the ICs. This enables to quantitatively estimate the convergence time, which is extremely useful in view of the practical applications to the solution in real time of signal processing problems. It is also noticed that exponential convergence holds even if a SMNN has a continuum of EPs (see Section II-C).
4) We stress that the approach based on LDS matrices in Theorems 3 and 4 enables to single out entire classes of matrices and ranges of interconnections parameters for which a SMNN is CS. Furthermore, such results hold for a general SMNN dimension $n$. We will further illustrate these features in Section IV. Theorem 4 can

be immediately extended to any other class of matrices enjoying the LDS property [52].

5) Theorems 1–4 ensure that exponential stability and CS of SMNNs enjoy fundamental robustness properties. First of all, exponential stability and CS are insensitive to the presence of arbitrary constant delays. This is of potential interest for the applications, because delays are subject to large uncertainties due to the difficulties to accurately measure them [57]. Moreover, the condition $D_m - A$ is LDS is robust with respect to small variations of $A$. Finally, if $D_m - A$ is LDS, Theorem 3 shows that there is a robustness margin with respect to the norm of the delayed interconnection matrix $A^\tau$.
6) Due to Theorem 1, we have that $T_i(t) \to T_{\text{amb}}$ and $v_i(t) \to 0$, $i = 1, \dots, n$, as $t \to \infty$. This means that, when a steady state is reached, the memristor is at the ambient temperature, moreover, voltages, currents and power in the SMNN vanish. This is a remarkable advantage over traditional NNs as cellular NNs or Hopfield NNs where the asymptotic values of voltages are not zero and the NNs consume power also at a steady state. It is worth to stress that, although voltages vanish, yet the nonvolatile memristors can retain in memory the result of the computation, i.e., the asymptotic values $\bar{g}_i$, $i = 1, \dots, n$, of the gaps, according to the principle of in-memory computing. In practice, the values $\bar{g}_i$ can be measured by a standard technique for evaluating the small signal memductances $(I_0/V_0)e^{-\bar{g}_i/\tilde{g}}$ displayed by the memristors [58].
7) CS of a NN, i.e., convergence of each trajectory toward an EP, is a peculiar dynamical property that can be guaranteed under suitable hypotheses on the activations and neuron interconnections (see Assumptions 1 and 2 in Theorem 1). On the other hand, in the general case, NNs do not enjoy CS since they can display sustained oscillations, traveling waves and even chaos and hyperchaos, see, e.g., [29], [59], [60], [61], [62], [63], [64], [65], [66], and references therein.

## IV. Numerical Simulations

As an application example of the bound (39) for CS of SMNNs, we focus on the case where $D_m - A$ belongs to the following class of matrices with a cyclic structure

$$C_{r,s} = \begin{pmatrix} r & 0 & \cdots & 0 & s \\ -s & r & \ddots & & 0 \\ 0 & -s & r & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -s & r \end{pmatrix} \in \mathbb{R}^{n\times n} \tag{41}$$

where $r > 0$, $s > 0$. Matrices in this class play an important role in modeling cyclic dynamical systems, food chains, chemical reactions, and also to investigate nonlinear dynamic phenomena and the potentials and limitations for information processing of cellular NN arrays [67], [68].

Next, we report known conditions ensuring that these matrices are LDS (see [69]).

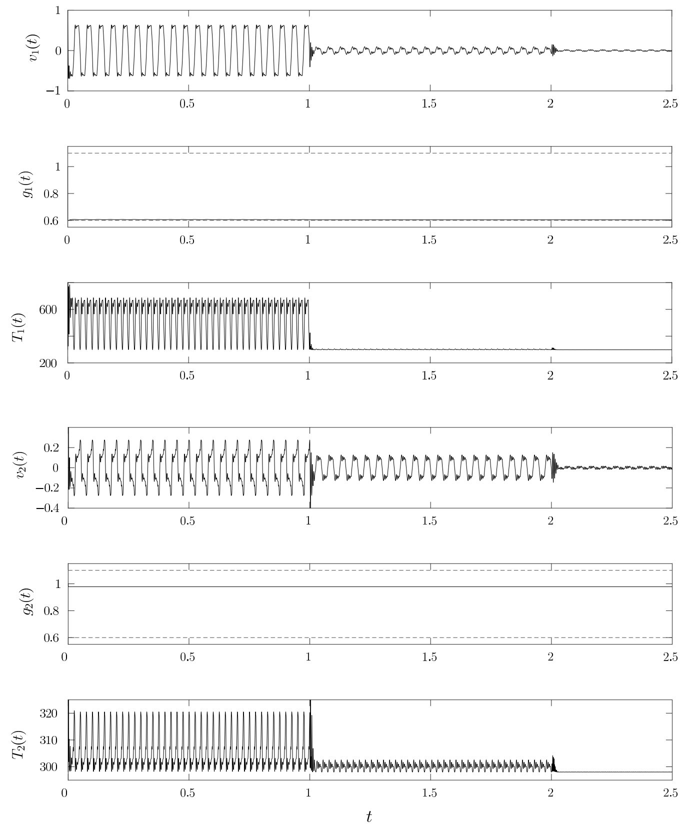


Fig. 1. Time evolution of memristor voltages $v_i$ [V], gaps $g_i$ [nm] and temperatures $T_i$ [K], $i = 1, 2$ for a 5-neuron SMNN with a cyclic interconnecting structure. Time $t$ in $\mu$s.

*Lemma 3:* Matrix $C_{r,s}$ is LDS if and only if

$$r - s\cos\left(\frac{\pi}{n}\right) > 0. \tag{42}$$

Exploiting this condition, we obtain the following result as a consequence of Theorem 3.

