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Optimal portfolio choice with path dependent benchmarked labor income: a mean field model

Boualem Djehiche*, Fausto Gozzi†, Giovanni Zanco‡ and Margherita Zanella§

Abstract

We consider the life-cycle optimal portfolio choice problem faced by an agent receiving labor income and allocating her wealth to risky assets and a riskless bond subject to a borrowing constraint. In this paper, to reflect a realistic economic setting, we propose a model where the dynamics of the labor income has two main features. First, labor income adjust slowly to financial market shocks, a feature already considered in Biffis et al. (2015) [10]. Second, the labor income y_i of an agent i is benchmarked against the labor incomes of a population $y^n := (y_1, y_2, \dots, y_n)$ of n agents with comparable tasks and/or ranks. This last feature has not been considered yet in the literature and is faced taking the limit when $n \rightarrow +\infty$ so that the problem falls into the family of optimal control of infinite dimensional McKean-Vlasov Dynamics, which is a completely new and challenging research field.

We study the problem in a simplified case where, adding a suitable new variable, we are able to find explicitly the solution of the associated HJB equation and find the optimal feedback controls. The techniques are a careful and nontrivial extension of the ones introduced in the previous papers of Biffis et al., [10, 9].

Key words: Life-cycle optimal portfolio with labor income following path dependent and law dependent dynamics; Dynamic programming/optimal control of SDEs in infinite dimension with McKean-Vlasov dynamics and state constraints; Second order Hamilton-Jacobi-Bellman equations in infinite dimension; Verification theorems and optimal feedback controls.

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1 Introduction

We consider the life-cycle optimal portfolio choice problem faced by an agent receiving labor income and allocating her wealth to risky assets and a riskless bond subject to a borrowing constraint. In line with the empirical findings and best practice, to reflect a realistic economic setting, the dynamics of labor incomes should include two main features. Firstly, labor incomes adjust slowly to financial market shocks, and income shocks have modest persistency when individuals can learn about their earning potential (see, e.g., [33], [17], [35]). This suggests that delayed dynamics may represent a very tractable way of modelling wages that adjust slowly to financial market shocks (e.g., [19], section 6). This aspect has been considered in the recent paper [10] (see also [8], [7]) in which the dynamics of labor income are modeled as a path-dependent delayed diffusion process of the form

$$dy(t) = \left[\mu_y y(t) + \int_{-d}^0 \phi(s) y(t+s) ds \right] dt + y(t) \sigma_y dZ(t), \quad (1) \quad \boxed{\text{delay}}$$

where Z is a Brownian motion (see Section 2 below for further details). The resulting optimal control problem which entails maximization of the expected power utility from lifetime consumption and bequest, subject to a linear state equation containing delay, as well as a state constraint (which is well known to make the problem considerably harder to solve), is infinite-dimensional, and can be seen as an infinite-dimensional generalization of Merton's optimal portfolio problem. In [10] the authors were able to solve it completely obtaining the optimal controls in feedback form (Theorem 4.12), which can be considered as the infinite dimensional generalization of the explicit solution to Merton's optimal portfolio problem which furthermore allows to fully understand the economic implications of the setting.

Secondly, and this is the novelty of this paper, the labor income y_i of an agent i is benchmarked against the labor incomes of a population $y^n := (y_1, y_2, \dots, y_n)$ of n agents with comparable tasks or ranks among the profession such as the level of full professor, associate professor, actuary, trader, risk manager etc., where one usually uses some wage level $b(y^n)$ as a reference to declare whether that agent has a superior, fair or inferior labor income compared with her peers. Typically, the labor income y_i 'mean-reverts' to the benchmark $b(y^n)$ with some mean reversion speed ϵ (see e.g. [19, §6] or [6] for the introduction of mean reverting terms in modeling labor income dynamics). Moreover,

such a benchmark $b(y^n)$ should reflect some 'consensus' labor income of an indistinguishable agent within the peer group. Many corporations use the average $\bar{y}_n(t) := \frac{1}{n-1} \sum_{j=1, j \neq i}^n y_j(t)$, the median wage or the truncated average above a certain level ℓ within the company or even within the profession, $y_n^\ell(t) := \frac{1}{n-1} \sum_{j=1, j \neq i}^n y_j(t) I_{\{y_j(t) \geq \ell\}}$, used as incentive to keep attractive agents within the company, as benchmark. These measures reflect some aggregation mechanism of some or all of the agents' labor incomes.

Let the benchmark $b(y^n)$ be the average income of a population of n individuals at time t , $b(y^n) := \bar{y}_n(t) := \frac{1}{n-1} \sum_{j=1, j \neq i}^n y_j(t)$. The dynamics of the i -th agent's labor income which includes the above mentioned aspects, i.e. path-dependency and benchmarking, can be modeled as

$$dy_i(t) = \left[\epsilon(y_i(t) - \bar{y}_n(t)) + \int_{-d}^0 \phi(s) y_i(t+s) ds \right] dt + y_i(t) \sigma_y dZ_i(t) \quad (2) \quad \boxed{\text{n-agents}}$$

where the Z_i 's are independent Brownian motions. It may be possible to generalize such model introducing a common macroeconomic noise but this would be out of the scope of the paper. Thus, $y^n = (y_1, y_2, \dots, y_n)$ solves a system of interacting diffusions which are statistically indistinguishable i.e. have exchangeable joint laws. By the propagation of chaos property (see e.g. [30], Theorem 1.3), in the limit $n \rightarrow \infty$, the dynamics of the labor income of the representative agent is of McKean-Vlasov or mean-field type and reads

$$dy(t) = \left[\epsilon(y(t) - \mathbb{E}[y(t)]) + \int_{-d}^0 \phi(s) y(t+s) ds \right] dt + y(t) \sigma_y dZ(t). \quad (3) \quad \boxed{\text{intro-mean}}$$

Thus, to extend the infinite dimensional generalization of Merton's optimal portfolio problem of [10] to benchmarked labor income dynamics, the mean field delayed SDE (3) can be used as the dynamics of the labor income of the representative agent instead of the system of n interacting diffusions for arbitrarily large n agents.

In the general case, to reflect consensus and aggregation, the benchmark wage $b(y^n)$ should be chosen such that y_1, y_2, \dots, y_n which solve a system of interacting diffusions of the form

$$dy_i(t) = \left[\epsilon(y_i(t) - b(\nu^n(t))) + \int_{-d}^0 \phi(s) y_i(t+s) ds \right] dt + y_i(t) \sigma_y dZ_i(t) \quad (4) \quad \boxed{\text{b-n-agents}}$$

are statistically indistinguishable ($\nu^n(t)$ denoting here the empirical measure of $y^n(t)$), in which case, in the limit $n \rightarrow \infty$, the dynamics of the representative agent's labor income satisfies the mean field type dynamics

$$dy(t) = \left[\epsilon(y(t) - b(\text{law}(y(t)))) + \int_{-d}^0 \phi(s) y(t+s) ds \right] dt + y(t) \sigma_y dZ(t). \quad (5) \quad \boxed{\text{intro-mf}}$$

It follows that the resulting optimal control problem adds a 'mean-field aspect' to the infinite-dimensional generalization of Merton's optimal portfolio problem studied in [10]. To be precise this problem falls into both families studied in this area: the 'Mean-Field Games' where we look for a Nash equilibrium of a game with many players; the 'Optimal Control of McKean-Vlasov Dynamics' (or 'Mean Field Control'), where a unique representative agent (the 'planner') takes the decisions.

In general the above two problems are different and give different results (see e.g. [14, §6.1]) but in our case, since the labor income is not influenced by the choice of the agents, they turn out to be the same. This means that the optimal strategies found in this paper can be interpreted in both ways: strategies of the players in a Nash equilibrium of a MFG or strategies adopted by the planner. Here our goal is mainly to develop the theoretical machinery to find the solution while we leave the analysis of its financial consequences for a subsequent paper.

In this paper we explicitly solve the mean-field and path-dependent generalization of Merton's optimal portfolio problem under the labor income dynamics (3). We are able to do this using a suitable infinite-dimensional general problem (coming from a change of variable introduced in

Remark 2.4) whose associated HJB equation admits an explicit solution \tilde{v} which allows to find the optimal control strategies in an explicit feedback/closed-loop form. This result may be exploited to perform a deep analysis of the practical consequences of the model and to fit it with real data. This is beyond the scope of this paper and is the subject of our current research. We give some hints of what can be done in Section 6.

Under the general dynamics (5), an explicit solution of the associated infinite dimensional HJB equation is however out of reach. Even establishing existence and uniqueness and deriving qualitative properties of the solution of the associated HJB seems a hard problem to solve for the time being. The main issue here is that, even in the finite-dimensional non-path-dependent case, the theory for HJB equations arising in the optimal control of McKean-Vlasov dynamics is at a very initial stage: only few results on viscosity solutions are available and no regularity theorem is proved up to now, except in very specific settings, like the linear quadratic one. Concerning the finite-dimensional non-path-dependent case, one can see e.g. the book [14] for an account of the theory, and the papers [13, 15, 43] for some recent results. Concerning instead the finite-dimensional path-dependent case one can see the paper [49] for some results on viscosity solutions of the HJB equations. Finally, up to now, the only paper treating the optimal control of infinite-dimensional path-dependent McKean-Vlasov equations is [12] where only viscosity solutions are considered, hence without any regularity results. On the other hand for mean-field games in infinite dimension we only know the linear quadratic model of [23].

