Grassmannians of codes

Grassmannians of codes

I. Cardinali, L. Giuzzi

September 12, 2023

Abstract

Consider the point line-geometry $\mathscr{P}_t(n,k)$ having as points all the $[n,k]$-linear codes having minimum dual distance at least $t+1$ and where two points X and Y are collinear whenever $X \cap Y$ is a $[n,k-1]$-linear code having minimum dual distance at least $t+1$. We are interested in the collinearity graph $\Lambda_t(n,k)$ of $\mathscr{P}_t(n,k)$. The graph $\Lambda_t(n,k)$ is a subgraph of the Grassmann graph and also a subgraph of the graph $\Delta_t(n,k)$ of the linear codes having minimum dual distance at least $t+1$ introduced in [9]. We shall study the structure of $\Lambda_t(n,k)$ in relation to that of $\Delta_t(n,k)$ and we will characterize the set of its isolated vertices. We will then focus on $\Lambda_1(n,k)$ and $\Lambda_2(n,k)$ providing necessary and sufficient conditions for them to be connected.

MSC: 51E22, 94B27
Keywords: Grassmann graph, Linear codes, Point-Line geometry, Connectivity

1 Introduction

Let $V := V(n,q)$ be an n-dimensional vector space over the finite field $\mathbb{F}_q$ with q elements and $0 < k < n$; let $\Gamma(n,k)$ be the graph whose vertices are the k-subspaces of V and where two vertices X, Y are adjacent if and only if $\dim(X \cap Y) = k-1$. The graph $\Gamma(n,k)$ is called the k-Grassmann graph of V.

An $[n,k]$-linear code is just a vertex of $\Gamma(n,k)$. We say that a $[n,k]$-linear code C has dual minimum distance at least $t+1$ if, given a generator matrix G for C, any set of t columns of G is linearly independent. It is easy to see that this condition does not depend on the particular generator matrix chosen for C and that this is the same as to require that the dual code of C has minimum Hamming distance at least $t+1$, whence the name.

Denote by $\mathscr{C}_t(n,k)$ the set of all $[n,k]$-codes with dual minimum distance at least $t+1$ over $\mathbb{F}_q$. By construction, the elements of $\mathscr{C}_t(n,k)$ can be regarded as the vertices of the induced subgraph $\Delta_t(n,k)$ of $\Gamma(n,k)$, where two codes X, Y, considered as two k-dimensional subspaces, are adjacent ($X \sim_\Delta Y$) if and only if $\dim(X \cap Y) = k-1$.

The graph $\Delta_t(n,k)$ has been called the *Grassmann graph of the linear codes with dual minimum distance at least* $t+1$. Clearly, $0 \le t \le k$. For $t=0$, $\mathscr{C}_0(n,k)$ consists of the class of all k-subspaces of V and $\Delta_0(n,k) = \Gamma(n,k)$; for $t=1$, $\mathscr{C}_1(n,k)$ is called the class of the *non-degenerate* $[n,k]$*-linear codes* and for $t=2$, $\mathscr{C}_2(n,k)$ is called the class of the *projective* $[n,k]$*-linear codes.* The case $t=k$ corresponds to codes whose dual is MDS (*maximum distance separable*). In particular as the duals of MDS codes are in turn MDS (see [12]), the elements of $\mathscr{C}_t(n,k)$ are themselves MDS.

The graphs $\Delta_t(n,k)$ for $t=1$ (non-degenerate codes) and for $t=2$ (projective codes) have been studied respectively in [9] and [11]. The graph $\Delta_t(n,k)$ for $t \ge 3$ has been introduced and studied in [4].

The following theorem synthetically reports on the main results of interest here regarding the graph $\Delta_t(n,k)$.

Theorem 1.1 ([4, 9, 11])**.** *1. $\Delta_1(n,k)$ is connected for any q; furthermore $\Delta_1(n,k)$ is isometrically embedded in the k-Grassmann graph $\Gamma(n,k)$ if and only if $n < (q+1)^2 + k - 2$.*

2. If $q \ge \binom{n}{2}$ then $\Delta_2(n,k)$ is connected and it is isometrically embedded in the k-Grassmann graph $\Gamma(n,k)$; furthermore, $\Delta_2(n,k)$ and $\Gamma(n,k)$ have the same diameter.

3. If $t > 2$ and $q \ge \binom{n}{t}$ then $\Delta_t(n,k)$ is connected and it is isometrically embedded in the k-Grassmann graph $\Gamma(n,k)$; furthermore, $\Delta_t(n,k)$ and $\Gamma(n,k)$ have the same diameter.

In this paper, we shall consider a subgraph of $\Delta_t(n,k)$, which we will denote by $\Lambda_t(n,k)$, defined as follows: the vertices of $\Lambda_t(n,k)$ are all the elements of $\mathscr{C}_t(n,k)$ and two vertices X and Y are adjacent in $\Lambda_t(n,k)$ ($X \sim_\Lambda Y$) whenever their intersection $X \cap Y$ belongs to $\mathscr{C}_t(n,k-1)$. In other words, the vertices of $\Lambda_t(n,k)$ are the same as the vertices of $\Delta_t(n,k)$, but the condition for an edge to exist is stronger.

Consider now the point-line geometry $\mathscr{P}_t(n,k) := (\mathscr{C}_t(n,k), \mathscr{L}_t(n,k))$ where the points are the elements of $\mathscr{C}_t(n,k)$ and the lines are defined as:

- $\ell_{X,Y} := \{Z \in \mathscr{C}_t(n,k) : X \subset Z \subset Y\}$ with $X \in \mathscr{C}_t(n,k-1)$, $Y \in \mathscr{C}_t(n,k+1)$ if $k < n-1$;
- $\ell_X := \{Z \in \mathscr{C}_t(n,n-1) : X \subset Z\}$ with $X \in \mathscr{C}_t(n,n-2)$ if $k = n-1$.

Observe that if Z is a k- dimensional vector subspace and it contains a subspace $D \in \mathscr{C}_t(n, k')$ with $k' < k$, then $Z \in \mathscr{C}_t(n, k)$. In particular, the line $\ell_{X,Y} \in \mathscr{L}_t(n, k)$ can also be described as $\ell_{X,Y} := \{Z : X \subset Z \subset Y\}$ with $X \in \mathscr{C}_t(n, k-1)$ and $\dim(Y) = k + 1$.

The geometry $\mathscr{P}_t(n, k)$ is a subgeometry of the *k-Grassmann geometry* $\mathscr{G}(n, k)$ (see Section 2) and the collinearity graph of $\mathscr{P}_t(n, k)$ is precisely the graph $\Lambda_t(n, k)$.

In this paper, we shall study the structure of $\Lambda_t(n, k)$ and the interplay between the geometry $\mathscr{P}_t(n, k)$ and the geometry $\mathscr{G}(n, k)$. We point out that from a more applied point of view, the study of the graph $\Lambda_t(n, k)$ is related to some code density problem, but we leave its investigation to further works.

Before stating our main results we need to give the following definitions.

Definition 1.2. A code $C \in \mathscr{C}_t(n, k)$ is *isolated* if C does not contain any proper subcode $D \in \mathscr{C}_t(n, k-1)$. We denote the set of all isolated codes in $\mathscr{C}_t(n, k)$ by the symbol $\mathscr{I}_t(n, k)$.

If a code C is not isolated, then there exists at least one code $C' \in \mathscr{C}_t(n, k)$ with $C' \neq C$ and such that $C \cap C' \in \mathscr{C}_t(n, k-1)$, that is $C' \sim_\Lambda C$. So, it follows readily that a vertex of $\Lambda_t(n, k)$ is *isolated* if and only if it corresponds to an isolated code of $\mathscr{C}_t(n, k)$. On the other hand, the graph induced by $\Lambda_t(n, k)$ on $\mathscr{I}_t(n, k)$ is *totally disconnected*, i.e. it contains no edge. We will use the same symbol $\mathscr{I}_t(n, k)$ to denote both the set of isolated codes of $\mathscr{P}_t(n, k)$ and the subgraph $(\mathscr{I}_t(n, k), \emptyset)$ of isolated vertices of $\Lambda_t(n, k)$.

Given a vector space of dimension k over $\mathbb{F}_q$, we denote by $\mathrm{PG}(k-1, q)$ the associated projective space; if X is a set of points of $\mathrm{PG}(k-1, q)$, then $\langle X \rangle$ denotes the projective subspace spanned by X. The graph $\Lambda_t(n, k)$ is related to interesting and well studied configurations of points of a projective space such as the *t-saturating sets* whose definition we report below.