*Corollary 1:* Consider a SMMN where $D_m - A = C_{r,s}$ satisfies (42) and suppose that Assumption 1 holds. Then, the SMNN is CS for all $A^\tau$ such that

$$\|A^\tau\|_2 < r - s\cos\left(\frac{\pi}{n}\right). \tag{43}$$

*Proof:* The proof follows by computing the right-side of (39) for $D_m - A = C_{r,s}$ and $P^* = I_n$. Indeed, we have $\Lambda_M(P^*) = \Lambda_m(P^*) = 1$ and

$$d^* = \Lambda_m\left(\frac{1}{2}\left(C_{r,s}^\top + C_{r,s}\right)\right).$$

Exploiting the eigenvalues-eigenvectors properties of $C_{r,s}$ we obtain

$$\Lambda_m\left(\frac{1}{2}\left(C_{r,s}^\top + C_{r,s}\right)\right) = r - s\cos\left(\frac{\pi}{n}\right)$$

which completes the proof. ■

It is worth noting that (43) provides the tightest bound on $\|A^\tau\|_2$ for which Assumption 2 holds, when $D_m - A = C_{r,s}$. In fact, consider $A^\tau = \lambda I_n$ with $\lambda > 0$. We have

$$D_m - A - A^\tau = C_{r-\lambda,s}$$

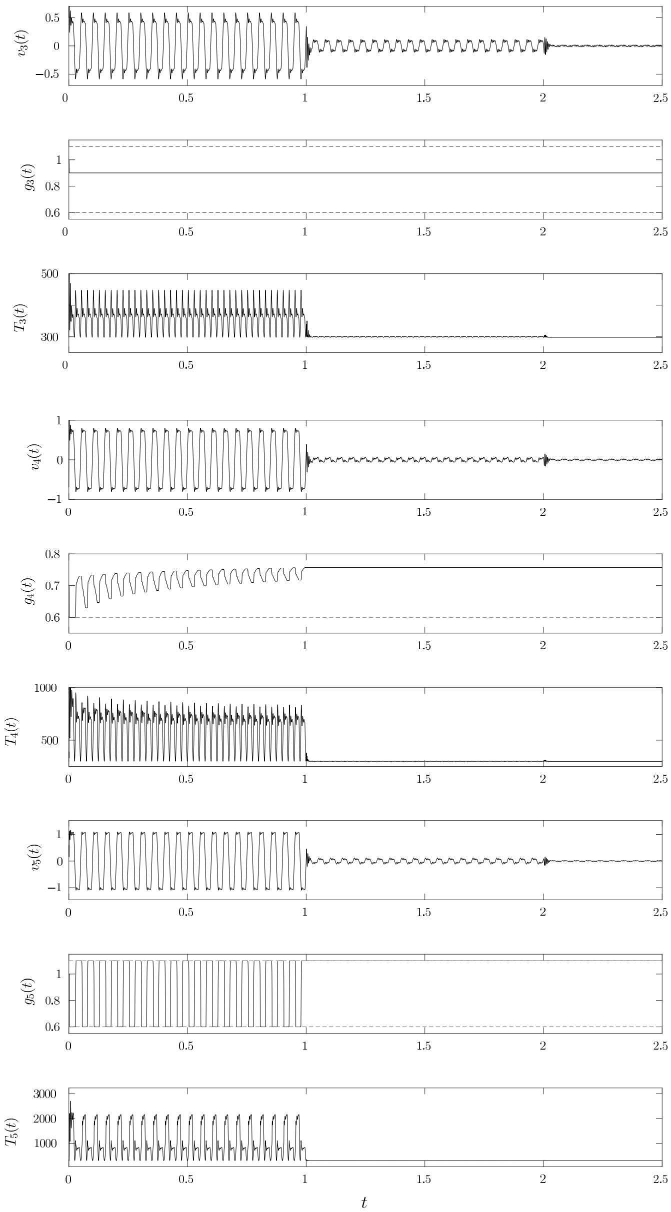


Fig. 2. Time evolution of memristor voltages $v_i$ [V], gaps $g_i$ [nm] and temperatures $T_i$ [K], $i = 3, 4, 5$, for a 5-neuron SMNN with a cyclic interconnecting structure. Time $t$ in $\mu$s.

and hence, according to Lemma 3, the necessary condition of Proposition 2 is satisfied if and only if

$$\lambda < r - s\cos\left(\frac{\pi}{n}\right).$$

Since $\|A^\tau\|_2 = \lambda$, Assumption 2 holds if and only if (43) is satisfied.

For illustration purposes, choose $A^\tau$ with the following structure:

$$A^\tau = \begin{pmatrix} u & 0 & \cdots & 0 & u \\ u & u & \ddots & & 0 \\ 0 & u & u & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & u & u \end{pmatrix} \in \mathbb{R}^{n\times n}. \quad (44)$$

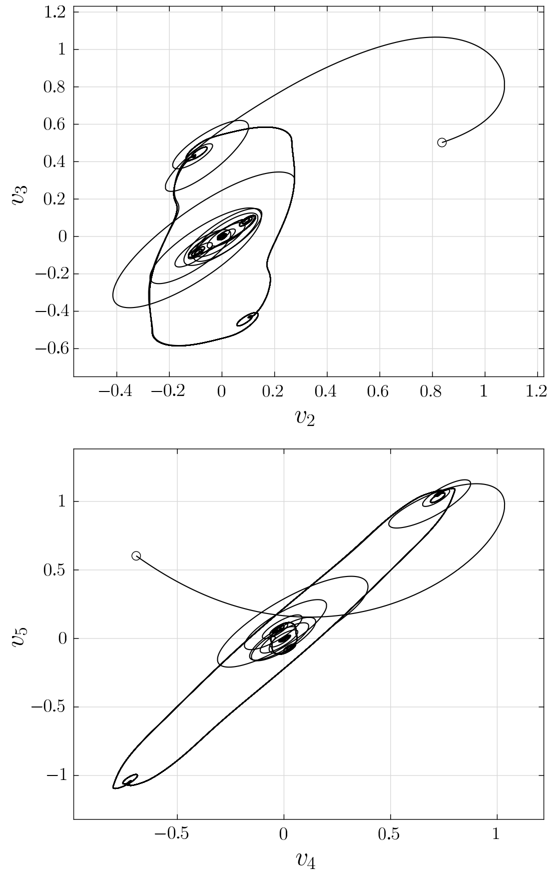


Fig. 3. Voltage $v_3$ as a function of $v_2$ (upper plot) and $v_5$ as a function of $v_4$ (lower plot) for a 5-neuron SMNN. The IC is marked with a circle. Voltages in V.