The structure of the paper is as follows. In Section 2 we outline the model and, in Remark 2.4, introduce the change of variable which we use to rewrite it in a more treatable form. Section 3 is devoted to the non-trivial task of rewriting the no-borrowing constraint (see (18) below) in our case. In Section 4 we first write our general problem (Problem 4.3) in a suitable infinite-dimensional setting (Subsection 4.1). Then, in Subsection 4.2, we write and solve the associated HJB equation (Theorem 4.7). In Section 5, we solve the general problem. First, in Subsection 5.1, we provide a lemma to understand what happens to admissible strategies when the boundary of the constraint set is reached, a key feature in dealing with state constraints problems. Then, in Subsections 5.2-5.3, we prove the fundamental identity and the verification theorem, which allow us to find the optimal strategies in feedback form. Here, for brevity, we consider mainly the case $\gamma \in (0, 1)$ simply recalling how to deal with the case $\gamma > 1$. Finally, Section 6 summarizes the main results of the paper for the original problem, with a short discussion.

2 Problem formulation

Problem formulation

To formulate properly the problem we first introduce the equation satisfied by the agent's wealth process W ; the basic setting is borrowed from [19] and [10].

Consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where we define the \mathbb{F} -adapted vector valued process (S_0, S) representing the price evolution of a riskless asset, S_0 , and n risky assets, $S = (S_1, \dots, S_n)^\top$, with dynamics

$$\begin{cases} dS_0(t) = S_0(t)r dt \\ dS(t) = \text{diag}(S(t))(\mu dt + \sigma dZ(t)) \\ S_0(0) = 1 \\ S(0) \in \mathbb{R}_+^n, \end{cases} \quad (6)$$

where we assume the following.

hp:S

Assumption 2.1.

- (i) Z is a n -dimensional Brownian motion. The filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ is the one generated by Z , augmented with the \mathbb{P} -null sets.
- (ii) $\mu \in \mathbb{R}^n$, and the matrix $\sigma \in \mathbb{R}^{n \times n}$ is invertible.

An agent is endowed with initial wealth $w \geq 0$, and receives labor income y until the stopping time $\tau_\delta > 0$, which represents the agent's random time of death. We assume the following.

hp:tau

Assumption 2.2.

- (i) τ_δ is independent of Z , and it has exponential law with parameter $\delta > 0$.
- (ii) The reference filtration is accordingly given by the enlarged filtration $\mathbb{G} := (\mathcal{G}_t)_{t \geq 0}$, where each sigma-field \mathcal{G}_t is defined as

$$\mathcal{G}_t := \cap_{u > t} (\mathcal{F}_u \vee \sigma_g(\tau_\delta \wedge u)),$$

augmented with the \mathbb{P} -null sets. Here by $\sigma_g(U)$ we denote the sigma-field generated by the random variable U .

Note that, with the above choice, \mathbb{G} is the minimal enlargement of the Brownian filtration satisfying the usual assumptions and making τ_δ a stopping time (see [44, Section VI.3, p.370] or [29, Section 7.3.3, p.420]). Moreover, see [2, Proposition 2.11-(b)], we have the following result. If a process A is \mathbb{G} -predictable then there exists a process a which is \mathbb{F} -predictable and such that

$$A(s, \omega) = a(s, \omega), \quad \forall \omega \in \Omega, \forall s \in [0, \tau_\delta(\omega)]. \quad (7)$$

eq:GFpred

Therefore, arguing as in [10, Section 2], we can reduce the problem (which is initially relative to the larger filtration \mathbb{G}) to the “pre-death” one (where we work with \mathbb{F} -predictable processes). Hence, from now on we express the problem in terms of \mathbb{F} -predictable processes.

The agent can invest her resources in the riskless and risky assets, and can consume at rate $c(t) \geq 0$. We denote by $\theta(t) \in \mathbb{R}^n$ the amounts allocated to the risky assets at each time $t \geq 0$. The agent can also purchase life insurance to reach a bequest target $B(\tau_\delta)$ at death, where $B(\cdot) \geq 0$ is also chosen by the agent. We let the agent pay an insurance premium of amount $\delta(B(t) - W(t))$ to purchase coverage of face value $B(t) - W(t)$, for $t < \tau_\delta$. As in [19], we interpret a negative face value $B(t) - W(t) < 0$ as a life annuity trading wealth at death for a positive income flow $\delta(W(t) - B(t))$ while living. The controls c, θ , and B are for the moment assumed to belong to the following set:

$$\Pi^0 := \left\{ \mathbb{F}\text{-predictable } c(\cdot), B(\cdot), \theta(\cdot) : c(\cdot), B(\cdot) \in L^1(\Omega \times [0, +\infty); \mathbb{R}_+), \theta(\cdot) \in L^2(\Omega \times \mathbb{R}; \mathbb{R}^n) \right\}. \quad (8)$$

DEF_PIO_FIRST_DEFI

The agent’s wealth (before death) is assumed to obey to the standard dynamic budget constraint of the Merton portfolio model, but with the labor income and insurance premium terms added in the drift, exactly as in [19] and [10]; thus the agent’s wealth W satisfies the SDE

$$dW(t) = [W(t)r + \theta(t) \cdot (\mu - r\mathbf{1}) + y(t) - c(t) - \delta(B(t) - W(t))] dt + \theta(t) \cdot \sigma dZ(t). \quad (9)$$

eq:W_single

On the other hand the evolution of the labor income y here is new. The main novelty here is that, as opposed to standard bilinear SDEs (as in, e.g., [19]) and to bilinear path-dependent SDEs (as in [10]), we assume the labor income y to follow the bilinear SDE (3), where the drift contains not only a path-dependent term but also a mean reverting term. Hence, the dynamics of the state variables (W, y) are

TH_LABOR_INCOME

$$\begin{cases} dW(t) = [W(t)r + \theta(t) \cdot (\mu - r\mathbf{1}) + y(t) - c(t) - \delta(B(t) - W(t))] dt + \theta(t) \cdot \sigma dZ(t) \end{cases} \quad (10a)$$

DYNAMICS_WEALTH_LAI

$$\begin{cases} dy(t) = \left[\epsilon(y(t) - \mathbb{E}[y(t)]) + \mu_y y(t) + \int_{-d}^0 \phi(s) y(t+s) ds \right] dt + y(t) \sigma_y \cdot dZ(t), \end{cases} \quad (10b)$$

DYNAMICS_WEALTH_LAI

$$\begin{cases} W(0) = w, \end{cases} \quad (10c)$$

DYNAMICS_WEALTH_LAI

$$\begin{cases} y(0) = x_0, \quad y(s) = x_1(s) \text{ for } s \in [-d, 0), \end{cases} \quad (10d)$$

DYNAMICS_WEALTH_LAI

where $(c, B, \theta) \in \Pi^0(w, x_0, x_1)$, $\mu_y, \epsilon \in \mathbb{R}$, $\sigma_y \in \mathbb{R}^n$, $\mathbf{1} = (1, \dots, 1)^\top$ is the unitary vector in \mathbb{R}^n , \cdot denotes the canonical inner product of \mathbb{R}^n and the functions $\phi(\cdot), x_1(\cdot)$ belong to $L^2(-d, 0; \mathbb{R})$.

Remark 2.3. From an economic point of view, as in [9], the term $\mu_y y(t)$ in the dynamics of y in (10b) models a discounting effect at rate μ_y to account for a possible inflationary ($\mu_y < 0$)/deflationary ($\mu_y > 0$) regime. Moreover, in the mean reverting term, it is standard to choose $\epsilon < 0$. Here we take generic $\epsilon \in \mathbb{R}$ since our method of solution works also in this case.

Once the control strategies $(c, B, \theta) \in \Pi^0(w, x_0, x_1)$ are fixed and the process $y \in L^1(\Omega \times [0, +\infty); \mathbb{R}_+)$ is given, existence and uniqueness of a strong solution to the SDE for W are ensured, e.g., by the results of [31, Chapter 5.6].

On the other hand existence and uniqueness of a solution for the equation for y is more delicate. When $\epsilon = 0$, [40, Theorem I.1 and Remark I.3(iv)] ensure existence and uniqueness of a solution with \mathbb{P} -a.s. continuous paths. The case when $\epsilon \neq 0$ can be treated as in [30] when there is no path-dependency, while the present case can be treated similarly to [49, Subsection 5.1].

rm:changevar

Remark 2.4. The equation for y can be rewritten by introducing the new variable

$$e(t) = \mathbb{E}[y(t)].$$

Taking expectation in the equation for y above we get that e satisfies the delay equation

$$de(t) = \left[\mu_y e(t) + \int_{-d}^0 \phi(s) e(t+s) ds \right] dt, \quad (11) \quad \text{eq:efirst}$$

while the equation for y becomes

$$dy(t) = \left[\epsilon(y(t) - e(t)) + \mu_y y(t) + \int_{-d}^0 \phi(s) y(t+s) ds \right] dt + y(t) \sigma_y \cdot dZ(t). \quad (12) \quad \text{eq:yefirst}$$

Now, thanks to [40, Theorem I.1 and Remark I.3(iv)] the system made of (11)-(12) admits a unique strong solution (e, y) with \mathbb{P} -a.s. continuous paths for $t \geq 0$ for every initial datum and such that $y \in L^2(\Omega; L^2(-d, T))$ for any $T > 0$. It is not difficult to prove that, when the initial data are chosen so that

$$\begin{aligned} y(0) &= x_0, \quad y(s) = x_1(s) \text{ for } s \in [-d, 0), \\ e(0) &= \mathbb{E}[x_0] = x_0, \quad e(s) = \mathbb{E}[x_1(s)] = x_1(s) \text{ for } s \in [-d, 0), \end{aligned}$$

then the component y of such solution is also a strong solution of (10b) and that, vice versa, given a strong solution to the labor income equation (10b), the couple $(\mathbb{E}[y], y)$ solves the system (11)-(12).