Definition 1.3. Let t be an integer $0 \leq t \leq k$. For any $\Omega \subseteq \mathrm{PG}(k-1, q)$ let

$$\mathscr{S}_t(\Omega) := \bigcup_{\substack{X \subseteq \Omega \\ |X| = t+1}} \langle X \rangle$$

be the set of all points of $\mathrm{PG}(k-1, q)$ on subspaces spanned by $t+1$ points of Ω. The set Ω is *t-saturating* if $\mathscr{S}_t(\Omega) = \mathrm{PG}(k-1, q)$ and $\mathscr{S}_{t-1}(\Omega) \neq \mathrm{PG}(k-1, q)$.

If $t = 0$, then the only 0-*saturating* set of $\mathrm{PG}(k-1, q)$ is the point set of $\mathrm{PG}(k-1, q)$. If $t = 1$, then a 1-*saturating* of $\mathrm{PG}(k-1, q)$ is a set Ω whose 2-secants cover $\mathrm{PG}(k-1, q)$, i.e. for any $P \in \mathrm{PG}(k-1, q)$ there are $X, Y \in \Omega$ such that $P \in \langle X, Y \rangle$.

We refer to [5, 6, 7] and the references therein for more information on saturating sets and for the known bounds on the minimal cardinality of a t-saturating set and related constructions.

1.1 Main Results

The following are the main results of this paper.

Theorem 1.4. *The graph* $\Delta_t(n, k-1)$ *is connected if and only if the subgraph* $\widetilde{\Lambda}_t(n, k)$ *of* $\Lambda_t(n, k)$ *induced by all non-isolated codes in* $\mathscr{C}_t(n, k) \setminus \mathscr{I}_t(n, k)$ *is connected. In particular,* $\Lambda_t(n, k) = \widetilde{\Lambda}_t(n, k) \cup \mathscr{I}_t(n, k)$.

Remark 1.5. If $\mathscr{C}_t(n, k-1) = \emptyset$, then $\Delta_t(n, k-1)$ is an empty graph and $\Lambda_t(n, k) = \mathscr{I}_t(n, k)$ consists only of isolated vertices. If we take by convention the empty graph to be connected, then Theorem 1.4 is true also in this case. Also the converse holds, i.e. if $\Lambda_t(n, k) = \mathscr{I}_t(n, k)$, then $\mathscr{C}_t(n, k-1) = \emptyset$. By Theorem 1.9, this holds for $t = k$.

Theorem 1.6. *A code* $C \in \mathscr{C}_t(n, k)$ *is isolated if and only if the columns of any generator matrix of* C *are vector representatives of a* $(t-1)$*-saturating set of* $\mathrm{PG}(k-1, q)$.

The following corollaries, holding for $t = 1$ and $t = 2$, are consequences of Theorem 1.1, Theorem 1.4 and Theorem 1.6. For Point 2. of Corollary 1.7, see also [10, Proposition 4].

Corollary 1.7. *The following hold.*

1. $\Lambda_1(n, k) = \widetilde{\Lambda}_1(n, k) \cup \mathscr{I}_1(n, k)$ *where* $\widetilde{\Lambda}_1(n, k)$ *is connected.*
2. *The graph* $\Lambda_1(n, k)$ *is connected if and only if* $n < \frac{q^k-1}{q-1}$.
3. *A code* $C \in \mathscr{C}_1(n, k)$ *is isolated if and only if the columns of any generator matrix of* C *are vector representatives of all the points of a* $(k-1)$*-dimensional projective space.*

Corollary 1.8. *The following hold.*

1. $\Lambda_2(n, k) = \widetilde{\Lambda}_2(n, k) \cup \mathscr{I}_2(n, k)$ *where* $\widetilde{\Lambda}_2(n, k)$ *is connected.*
2. *The graph* $\Lambda_2(n, k)$ *is connected if and only if*

$$n < \min\{|\Omega| \colon \Omega \textit{ is a } 1\textit{-saturating set of } \mathrm{PG}(k-1, q)\}.$$

3. *A code $C \in \mathscr{C}_2(n,k)$ is isolated if and only if the 2-secants of the projective set determined by the columns of any generator matrix of C cover all the points of a $(k-1)$-dimensional projective space.*

Theorem 1.9. *If $q > \binom{n}{t}$ and $t < k$, then the graph $\Lambda_t(n,k)$ is connected. For $t = k$, the graph $\Lambda_k(n,k)$ is totally disconnected.*

In light of Theorem 1.9 it makes sense to ask for given t, k and q what is the structure of the graph $\Lambda_t(n,k)$, as n grows, under the condition $\mathscr{C}_t(n,k) \neq \emptyset$. Keeping in mind Definition 1.2 of isolated code, we can consider the following two parameters

$$\nu_t(k;q) := \min\{n : \Lambda_t(n,k) \text{ is disconnected}\},$$

$$\nu_t^+(k;q) := \min\{n : \mathscr{I}_t(n,k) \neq \emptyset\}.$$

If a graph has an isolated vertex then it is disconnected, so we have

$$\nu_t(k;q) \leq \nu_t^+(k;q).$$

Clearly, if $\nu_t(k;q) = \nu_t^+(k;q)$ then the graph $\Lambda_t(n,k)$ is disconnected if and only if it has isolated vertices. We conjecture the following.

Conjecture 1.10. *For any q, k and t with $k \geq t$, $\nu_t(k;q) = \nu_t^+(k;q)$.*

We observe that Conjecture 1.10 holds true for $t = 1$ by Corollary 1.7 and for $t = 2$ by Corollary 1.8.

Moving to the geometry $\mathscr{P}_t(n,k)$ whose collinearity graph is $\Lambda_t(n,k)$, we shall prove that the inclusion map ι of $\mathscr{P}_t(n,k)$ in the Grassmann geometry $\mathscr{G}(n,k)$ has the property that lines of $\mathscr{P}_t(n,k)$ are mapped into lines of $\mathscr{G}(n,k)$ and the preimage of any line of $\mathscr{G}(n,k)$ contained in the image of ι is still a line of $\mathscr{P}_t(n,k)$. According to [3], this is exactly the definition for an embedding to be *transparent* (see Section 4.4).

Theorem 1.11. *The inclusion map $\iota : \mathscr{P}_t(n,k) \to \mathscr{G}(n,k)$ is a transparent embedding.*

Since $\mathscr{G}(n,k)$ is *projectively embeddable* by means of the Plücker embedding ε_k in the projective space $\mathrm{PG}(\bigwedge^k V)$, Theorem 1.11 implies that the geometry $\mathscr{P}_t(n,k)$ is also projectively embeddable in $\mathrm{PG}(\bigwedge^k V)$ by means of the restriction of ε_k to $\mathscr{P}_t(n,k)$, and this embedding is transparent.

The resulting point set $\mathscr{V}_t := \varepsilon_k(\mathscr{C}_t(n,k))$ is a subset of the Grassmann variety in $\mathrm{PG}(\bigwedge^k V)$ and, by Theorem 1.11 and the transparency of the Plücker embedding ε_k, it has the property that for any two distinct points $P, Q \in \mathscr{V}_t$, the projective line joining P and Q is fully contained in $\mathscr{V}_t$ if and only if $\varepsilon_k^{-1}(P)$ and $\varepsilon_k^{-1}(Q)$ are adjacent in $\Lambda_t(n,k)$.

In particular, it is possible to reconstruct the graph $\Lambda_t(n,k)$ from just looking at the point set $\mathscr{V}_t$. We observe that this property does not hold for the graph $\Delta_t(n,k)$, i.e. it can happen that X and Y are adjacent in $\Delta_t(n,k)$ but the projective line ℓ joining $\varepsilon_k(X)$ and $\varepsilon_k(Y)$ is not fully contained in $\mathscr{V}_t$, i.e. there exists a point Z on ℓ such that $\varepsilon^{-1}(Z) \notin \mathscr{C}_t(n,k)$.

Structure of the paper In Section 2 we will set the notation and recall the basic results regarding the objects of our interest. In Section 3 we shall focus on the graph $\Lambda_t(n,k)$ proving some general results for arbitrary values of t,k and n. In Section 4 we will prove our main theorems; in particular we will prove Corollary 1.7 in Subsection 4.1 and Corollary 1.8 in Subsection 4.2. The general case is treated in Subsection 4.3 and Theorem 1.11 is proved in Section 4.4.