It can be easily checked that $\|A^\tau\|_2 = 2|u|$, hence due to Corollary 1, we have that Assumption 2 is satisfied and the SMNN is CS if (42) holds and in addition

$$|u| < u_M \doteq \frac{r}{2} - \frac{s}{2}\cos\left(\frac{\pi}{n}\right).$$

This demonstrates that it is easy to find in an analytic way, via the developed LDS approach, conditions for CS that hold for an entire class of $A$ and $A^\tau$, for open ranges of interconnection parameters and for any dimension $n$.

Let $n = 5$ neurons, $\tau = 10^{-6}$ s, $C = 10^{-9}$ F and consider the neuron activation $f(\gamma) = (1/2)(|\gamma+5| - |\gamma-5|)$, which satisfies Assumption 1. Also, refer to the memristor parameters in Table I. Note in particular that the gap $g \in [g_m, g_M] = [0.6, 1.1]$ nm.

The EPs of the SMNN are such that the voltages $\bar{v}_i = 0$, the temperatures $\bar{T}_i = T_{\text{amb}}$, while the gap $\bar{g}_i$ can assume any value in $[g_m, g_M]$, $i = 1, \ldots, 5$. Note that there is a continuum of nonisolated EPs.

Let $A = -C_{p,q}$ with $p = 1.83, q = 2$. We have $D_m = 2.84\cdot 10^{-4} I_5$, hence $D_m - A = C_{r,s}$ with $r = 1.8303$, $s = 2$ and so $u_M = 0.2123$. Next, we fix $u = 0.1$, so that $2u < u_M$. We have simulated the SMNN using MATLAB routine `dde23` for delayed differential equations. The hard constraints $g_i \in [g_m, g_M]$ for the neuron gaps are imposed numerically using the "Events" option in MATLAB environment (see [70]). The ICs $v_i(t) \in C([-10^{-6}, 0], \mathbb{R})$ for the neuron voltages are

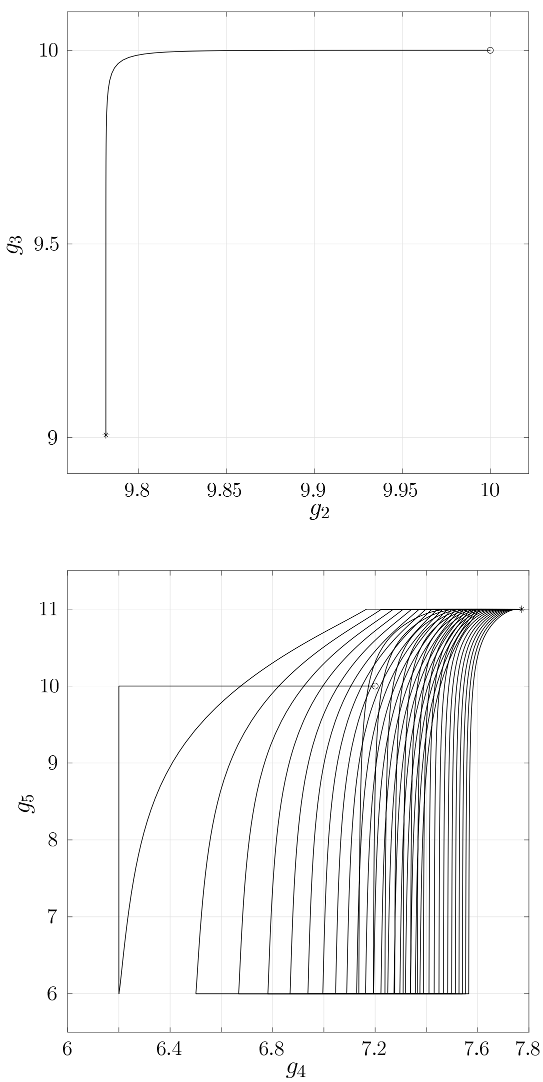


Fig. 4. Gap $g_3$ as a function of $g_2$ (upper plot) and $g_5$ as a function of $g_4$ (lower plot) for a 5-neuron SMNN. The IC is marked with a circle and the final state with an asterisk. Gaps in nm.

sinusoidal functions with angular frequency $20/\tau$ and with various phases and amplitudes (details are omitted). The ICs of the gaps are $g_1(0) = g_2(0) = g_3(0) = g_5(0) = 1$ nm, while $g_4(0) = 0.7$ nm. The ICs of the temperatures are $T_i(0) = T_{\mathrm{amb}} = 298$ K for $i = 1, \ldots, 5$. Figs. 1 and 2 depict the time-domain behavior of memristor voltages $v_i$, gaps $g_i$ and temperatures $T_i$, $i = 1, \ldots, 5$, in the time interval $[0, 2.5 \times 10^{-6}] = [0, 2.5\tau]$. Fig. 3 reports $v_3$ as a function of $v_2$ and $v_5$ as a function of $v_4$, while Fig. 4 shows $g_3$ as a function of $g_2$ and $g_5$ as a function of $g_4$. It is seen that after some transient oscillations $v_i$, $g_i$ and $T_i$ tend to constant values. In particular, voltages tend to 0, while temperatures tend to $T_{\mathrm{amb}}$. The gaps tend to constant values $\bar{g}_1 = 0.605$, $\bar{g}_2 = 0.978$, $\bar{g}_3 = 0.901$, $\bar{g}_4 = 0.757$ and $\bar{g}_5 = 1.1$ nm. The asymptotic values $\bar{g}_i$, which are retained in memory by the nonvolatile memristors, are the result of the SMNN computation. The observed behavior is in accordance with that predicted by Corollary 1.

## V. Conclusion

The article has proved some fundamental results on CS for a class of delayed NNs with nonvolatile extended memristors obeying the popular and widely used Stanford model. The conditions for CS, which can be effectively checked numerically via an LMI approach, or analytically, using the concept of LDS matrices, hold for any value of the concentrated delay and they are robust with respect to variations of the interconnections. Using the LDS approach, several classes of interconnection matrices have been singled out for which CS holds. The obtained results not only ensure CS but also exponential convergence of each trajectory with a known convergence rate which is independent of the ICs. Furthermore, at the end of the transient power in the SMNN vanishes, while nonvolatile memristors retain in memory the result of computation in accordance with the in-memory computing principle. Future work will be devoted to study whether the techniques developed in the article can be applied or extended to study CS of NNs with other memristors used for modeling real devices in nanotechnology.