Hence, system (10) can be rewritten, in the variables (W, y, e) as

$$\begin{cases} dW(t) = [W(t)r + \theta(t) \cdot (\mu - r\mathbf{1}) + y(t) - c(t) - \delta(B(t) - W(t))] dt + \theta(t) \cdot \sigma dZ(t) \\ dy(t) = \left[\epsilon(y(t) - e(t)) + \mu_y y(t) + \int_{-d}^0 \phi(s) y(t+s) ds \right] dt + y(t) \sigma_y \cdot dZ(t), \\ de(t) = \left[\mu_y e(t) + \int_{-d}^0 \phi(s) e(t+s) ds \right] dt, \\ W(0) = w, \\ y(0) = x_0, \quad y(s) = x_1(s) \text{ for } s \in [-d, 0), \\ e(0) = \mathbb{E}[x_0] = x_0, \quad e(s) = \mathbb{E}[x_1(s)] = x_1(s) \text{ for } s \in [-d, 0); \end{cases} \quad (13) \quad \text{DYNAMICS_WEALTH_LABOR}$$

see also [5] for related observations. ■

We aim to maximize the expected utility from lifetime consumption and bequest,

$$\mathbb{E} \left(\int_0^{\tau_\delta} e^{-\rho t} \frac{c(t)^{1-\gamma}}{1-\gamma} dt + e^{-\rho \tau_\delta} \frac{(kB(\tau_\delta))^{1-\gamma}}{1-\gamma} \right), \quad (14)$$

over all triplets $(c, \theta, B) \in \Pi^0$ satisfying a suitable no-borrowing state constraint introduced below in (18). In the above, $k > 0$ measures the intensity of preference for leaving a bequest, $\gamma \in (0, 1) \cup (1, +\infty)$ is the risk-aversion coefficient and $\rho > 0$ is the discount rate. As the death time is

independent of Z and exponentially distributed, we can rewrite the objective functional as follows (e.g., [10, Section 2] or [42, Section 3.6.2]):

$$J(c, B) := \mathbb{E} \left(\int_0^{+\infty} e^{-(\rho+\delta)t} \left(\frac{c(t)^{1-\gamma}}{1-\gamma} + \delta \frac{(kB(t))^{1-\gamma}}{1-\gamma} \right) dt \right). \quad (15)$$

OBJECTIVE_FUNCTION

The announced state constraint is the same as in [10], which is also considered in [19]. First of all recall that, given the financial market described by (6), the pre-death state-price density of the agent obeys the stochastic differential equation

$$\begin{cases} d\xi(t) &= -\xi(t)(r + \delta)dt - \xi(t)\kappa \cdot dZ(t), \\ \xi(0) &= 1. \end{cases} \quad (16)$$

DYN_STATE_PRICE_DE

where κ is the market price of risk and is defined as follows (e.g., [32]):

$$\kappa := (\sigma)^{-1}(\mu - r\mathbf{1}). \quad (17)$$

DEF_KAPPA

We require the agent to satisfy the following constraint

$$W(t) + \xi^{-1}(t)\mathbb{E} \left(\int_t^{+\infty} \xi(u)y(u)du \middle| \mathcal{F}_t \right) \geq 0, \quad (18)$$

NO_BORROWING_WITHOUT

which is a no-borrowing-without-repayment constraint as the second term in (18) represents the agent's market value of human capital at time t . In other words, human capital can be pledged as collateral, and represents the agent's maximum borrowing capacity. We note that the agent cannot default on his/her debt upon death, as the bequest target B is nonnegative.

Let us denote by $(W^{w,x_0,x_1}(t; c, B, \theta), y^{x_0,x_1}(t))$ the solution at time t of system (10), where we emphasize the dependence of the solution on the initial conditions (w, x_0, x_1) and strategies (c, B, θ) . We can then define the set of admissible controls as follows:

$$\Pi(w, x_0, x_1) := \left\{ c(\cdot), B(\cdot), \theta(\cdot) \in \Pi^0(w, x_0, x_1), \text{ such that:} \right. \\ \left. W^{w,x_0,x_1}(t; c, B, \theta) + \xi^{-1}(t)\mathbb{E} \left(\int_t^{+\infty} \xi(u)y^{x_0,x_1}(u)du \middle| \mathcal{F}_t \right) \geq 0 \quad \forall t \geq 0 \right\}. \quad (19)$$

DEF_PI_FIRST_DEFIN

Our problem is then to maximize the functional given in (15) over all controls in $\Pi(w, x_0, x_1)$.

We introduce two assumptions that will hold throughout the whole paper:

Hyp_K

Assumption 2.5.

$$\begin{cases} r + \delta - (\epsilon + \mu_y - \sigma_y \cdot \kappa) - \int_{-d}^0 e^{(r+\delta)s} |\phi(s)| ds > 0 & \text{if } \epsilon - \sigma_y \cdot \kappa < 0, \\ r + \delta - \mu_y - \int_{-d}^0 e^{(r+\delta)s} |\phi(s)| ds > 0 & \text{if } \epsilon - \sigma_y \cdot \kappa \geq 0. \end{cases}$$

Remark 2.6.

- (a) Assumption 2.5 is needed to rewrite in a convenient way (as we do in Section 3) the constraint (18), and will be carefully explained in Subsection 3.5. Here we only observe that this condition is a refinement of the one provided in [10, Hypothesis 2.4] and that in the interesting case $\epsilon < 0$, only the first formula holds, which reduces, when $\epsilon = 0$, to Hypothesis 2.4 of [10].
- (b) Mimicking the method of the proof of Proposition 2.2 in [10], we obtain the following representation of the labor income process y :

$$y(t) = E(t)(x_0 + I(t))$$

where

$$\begin{aligned} E(t) &= e^{(\epsilon + \mu_y - \frac{1}{2}|\sigma_y|^2)t + \sigma_y \cdot Z(t)}, \\ I(t) &= \int_0^t E^{-1}(u) \left(-\epsilon e(u) + \int_{-d}^0 \phi(s) y(s+u) ds \right) du. \end{aligned}$$

Moreover, if $x_0 > 0, x_1 \geq 0$ a.s., $\phi \geq 0$ a.e. and $\epsilon < 0$, then it must be that $y(t) > 0$ \mathbb{P} -a.s..

Hyp_gamma

Assumption 2.7.

$$\rho + \delta - (1 - \gamma) \left(r + \delta + \frac{|\kappa|^2}{2\gamma} \right) > 0.$$

rm:hp2

Remark 2.8. Assumption 2.7 is required to ensure that the candidate solution of our HJB equation in Section 4 is well defined and finite, see Theorem 4.7. In similar simple cases it can actually be proved that, when $\gamma \in (0, 1)$ and

$$\rho + \delta - (1 - \gamma) \left(r + \delta + \frac{|\kappa|^2}{2\gamma} \right) < 0,$$

the value function is infinite; for example, see [24] for the deterministic case.

3 Reformulation of the constraint

Within this section we assume that equation (10b) has a unique continuous \mathbb{F} -adapted solution y . We find an equivalent expression for the human capital defined as

$$HC(t_0) := \xi(t_0)^{-1} \mathbb{E} \left[\int_{t_0}^{\infty} \xi(u) y(u) du | \mathcal{F}_{t_0} \right], \quad (20) \quad \text{def_HC}$$

which is the secon summand in the left hand side of the constraint (18).

Following the idea of [9], we incorporate the discount factor ξ in an equivalent probability measure $\tilde{\mathbb{P}}$ (Subsection 3.1) and rewrite the dynamics of y under $\tilde{\mathbb{P}}$ in a suitable Hilbert space, using the so-called *product-space framework* for path-dependent equations (Subsection 3.2). Exploiting some spectral properties of the operators that appear in this formulation we can finally obtain the mentioned equivalent expression for $HC(t_0)$ (Subsections 3.3 and 3.4). We then comment on the relation between the spectral properties used herein and our Assumption 2.5 (Subsection 3.5).

3.1 Equivalent probability measure

We start by considering the equivalent probability measure $\tilde{\mathbb{P}}_s$ on \mathcal{F}_s such that

$$\frac{d\tilde{\mathbb{P}}_s}{d\mathbb{P}} = \exp \left(-\frac{1}{2} |\kappa|^2 s - \kappa \cdot Z(s) \right) = e^{(r+\delta)s} \xi(s); \quad (21) \quad \text{Ptilde}$$

by [31, Lemma 3.5.3] we can write

$$\mathbb{E} [\xi(s) y(s) | \mathcal{F}_{t_0}] = \xi(t_0) e^{-(r+\delta)(s-t_0)} \tilde{\mathbb{E}}_s [y(s) | \mathcal{F}_{t_0}],$$

and thus

$$\int_{t_0}^{+\infty} \mathbb{E} [\xi(s) y(s) | \mathcal{F}_{t_0}] ds = \xi(t_0) e^{(r+\delta)t_0} \int_{t_0}^{+\infty} e^{-(r+\delta)s} \tilde{\mathbb{E}}_s [y(s) | \mathcal{F}_{t_0}] ds. \quad (22) \quad \text{EXPRESSION_II}$$

The idea is now to understand what kind of SDE the quantity $\tilde{\mathbb{E}} [y(s) | \mathcal{F}_{t_0}] = \tilde{\mathbb{E}}_s [y(s) | \mathcal{F}_{t_0}]$ satisfies. Let $\tilde{\mathbb{P}}$ the measure such that $\tilde{\mathbb{P}}|_{\mathcal{F}_s} = \tilde{\mathbb{P}}_s$ for all $s \geq 0$. By the Girsanov Theorem the

process $\tilde{Z}(t) = Z(t) + \kappa t$ is an n -dim. Brownian motion under $\tilde{\mathbb{P}}$. The dynamics of y under $\tilde{\mathbb{P}}$ is then

$$\begin{cases} dy(t) = \left[(\epsilon + \mu_y - \sigma_y \cdot \kappa) y(t) - \epsilon e(t) + \int_{-d}^0 \phi(s) y(t+s) ds \right] dt + y(t) \sigma_y \cdot d\tilde{Z}(t), \\ de(t) = \left[\mu_y e(t) + \int_{-d}^0 \phi(s) e(t+s) ds \right] dt, \\ y(0) = x_0, \quad y(s) = x_1(s) \text{ for } s \in [-d, 0), \\ e(0) = \mathbb{E}[x_0] = x_0, \quad e(s) = \mathbb{E}[x_1(s)] = x_1(s) \text{ for } s \in [-d, 0). \end{cases} \quad (23)$$

Remark 3.1. Under the equivalent probability measure $\tilde{\mathbb{P}}$ the DDE satisfied by e remains the same.