2 Preliminaries

As in Introduction, let $V := V(n,q)$ be a n-dimensional vector space over the finite field $\mathbb{F}_q$ and $0 < k < n$. The *k-Grassmann geometry* is defined as the point-line geometry $\mathscr{G}(n,k)$ whose points are the k-dimensional subspaces of V and whose lines are the sets

- $\ell_{X,Y} := \{Z : X \subset Z \subset Y\}$ with $\dim(X) = k-1$, $\dim(Y) = k+1$ if $k < n-1$;
- $\ell_X := \{Z : X \subset Z\}$ with $\dim(X) = n-2$ if $k = n-1$.

This geometry has been widely investigated (see for example [15]) and its collinearity graph is the Grassmann graph $\Gamma(n,k)$ as defined in Introduction.

Suppose that $\mathfrak{B} = (\mathbf{e}_1, \ldots, \mathbf{e}_n)$ is a given fixed basis of V; henceforth we shall always write the coordinates of the vectors in V with respect to $\mathfrak{B}$.

Given two vectors $\mathbf{x} = \sum_{i=1}^n x_i \mathbf{e}_i$ and $\mathbf{y} = \sum_{i=1}^n y_i \mathbf{e}_i$, the Hamming distance (with respect to the basis $\mathfrak{B}$) between $\mathbf{x}$ and $\mathbf{y}$ is $d(\mathbf{x}, \mathbf{y}) := |\{i : x_i \neq y_i\}|$. In this setting, a *$[n,k]$-linear code* C is just a k-dimensional vector subspace of V together with the restriction to $C \times C$ of the Hamming distance induced by $\mathfrak{B}$.

If B_C is an ordered basis of C, a *generator matrix* G_C for C is the $k \times n$ matrix whose rows are the components of the elements of B_C with respect to

$\mathfrak{B}$. Given a $[n,k]$-linear code C, its *dual code* is the $[n,n-k]$-linear code $C^{\perp}$ given by
$$C^{\perp} := \{\mathbf{v} \in V : \forall \mathbf{c} \in C, \mathbf{v} \cdot \mathbf{c} = 0\}$$
where $\cdot$ denotes the standard symmetric bilinear form on V given by
$$(v_1\mathbf{e}_1 + \cdots + v_n\mathbf{e}_n) \cdot (c_1\mathbf{e}_1 + \cdots + c_n\mathbf{e}_n) = v_1c_1 + \cdots + v_nc_n.$$
Since the bilinear form "$\cdot$" is non-degenerate, $C^{\perp\perp} = C$. We say that C has *dual minimum distance* at least $t+1$ if and only if the minimum Hamming distance of the dual code $C^{\perp}$ of C is at least $t+1$. As mentioned in Introduction, this condition is equivalent to saying that for any generator matrix G_C of C, any set of t columns of G_C is linearly independent.

For $t \in \mathbb{N}$ we already defined the set $\mathscr{C}_t(n,k)$ of all $[n,k]$-linear codes with dual minimum distance at least $t+1$ and the point-line geometry $\mathscr{P}_t(n,k) = (\mathscr{C}_t(n,k), \mathscr{L}_t(n,k))$ whose collinearity graph is precisely $\Lambda_t(n,k)$. The geometry $\mathscr{P}_t(n,k)$, clearly, can be regarded as a subgeometry of $\mathscr{G}(n,k)$.

We recall that a graph Γ is *connected* whenever given any two of its vertices (say P and Q with $P \neq Q$) there exists a path in Γ, i.e. a sequence of non-repeated adjacent vertices, starting at P and ending at Q. We consider the empty graph to be connected. The length of a path between P and Q is defined as the smallest cardinality of a path connecting P and Q diminished by 1; the distance $d_\Gamma(P,Q)$ between P and Q in Γ is the length of a shortest path between P and Q; a connected component of a graph is a maximal non-empty subset of its vertices such that the subgraph induced on it is connected. The diameter of a graph is the maximum distance between any two vertices; see [1] for a general reference about graph theory. In general we say that a subgraph Γ' is isometrically embedded in a larger graph Γ if there exists a distance-preserving map $\Gamma' \to \Gamma$; see [8].

We already mentioned in Introduction the subgraph $\Delta_t(n,k)$ of the Grassmann graph having as vertices the $[n,k]$-linear codes $\mathscr{C}_t(n,k)$ with dual minimum distance at least $t+1$. The graph $\Delta_t(n,k)$ was first introduced in [9] where the authors focused mostly on the case of non-degenerate linear codes, i.e. on $\Delta_1(n,k)$. In [11] these results, among others, were extended to the graph $\Delta_2(n,k)$ of projective linear codes. In [4] we considered properties of the graph $\Delta_t(n,k)$ for arbitrary t.

Comparing the graph $\Lambda_t(n,k)$ with the graph $\Delta_t(n,k)$, we observe that $\Lambda_t(n,k)$ has the same vertices as $\Delta_t(n,k)$, but less edges. In particular, there exist codes $A,B \in \mathscr{C}_t(n,k)$ such that $A \sim_\Delta B$, but $A \not\sim_\Lambda B$. Thus $d_\Lambda(A,B) > d_\Delta(A,B)$ and $\Lambda_t(n,k)$ is not isometrically embedded in $\Delta_t(n,k)$; consequently, $\Lambda_t(n,k)$ is not isometrically embedded in the Grassmann graph either.

2.1 Equivalent codes

Given the basis $\mathfrak{B}$, the *monomial group* $\mathscr{M}(V)$ of V consists of all linear transformations of V which map the set of subspaces $\{\langle \mathbf{e}_1\rangle, \dots, \langle \mathbf{e}_n\rangle\}$ in itself. It is straightforward to see that $\mathscr{M}(V) \cong \mathbb{F}_q^\times \wr S_n$, where $\wr$ denotes the wreath product and S_n is the symmetric group of order n; see [12, Chapter 8,§5] for more details.

Definition 2.1. Two $[n,k]$-linear codes X and Y are *equivalent* if there exists a monomial transformation $\rho \in \mathscr{M}(V)$ such that $X = \rho(Y)$.

Equivalence between linear codes is an equivalence relation and the equivalence class of a code X corresponds to the orbit of X under the action of $\mathscr{M}(V)$ on the subspaces of V.

A monomial transformation ρ is given by a linear transformation of V which sends the code Y (regarded as a k-dimensional subspace of V), into the code $X = \rho(Y)$. In particular, if G_Y is a generator matrix for Y, then the matrix $\rho(G_Y)$ obtained from G_Y by applying ρ to each of its rows, is a generator matrix for X. We put $G_{\rho(Y)} := \rho(G_Y)$. Observe that the transformation induced by ρ acts on the columns of G_Y. Indeed, if ρ is represented by a $n \times n$ matrix R with respect to the basis $\mathfrak{B}$, then $G_{\rho(Y)} = G_Y \cdot R$.

The matrix R factors as a product $R = PD$, where P is a permutation matrix and D is a non-singular diagonal matrix. In the rest of this paper, we shall denote, with a slight abuse of notation, by the same symbol ρ not only the isometry $\rho : V \to V$, but also the corresponding map acting on the generator matrices of the codes.

If X is a $[n,k]$-linear code with generator matrix G_X and $A \in \mathrm{GL}(k, \mathbb{F}_q)$, then $G'_X = AG_X$ is also a generator matrix for X.

It follows that two $[n,k]$-linear codes X and Y with generator matrices respectively G_X and G_Y are *equivalent* if there exists $A \in \mathrm{GL}(k, \mathbb{F}_q)$, a permutation matrix $P \in \mathrm{GL}(n, \mathbb{F}_q)$ and a diagonal matrix $D \in \mathrm{GL}(n, \mathbb{F}_q)$ such that

$$G_X = AG_Y(PD).$$

In particular, two codes are equivalent if and only if any two of their generator matrices belong to the same orbit under the action of the group $\mathrm{GL}(k, \mathbb{F}_q) : (\mathbb{F}_q^\times \wr S_n)$, where $\mathrm{GL}(k, \mathbb{F}_q)$ acts on the left of the generator matrix and fixes each code, regarded as a subspace, while $\mathbb{F}_q^\times \wr S_n$ acts on the right.