## Appendix A
## Proof of Lemma 1

Consider an IVP for (19)–(21) and suppose without loss of generality $C = 1$. We first show that there exists $t'_f < \infty$ such that $v(t) \in \Xi$ for $t \geq t'_f$. Consider $v(0) \in \mathbb{R}^n$, $\|v(0)\|_\infty > v_M$. Let $\hat{\imath}(t) \in \{1, \ldots, n\}$ be such that $|v_{\hat{\imath}(t)}| = \|v(t)\|_\infty$. Let us first consider $t \in [0, \tau]$. Function $\|v(t)\|_\infty$ is absolutely continuous on $[0, \tau]$, hence it is differentiable for a.a. $t \in [0, \tau]$. By proceeding as in [38, Appendix A], we have for a.a. $t \in [0, \tau]$

$$\frac{d\|v(t)\|_\infty}{dt} = \mathrm{sgn}(v_{\hat{\imath}(t)}(t))\Bigg(-I_0 e^{-\frac{g_{\hat{\imath}(t)}(t)}{\bar{g}}} \sinh\left(\frac{v_{\hat{\imath}(t)}(t)}{V_0}\right) + \sum_{j=1}^{n} a_{\hat{\imath}(t)j} f(v_j(t)) + \sum_{k=1}^{n} a^{\tau}_{\hat{\imath}(t)k} f(v_k(t-\tau))\Bigg)$$

where sgn denotes the signum function. As long as $\|v(t)\|_\infty > v_M$, we have $|v_{\hat{\imath}(t)}(t)| > v_M \geq 1$. Considering that $|f(v_j(t))| \leq |v_j(t)|$ and $|(v_j(t-\tau))| < V_F$ for any $j \in \{1, \ldots, n\}$ and $t \in [0, \tau]$, we have

$$\frac{d\|v(t)\|_\infty}{dt} \leq |v_{\hat{\imath}(t)}(t)|\Bigg(-\frac{I_0}{V_0} e^{-\frac{g_M}{\bar{g}}}\left(1 + \frac{v_M^2}{6V_0^2}\right) + \sum_{j=1}^{n} |a_{\hat{\imath}(t)j}| + \sum_{k=1}^{n} |a^{\tau}_{\hat{\imath}(t)k}| V_F\Bigg).$$

Taking into account (22) the inequality boils down to

$$\frac{d\|v(t)\|_\infty}{dt} \leq -|v_{\hat{\imath}(t)}|\varepsilon \leq -\varepsilon. \tag{45}$$

As a consequence, in $[0, \tau]$, $\|v(t)\|_\infty$ decreases toward $v_M$ with velocity not smaller than $\varepsilon$. Moreover, since $\|v(t)\|_\infty$ is monotone decreasing, $\max_{t\in[0,\tau]} |v_{\hat{\imath}(t)}| \leq \max_{t\in[-\tau,0]} \|v(t)\|_\infty = V_F$ and inequality (45) holds true for any $t \in [k\tau, (k+1)\tau]$, $k = 1, 2, \ldots$, as long as $\|v(t)\|_\infty \geq v_M$. This implies that $\Xi$ is globally attracting and positively invariant. Moreover, $v(t) \in \Xi$ for $t \geq t'_f = (\|v_i(0)\|_\infty - v_M)/\varepsilon$.

Now, assume that $T(t'_f) \notin \Upsilon$ and let $\eta_T \neq \emptyset$ be the set of indexes $i \in \{1, \ldots, n\}$ such that $T_i(t'_f) > T_M$. It can be checked that the following inequality holds if $T_i(t) \geq T_M$

$$\dot{T}_i(t) \leq -\frac{T_M - T_{\text{amb}}}{\Theta_{\text{th}}} + I_0 e^{-\frac{g_m}{\bar{g}}} \sinh\left(\frac{v_M}{V_0}\right) v_M \leq -\varepsilon.$$

Arguing as before we conclude that $\Upsilon$ is globally attracting and positively invariant. Moreover, $T(t) \in \Upsilon$ for $t \geq t_f = t'_f + t''_f$ where $t''_f = \max_{i \in \eta_T}(|T_i(t'_f)| - T_M)/\varepsilon$.

From the proof it also follows that $\Xi \times \Gamma \times \Upsilon$ is positively invariant for the dynamics of (19)–(21). ■

## Appendix B
## Proof of Proposition 1

We proceed by using the method of steps. First, we show the existence of a solution for an IVP associated with (24)–(26) for $t \in [-\tau, \tau]$. Consider the following IVP for a DVI without delay

$$\begin{pmatrix} \dot{w} \\ \dot{h} \\ \dot{u} \\ \dot{\theta} \end{pmatrix} \in \begin{pmatrix} -\mathfrak{D}(w,h) + A\mathfrak{F}(w) + A^\tau \mathfrak{F}(v(\theta)) \\ \mathfrak{G}(w,u) \\ \mathfrak{T}(w,h) - \frac{u - \tilde{T}_{\text{amb}}}{\Theta_{\text{th}}} \\ 1 \end{pmatrix} - \begin{pmatrix} N_\Xi(w) \\ N_\Gamma(h) \\ N_\Upsilon(u) \\ N_{[-\tau,0]}(\theta) \end{pmatrix} = F(w,h,u,\theta) - N_{\Xi\times\Gamma\times\Upsilon\times[-\tau,0]}(w,h,u,\theta) \quad (46)$$

where $t \in [0, \tau]$ and the ICs are $w(0) = v(0), h(0) = g_0, u(0) = T_0$ and $\theta(0) = -\tau$. Since $\Xi \times \Gamma \times \Upsilon \times [-\tau, 0] \subset \mathbb{R}^{3n+1}$ is a nonempty compact convex set, $F : \Xi \times \Gamma \times \Upsilon \times [-\tau, 0] \to R^{3n+1}$ is continuous in $\Xi \times \Gamma \times \Upsilon \times [-\tau, 0]$ and the ICs belong to $\Xi \times \Gamma \times \Upsilon \times [-\tau, 0]$, due to Property 1 there exists at least one solution to the IVP (46). The solution is bounded for $t \geq -\tau$ due to Lemma 1.