Therefore, the quantity $\tilde{\mathbb{E}}[y(t)|\mathcal{F}_{t_0}]$ satisfies the equation

$$\begin{aligned} \tilde{\mathbb{E}}[y(t)|\mathcal{F}_{t_0}] &= y(t_0) + (\epsilon + \mu_y - \sigma_y \cdot \kappa) \int_{t_0}^t \tilde{\mathbb{E}}[y(s)|\mathcal{F}_{t_0}] ds - \epsilon \int_{t_0}^t \tilde{\mathbb{E}}[e(s)|\mathcal{F}_{t_0}] ds \\ &\quad + \int_{t_0}^t \int_{-d}^0 \tilde{\mathbb{E}}[y(s+\tau)|\mathcal{F}_{t_0}] \phi(\tau) d\tau ds \\ &= y(t_0) + (\epsilon + \mu_y - \sigma_y \cdot \kappa) \int_{t_0}^t \tilde{\mathbb{E}}[y(s)|\mathcal{F}_{t_0}] ds - \epsilon \int_{t_0}^t e(s) ds \\ &\quad + \int_{t_0}^t \int_{-d}^0 \tilde{\mathbb{E}}[y(s+\tau)|\mathcal{F}_{t_0}] \phi(\tau) d\tau ds, \end{aligned}$$

where in the last equality we exploit the fact that e satisfies a deterministic equation, thus $\tilde{\mathbb{E}}[e(s)|\mathcal{F}_{t_0}] = e(s)$. Notice that the stochastic integral with respect to \tilde{Z} is a martingale, and has zero mean, hence $\tilde{\mathbb{E}}\left[\int_{t_0}^t y(s) \sigma_y \cdot d\tilde{Z}(s) | \mathcal{F}_{t_0}\right] = 0$ (for more details see [9, Lemma 4.6]).

Therefore, defining $M_{t_0}(t) := \tilde{\mathbb{E}}[y(t)|\mathcal{F}_{t_0}]$, we have that the couple (M_{t_0}, e) satisfies for $t \geq t_0$ the system (with random initial conditions)

$$\begin{cases} dM_{t_0}(t) = \left[(\epsilon + \mu_y - \sigma_y \cdot \kappa) M_{t_0}(t) - \epsilon e(t) + \int_{-d}^0 M_{t_0}(t+s) \phi(s) ds \right] dt \\ de(t) = \mu_y e(t) dt + \int_{-d}^0 e(t+s) \phi(s) ds dt \\ M_{t_0}(t_0) = y(t_0), \quad M_{t_0}(t_0+s) = y(t_0+s), \quad s \in [-d, 0], \\ e(t_0) = \mathbb{E}[y(t_0)], \quad e(s) = \mathbb{E}[y(t_0+s)], \quad s \in [-d, 0]. \end{cases} \quad (24) \quad \boxed{\text{system1}}$$

3.2 Reformulation of the problem in an infinite-dimensional framework

We introduce first the space

$$\mathcal{M}_2 = \mathbb{R} \times L^2(-d, 0; \mathbb{R})$$

whose elements are denoted as $\bar{x} = (x_0, x_1)$. \mathcal{M}_2 is a Hilbert space when endowed with the inner product $\langle (x_0, x_1), (y_0, y_1) \rangle_{\mathcal{M}_2} = x_0 y_0 + \langle x_1, y_1 \rangle$, the latter being the usual inner product of $L^2(-d, 0; \mathbb{R})$, that is

$$\langle x_1, y_1 \rangle = \int_{-d}^0 x_1(s) y_1(s) ds.$$

We will denote vectors in \mathbb{R}^2 with boldface letters: $\mathbf{a} = \begin{pmatrix} a \\ c \end{pmatrix}$; similarly we will write \mathbb{R}^2 -valued functions as $\mathbf{f}(\cdot) = \begin{pmatrix} g(\cdot) \\ h(\cdot) \end{pmatrix}$.

The state space for the reformulation needed within this section is $\mathcal{M}_2^2 = (\mathbb{R} \times L^2(-d, 0; \mathbb{R}))^{\oplus 2} \cong \mathbb{R}^2 \times L^2(-d, 0; \mathbb{R}^2)$. Elements of \mathcal{M}_2^2 will be written in any of the following equivalent ways:

$$\bar{\mathbf{x}} = \begin{pmatrix} (x_0^{(1)}, x_1^{(1)}) \\ (x_0^{(2)}, x_1^{(2)}) \end{pmatrix} \text{ with } (x_0^{(1)}, x_1^{(1)}), (x_0^{(2)}, x_1^{(2)}) \in \mathcal{M}_2, \quad (25) \quad \boxed{\text{notation1}}$$

$$\bar{\mathbf{x}} = \begin{pmatrix} \bar{x}^{(1)} \\ \bar{x}^{(2)} \end{pmatrix} \text{ with } \bar{x}^{(1)}, \bar{x}^{(2)} \in \mathcal{M}_2, \quad (26) \quad \text{notation2}$$

$$\bar{\mathbf{x}} = \left(\begin{pmatrix} x_0^{(1)} \\ x_0^{(2)} \end{pmatrix}, \begin{pmatrix} x_1^{(1)} \\ x_1^{(2)} \end{pmatrix} \right) \text{ with } \begin{pmatrix} x_0^{(1)} \\ x_0^{(2)} \end{pmatrix} \in \mathbb{R}^2, \begin{pmatrix} x_1^{(1)} \\ x_1^{(2)} \end{pmatrix} \in L^2(-d, 0; \mathbb{R}^2), \quad (27) \quad \text{notation3}$$

$$\bar{\mathbf{x}} = (\mathbf{x}_0, \mathbf{x}_1) \text{ with } \mathbf{x}_0 \in \mathbb{R}^2, \mathbf{x}_1 \in L^2(-d, 0; \mathbb{R}^2); \quad (28) \quad \text{notation4}$$

We then rewrite system (24) in a more compact form. First for any fixed \mathcal{F}_{t_0} -measurable \mathcal{M}_2^2 -valued random variable $\bar{\mathbf{m}} = (\mathbf{m}_0, \mathbf{m}_1)$ we consider the 2-dimensional system

$$\begin{cases} d\mathbf{n}(t_0; t) = \left[C_0 \mathbf{n}(t_0; t) + \int_{-d}^0 \phi(s) \mathbf{n}(t_0; t+s) ds \right] dt \\ \mathbf{n}(t_0; t_0) = \mathbf{m}_0, \\ \mathbf{n}(t_0; t_0 + s) = \mathbf{m}_1(s), \quad s \in [-d, 0]. \end{cases} \quad (29) \quad \text{system2}$$

where

$$C_0 := \begin{pmatrix} \epsilon + \mu_y - \sigma_y \cdot \kappa & -\epsilon \\ 0 & \mu_y \end{pmatrix}.$$

The following is a simple generalization of [3, Part II, Chapter 4, Theorem 3.2] to random initial conditions.

Lemma 3.2. *Given any fixed \mathcal{F}_{t_0} -measurable \mathcal{M}_2^2 -valued random variable $\bar{\mathbf{m}}$, the Cauchy problem (29) has a unique absolutely continuous solution. Moreover system (24) is equivalent to (29) above when we choose*

$$\mathbf{m}_0 = \begin{pmatrix} y(t_0) \\ \mathbb{E}[y(t_0)] \end{pmatrix}, \mathbf{m}_1 = \begin{pmatrix} y(t_0 + \cdot) \\ \mathbb{E}[y(t_0 + \cdot)] \end{pmatrix}; \quad (30) \quad \text{vector_conditions}$$

indeed in this case $\mathbf{n}(t_0; t) = (M_{t_0}^t)(t)$ for every $t \in [t_0 - d, +\infty)$.

We define next the operator $A_0 : \mathcal{D}(A_0) \subset \mathcal{M}_2^2 \rightarrow \mathcal{M}_2^2$ as

$$\begin{aligned} \mathcal{D}(A_0) &:= \{(\mathbf{x}_0, \mathbf{x}_1) \in \mathcal{M}_2^2 : \mathbf{x}_1 \in W^{1,2}(-d, 0; \mathbb{R}^2), \mathbf{x}_1(0) = \mathbf{x}_0\}, \\ A_0(\mathbf{x}_0, \mathbf{x}_1) &:= \left(C_0 \mathbf{x}_0 + \int_{-d}^0 \phi(s) \mathbf{x}_1(s) ds, \frac{d}{ds} \mathbf{x}_1 \right). \end{aligned} \quad (31) \quad \text{defA0}$$

We can then reformulate system (29) above as an evolution equation in \mathcal{M}_2^2 . Consider, again for any fixed \mathcal{F}_{t_0} -measurable \mathcal{M}_2^2 -valued random variable $\bar{\mathbf{m}} = (\mathbf{m}_0, \mathbf{m}_1)$, the \mathcal{M}_2^2 -valued process $\bar{\mathbf{N}}(t_0; \cdot)$ that is the solution on $[t_0, +\infty)$ of

$$\begin{cases} d\bar{\mathbf{N}}(t_0; t) = A_0 \bar{\mathbf{N}}(t_0; t) dt, \\ \bar{\mathbf{N}}(t_0; t_0) = \bar{\mathbf{m}}. \end{cases} \quad (32) \quad \text{INFINITE_DIMENSION}$$

We collect now some useful results about the above equation, also for later reference; definitions of strict and weak solutions can be found for example in [16, Appendix A]. Proofs are given in the Appendix.