The following lemma provides explicit generators for the group $\mathscr{M}(V)$ and describe their action on the set of the linear codes.

Lemma 2.2. *Let $\mathfrak{B} = (\mathbf{e}_1, \dots, \mathbf{e}_n)$ be a fixed basis of V and α a generator of $\mathbb{F}_q^\times$. A set of generators for the monomial group $\mathscr{M}(V)$ with respect to $\mathfrak{B}$ is given by all the linear functions $\tau_{ij} : V \to V$ with $1 \le i < j \le n$ and $\mu : V \to V$ such that*

$$\tau_{ij}(\mathbf{e}_k) := \begin{cases} \mathbf{e}_i & \text{if } k = j \\ \mathbf{e}_j & \text{if } k = i \\ \mathbf{e}_k & \text{if } k \notin \{i,j\}, \end{cases} \qquad \mu(\mathbf{e}_k) := \begin{cases} \alpha\mathbf{e}_1 & \text{if } k = 1 \\ \mathbf{e}_k & \text{if } k \neq 1. \end{cases}$$

In particular, the map τ_{ij} is represented with respect to $\mathfrak{B}$ by a permutation matrix, while μ is represented by the matrix $\mathrm{diag}(\alpha, 1, 1, \dots, 1)$.

Proof. By construction $\mathscr{M}(V)$ is generated by the generators of the constituent groups of the wreath product $\mathbb{F}_q^\times \wr S_n$. As $\mathbb{F}_q^\times$ is cyclic generated by α and S_n is generated by the swaps, it follows that the above set is enough to generate $\mathscr{M}(V)$. □

Theorem 2.3. *Take $C \in \mathscr{C}_t(n,k)$. The orbit of C under the action of $\mathscr{M}(V)$ is contained in a connected component of $\Delta_t(n,k)$ for any $t \le k$.*

Proof. Let $C \in \mathscr{C}_t(n,k)$. We claim that $C \sim_\Delta \rho(C)$, for $\rho \in \mathscr{M}(V)$.

By Lemma 2.2, the group $\mathscr{M}(V)$ is generated by transpositions τ_{ij} ($1 \le i < j \le n$) and the diagonal transformation μ, so it is enough prove the claim above for $\rho = \tau_{ij}$ and $\rho = \mu$.

Suppose $G_C = (P_1, \dots, P_i, \dots, P_n)$ is a given generator matrix of C, where $P_1, \dots, P_n$ are the columns, hence vectors of $\mathbb{F}_q^k$. Let $G_{\mu(C)}$ be the generator matrix of the code $\mu(C)$, hence $G_{\mu(C)} = G_C \cdot \mathrm{diag}(\alpha, 1, 1, \dots, 1) = (\alpha P_1, \dots, P_i, \dots, P_n)$, since μ is represented by $\mathrm{diag}(\alpha, 1, 1, \dots, 1)$, with $0 \neq \alpha \in \mathbb{F}_q$.

We study the rank of the matrix $\begin{pmatrix} G_C \\ G_{\mu(C)} \end{pmatrix}$. We have

$$\mathrm{rank}\begin{pmatrix} P_1 & P_2 & \dots & P_i & \dots & P_n \\ \alpha P_1 & P_2 & \dots & P_i & \dots & P_n \end{pmatrix} = \mathrm{rank}\begin{pmatrix} P_1 & P_2 & \dots & P_i & \dots & P_n \\ (\alpha - 1)P_1 & 0 & \dots & 0 & \dots & 0 \end{pmatrix} \le k+1.$$

In particular, either $\mathrm{rank}\begin{pmatrix} G_C \\ G_{\mu(C)} \end{pmatrix} = k$ or $\mathrm{rank}\begin{pmatrix} G_C \\ G_{\mu(C)} \end{pmatrix} = k+1$. In the former case, $C = \mu(C)$ (and there is nothing to prove); in the latter, $\dim(C \cap \mu(C)) = k - 1$ and C and $\mu(C)$ are adjacent in the graph $\Delta_t(n,k)$.

We consider now the action of swaps τ_{ij}, $1 \le i < j \le n$, on G_C. If $P_i = P_j$, then $\tau_{ij}(C) = C$ and there is nothing to say. If $P_i \ne P_j$, then the code $\tau_{ij}(C)$ has generator matrix $G_{\tau_{ij}(C)} = \begin{pmatrix} P_1 & \dots & P_j & \dots & P_i & \dots & P_n \end{pmatrix}$.

Then

$$\operatorname{rank}\begin{pmatrix} G_C \\ G_{\tau_{ij}(C)} \end{pmatrix} = \operatorname{rank}\begin{pmatrix} P_1 & \dots & P_i & \dots & P_j & \dots & P_n \\ P_1 & \dots & P_j & \dots & P_i & \dots & P_n \end{pmatrix} =$$
$$\operatorname{rank}\begin{pmatrix} P_1 & \dots & P_i & \dots & P_j & \dots & P_n \\ 0 & \dots & P_j - P_i & \dots & P_i - P_j & \dots & 0 \end{pmatrix} =$$
$$\operatorname{rank}\begin{pmatrix} P_1 & \dots & P_i + P_j & \dots & P_j & \dots & P_n \\ 0 & \dots & 0 & \dots & P_i - P_j & \dots & 0 \end{pmatrix} \le k+1.$$

Proceeding as before, we have that either $C = \tau_{ij}(C)$ or $C \sim_\Delta \tau_{ij}(C)$. $\square$

It has been pointed out to the authors that Proposition 2 of [14] is equivalent to Theorem 2.3.

3 General results on $\mathscr{C}_t(n, k)$ and the graph $\Lambda_t(n, k)$

Let C be a $[n, k]$-linear code and $G_C = (P_1, P_2, \dots, P_n)$ a given generator matrix of C, where $P_1, \dots, P_n$ denote column vectors in $\mathbb{F}_q^k$.

Lemma 3.1. *All $[n, k-1]$-subcodes of C can be represented by generator matrices $G_C^\varphi := (\varphi(P_1), \varphi(P_2), \dots \varphi(P_n))$, as φ varies in all possible ways in the set $\mathscr{M}$ of all surjective linear functions from $\mathbb{F}_q^k$ to $\mathbb{F}_q^{k-1}$.*

Proof. Let D be an $[n, k-1]$-subcode of C. Then a generator matrix G_D for D has, as rows, $k-1$ linearly independent vectors of $\mathbb{F}_q^n$, and each of these vectors is a linear combination of the rows of G_C. So, there exists a $(k-1) \times k$ matrix F of rank $k-1$ such that $G_D = FG_C$. If φ is the linear function $\mathbb{F}_q^k \to \mathbb{F}_q^{k-1}$ with matrix F with respect to the canonical bases of $\mathbb{F}_q^k$ and $\mathbb{F}_q^{k-1}$, then φ is surjective. Define $G_C^\varphi := (\varphi(P_1), \dots, \varphi(P_n))$ as the matrix having as columns the images under φ of the columns $P_1, \dots, P_n$. Then $G_C^\varphi = G_D$. Conversely, if $\varphi : \mathbb{F}_q^k \to \mathbb{F}_q^{k-1}$ is a surjective linear function, then put $G_C^\varphi := FG_C$ for F a $(k-1) \times k$ matrix (of rank $k-1$) representing φ. Clearly, $\operatorname{rank}(G_C^\varphi) = k-1$ and all rows of G_C^φ are linear combinations of rows of G_C; so G_C^φ is the generator matrix of a subcode of C. $\square$

Remark 3.2. Any map φ of the previous lemma can be regarded as a function which sends all k-dimensional subspaces C of V into $(k-1)$-dimensional subspaces D with $D \subset C$. Its explicit action depends on the choice of a basis for each k-dimensional subspace C of V, i.e. on the choice of a generator matrix G_C for C. In particular, φ can be regarded as a map $\mathbb{F}_q^{k,n} \to \mathbb{F}_q^{k-1,n}$ sending a $k \times n$ matrix G into a $(k-1) \times n$ matrix G', $G \mapsto G' = FG$, where F is the representative matrix of φ (with respect to given bases). In any case, by Lemma 3.1, the set of all subcodes of C does not depend on the specific choice of G_C for C since φ varies among all surjective linear functions $\mathbb{F}_q^k \to \mathbb{F}_q^{k-1}$.