Solving the IVP for $\theta$, we obtain $\theta(t) = t - \tau$ in $t \in [0, \tau]$. Substituting in (46), we can check that

$$(v, g, T) = \begin{cases} (v, g_0, T_0), & t \in [-\tau, 0] \\ (w, h, u), & t \in [0, \tau] \end{cases} \quad (47)$$

is a solution of (24)–(26) for $t \in [-\tau, \tau]$.

Now, it is possible to proceed in a way analogous to that used to prove Property 4 in [71] to show that (24)–(26) admits at least one solution in the interval $[-\tau, (m+1)\tau]$ for any integer $m > 1$, thus completing the proof of the existence part. The solution is bounded due to Lemma 1.

To prove the uniqueness of the solution, suppose for contradiction that there exist two solutions $(v_a, g_a, T_a)$ and $(v_b, g_b, T_b)$ of an IVP for (24)–(26) and define their distance

$$\Delta = \frac{1}{2}\left(\|v_a - v_b\|^2 + \|g_a - g_b\|^2 + \|T_a - T_b\|^2\right). \quad (48)$$

We want to prove that $\Delta(t) = 0$ for any $t \in [-\tau, m\tau]$, where $m \geq 0$ is an integer. This result holds for $m = 0$, since for $t \in [-\tau, 0]$ both solutions coincide with $(v, g_0, T_0)$, therefore $\Delta(t) = 0$. To prove the uniqueness for $m > 0$, we apply again the method of steps. Suppose that $\Delta(t) = 0$ for $t \in [-\tau, m\tau]$. For $t \in [-m\tau, (m+1)\tau]$ we have

$$\begin{cases} \dot{v}_a(t) = -\mathfrak{D}(v_a(t), g_a(t)) + A\mathfrak{F}(v_a(t)) \\ \qquad + A^\tau \mathfrak{F}(v_a(t-\tau)) - n_{v_a,t} \\ \dot{g}_a(t) = \mathfrak{G}(v_a(t), T_a(t)) - n_{g_a,t} \\ \dot{T}_a(t) = \mathfrak{T}(v_a(t), g_a(t)) - \dfrac{T_a(t) - \tilde{T}_{\text{amb}}}{\Theta_{\text{th}}} - n_{T_a,t} \end{cases}$$
$$\begin{cases} \dot{v}_b(t) = -\mathfrak{D}(v_b(t), g_b(t)) + A\mathfrak{F}(v_b(t)) \\ \qquad + A^\tau \mathfrak{F}(v_b(t-\tau)) - n_{v_b,t} \\ \dot{g}_b(t) = \mathfrak{G}(v_b(t), T_b(t)) - n_{g_b,t} \\ \dot{T}_b(t) = \mathfrak{T}(v_b(t), g_b(t)) - \dfrac{T_b(t) - \tilde{T}_{\text{amb}}}{\Theta_{\text{th}}} - n_{T_b,t} \end{cases} \quad (49)$$

where $n_{v_a,t} \in N_\Xi(v_a(t))$, $n_{v_b,t} \in N_\Xi(v_b(t))$, $n_{g_a,t} \in N_\Gamma(g_a(t))$, $n_{g_b,t} \in N_\Gamma(g_b(t))$, $n_{T_a,t}, n_{T_b,t} \in N_\Upsilon(T_a(t))$.

We obtain

$$\begin{aligned} \dot{\Delta}(t) &= \langle v_a(t) - v_b(t), \dot{v}_a(t) - \dot{v}_b(t) \rangle \\ &\quad + \langle g_a(t) - g_b(t), \dot{g}_a(t) - \dot{g}_b(t) \rangle \\ &\quad + \langle T_a(t) - T_b(t), \dot{T}_a(t) - \dot{T}_b(t) \rangle \end{aligned} \quad (50)$$

where

$$\begin{aligned} &\langle v_a(t) - v_b(t), \dot{v}_a(t) - \dot{v}_b(t) \rangle \\ &\quad = -\langle v_a(t) - v_b(t), \mathfrak{D}(v_a(t), g_a(t)) - \mathfrak{D}(v_b(t), g_b(t)) \rangle \\ &\qquad + \langle v_a(t) - v_b(t), A\mathfrak{F}(v_a(t)) - A\mathfrak{F}(v_b(t)) \rangle \\ &\qquad + \langle v_a(t) - v_b(t), A^\tau \mathfrak{F}(v_a(t-\tau)) - A^\tau \mathfrak{F}(v_b(t-\tau)) \rangle \\ &\qquad - \langle v_a(t) - v_b(t), n_{v_a,t} - n_{v_b,t} \rangle \end{aligned}$$

moreover

$$\begin{aligned} &\langle g_a(t) - g_b(t), \dot{g}_a(t) - \dot{g}_b(t) \rangle \\ &\quad = \langle g_a(t) - g_b(t), \mathfrak{G}(v_a(t), T_a(t)) - \mathfrak{G}(v_b(t), T_b(t)) \rangle \\ &\qquad - \langle g_a(t) - g_b(t), n_{g_a,t} - n_{g_b,t} \rangle \end{aligned}$$

and, finally,

$$\begin{aligned} &\langle T_a(t) - T_b(t), \dot{T}_a(t) - \dot{T}_b(t) \rangle \\ &\quad = \langle T_a(t) - T_b(t), \mathfrak{T}(v_a(t), g_a(t)) - \mathfrak{T}(v_b(t), g_b(t)) \rangle \\ &\qquad - \left\langle T_a(t) - T_b(t), \frac{T_a(t) - T_b(t)}{\Theta_{\text{th}}} \right\rangle \\ &\qquad - \langle T_a(t) - T_b(t), n_{T_a,t} - n_{T_b,t} \rangle. \end{aligned}$$