Proposition 3.3. (i) *The operator A_0 generates a strongly continuous semigroup $\{S(t)\}_{t \geq 0}$ in \mathcal{M}_2^2 .*

(ii) *$S(t)$ is a compact operator for every $t \geq d$.*

(iii) *For every \mathcal{F}_{t_0} -measurable \mathcal{M}_2^2 -valued random variable $\bar{\mathbf{m}} \in \mathcal{M}_2^2$ the process*

$$S(t - t_0) \bar{\mathbf{m}}; \quad (33) \quad \text{semigroup_1}$$

is the unique weak (in distributional sense) solution of (32); in particular

$$\bar{\mathbf{N}}(t_0; t) = \bar{\mathbf{N}}(0; t - t_0). \quad (34) \quad \text{semigroup_2}$$

Moreover if $\bar{\mathbf{m}} \in \mathcal{D}(A_0)$ a.s. then the solution is actually strict.

(iv) The Cauchy problem (32) is equivalent to (29).

item:5

(v) Let y be a solution of (10b) on $[0, t_0]$; when choosing $\bar{\mathbf{m}}$ as in (30), (32) is equivalent to (24) and in this case we have

$$\bar{\mathbf{N}}(t_0; t) = S(t - t_0)\bar{\mathbf{m}} = (\mathbf{n}(t_0; t), \mathbf{n}(t_0; t + \cdot)) = \left(\begin{pmatrix} M_{t_0}(t) \\ e(t) \end{pmatrix}, \begin{pmatrix} \{M_{t_0}(t + s)\}_{s \in [-d, 0]} \\ \{e(t + s)\}_{s \in [-d, 0]} \end{pmatrix} \right).$$

3.3 Spectral properties of A_0

subsec:spectral

The following result is an immediate consequence of [27, Chapter 7, Lemma 2.1 and Theorem 4.2]

lemma_speck

Lemma 3.4. (i) The spectrum of the operator A_0 is given by

$$\{\lambda \in \mathbb{C} : K(\lambda) = 0\},$$

where

$$K(\lambda) := K_1(\lambda)K_2(\lambda),$$

with

$$K_1(\lambda) := \lambda - (\epsilon + \mu_y - \sigma_y \cdot \kappa) - \int_{-d}^0 e^{\lambda s} \phi(s) ds,$$

$$K_2(\lambda) := \lambda - \mu_y - \int_{-d}^0 e^{\lambda s} \phi(s) ds = K_1(\lambda) - (\epsilon - \sigma_y \cdot \kappa).$$

In particular the spectral bound of A_0 is

$$\lambda_0 = \sup \{\operatorname{Re} \lambda : K(\lambda) = 0\}.$$

(ii) The spectrum of A_0 coincides with its point spectrum and is a discrete (thus countable) set.

We can explicitly compute the resolvent operator of A_0 (a proof is sketched in the Appendix):

lemma_resolvent

Lemma 3.5. Let $R(A_0)$ denote the resolvent set of A_0 and let $\lambda \in \mathbb{R} \cap R(A_0)$; then the resolvent operator of A_0 at λ is given by

$$R(\lambda, A_0) \left(\begin{pmatrix} m_0 \\ e_0 \end{pmatrix}, \begin{pmatrix} m_1 \\ e_1 \end{pmatrix} \right) = \left(\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}, \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} \right),$$

with

$$\begin{aligned} v_0 &= \frac{1}{K_2(\lambda)} \left[e_0 + \int_{-d}^0 \int_{-d}^s e^{-\lambda(s-\tau)} \phi(\tau) d\tau e_1(s) ds \right], \\ u_0 &= \frac{1}{K_1(\lambda)} \left[m_0 + \int_{-d}^0 \int_{-d}^s e^{-\lambda(s-\tau)} \phi(\tau) d\tau m_1(s) ds \right] \\ &\quad - \frac{\epsilon}{K(\lambda)} \left[e_0 + \int_{-d}^0 \int_{-d}^s e^{-\lambda(s-\tau)} \phi(\tau) d\tau e_1(s) ds \right], \\ u_1(s) &= u_0 e^{\lambda s} + \int_s^0 e^{-\lambda(\tau-s)} m_1(\tau) d\tau, \\ v_1(s) &= v_0 e^{\lambda s} + \int_s^0 e^{-\lambda(\tau-s)} e_1(\tau) d\tau. \end{aligned} \tag{35}$$

u0

A crucial tool will be the following well known fact:

lem_laplace

Lemma 3.6. For every real λ such that $\lambda > \lambda_0$ and every $\bar{\mathbf{m}} \in \mathcal{M}_2^2$ we have

$$\int_0^{+\infty} e^{-\lambda t} S(t) \bar{\mathbf{m}} dt = R(\lambda, A_0) \bar{\mathbf{m}}. \tag{36}$$

LAPLACE_SEMIGROUP

Proof. Identity (36) is well known to hold for all real λ that are larger than the type of $S(t)$. Since $S(t)$ is compact for every $t \geq d$, its type is actually equal to its spectral radius λ_0 , see for example [3, Part II, Chapter 1, Corollary 2.5]. \square

Note asking $\lambda > \lambda_0$ is more restrictive than just asking that $\lambda \in \mathbb{R} \cap R(A_0)$.

3.4 A formula for the human capital

Theorem 3.7. Assume that $r + \delta > \lambda_0$; then

$$HC(t_0) = \frac{1}{K_1(r + \delta)} \left[y(t_0) + \int_{-d}^0 \int_{-d}^s e^{-(r+\delta)(s-\tau)} \phi(\tau) d\tau y(t_0 + s) ds \right] \\ - \frac{\epsilon}{K(r + \delta)} \left[e(t_0) + \int_{-d}^0 \int_{-d}^s e^{-(r+\delta)(s-\tau)} \phi(\tau) d\tau e(t_0 + s) ds \right].$$

Proof. Recall (20) and let $\bar{\mathbf{m}} = (\mathbf{m}_0, \mathbf{m}_1)$ be given by (30). We can rewrite the Human Capital as follows; we denote here by $P_{1,0}$ the projection on the first finite dimensional component of \mathcal{M}_2^2 , i.e.

$$P_{1,0} \left(\begin{pmatrix} x_0^{(1)}, x_1^{(1)} \\ x_0^{(2)}, x_1^{(2)} \end{pmatrix} \right) = x_0^{(1)}.$$

Starting from (22) we write

$$\begin{aligned} \frac{1}{\xi(t_0)} \int_{t_0}^{\infty} \mathbb{E}[\xi(s)y(s)|\mathcal{F}_{t_0}] ds &= e^{(r+\delta)t_0} \int_{t_0}^{\infty} e^{-(r+\delta)s} \tilde{\mathbb{E}}[y(s)|\mathcal{F}_{t_0}] ds & (37) \quad \boxed{\text{new1}} \\ &= e^{(r+\delta)t_0} \int_{t_0}^{\infty} e^{-(r+\delta)s} M_{t_0}(s) ds \\ &= e^{(r+\delta)t_0} \int_{t_0}^{\infty} e^{-(r+\delta)s} P_{1,0} [\bar{\mathbf{N}}(t_0; s)] ds & (\text{by Prop. 3.3 - (v)}) \\ &= e^{(r+\delta)t_0} \int_0^{\infty} e^{-(r+\delta)t_0} e^{-(r+\delta)s} P_{1,0} [\bar{\mathbf{N}}(0; s)] ds & (\text{by (34)}) \\ &= \int_0^{\infty} e^{-(r+\delta)s} P_{1,0} [S(s)\bar{\mathbf{m}}] ds & (\text{by (33)}) \\ &= P_{1,0} [R(r + \delta, A_0)\bar{\mathbf{m}}] & (\text{by Lemma 3.6}) \\ &= \frac{1}{K_1(r + \delta)} \left[y(t_0) + \int_{-d}^0 \int_{-d}^s e^{-(r+\delta)(s-\tau)} \phi(\tau) d\tau y(t_0 + s) ds \right] \\ &\quad - \frac{\epsilon}{K(r + \delta)} \left[e(t_0) + \int_{-d}^0 \int_{-d}^s e^{-(r+\delta)(s-\tau)} \phi(\tau) d\tau e(t_0 + s) ds \right] & (\text{by Lemma 3.5}). \end{aligned}$$

From the above equality and Remark 2.4 we infer the \mathbb{P} integrability of the term $\int_{t_0}^{\infty} \mathbb{E}[\xi(s)y(s)|\mathcal{F}_{t_0}] ds$. This justifies the equality

$$\mathbb{E} \left[\int_{t_0}^{\infty} \xi(s)y(s)|\mathcal{F}_{t_0} ds \right] = \int_{t_0}^{\infty} \mathbb{E}[\xi(s)y(s)|\mathcal{F}_{t_0}] ds. \quad (38) \quad \boxed{\text{new2}}$$

In fact by the characteristic property of the conditional mean and Fubini's Theorem we have that, for any $F \in \mathcal{F}_{t_0}$,

$$\begin{aligned} \int_F \int_{t_0}^{+\infty} \mathbb{E}[\xi(s)y(s) | \mathcal{F}_{t_0}] ds d\mathbb{P} &= \int_{t_0}^{+\infty} \int_F \mathbb{E}[\xi(s)y(s) | \mathcal{F}_{t_0}] d\mathbb{P} ds \\ &= \int_{t_0}^{+\infty} \int_F \xi(s)y(s) d\mathbb{P} ds = \int_F \int_{t_0}^{+\infty} \xi(s)y(s) ds d\mathbb{P} \\ &= \int_F \mathbb{E} \left[\int_{t_0}^{+\infty} \xi(s)y(s) ds | \mathcal{F}_{t_0} \right] d\mathbb{P}. \end{aligned}$$