Lemma 3.3. *Let $C \in \mathscr{C}_t(n,k) \setminus \mathscr{I}_t(n,k)$ and C' be equivalent $[n,k]$-codes. Then there exist equivalent $[n,k-1]$-subcodes $D, D' \in \mathscr{C}_t(n,k-1)$ with $D \subset C$ and $D' \subset C'$ respectively. In particular, $C' \in \mathscr{C}_t(n,k) \setminus \mathscr{I}_t(n,k)$.*

Proof. Since C and C' are equivalent $[n,k]$-codes, then $C' \in \mathscr{C}_t(n,k)$, and there exists $\rho \in \mathscr{M}(V)$ such that $\rho(C) = C'$. Take $D \in \mathscr{C}_t(n,k-1)$ as a subcode of C (D exists because C is not isolated) and let G_C be a generator matrix for C. As seen in Section 2, $G_{\rho(C)} (= \rho(G_C))$ is a generator matrix of C'.

Denote by $\varphi : \mathbb{F}_q^k \to \mathbb{F}_q^{k-1}$ the surjective map associated to D (see Lemma 3.1), so G_C^φ is a generator matrix of D. We claim that D is equivalent to the subcode of $\rho(C)$ determined by φ, i.e.

$$G^\varphi_{\rho(C)} = \rho(G^\varphi_C). \tag{1}$$

By Lemma 2.2, it is enough to prove Claim (1) for $\rho = \tau_{ij}$ and $\rho = \mu$.

Write $G_C = (P_1, \ldots, P_i, \ldots, P_j, \ldots, P_n)$. If τ_{ij} acts as a transposition on the columns P_i and P_j of G_C, then

$$\begin{gathered}\tau_{ij}(G^\varphi_C) = (\tau_{ij} \circ \varphi)(G_C) = (\tau_{ij} \circ \varphi)((P_1, \ldots, P_i, \ldots, P_j, \ldots P_n)) = \\ \tau_{ij}(\varphi(P_1), \ldots, \varphi(P_i), \ldots, \varphi(P_j), \ldots, \varphi(P_n)) = \\ (\varphi(P_1), \ldots, \varphi(P_j), \ldots, \varphi(P_i), \ldots, \varphi(P_n)) = \varphi((P_1, \ldots, P_j, \ldots, P_i, \ldots, P_n)) = \\ \varphi(\tau_{ij}(P_1, \ldots, P_i, \ldots, P_j, \ldots P_n)) = \varphi(G_{\tau_{ij}(C)}) = G^\varphi_{\tau_{ij}(C)}.\end{gathered}$$

Likewise, if μ acts by multiplying the column P_1 by a primitive element α of $\mathbb{F}_q^\times$, then

$$\begin{gathered}\mu(G^\varphi_C) = (\mu \circ \varphi)(G_C) = \mu(\varphi(P_1), \varphi(P_2), \ldots, \varphi(P_n)) = \\ (\alpha\varphi(P_1), \varphi(P_2), \ldots, \varphi(P_n)) = (\varphi(\alpha P_1), \varphi(P_2) \ldots, \varphi(P_n)) = \\ \varphi((\alpha P_1, P_2, \ldots, P_n)) = \varphi(\mu(P_1, P_2, \ldots, P_n)) = \varphi(G_{\mu(C)}) = G^\varphi_{\mu(C)}.\end{gathered}$$

Thus, the Claim (1) is proved. In particular, $D' = \rho(D) \in \mathscr{C}_t(n,k-1)$ and $D' \subseteq C'$ so $C' \in \mathscr{C}_t(n,k) \setminus \mathscr{I}_t(n,k)$. $\square$

Lemma 3.4. *Let $C, C' \in \mathscr{C}_t(n,k) \setminus \mathscr{I}_t(n,k)$ and suppose that D and D' are two $[n, k-1]$-linear subcodes of C and C' ,respectively. If D, D' are in the same connected component of $\Delta_t(n, k-1)$, then C and C' are in the same connected component of $\Lambda_t(n,k)$.*

Proof. By hypothesis, there exists a path $D_0 = D \sim_\Delta D_1 \sim_\Delta \cdots \sim_\Delta D_w = D'$ in $\Delta_t(n, k-1)$ from D to D'. For any $i = 0, \ldots, w-1$ we have $\dim(D_i \cap D_{i+1}) = k-2$ and $\dim(\langle D_i, D_{i+1}\rangle) = k$. Consider the codes $E_i := \langle D_i, D_{i+1}\rangle \in \mathscr{C}_t(n,k)$. Since $E_i \cap E_{i+1} = D_{i+1} \in \mathscr{C}_t(n, k-1)$ for all i, they are the vertices of a path $E_0 \sim_\Lambda E_1 \sim_\Lambda \cdots \sim_\Lambda E_{w-1}$ in $\Lambda_t(n,k)$. Furthermore, C is adjacent to E_0 in $\Lambda_t(n,k)$ since $D \subseteq C \cap E_0$ and likewise C' is adjacent to E_{w-1}. It follows that C and C' are in the same connected component of $\Lambda_t(n,k)$. $\square$

Recall that $\widetilde{\Lambda}_t(n,k)$ is the graph induced by $\Lambda_t(n,k)$ on the codes in $\mathscr{C}_t(n,k) \setminus \mathscr{I}_t(n,k)$. The following is immediate.

Corollary 3.5. *Suppose that the graph $\Delta_t(n, k-1)$ is connected. Then the following conditions hold true:*

1. *All elements of $\mathscr{C}_t(n,k) \setminus \mathscr{I}_t(n,k)$ belong to the same connected component $\widetilde{\Lambda}_t(n,k)$ of $\Lambda_t(n,k)$. In particular, if the graph $\Delta_t(n, k-1)$ is connected, then $\widetilde{\Lambda}_t(n,k)$ is connected.*

2. *Any two codes in $\mathscr{C}_t(n,k) \setminus \mathscr{I}_t(n,k)$ are at distance at most* $\mathrm{diam}(\Delta_t(n, k-1)) + 1$.

Corollary 3.6. *Suppose $C, C' \in \mathscr{C}_t(n,k) \setminus \mathscr{I}_t(n,k)$ are equivalent codes. Then C and C' belong to the same connected component of $\Lambda_t(n,k)$.*

Proof. By Lemma 3.3, there are equivalent subcodes D, D' of C and C' (respectively) with $D, D' \in \mathscr{C}_t(n, k-1)$. By Theorem 2.3, D and D' belong to the same connected component of $\Delta_t(n, k-1)$. The thesis now follows directly from Lemma 3.4. $\square$

In Lemma 3.8 we shall make use of the following remark.

Remark 3.7. For any $D \in \mathscr{C}_t(n, k-1)$ and for any $[n,k]$-linear code C with $D \subset C$ we have $C \in \mathscr{C}_t(n,k)$; consequently, for any element $D \in \mathscr{C}_t(n, k-1)$ with $k - 1 < n$ there exists at least one $C \in \mathscr{C}_t(n,k)$ with $D \subset C$.

Lemma 3.8. *Suppose $\Delta_t(n, k-1)$ not to be connected. Then the graph $\widetilde{\Lambda}_t(n,k)$ is not connected.*

Proof. Take two codes $D, D' \in \mathscr{C}_t(n,k-1)$ which belong to different connected components of $\Delta_t(n,k-1)$ and let C, C' be two codes in $\mathscr{C}_t(n,k)$ having D, D' as subcodes respectively. These codes exist by Remark 3.7. By contradiction, suppose there exists a path $C = C_0 \sim_\Lambda C_1 \sim_\Lambda \cdots \sim_\Lambda C_w = C'$ in $\widetilde{\Lambda}_t(n,k)$ joining C and C'. So, $D_i = C_i \cap C_{i+1} \in \mathscr{C}_t(n,k-1)$ for $i = 0,\ldots,w-1$. Since both D_i and D_{i+1} are contained in C_{i+1}, we have either $D_i = D_{i+1}$ or $\dim(D_i \cap D_{i+1}) = k-2$. In either case, we have a collection of subcodes $D = D_{i_0} \sim_\Delta D_{i_1} \sim_\Delta \cdots \sim_\Delta D_{i_r} = D'$ in $\mathscr{C}_t(n,k-1)$ such that $\dim(D_{i_j} \cap D_{i_{j+1}}) = k-2$, i.e. a path in $\Delta_t(n,k-1)$. This is a contradiction, since we assumed D and D' to be in distinct connected components of $\Delta_t(n,k-1)$. $\square$

From Corollary 3.5 and Lemma 3.8, we have the following.