The normal cone to a set $Q$ is a monotone operator (see Section I-A1). Moreover, by assumption, $\Delta(t) = 0$, i.e., $v_a(t-\tau) = v_b(t-\tau)$, for $t \in [-\tau, m\tau]$. Then, from (50)

$$\dot{\Delta} \leq \left\langle \begin{pmatrix} v_a - v_b \\ g_a - g_b \\ T_a - T_b \end{pmatrix}, \begin{pmatrix} -\mathfrak{D}(v_a, g_a) + A\mathfrak{F}(v_a) + \mathfrak{D}(v_b, g_b) - A\mathfrak{F}(v_b) \\ \mathfrak{G}(v_a, T_a) - \mathfrak{G}(v_b, T_b) \\ \mathfrak{T}(v_a, g_a) - \mathfrak{T}(v_b, g_b) \end{pmatrix} \right\rangle. \quad (51)$$

By assumption, $\mathfrak{F}(\cdot)$ is Lipschitz continuous. From (15)–(17), also $\mathfrak{D}(\cdot,\cdot)$, $\mathfrak{G}(\cdot,\cdot)$, $\mathfrak{T}(\cdot,\cdot)$ are Lipschitz continuous in $\Xi \times \Gamma \times \Upsilon$. Then, there exists $0 < \xi < +\infty$ such that

$$\dot{\Delta}(t) \leq 2\xi\Delta(t).$$

Therefore, by the Gronwall lemma, we obtain

$$0 \leq \Delta(t) \leq \Delta(m\tau)\exp(2\xi(t-m\tau)) = 0 \tag{52}$$

i.e., $\Delta(t) = 0$, when $t \in [-\tau, m\tau]$. This shows the uniqueness of the solution for any $t \geq -\tau$. ■

## Appendix C

To prove Theorem 1 we first establish an algebraic property and a fundamental result on exponential convergence of memristor voltages.

*Proposition 5:* If Assumptions 1 and 2 hold, we have

$$S_k \doteq \begin{pmatrix} 2PD_m - PA - A^\top P - Z - \frac{2kP}{\gamma^2(v_M)} & -PA^\tau \\ -(PA^\tau)^\top & e^{-2k\tau}Z \end{pmatrix} > 0$$

where $k$ is as in (28).

*Proof:* Since $Z > 0$, from Schur's lemma [51], $S_k > 0$ if and only if

$$\begin{aligned}\Omega_k &= 2PD_m - PA - A^\top P - Z - \frac{2kP}{\gamma^2(v_M)} \\ &\quad - e^{2k\tau}PA^\tau Z^{-1}(PA^\tau)^\top \\ &= \Omega - \frac{2kP}{\gamma^2(v_M)} - (e^{2k\tau}-1)PA^\tau Z^{-1}(PA^\tau)^\top > 0\end{aligned}$$

where we have taken into account (29). Since $k > 0$, $\gamma^2(v_M) > 0$, $P > 0$ and $PA^\tau Z^{-1}(PA^\tau)^\top > 0$, we obtain

$$\begin{aligned}\Lambda_m(\Omega_k) \geq \Lambda_m(\Omega) - \frac{2k}{\gamma^2(v_M)}\Lambda_M(P) \\ -(e^{2k\tau}-1)\Lambda_M(PA^\tau Z^{-1}(PA^\tau)^\top).\end{aligned} \tag{53}$$

If we choose $k$ as in (28), we have $\Lambda_m(\Omega_k) > 0$, which in turn implies $\Omega_k > 0$ and $S_k > 0$. ■

*Lemma 4:* Suppose that Assumptions 1 and 2 hold. Then $v(t) \to 0$ exponentially as $t \to +\infty$ with convergence rate $k$ as in (28), i.e., $\|v(t)\| \leq M_v e^{-kt}$, $t \geq 0$, where

$$M_v = \sqrt{\frac{\Lambda_M(P) + \Lambda_M(Z)\frac{1-e^{-2k\tau}}{2k}}{\Lambda_m(P)\gamma(v_M)}} \max_{-\tau \leq \theta \leq 0} \|v(\theta)\|. \tag{54}$$

*Proof:* Suppose for simplicity $C = 1$. Consider for (24)–(26) the following candidate Lyapunov function:

$$\begin{aligned}W(v(t),t) = e^{2kt}\sum_{i=1}^{n} 2p_i \int_0^{v_i(t)} f(\sigma)d\sigma \\ + \int_{t-\tau}^{t} e^{2ks}\mathfrak{F}^\top(v(s))Z\mathfrak{F}(v(s))ds.\end{aligned} \tag{55}$$

Its time derivative along the solutions of (24)–(26) is given by

$$\begin{aligned}\dot{W}(v(t),t) &= 2ke^{2kt}\sum_{i=1}^{n} 2p_i \int_0^{v_i(t)} f(\sigma)d\sigma \\ &\quad + e^{2kt}\sum_{i=1}^{n} 2p_i f(v_i(t))\dot{v}_i(t) \\ &\quad + e^{2kt}\mathfrak{F}^\top(v(t))Z\mathfrak{F}(v(t)) \\ &\quad - e^{2k(t-\tau)}\mathfrak{F}^\top(v(t-\tau))Z\mathfrak{F}(v(t-\tau)) \\ &= e^{2kt}\Big\{2k\sum_{i=1}^{n} 2p_i \int_0^{v_i(t)} f(\sigma)d\sigma \\ &\quad - 2\mathfrak{F}^\top(v(t))P\mathfrak{D}(v(t),g(t)) \\ &\quad + 2\mathfrak{F}^\top(v(t))PA\mathfrak{F}(v(t)) \\ &\quad + 2\mathfrak{F}^\top(v(t))PA^\tau\mathfrak{F}(v(t-\tau)) \\ &\quad + \mathfrak{F}^\top(v(t))Z\mathfrak{F}(v(t)) \\ &\quad - e^{-2k\tau}\mathfrak{F}^\top(v(t-\tau))Z\mathfrak{F}(v(t-\tau))\Big\}\end{aligned} \tag{56}$$

for a.a. $t \geq 0$. Considering Assumption 1, we have

$$\begin{aligned}\sum_{i=1}^{n} 2\, p_i \int_0^{v_i(t)} f(\sigma)d\sigma &\leq \sum_{i=1}^{n} 2\, p_i \int_0^{v_i(t)} \sigma d\sigma \\ &= \sum_{i=1}^{n} p_i v_i^2(t) \leq \mathfrak{F}^\top(v(t))\frac{P}{\gamma^2(v_M)}\mathfrak{F}(v(t)).\end{aligned} \tag{57}$$