Recalling (20), from (37) and (38) the thesis immediately follows. \square

Defining now

$$\begin{aligned}
K_1 &:= K_1(r + \delta) = r + \delta - (\epsilon + \mu_y - \sigma_y \cdot \kappa) - \int_{-d}^0 e^{(r+\delta)s} \phi(s) \, ds, \\
K_2 &:= K_2(r + \delta) = K_1(r + \delta) + (\epsilon - \sigma_y \cdot \kappa) = r + \delta - \mu_y - \int_{-d}^0 e^{(r+\delta)s} \phi(s) \, ds, \\
g_\infty &= \frac{1}{K_1}, \quad i_\infty = \frac{\epsilon}{K_2},
\end{aligned} \tag{39}$$

$$G(s) := \int_{-d}^s e^{-(r+\delta)(s-\tau)} \phi(\tau) \, d\tau \text{ and } h_\infty(s) := g_\infty G(s) \tag{40}$$

we finally obtain that, if $r + \delta > \lambda_0$,

$$HC(t_0) = g_\infty \left[y(t_0) + \int_{-d}^0 G(s) y(t_0 + s) \, ds \right] - g_\infty i_\infty \left[e(t_0) + \int_{-d}^0 G(s) e(t_0 + s) \, ds \right] \tag{41}$$

$$= \langle (g_\infty, h_\infty), (y(t_0), y(t_0 + \cdot)) \rangle_{\mathcal{M}_2} - i_\infty \langle (g_\infty, h_\infty), (e(t_0), e(t_0 + \cdot)) \rangle_{\mathcal{M}_2}. \tag{42}$$

The above representation allows to rewrite the constraint (18) as

$$W(t) + \langle (g_\infty, h_\infty), (y(t), y(t + \cdot)) \rangle_{\mathcal{M}_2} - i_\infty \langle (g_\infty, h_\infty), (e(t), e(t + \cdot)) \rangle_{\mathcal{M}_2} \geq 0, \tag{43}$$

which in turn suggests to set

$$\bar{l}_\infty = (g_\infty, h_\infty) \in \mathcal{M}_2, \quad \bar{\mathbf{l}}_\infty = \begin{pmatrix} i_\infty \\ -i_\infty \bar{l}_\infty \end{pmatrix} \in \mathcal{M}_2^2$$

and to define the function $\mathbb{R} \times \mathcal{M}_2^2 \rightarrow \mathbb{R}$

$$\Gamma_\infty(w, \bar{\mathbf{x}}) := w + \langle \bar{\mathbf{l}}_\infty, \bar{\mathbf{x}} \rangle_{\mathcal{M}_2^2} \tag{44a}$$

$$\begin{aligned}
&= w + \langle \bar{l}_\infty, \bar{x}^{(1)} \rangle_{\mathcal{M}_2} - i_\infty \langle \bar{l}_\infty, \bar{x}^{(2)} \rangle_{\mathcal{M}_2} \\
&= w + g_\infty x_0^{(1)} + \langle h_\infty, x_1^{(1)} \rangle - i_\infty \left[g_\infty x_0^{(2)} + \langle h_\infty, x_1^{(2)} \rangle \right]
\end{aligned} \tag{44b}$$

$$= w + g_\infty \left[x_0^{(1)} + \langle G, x_1^{(1)} \rangle \right] - i_\infty g_\infty \left[x_0^{(2)} + \langle G, x_1^{(2)} \rangle \right] \tag{44c}$$

for $w \in \mathbb{R}$, $\bar{\mathbf{x}} \in \mathcal{M}^2$, and to consider the sets

$$\begin{aligned}
\mathcal{H} &:= \mathbb{R} \times \mathcal{M}_2^2, \\
\mathcal{H}_+ &= \{(w, \bar{\mathbf{x}}) \in \mathcal{H} : \Gamma_\infty(w, \bar{\mathbf{x}}) \geq 0\}, \\
\mathcal{H}_{++} &= \{(w, \bar{\mathbf{x}}) \in \mathcal{H} : \Gamma_\infty(w, \bar{\mathbf{x}}) > 0\}.
\end{aligned} \tag{45}$$

\mathcal{H} is naturally a Hilbert space when endowed with the inner product

$$\langle (a, \bar{\mathbf{x}}), (b, \bar{\mathbf{y}}) \rangle_{\mathcal{H}} = \left\langle \begin{pmatrix} a \\ \bar{x}^{(1)}, \bar{x}^{(2)} \end{pmatrix}, \begin{pmatrix} b \\ \bar{y}^{(1)}, \bar{y}^{(2)} \end{pmatrix} \right\rangle_{\mathcal{H}} := ab + \left\langle \bar{x}^{(1)}, \bar{y}^{(1)} \right\rangle_{\mathcal{M}_2} + \left\langle \bar{x}^{(2)}, \bar{y}^{(2)} \right\rangle_{\mathcal{M}_2}.$$

Equations (43) and (44b) emphasize the differences between the present setting and the one in [10].

3.5 Why Assumption 2.5

The requirement

$$r + \delta > \lambda_0$$

in Theorem 3.7 is difficult to check in practice, as it requires an explicit computation of λ_0 . Therefore we look for some sufficient condition, possibly easier to check, for such requirement to be satisfied. Set for $\lambda \in \mathbb{C}$

$$\begin{aligned}\tilde{K}_1(\lambda) &:= \lambda - (\epsilon + \mu_y - \sigma_y \cdot \kappa) - \int_{-d}^0 e^{\lambda s} |\phi(s)| ds, \\ \tilde{K}_2(\lambda) &:= \lambda - \mu_y - \int_{-d}^0 e^{\lambda s} |\phi(s)| ds = \tilde{K}_1(\lambda) - (\epsilon - \sigma_y \cdot \kappa), \\ \tilde{K}(\lambda) &:= \tilde{K}_1(\lambda) \tilde{K}_2(\lambda).\end{aligned}$$

Finally set

$$\tilde{\lambda}_0 = \sup \left\{ \Re(\lambda) : \tilde{K}(\lambda) = 0 \right\}.$$

Note that $\tilde{\lambda}_0$ is the spectral radius of the operator $\tilde{A}_0 : \mathcal{D}(\tilde{A}_0) \subset \mathcal{M}_2^2 \rightarrow \mathcal{M}_2^2$,

$$\begin{aligned}\mathcal{D}(\tilde{A}_0) &:= \left\{ (\mathbf{x}_0, \mathbf{x}_1) \in \mathcal{M}_2^2 : \mathbf{x}_1 \in W^{1,2}(-d, 0; \mathbb{R}^2), \mathbf{x}_1(0) = \mathbf{x}_0 \right\}, \\ \tilde{A}_0(\mathbf{x}_0, \mathbf{x}_1) &:= \left(C_0 \mathbf{x}_0 + \int_{-d}^0 |\phi(s)| \mathbf{x}_1(s) ds, \frac{d}{ds} \mathbf{x}_1 \right).\end{aligned}$$

lemma2

Lemma 3.8. *The functions \tilde{K}_1 and \tilde{K}_2 , restricted to the real numbers, are strictly increasing. Moreover the function \tilde{K} restricted to the real numbers is continuous and such that:*

- (i) $\lim_{\xi \rightarrow \pm\infty} \tilde{K}(\xi) = +\infty$;
- (ii) *the equation $\tilde{K}(\xi) = 0$ admits exactly two real solutions ξ_1 and ξ_2 with $\tilde{K}_1(\xi_1) = 0$ and $\tilde{K}_2(\xi_2) = 0$, and $\xi_1 = \xi_2$ if and only if $\epsilon - \sigma_y \cdot \kappa = 0$;*
- (iii) *setting $\xi_0 := \max(\xi_1, \xi_2)$ we have*

$$\xi_0 = \begin{cases} \xi_2 & \text{if } \epsilon - \sigma_y \cdot \kappa > 0, \\ \xi_1 & \text{if } \epsilon - \sigma_y \cdot \kappa < 0 \end{cases}$$

and eventually it holds $\tilde{\lambda}_0 = \xi_0$.

Proof. By definition $\tilde{K}(\xi) = 0$ if either $\tilde{K}_1(\xi) = 0$ or $\tilde{K}_2(\xi) = 0$. It is immediate to check that both \tilde{K}_1 and \tilde{K}_2 are continuous strictly increasing functions on \mathbb{R} with $\lim_{\xi \rightarrow \pm\infty} \tilde{K}_j(\xi) = \pm\infty$, $j = 1, 2$. Therefore \tilde{K} is continuous on \mathbb{R} and there exists exactly one value $\xi_1 \in \mathbb{R}$ such that $\tilde{K}_1(\xi_1) = 0$ and exactly one value $\xi_2 \in \mathbb{R}$ such that $\tilde{K}_2(\xi_2) = 0$. Since $\tilde{K}_2(\xi) = \tilde{K}_1(\xi) - (\epsilon - \sigma_y \cdot \kappa)$, $\xi_1 = \xi_2$ if and only if $\epsilon - \sigma_y \cdot \kappa = 0$ and

$$\max(\xi_1, \xi_2) = \begin{cases} \xi_2 & \text{if } \epsilon - \sigma_y \cdot \kappa > 0, \\ \xi_1 & \text{if } \epsilon - \sigma_y \cdot \kappa < 0. \end{cases}$$

Let us now prove that $\tilde{\lambda}_0 = \xi_0$. By definition we have $\xi_0 \leq \tilde{\lambda}_0$. In order to prove that $\xi_0 \geq \tilde{\lambda}_0$, let us consider an arbitrary $\lambda = a + ib \in \mathbb{C}$ such that $\tilde{K}(\lambda) = 0$. Suppose first that $\tilde{K}_1(\lambda) = 0$, so that in particular

$$\begin{aligned}0 &= \Re \left(\tilde{K}_1(\lambda) \right) = a - (\epsilon + \mu_y - \sigma_y \cdot \kappa) - \int_{-d}^0 e^{as} \cos(bs) |\phi(s)| ds \\ &\geq a - (\epsilon + \mu_y - \sigma_y \cdot \kappa) - \int_{-d}^0 e^{as} |\phi(s)| ds = \tilde{K}_1(a).\end{aligned}$$

Therefore $a \leq \xi_1 \leq \xi_0$. If instead $\tilde{K}_2(\lambda) = 0$ then

$$0 = \Re(\tilde{K}_2(\lambda)) = a - \mu_y - \int_{-d}^0 e^{as} \cos(bs) |\phi(s)| ds \geq a - \mu_y - \int_{-d}^0 e^{as} |\phi(s)| ds = \tilde{K}_2(a)$$

hence $a \leq \xi_2 \leq \xi_0$. In both cases taking the supremum in the definition of $\tilde{\lambda}_0$ yields $\tilde{\lambda}_0 \leq \xi_0$. \square

The convenience of introducing $\tilde{\lambda}_0$ is clarified by its relation with λ_0 .