Corollary 3.9. *The graph $\widetilde{\Lambda}_t(n,k)$ is connected if and only if $\Delta_t(n,k-1)$ is connected.*

4 Proof of the main theorems

As in Lemma 3.1, put

$$\mathscr{M} = \{\varphi\colon \mathbb{F}_q^k \to \mathbb{F}_q^{k-1}\colon\ \varphi \text{ is linear and } \mathrm{Im}(\varphi) = \mathbb{F}_q^{k-1}\} \text{ and}$$

$$\mathscr{K} = \{\ker(\varphi)\colon \varphi \in \mathscr{M}\}.$$

Clearly, $\mathscr{K} \cong \mathrm{PG}(k-1,q)$, hence $|\mathscr{K}| = (q^k-1)/(q-1)$.

Proof of Theorem 1.4. Since $\Lambda_t(n,k) = \widetilde{\Lambda}_t(n,k) \cup \mathscr{I}_t(n,k)$ and the elements of $\mathscr{I}_t(n,k)$ are isolated vertices of $\Lambda_t(n,k)$, Theorem 1.4 is a direct consequence of Corollary 3.9. $\square$

Proof of Theorem 1.6 We have that $C \in \mathscr{C}_t(n,k)$ is an isolated code if and only if every $[n,k-1]$-subcode of C does not belong to $\mathscr{C}_t(n,k-1)$. This is the same as to say that for every $[n,k-1]$-subcode D of C there exist at least t columns in any of its generator matrices which are linearly dependent. By Lemma 3.1, any subcode D has a generator matrix of the form $G_C^\varphi = (\varphi(P_1),\ldots,\varphi(P_n))$, where $G_C = (P_1,\ldots,P_n)$ is a given generator matrix for C with columns $P_1,\ldots,P_n$ and φ is an element of $\mathscr{M}$. For C to be isolated we require that for every $\varphi \in \mathscr{M}$ there is at least one set of t columns of G_C^φ which are linearly dependent. This is equivalent to saying that there exist $i_1,\ldots i_t \in \{1,\ldots,n\}$ such that $\lambda_{i_1}P_{i_1} + \ldots \lambda_{i_t}P_{i_t} \in \ker(\varphi)$, i.e. the projective point given by $\ker(\varphi)(\in \mathscr{K})$ is in the $(t-1)$-dimensional projective space of $\mathrm{PG}(k-1,q)$ spanned by the points $\langle P_{i_1}\rangle,\ldots,\langle P_{i_t}\rangle$. Since this needs to be true for all $\varphi\in\mathscr{M}$ we have that the columns of the generator matrix G_C of C must form a $(t-1)$-saturating set of $\mathrm{PG}(k-1,q)$. Theorem 1.6 follows. $\square$

4.1 Non-degenerate codes

By Theorem 1.6, we have that $C\in\mathscr{C}_1(n,k)$, with generator matrix $G_C=(P_1,\ldots,P_n)$, is an isolated code if and only if for every $\varphi\in\mathscr{M}$ there exists $i\in\{1,\ldots,n\}$ such that $\langle P_i\rangle=\ker(\varphi)$, i.e. $\forall v\in\mathscr{K}$ there exists $i\in\{1,\ldots,n\}$ such that $\langle P_i\rangle=v$.

The following lemma is a direct consequence of Theorem 1.1 and Theorem 1.4. We also propose an alternative direct proof which holds only for $t=1$.

Lemma 4.1. $\Lambda_1(n,k)=\mathscr{I}_1(n,k)\cup\widetilde{\Lambda}_1(n,k)$, *where* $\widetilde{\Lambda}_1(n,k)$ *is a connected component of* $\Lambda_1(n,k)$. *The diameter of* $\widetilde{\Lambda}_1(n,k)$ *is at most* $k+1$.

Proof. It is enough to show that if a code is not isolated, then it belongs to the same connected component as any other non-isolated code. First observe that a code which contains the all-1 vector $\mathbf{1}$, is not isolated. Let now C_1 and C_2 be two non-isolated codes and suppose that D_1 and D_2 are two non-degenerate $[n,k-1]$-subcodes of C_1 and C_2, respectively. If $\mathbf{1}\in C_i$, then put $C_i':=C_i$ for $i=1,2$. If $\mathbf{1}\notin C_i$, then put $C_i'=\langle D_i,\mathbf{1}\rangle$. Observe that $d(C_i,C_i')\leq 1$ in $\Lambda_1(n,k)$. Since $\Delta_1(n,k-1)$ is connected (Theorem 1.1) we can construct a path in $\Lambda_1(n,k)$ from C_1' to C_2' of codes all containing the vector $\mathbf{1}$. So, there is a path from C_1 to C_1' and then from C_1' to C_2' and finally to C_2' to C_2, i.e. C_1 and C_2 belong to the same connected component. Furthermore, observe that $d(C_1,C_2)\leq d(C_1,C_1')+d(C_1',C_2')+d(C_2',C_2)\leq 2+d(C_1',C_2')\leq 2+k-1=k+1$. $\square$

Lemma 4.2. $\Lambda_1(n,k)$ *is connected if and only if* $n<(q^k-1)/(q-1)$.

Proof. By Lemma 4.1, the graph $\Lambda_1(n,k)$ is connected if and only if there exists no isolated vertex in it, i.e. $\mathscr{I}_1(n,k)=\emptyset$. This is equivalent to saying that for every $C\in\mathscr{C}_1(n,k)$, there exists at least a $[n,k-1]$-subcode D which is non-degenerate. By Lemma 3.1, we have that C admits at least a non-degenerate subcode if and only if there is at least one function in $\mathscr{M}$ whose kernel does not contain any column of G_C. This is always guaranteed if the number of surjective linear functions from $\mathbb{F}_q^k$ onto $\mathbb{F}_q^{k-1}$, and hence the number of their kernels, is strictly greater than the number of columns of a generator matrix of C (i.e. the length of C), hence for $|\mathscr{K}| = \frac{q^k-1}{q-1} > n$. On the other hand, if $n = \frac{q^k-1}{q-1}$ the code whose generator matrix has a column set consisting of all the distinct vector representatives of the 1-dimensional subspaces of $\mathbb{F}_q^k$ is definitely in $\mathscr{I}_1(n,k)$ and so $\Lambda_1(n,k)$ is disconnected. $\square$

Corollary 1.7 follows from Lemma 4.1 and Lemma 4.2 or, alternatively, Theorem 1.6.

4.2 Projective codes

We first prove that the Grassmann graph of the projective codes $\Delta_2(n,k)$ is always connected. Observe that, in any case, there is a natural bound on n, that is $n \le \frac{q^k-1}{q-1}$ as the maximum length of a projective code of dimension k is the same as the number of points of $\mathrm{PG}(k-1,q)$.

Lemma 4.3. *The graph $\Delta_2(n,k)$ is connected for any q and any $2 \le k \le n$.*

For a different proof of this lemma see also [14, Theorem 1].

Proof. Let C and C' be two projective codes with generator matrices respectively $G_C = (P_1, P_2, \dots, P_n)$ and $G_{C'} = (P'_1, P'_2, \dots, P'_n)$, where $P_i, P'_i \in \mathbb{F}_q^k$ are column vectors. By Theorem 2.3, we can assume, up to code equivalence, that if a column P_i in G_C is proportional to a column P'_j in $G_{C'}$, then $i = j$. Also by Theorem 2.3, we can assume that if P_i in G_C is proportional to P'_i in $G_{C'}$, then $P_i = P'_i$ (i.e. the coefficient of proportionality is 1 and the columns are the same). If under these assumptions $G_C = G_{C'}$, then C and C' are equivalent codes and, by Theorem 2.3 they belong to the same connected component of $\Delta_2(n,k)$. Otherwise we may construct a path in $\Delta_2(n,k)$ joining C and C' by considering the codes $C_0 := C \sim_\Delta C_1 \sim_\Delta \cdots \sim_\Delta C_w =: C'$, where the code C_i has as generator matrix, the matrix where the first i columns are the first i columns of $G_{C'}$ and the remaining columns are the corresponding columns of G_C. Indeed, $\operatorname{rank}\begin{pmatrix} G_{C_i} \\ G_{C_{i+1}} \end{pmatrix} = k$ or $\operatorname{rank}\begin{pmatrix} G_{C_i} \\ G_{C_{i+1}} \end{pmatrix} = k+1$ so, either $C_i = C_{i+1}$ or $C_i \sim_\Delta C_{i+1}$. $\square$

Part 1. of Corollary 1.8 is a direct consequence of Lemma 4.3 and Theorem 1.4. The following is part 2. of Corollary 1.8. Part 3. of Corollary 1.8 is Theorem 1.6.