Additionally, it can be verified that

$$\begin{aligned}-\mathfrak{F}^\top(v(t))P\mathfrak{D}(v(t),g(t)) &= -\mathfrak{F}^\top(v(t))PD(g(t))V_0\sinh\left(\frac{v(t)}{V_0}\right) \\ &\leq -\mathfrak{F}^\top(v(t))PD(g(t))v(t)\end{aligned} \tag{58}$$

where we have let

$$D(g(t)) = \frac{I_0}{V_0}\operatorname{diag}\left(\exp\left(-\frac{g_1(t)}{\tilde{g}}\right), \ldots, \exp\left(-\frac{g_n(t)}{\tilde{g}}\right)\right).$$

As a consequence, we have

$$\begin{aligned}\dot{W}(v(t),t) \leq e^{2kt}\Big\{&\mathfrak{F}^\top(v(t))\frac{2kP}{\gamma^2(v_M)}\mathfrak{F}(v(t)) \\ &- \mathfrak{F}^\top(v(t))2\,PD(g(t))v(t) \\ &+ 2\mathfrak{F}^\top(v(t))PA\mathfrak{F}(v(t)) \\ &+ 2\mathfrak{F}^\top(v(t))PA^\tau\mathfrak{F}(v(t-\tau)) \\ &+ \mathfrak{F}^\top(v(t))Z\mathfrak{F}(v(t)) \\ &- e^{-2k\tau}\mathfrak{F}^\top(v(t-\tau))Z\mathfrak{F}(v(t-\tau))\Big\}.\end{aligned}$$

Let

$$\mathcal{V}(t,\tau) = \begin{pmatrix}\mathfrak{F}(v(t)) \\ \mathfrak{F}(v(t-\tau))\end{pmatrix} \in \mathbb{R}^{2n}$$

and

$$S(g(t)) = \begin{pmatrix} 2PD(g(t)) - PA - A^\top P - Z - \frac{2kP}{\gamma^2(v_M)} & -PA^\tau \\ -(PA^\tau)^\top & e^{-2k\tau}Z \end{pmatrix}.$$

We have $2PD(g(t)) > 0$ and, due to Assumption 1, $\mathfrak{F}^\top(v(t))PD(g(t))(\mathfrak{F}(v(t)) - v(t)) \leq 0$. Therefore we can rewrite the last inequality as

$$\begin{aligned}\dot{W}(v(t),t) \leq -e^{2kt}\Big\{&\mathcal{V}^\top(t,\tau)S(g(t))\mathcal{V}(t,\tau) \\ &+ 2\mathfrak{F}^\top(v(t))PD(g(t))(\mathfrak{F}(v(t)) - v(t))\Big\}\end{aligned}$$

$$\leq -e^{2kt}\mathcal{V}^\top(t,\tau)S(g(t))\mathcal{V}(t,\tau). \tag{59}$$

Since

$$S(g(t)) = S_k + \begin{pmatrix} 2P(D(g(t)) - D_m) & 0 \\ 0 & 0 \end{pmatrix} \tag{60}$$

and, from (18), $D(g(t)) - D_m \geq 0$, we have $S(g(t)) \geq S_k$. As a consequence, from Proposition 5, $S(g(t)) \geq S_k > 0$. In turn, this implies (see Assumption 1)

$$\begin{aligned}\dot{W}(v(t),t) &\leq -e^{2kt}\Lambda_M\|\mathfrak{F}(v(t))\|^2\\ &\leq -e^{2kt}\frac{\Lambda_M}{\gamma^2(v_M)}\|(v(t))\|^2 \leq 0.\end{aligned} \tag{61}$$

We are now in a position to prove that $v(\cdot)$ converges exponentially to 0. First of all, note that (61) implies that $W(v(t),t) \leq W(v(0),0)$ for any $t \geq 0$. Then, we have

$$\begin{aligned}W(v(0),0) &= \sum_{i=1}^n 2p_i\int_0^{v_i(0)} f(\sigma)d\sigma\\ &\quad+\int_{-\tau}^0 e^{2ks}\mathfrak{F}^\top(v(s))Z\mathfrak{F}(v(s))ds\\ &\leq v^\top(0)Pv(0)\\ &\quad+\int_{-\tau}^0 e^{2ks}\mathfrak{F}^\top(v(s))Z\mathfrak{F}(v(s))ds\\ &\leq \Lambda_M(P)\|v(0)\|^2\\ &\quad+\Lambda_M(Z)\frac{1-e^{-2k\tau}}{2k}\max_{-\tau\leq\theta\leq 0}\|v(\theta)\|^2\end{aligned} \tag{62}$$

and

$$\begin{aligned}W(v(t),t) &\geq e^{2kt}\sum_{i=1}^n 2p_i\int_0^{v_i(t)} f(\sigma)d\sigma\\ &\geq e^{2kt}\sum_{i=1}^n 2p_i\int_0^{v_i(t)}\gamma(v_M)\sigma d\sigma\\ &\geq e^{2kt}\Lambda_m(P)\gamma(v_M)\|v(t)\|^2.\end{aligned} \tag{63}$$

Inequalities (62) and (63) can be combined into

$$\begin{aligned}&e^{2kt}\Lambda_m(P)\gamma(v_M)\|v(t)\|^2\\ &\quad\leq\left(\Lambda_M(P)+\Lambda_M(Z)\frac{1-e^{-2k\tau}}{2k}\right)\max_{-\tau\leq\theta\leq 0}\|v(\theta)\|^2\end{aligned}$$

which yields the stated result. ∎

*Proof of Theorem 1*. Recalling Lemma 1 and noting that $\sinh(x)$ is convex for $x \geq 0$ we have from (16) and Lemma 4

$$\begin{aligned}|\mathfrak{g}(v_i(t),T_i(t))| &\leq v_0\left|\sinh\left(\frac{q_e a_0\psi}{\ell K T_{\text{amb}}}v_i(t)\right)\right|\\ &\leq \sinh\left(\frac{q_e a_0\psi}{\ell K T_{\text{amb}}}v_M\right)|v_i(t)|\\ &\leq v_0\sinh\left(\frac{q_e a_0\psi}{\ell K T_{\text{amb}}}v_M\right)M_v e^{-kt}\end{aligned}$$

for $t \geq 0$, where $M_v$ is given in (54). By applying Proposition 6 to (25) it can be easily verified that

$$\|g(t)-\bar{g}\| \leq M_g e^{-kt}, \quad t \geq 0$$

where

$$M_g = \sqrt{n}v_0\sinh\left(\frac{q_e a_0\psi}{\ell K T_{\text{amb}}}v_M\right)\frac{M_v}{k}.$$