Lemma 3.9. *We have*

$$\lambda_0 \leq \tilde{\lambda}_0 .$$

Proof. For every $\lambda = a + ib \in \mathbb{C}$ we have

$$\Re(K_1(\lambda)) = a - (\epsilon + \mu_y - \sigma_y \cdot \kappa) - \int_{-d}^0 e^{as} \cos(bs) \phi(s) ds , \quad (46) \quad \text{eq:kappa1}$$

$$\Re(K_2(\lambda)) = a - \mu_y - \int_{-d}^0 e^{as} \cos(bs) \phi(s) ds . \quad (47) \quad \text{eq:kappa2}$$

Suppose first that $\xi_0 = \xi_1$. Then $\epsilon - \sigma_y \cdot \kappa < 0$ and $\tilde{K}_1(\xi) < \tilde{K}_2(\xi)$ for every $\xi \in \mathbb{R}$. Therefore, recalling the definition of λ_0 , it is enough to show that for every number $\lambda = a + ib$ such that $K(\lambda) = 0$ we have $\tilde{K}_2(a) \leq 0$. So let λ be such a number; we have

$$\begin{aligned} \tilde{K}_2(a) &= a - \mu_y - \int_{-d}^0 e^{as} |\phi(s)| ds \\ &= a - \mu_y - \int_{-d}^0 e^{as} \cos(bs) \phi(s) ds - (\epsilon - \sigma_t \cdot \kappa) + (\epsilon - \sigma_y \cdot \kappa) \\ &\quad + \int_{-d}^0 e^{as} (\cos(bs) \phi(s) - |\phi(s)|) ds \\ &\leq \int_{-d}^0 e^{as} (\cos(bs) \phi(s) - |\phi(s)|) ds \\ &\leq 0 , \end{aligned}$$

where to deduce the second to last inequality one uses (46) together with the fact that $\epsilon - \sigma_y \cdot \kappa < 0$ if $K_1(\lambda) = 0$ and (47) if $K_2(\lambda) = 0$.

Similarly if $\xi_0 = \xi_2$ we have $\epsilon - \sigma_y \cdot \kappa \geq 0$ and it suffices to show that for any $\lambda = a + ib$ as above $\tilde{K}_1(a) \leq 0$. In this case we find

$$\begin{aligned} \tilde{K}_1(a) &= a - \mu_y - (\epsilon - \sigma_y \cdot \kappa) - \int_{-d}^0 e^{as} |\phi(s)| ds \\ &= a - \mu_y - (\epsilon - \sigma_y \cdot \kappa) - \int_{-d}^0 e^{as} \cos(bs) \phi(s) ds + \int_{-d}^0 e^{as} (\cos(bs) \phi(s) - |\phi(s)|) ds \\ &\leq \int_{-d}^0 e^{as} (\cos(bs) \phi(s) - |\phi(s)|) ds \\ &\leq 0 \end{aligned}$$

and the conclusion follows as before. \square

Thanks to this last result it becomes clearer why we make Assumption 2.5. If r and δ are such that Assumption 2.5 is satisfied, then in particular both $\tilde{K}_1(r + \delta)$ and $\tilde{K}_2(r + \delta)$ are positive and we have that

$$r + \delta > \xi_0 = \tilde{\lambda}_0 \geq \lambda_0;$$

the assumption of Theorem 3.7 is therefore satisfied, so that the constraint (18) takes the convenient formulation (43). Note that Assumption 2.5 is not equivalent to saying that $r + \delta > \lambda_0$ but only a sufficient condition. However it is usually much easier to verify the inequalities in Assumption 2.5 than computing explicitly λ_0 .

Remark 3.10. *Repeating the arguments above one can easily show that if there is no mean-reverting effect (i.e. $\epsilon = 0$) and ϕ is positive almost everywhere then $r + \delta > \lambda_0$ if and only if $K_1(r + \delta) > 0$.*

4 The general problem and the associated HJB equation

4.1 The infinite dimensional general problem

Here we rewrite, in a suitable infinite-dimensional setting, the problem of maximizing the functional (15) under the constraint (18) and under the transformed state equation (13). We call it 'general problem' since we consider it for generic initial data, so, not necessarily connected with our original control of McKean-Vlasov dynamics. In Section 6 we will go back to the original problem.

For any two Banach spaces E and E' we will denote by $L(E; E')$ the space of bounded linear operators from E to E' .

We proceed similarly to the previous section to reformulate system (13) in the infinite-dimensional space \mathcal{M}_2^2 . To begin with, we define the finite-dimensional operator C on \mathbb{R}^2 as

$$C := \begin{pmatrix} \epsilon + \mu_y & -\epsilon \\ 0 & \mu_y \end{pmatrix}$$

and the operator $A : \mathcal{D}(A) \subset \mathcal{M}_2^2 \rightarrow \mathcal{M}_2^2$ as

$$\begin{aligned} \mathcal{D}(A) &:= \{(\mathbf{x}_0, \mathbf{x}_1) \in \mathcal{M}_2^2 : \mathbf{x}_1 \in W^{1,2}(-d, 0; \mathbb{R}^2), \mathbf{x}_1(0) = \mathbf{x}_0\}, \\ A(\mathbf{x}_0, \mathbf{x}_1) &:= \left(C\mathbf{x}_0 + \int_{-d}^0 \phi(s)\mathbf{x}_1(s) ds, \frac{d}{ds}\mathbf{x}_1 \right) \end{aligned} \quad (48) \quad \text{defA}$$

(the only difference between A and A_0 defined in (31) lies in the use of C in place of C_0). By Proposition 3.3 - (i) the operator A is the infinitesimal generator of a strongly continuous semigroup $U(t)$ in \mathcal{M}_2^2 .

Furthermore, we need to define the linear bounded operator $F : \mathcal{M}_2^2 \rightarrow L(\mathbb{R}^n; \mathcal{M}_2^2)$ in the following way: for every $\bar{\mathbf{x}} = \left(\begin{pmatrix} x_0^{(1)} \\ x_0^{(2)} \end{pmatrix}, \begin{pmatrix} x_1^{(1)} \\ x_1^{(2)} \end{pmatrix} \right) \in \mathcal{M}_2^2$, $F(\bar{\mathbf{x}})$ is the linear map

$$z \mapsto (P_{1,0}\bar{\mathbf{x}}) \left(\begin{pmatrix} \sigma_y \cdot z \\ 0 \end{pmatrix}, \begin{pmatrix} 0_{L^2} \\ 0_{L^2} \end{pmatrix} \right) = \left(\begin{pmatrix} x_0^{(1)} \sigma_y \cdot z \\ 0 \end{pmatrix}, \begin{pmatrix} 0_{L^2} \\ 0_{L^2} \end{pmatrix} \right), \quad (49) \quad \text{defF}$$

where 0_{L^2} denotes the null function in $L^2(-d, 0; \mathbb{R})$.

Consider the equation in \mathcal{M}_2^2

$$\begin{cases} d\bar{\mathbf{Y}}(t) = A\bar{\mathbf{Y}}(t)dt + F(\bar{\mathbf{Y}}(t)) dZ(t), & t \in [0, +\infty), \\ \bar{\mathbf{Y}}(0) = \bar{\mathbf{x}}. \end{cases} \quad (50) \quad \text{Y_condensed}$$

Since only the first finite-dimensional component of $F(\bar{\mathbf{Y}}(t))$ is nonzero, the stochastic integral above is well-defined without the need to recur to any infinite-dimensional stochastic integration theory. As paths of $\bar{\mathbf{Y}}$ have almost surely the regularity of Brownian paths, $\bar{\mathbf{Y}}(t)$ is almost surely not in $\mathcal{D}(A)$ for every t ; therefore as a solution of the above equation we mean a *mild* solution, that is, almost surely $\bar{\mathbf{Y}}(t)$ should satisfy

$$\bar{\mathbf{Y}}(t) = U(t)\bar{\mathbf{x}} + \int_0^t U(t-s)F(\bar{\mathbf{Y}}(s)) dZ(s) \quad (51) \quad \text{sol_semig}$$

for every $t \in [0, +\infty)$.

Proposition 4.1. *For every initial condition $\bar{\mathbf{x}}$ equation (50) has a unique mild solution, that is also a weak solution, given by (51), with continuous trajectories almost surely. When*

$$\bar{\mathbf{x}} = \begin{pmatrix} \bar{x} \\ \bar{x} \end{pmatrix} = \begin{pmatrix} x_0 \\ x_0 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_1 \end{pmatrix}$$

and x_0, x_1 are as in (10), equation (50) and equation (10b) are equivalent, meaning that if y solves (10b) then

$$\bar{\mathbf{Y}}(t) = \left(\begin{pmatrix} y(t) \\ \mathbb{E}[y(t)] \end{pmatrix}, \begin{pmatrix} \{y(t+s)\}_{s \in [-d, 0]} \\ \{\mathbb{E}[y(t+s)]\}_{s \in [-d, 0]} \end{pmatrix} \right) ; \quad (52) \quad \text{eq:Yy}$$

is a mild solution of (50), and conversely if $\bar{\mathbf{Y}}$ is a solution of (50) then $P_{1,0}\bar{\mathbf{Y}}$ is a solution of (10b). In particular there exists a unique (probabilistically) strong solution y of (10b) with initial condition $\bar{x} = (x_0, x_1)$.