Lemma 4.4. *$\Lambda_2(n,k)$ is connected if and only if n is strictly less than the minimum size μ_1 of a 1-saturating set for $\mathrm{PG}(k-1,q)$.*

Proof. By Corollary 1.8 part 1., the graph $\Lambda_2(n,k)$ is connected if and only if there exists no isolated vertex in it. This is equivalent to saying that for every $C \in \mathscr{C}_2(n,k)$, there exists at least a $[n,k-1]$-subcode D which is projective. By Lemma 3.1, we have that C admits at least a projective subcode if and only if there is at least one function in $\mathscr{M}$ whose kernel does not belong to any 2-space spanned by 2 columns of G_C. This is always guaranteed if the number of columns of any generator matrix of C (i.e. the length of C) is strictly less than the minimum size of a 1-saturating set of $\mathrm{PG}(k-1,q)$.

On the other hand, if $n \geq \mu_1$, where μ_1 is the minimum size of a 1-saturating set Ω of $\mathrm{PG}(k-1,q)$, the code whose generator matrix contains as columns vector representatives for all of the 1-dimensional subspaces of Ω is definitely in $\mathscr{I}_2(n,k)$ and so $\Lambda_2(n,k)$ is disconnected. $\square$

4.3 Proof of Theorem 1.9

Lemma 4.5. *If $t = k$, then $\mathscr{C}_k(n,k) = \mathscr{I}_k(n,k)$. If $t < k$ and $q > \binom{n}{t}^{1/(k-t)}$, then we have $\mathscr{C}_t(n,k) \neq \emptyset$ and $\mathscr{I}_t(n,k) = \emptyset$.*

Proof. If $k = t$, then each set of k columns of any generator matrix of a code $C \in \mathscr{C}_k(n,k)$ is linearly independent, so C is MDS. In particular, the dual minimum distance of each $[n,k-1]$-subcode of an MDS code is at most $k-1 < k$ (as its generator matrix has just $k-1$ rows). So, every code $C \in \mathscr{C}_k(n,k)$ is isolated and $\mathscr{C}_k(n,k) = \mathscr{I}_k(n,k)$.

Suppose now $t \leq k-1$ and $q > n$. Take $C \in \mathscr{C}_t(n,k)$ with generator matrix G_C. Let $\varphi : \mathbb{F}_q^k \to \mathbb{F}_q^{k-1}$ be a surjective linear map and denote by $D = \varphi(C)$ the subcode of C having as generator matrix G_C^{φ}. We have $D \notin \mathscr{C}_t(n,k-1)$ if and only if there is a set Ω of t linearly independent columns in G_C which is mapped into a set of linearly dependent columns in G_C^{φ}. In other words, this happens if and only if $\ker(\varphi) \in \langle\Omega\rangle$. As φ varies in all possible ways, there are exactly $(q^k-1)/(q-1)$ possible subspaces for $\ker(\varphi)$. On the other hand the number of possible t-subspaces spanned by t columns of G_C is $\binom{n}{t}$ and each of these contains at most $(q^t-1)/(q-1)$ distinct 1-dimensional subspaces. So, if

$$(q^k-1) > \binom{n}{t}(q^t-1),$$

then we have that there is at least one $\varphi \in \mathscr{M}$ such that the subcode of C determined by φ is in $\mathscr{C}_t(n,k-1)$ and, consequently, $C \notin \mathscr{I}_t(n,k)$. So, if

$$\frac{q^k-1}{q^t-1} > \binom{n}{t},$$

then $\mathscr{I}_t(n,k)=\emptyset$.

For $t<k$ we have the approximation $q^{k-t}<\frac{q^k-1}{q^t-1}$, since

$$q^{k-t}=\frac{q^k}{q^t}<\frac{q^k-1}{q^t-1}\Leftrightarrow q^k(q^t-1)<q^t(q^k-1)\Leftrightarrow q^k>q^t\Leftrightarrow k>t.$$

So, if we require $q^{k-t}>\binom{n}{t}$, then $\frac{q^k-1}{q^t-1}>\binom{n}{t}$. Hence $\mathscr{I}_t(n,k)=\emptyset$. $\square$

Theorem 1.9 now follows from Theorem 1.1, Theorem 1.4 and Lemma 4.5.

4.4 Transparent embeddings

Given two point-line geometries $\mathscr{G}_1$ and $\mathscr{G}_2$, we say that an injective map $\varepsilon:\mathscr{G}_1\to\mathscr{G}_2$ of the point-set of $\mathscr{G}_1$ into the point-set of $\mathscr{G}_2$ is an *embedding* of $\mathscr{G}_1$ into $\mathscr{G}_2$ if for any line ℓ of $\mathscr{G}_1$ the image $\varepsilon(\ell):=\{\varepsilon(p)\colon p\in\ell\}$ is a line of $\mathscr{G}_2$. If $\mathscr{G}_2$ is a projective geometry we say that an embedding of $\mathscr{G}_1$ into $\mathscr{G}_2$ is a projective embedding into the projective subspace $\langle\varepsilon(\mathscr{G}_1)\rangle$.

In [3] the notion of *transparency* for an embedding has been introduced. An embedding $\varepsilon:\mathscr{G}_1\to\mathscr{G}_2$ is transparent if the preimage of any line of $\mathscr{G}_2$ contained in the image of ε is a line of $\mathscr{G}_1$.

Let us focus now on the Grassmann geometry $\mathscr{G}(n,k)$ and the geometry $\mathscr{P}_t(n,k)$. Since $\mathscr{P}_t(n,k)$ is a subgeometry of $\mathscr{G}(n,k)$, the inclusion map $\iota:\mathscr{P}_t(n,k)\to\mathscr{G}(n,k)$ is an embedding.

There is a vast literature regarding projective embeddings of the Grassmann geometry. For the sake of our paper we stick only to the essential results referring the interested reader to, e.g., [2].

Proof of Theorem 1.11. By definition of $\mathscr{P}_t(n,k)$, any line of $\mathscr{P}_t(n,k)$ is also a line of $\mathscr{G}(n,k)$. In particular, the inclusion ι is an embedding $\iota:\mathscr{P}_t(n,k)\to\mathscr{G}(n,k)$.

To prove that ι is transparent we need to show that if a line ℓ of $\mathscr{G}(n,k)$ consists all of points of $\mathscr{C}_t(n,k)$, then for any two $X,Y\in\ell$ we have $X\sim_\Lambda Y$. This is equivalent to saying that if X and Y are not on a line of $\Lambda_t(n,k)$, then $\iota(X)$ and $\iota(Y)$ are not on a line contained in $\iota(\mathscr{C}_t(n,k))$, i.e. either X and Y are not on a line of $\mathscr{G}(n,k)$ or X and Y are on a line of $\mathscr{G}(n,k)$ whose points are not all images of points in $\mathscr{C}_t(n,k)$. If $\iota(X)$ and $\iota(Y)$ are not collinear in $\mathscr{G}(n,k)$, then X and Y are not collinear in $\mathscr{P}_t(n,k)$ as all lines of $\mathscr{P}_t(n,k)$ are lines of $\mathscr{G}(n,k)$. So, we have to deal only with the latter case. Observe that X and Y lie on a line of $\mathscr{G}(n,k)$ if and only if $X\sim_\Delta Y$.

So, suppose $X\sim_\Delta Y$ but $X\not\sim_\Lambda Y$ for $X,Y\in\mathscr{C}_t(n,k)$. To obtain the result it is enough to show that there exists at least one Z on the line ℓ of $\mathscr{G}(n,k)$

determined by X and Y with $Z \notin \mathscr{C}_t(n,k)$. So, the theorem is a consequence of the following lemma.