Arguing as before we have from (17)

$$|\mathfrak{t}(v_i,g_i)| \leq \tfrac{I_0}{C_{\text{th}}}\exp\left(-\tfrac{g_m}{\tilde{g}}\right)M_v^2\sinh\left(\tfrac{v_M}{V_0}\right)e^{-2kt}$$

for $t \geq 0$. By applying Proposition 7 to (26) we obtain

$$\|T(t)-\tilde{T}_{\text{amb}}\| \leq M_T e^{-k_T t}$$

for $t \geq 0$, where $k_T = \min\{k, 1/\Theta_{\text{th}}\}$ and on the basis of (64)–(66)

$$M_T = \sqrt{n}T(0)\left(1+\tilde{M}_T\frac{I_0}{C_{\text{th}}}\exp\left(-\frac{g_m}{\tilde{g}}\right)M_V^2\sinh\left(\frac{V_M}{V_0}\right)\right)$$

and

$$\tilde{M}_T = \max\left\{\frac{1}{|2k-\Theta_{\text{th}}|}, \frac{1}{ek}\right\}.$$

∎

## Appendix D

*Proposition 6:* Consider the scalar DVI

$$\dot{x}(t) \in a(t) - N_{[x_m,x_M]}(x(t))$$

for a.a. $t \geq 0$, where $x \in \mathbb{R}$, $x(0) \in [x_m, x_M]$ and $a(t): \mathbb{R} \to \mathbb{R}$ is continuous and tends to 0 exponentially, i.e., there exist $H, \beta > 0$ such that $|a(t)| \leq He^{-\beta t}$, $t \geq 0$. Then, $x(t) \to \bar{x} \in [x_m, x_M]$ exponentially, with convergence rate $\beta$, namely, we have $|x(t)-\bar{x}| \leq (H/\beta)e^{-\beta t}$ for $t \geq 0$.

*Proof:* For a.a. $t \geq 0$ we have $\dot{x}(t) = 0$ if $x(t) = x_M$ and $a(t) \geq 0$ or $x(t) = x_m$ and $a(t) \leq 0$. Otherwise, we have $\dot{x}(t) = a(t)$. Therefore, $|\dot{x}(t)| \leq |a(t)| \leq He^{-\beta t}$ for a.a. $t \geq 0$. Given $t_2 \geq t_1 \geq 0$, we have $|x(t_2)-x(t_1)| = |\int_{t_1}^{t_2}\dot{g}(t)dt| \leq \int_{t_1}^{t_2}|\dot{g}(t)|dt \leq \int_{t_1}^{t_2}He^{-\beta t}dt = (H/\beta)[e^{-\beta t_1} - e^{-\beta t_2}] \leq (2H/\beta)e^{-\beta t_1}$. Hence, for any $\epsilon > 0$ we have $|x(t_2)-x(t_1)| \leq \epsilon$ when $t_2 \geq t_1 \geq (1/\beta)\ln(2H/\beta\epsilon)$. By the Cauchy criterion on limit existence (sufficiency part), there exists the $\lim_{t\to\infty}x(t) = \bar{x}$. Since $x(t) \in [x_m, x_M]$ for all $t \geq 0$ we necessarily have $\bar{x} \in [x_m, x_M]$. Finally, note that $|x(t)-\bar{x}| = |x(t)-x(0)-(\bar{x}-x(0))| \leq \int_t^\infty |a(\sigma)|d\sigma \leq \int_t^\infty |He^{-\beta\sigma}|d\sigma = (H/\beta)e^{-\beta t}$ for $t \geq 0$. ∎

*Proposition 7:* Consider the scalar differential equation

$$\dot{x}(t) = a(t) - b(x(t)-\bar{x})$$

for $t \geq 0$, where $x, b \in \mathbb{R}$, $b > 0$, and $a: \mathbb{R} \to \mathbb{R}$ is continuous and tends to 0 exponentially, i.e., there exist $H, \beta > 0$ such that $|a(t)| \leq He^{-\beta t}$, $t \geq 0$. Then, $x(t) \to \bar{x}$ exponentially as $t \to \infty$ with convergence rate equal to $\min\{b,\beta\}$ if $b \neq \beta$ and $b/2 = \beta/2$ if $b = \beta$.

*Proof:* Let $y = x - \bar{x}$, so that $\dot{y}(t) = a(t) - by(t)$. By the variation of constants formula we have

$$y(t) = e^{-bt}y(0) + e^{-bt}\int_0^t e^{b\sigma}a(\sigma)d\sigma, \quad t \geq 0.$$

If $\beta < b$. We have

$$\left|\int_0^t e^{b\sigma}a(\sigma)d\sigma\right| \leq \left|H\int_0^t e^{b\sigma}e^{-\beta\sigma}d\sigma\right| = \frac{H}{b-\beta}\left(e^{(b-\beta)t}-1\right)$$

and

$$|y(t)| \leq e^{-bt}|y(0)| + \frac{H}{b-\beta}(e^{-\beta t}-e^{-bt})$$

$$\leq \left(\frac{H}{b-\beta} + |y(0)|\right)e^{-\beta t}, \quad t \geq 0. \tag{64}$$

Similarly, if $b < \beta$ we obtain

$$\|y(t)\| \leq |y(0)|\left(1 + \frac{H}{\beta - b}\right)e^{-bt}, \quad t \geq 0. \tag{65}$$

If $b = \beta$, we easily obtain

$$\begin{aligned}|y(t)| &\leq (|y(0)| + Ht)e^{-bt}\\ &\leq \left(|y(0)| + \frac{2H}{eb}\right)e^{-bt/2}, \quad t \geq 0. \end{aligned} \tag{66}$$

■

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