Lemma 4.6. *Suppose* $X, Y \in \mathscr{C}_t(n,k)$*,* $\dim(X \cap Y) = k-1$ *but* $X \cap Y \notin \mathscr{C}_t(n,k-1)$*. Then there exists* Z *such that* $X \cap Y \subset Z \subset \langle X, Y\rangle$*,* $\dim(Z) = k$ *and* $Z \notin \mathscr{C}_t(n,k)$*.*

Proof. Put $D = X \cap Y$. By hypothesis, $D \notin \mathscr{C}_t(n,k-1)$. Suppose by contradiction that for all $\mathbf{v} \in \langle X, Y\rangle \setminus D$ we have $Z_{\mathbf{v}} := \langle D, \mathbf{v}\rangle \in \mathscr{C}_t(n,k)$. Write $X = \langle D, \mathbf{x}\rangle$ and $Y = \langle D, \mathbf{y}\rangle$, where $\mathbf{x} = (x_1, \dots, x_n) \in X \setminus D$ and $\mathbf{y} = (y_1, \dots, y_n) \in Y \setminus D$. So, we can always take $\mathbf{v} = \alpha\mathbf{x} + \beta\mathbf{y}$, for α, $\beta \in \mathbb{F}_q$.

Since $D \notin \mathscr{C}_t(n,k-1)$, there is at least one set of t columns $P_{i_1}, \dots, P_{i_t}$ with $1 \le i_1 < \cdots < i_t \le n$ in any generator matrix G_D of D which are linearly dependent. Denote by $\overline{G_D}$ the $(k-1) \times t$ submatrix of G_D having $P_{i_1}, \dots, P_{i_t}$ as columns (and $R_1, \dots, R_{k-1}$ as rows). By construction, we have that $\operatorname{rank}(\overline{G_D}) \le t-1$.

Consider the $(k \times t)$-matrix

$$G_{\alpha,\beta} := \begin{pmatrix} P_{i_1} & P_{i_2} & \dots & P_{i_t} \\ \alpha x_{i_1} + \beta y_{i_1} & \alpha x_{i_2} + \beta y_{i_2} & \dots & \alpha x_{i_t} + \beta y_{i_t} \end{pmatrix}. \tag{2}$$

The matrix $G_{\alpha,\beta}$ can be regarded as the submatrix of a generator matrix of the generic code $Z_{\alpha\mathbf{x}+\beta\mathbf{y}}$ with $D \subset Z_{\alpha\mathbf{x}+\beta\mathbf{y}} \subset \langle X, Y\rangle$. If $Z_{\alpha\mathbf{x}+\beta\mathbf{y}} \in \mathscr{C}_t(n,k)$, then necessarily $\operatorname{rank}(G_{\alpha,\beta}) = t$.

On the other hand, since $\begin{pmatrix} P_{i_1} & P_{i_2} & \dots & P_{i_t} \\ x_{i_1} & x_{i_2} & \dots & x_{i_t} \end{pmatrix}$ is a $(k \times t)$-submatrix of a generator matrix of the code X and since $X \in \mathscr{C}_t(n,k)$, we have that $\operatorname{rank}\begin{pmatrix} P_{i_1} & P_{i_2} & \dots & P_{i_t} \\ x_{i_1} & x_{i_2} & \dots & x_{i_t} \end{pmatrix} = t$.

So,

$$\operatorname{rank}\begin{pmatrix} P_{i_1} & P_{i_2} & \dots & P_{i_t} \\ x_{i_1} & x_{i_2} & \dots & x_{i_t} \\ y_{i_1} & y_{i_2} & \dots & y_{i_t} \end{pmatrix} = t = \operatorname{rank}\begin{pmatrix} P_{i_1} & P_{i_2} & \dots & P_{i_t} \\ x_{i_1} & x_{i_2} & \dots & x_{i_t} \end{pmatrix}.$$

Thus, the vector $(y_{i_1}, \dots, y_{i_t})$ is a linear combination of the rows $R_1, \dots R_{k-1}$ of $\overline{G_D}$ and of the vector $(x_{i_1}, \dots, x_{i_t})$, where $(x_{i_1}, \dots, x_{i_t})$ appears with a non-zero coefficient (otherwise $(y_{i_1}, \dots, y_{i_t}) \in \langle R_1, \dots, R_{k-1}\rangle$ and consequently the columns $\begin{pmatrix} P_{i_1} \\ y_{i_1} \end{pmatrix}, \begin{pmatrix} P_{i_2} \\ y_{i_2} \end{pmatrix}, \dots, \begin{pmatrix} P_{i_t} \\ y_{i_t} \end{pmatrix}$ of the generator matrix $\begin{pmatrix} G_D \\ \mathbf{y} \end{pmatrix}$ of Y would be linearly dependent and thus $Y \notin \mathscr{C}_t(n,k)$, against the hypothesis). Hence, we can write $\lambda(y_{i_1},\dots,y_{i_t}) = (x_{i_1},\dots,x_{i_t}) + \sum_{j=1}^{k-1}\theta_j R_j$ for suitable $\theta_j, \lambda \in \mathbb{F}_q$ and $\lambda \neq 0$. So,

$$\lambda(y_{i_1},\dots,y_{i_t}) - (x_{i_1},\dots,x_{i_t}) = \sum_{j=1}^{k-1}\theta_j R_j,$$

and the rank of the matrix $G_{-1,\lambda}$ (obtained from (2) with $\alpha = -1$ and $\beta = \lambda$) is at most $t-1$. It follows that $Z_{-\mathbf{x}+\lambda\mathbf{y}} \notin \mathscr{C}_t(n,k)$. $\square$

Acknowledgments

Both authors are affiliated with GNSAGA of INdAM (Italy) whose support they kindly acknowledge.

References

[1] J.A. Bondy and U.S.R. Murty, *Graph theory and applications*, North-Holland, New York (1982).

[2] F. Buekenhout, Handbook of Incidence Geometry – Buildings and Foundations, Elsevier, Amsterdam (1995).

[3] I. Cardinali, L. Giuzzi, A.Pasini *On transparent embeddings of point-line geometries.* J. Combin. Theory Ser. A 155 (2018), 190–224.

[4] I. Cardinali, L. Giuzzi, M. Kwiatowski, *On the Grassmann graph of linear codes*, Finite Fields Appl. **75** (2021), 101895.

[5] A. A. Davydov, S. Marcugini, and F. Pambianco, *On saturating sets in projective spaces*, J. Combin. Th. A 103 (2003) 1–15.

[6] A. A. Davydov, S. Marcugini, and F. Pambianco, *New covering codes of radius R, codimension tR and* $tR+\frac{R}{2}$ *and saturating sets in projective spaces*, Des. Codes Cryptogr. **87** (2019), 2771–2792.

[7] M. Giulietti, *The geometry of covering codes: small complete caps and saturating sets in Galois spaces* in S.R. Blackburn, R. Holloway, M. Wildon, Surveys in Combinatorics 2013, London Math. Soc. Lec. Note Series, 409, 51–90, Cambridge University Press, Cambridge (2013).

[8] R.L. Graham and P. M. Winkler, *On Isometric Embeddings of Graphs*, Trans. Amer. Math. Soc. **288**:2 (1985), 527–536.

[9] M. Kwiatkowski, M. Pankov, *On the distance between linear codes*, Finite Fields Appl. **39** (2016), 251–263.

[10] M. Kwiatowski, M. Pankov, *Chow's theorem for linear codes*, Finite Fields Appl. **46** (2017), 147–162.

[11] M. Kwiatkowski, M. Pankov, A. Pasini, *The graphs of projective codes*, Finite Fields Appl. **54** (2018), 15–29.

[12] F.J. MacWilliams, N.J.A. Sloane, *The Theory of Error-Correcting Codes*, North-Holland Mathematical Library vol. 16, North-Holland Publishing Co., Amsterdam (1977).

[13] M. Pankov, *The graphs of non-degenerate linear codes*, J. Combin Th. A **195** (2023) 105720.

[14] M. Pankov, *Connectedness of projective codes in the Grassmann graph*, preprint, arXiv:1806.02729v2.

[15] E.E. Shult, *Points and lines. Characterizing the classical geometries*, Universitext, Springer, Heidelberg (2011).

Authors' addresses:

Ilaria Cardinali
Dep. Information Engineering and Mathematics
University of Siena
Via Roma 56, I-53100 Siena, Italy
ilaria.cardinali@unisi.it

Luca Giuzzi
D.I.C.A.T.A.M.
University of Brescia
Via Branze 43, I-25123 Brescia, Italy
luca.giuzzi@unibs.it