Proof. Existence and uniqueness of a continuous mild solution of (50) and the equivalence between mild and weak solutions follow from [16, Theorem 6.7]. The equivalence property has been proven in [11, Theorem 3.1]. \square

Remark 4.2. *In the setting of the above proposition the map $[-d, 0] \ni s \mapsto y(t+s)$ may fail to be continuous for small times, since the initial condition is only in L^2 . However this cannot happen if the initial condition x_1 is continuous and $x_0 = x_1(0)$. In this case everything can be formulated in spaces of continuous functions, see for example [22, Section 5] for a discussion.*

Using (50), for any given $(\theta, c, B) \in \Pi^0$ and $(w, \bar{\mathbf{x}}) \in \mathcal{H}$ system (10) can be formulated in the unknown $(W, \bar{\mathbf{Y}}) \in \mathcal{H} = \mathbb{R} \times \mathcal{M}_2^2$ as

$$\begin{cases} dW(t) = [(r + \delta)W(t) + \theta \cdot (\mu - r\mathbf{1}) - c(t) - \delta B(t) + P_{1,0}(\bar{\mathbf{Y}}(t))] dt + \theta(t) \cdot \sigma dZ(t) , \\ d\bar{\mathbf{Y}}(t) = A\bar{\mathbf{Y}}(t)dt + F(\bar{\mathbf{Y}}(t)) dZ(t) , \\ (W(0), \bar{\mathbf{Y}}(0)) = (w, \bar{\mathbf{x}}) \end{cases} \quad (53) \quad \text{systemH}$$

and by Proposition 4.1 it admits a unique strong solution (W, y) corresponding to the unique solution $(W, \bar{\mathbf{Y}})$ of the latter.

We will introduce in a moment a convenient formulation of our optimal control problem and the Hamilton-Jacobi-Bellman equation associated to it. To this end we condense further the notation defining two linear operators on $\mathcal{H} \times \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{R}_+$: \mathcal{B} is the unbounded operator with values in \mathcal{H} given by

$$\begin{aligned} D(\mathcal{B}) &= (\mathbb{R} \times D(A)) \times \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{R}_+ , \\ \mathcal{B}(w, \bar{\mathbf{x}}, \theta, c, B) &= ((r + \delta)w + \theta \cdot (\mu - r\mathbf{1}) - c - \delta B + P_{1,0}\bar{\mathbf{x}}, A\bar{\mathbf{x}}) , \end{aligned}$$

while \mathcal{S} is the bounded operator with values in $L(\mathbb{R}^n; \mathcal{H})$

$$\mathcal{S}(w, \bar{\mathbf{x}}, \theta, c, B) = [z \mapsto (\sigma^\top \theta \cdot z, F(\bar{\mathbf{x}})z)] .$$

For simplicity we let both of them depend also on the variables in $\mathcal{H} \times \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{R}_+$ that do not explicitly appear in their definition; in particular \mathcal{S} depends only on $x_0^{(1)}$ and θ .

It is not difficult to check that for every fixed $(w, \bar{\mathbf{x}}, \theta, c, B) \in \mathcal{H} \times \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{R}_+$ the adjoint of $\mathcal{S}(w, \bar{\mathbf{x}}, \theta, c, B)$ is the map $\mathcal{S}(w, \bar{\mathbf{x}}, \theta, c, B)^* \in L(\mathcal{H}; \mathbb{R}^n)$ given by

$$(u, \bar{\mathbf{p}}) \mapsto u\sigma^\top \theta + P_{1,0}(\bar{\mathbf{x}})P_{1,0}(\bar{\mathbf{p}})\sigma_y .$$

Set

$$\begin{aligned} \mathcal{I} &:= L^2(-d, 0; \mathbb{R}) \times \left(L^2(-d, 0; \mathbb{R}) \times L(L^2(-d, 0; \mathbb{R}); L^2(-d, 0; \mathbb{R})) \right) \times \\ &\quad \times \left(L^2(-d, 0; \mathbb{R}) \times L(L^2(-d, 0; \mathbb{R}); L^2(-d, 0; \mathbb{R})) \right) ; \end{aligned}$$

then $L(\mathcal{H}; \mathcal{H}) \cong \mathcal{N} := \mathcal{H} \times (\mathcal{H} \times \mathcal{I}) \times (\mathcal{H} \times \mathcal{I})$ and given an element

$$Q = H_0 \times (H_1 \times I_1) \times (H_2 \times I_2) \in \mathcal{N}$$

we can think of it as a column vector and index its entries as $Q_{11}, Q_{12}, \dots, Q_{15}, Q_{21}, \dots, Q_{25}, \dots, Q_{51}, \dots, Q_{55}$; here Q_{11}, \dots, Q_{15} are the elements of H_0 , in the order given by the definition of the space \mathcal{H} , and so on, so that

$$Q = \begin{pmatrix} H_0 \\ H_1 \\ I_1 \\ H_2 \\ I_2 \end{pmatrix} = \begin{pmatrix} Q_{11} & \dots & Q_{15} \\ Q_{21} & \dots & Q_{25} \\ \vdots & & \vdots \\ Q_{51} & \dots & Q_{55} \end{pmatrix}. \quad (54) \quad \boxed{\text{Q_MATRIX}}$$

Notice that if we formally take the tranpose of the matrix on the right-hand side of (54) we obtain again an element of \mathcal{N} , therefore through this interpretation we can define the space of symmetric elements in \mathcal{N} as

$$\mathcal{N}_{\text{sym}} := \{Q \in \mathcal{N} : Q_{ij} = Q_{ji}, i, j = 1, \dots, 5\}.$$

By simple computations we then have, for any $Q \in \mathcal{N}_{\text{sym}}$ and any $(w, \bar{x}, \theta, c, B) \in \mathcal{H} \times \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{R}_+$,

$$\begin{aligned} \text{Tr} [QS(w, \bar{x}, \theta, c, B) S(w, \bar{x}, \theta, c, B)^*] \\ = Q_{11} |\sigma^\top \theta|^2 + Q_{22} P_{1,0}(\bar{x})^2 |\sigma_y|^2 + 2Q_{12} P_{1,0}(\bar{x}) \sigma_y \cdot \sigma^\top \theta. \end{aligned} \quad (55) \quad \boxed{\text{eq:trace}}$$

In particular for any given function

$$f: \mathcal{H} \rightarrow \mathbb{R}$$

its second Fréchet derivative at a given point (w, \bar{x}) is an element $\nabla^2 f(w, \bar{x}) \in \mathcal{N}_{\text{sym}}$, and the above formula provides the second order term that will appear in our Hamilton-Jacobi-Bellman equation.

Recall the set Π^0 defined in (8). We denote a triple of controls $(\theta(\cdot), c(\cdot), B(\cdot)) \in \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{R}_+$ as $\pi(\cdot)$; for given $\pi(\cdot) \in \Pi^0$ and given initial condition (w, \bar{x}) we eventually rewrite system (53) for the unknown \mathcal{H} -valued process $\mathcal{X} = (W, \bar{\mathbf{Y}})$ as

$$\begin{cases} d\mathcal{X}(t) = \mathcal{B}(\mathcal{X}(t), \pi(t)) dt + \mathcal{S}(\mathcal{X}(t), \pi(t)) dZ(t), \\ \mathcal{X}(0) = (w, \bar{x}), \end{cases} \quad (56) \quad \boxed{\text{systemH_condensed}}$$

where again solutions are to be intended in mild sense, since almost surely $\mathcal{X} \notin D(\mathcal{B})$. By [31, Chapter 5.6] and Proposition 4.1 there is a unique mild solution of (56); we will denote such solution at time $t \geq 0$ as $\mathcal{X}^{w, \bar{x}}(t; \pi) = (W^{w, \bar{x}}(t; \pi), \bar{\mathbf{Y}}^{\bar{x}}(t))$; recall that we are interested only in initial conditions of the form

$$(w, \bar{x}) = (w, (\frac{\bar{x}}{x})) = (w, (\frac{x_0}{x_0}, (\frac{x_1}{x_1}))) . \quad (57) \quad \boxed{\text{innt_cond_sec_4}}$$

The dependence on the initial condition will be sometimes hidden if notationally convenient. Thanks to the results proved in Section 3.4 we can write the set of admissible controls as

$$\begin{aligned} \Pi(w, \bar{x}) &= \{\pi \in \Pi^0 : \mathcal{X}^{w, \bar{x}}(t; \pi) \in \mathcal{H}_+ \forall t \geq 0\} \\ &= \{\pi \in \Pi^0 : \Gamma_\infty(W^{w, \bar{x}}(t; \pi), \bar{\mathbf{Y}}^{\bar{x}}(t)) \geq 0 \forall t \geq 0\} \\ &= \{\pi \in \Pi^0 : W^{w, \bar{x}}(t) + \langle (g_\infty, h_\infty), (y^{\bar{x}}(t), y^{\bar{x}}(t + \cdot)) \rangle_{\mathcal{M}_2} + \\ &\quad - i_\infty \langle (g_\infty, h_\infty), (\mathbb{E}[y^{\bar{x}}(t)], \mathbb{E}[y^{\bar{x}}(t + \cdot)]) \rangle_{\mathcal{M}_2} \geq 0 \forall t \geq 0\} \\ &= \{\pi \in \Pi^0 : W^{w, \bar{x}}(t) + g_\infty y^{\bar{x}}(t) + \langle h_\infty, y^{\bar{x}}(t + \cdot) \rangle + \\ &\quad - i_\infty [g_\infty \mathbb{E}[y^{\bar{x}}(t)] + \langle h_\infty, \mathbb{E}[y^{\bar{x}}(t + \cdot)] \rangle] \geq 0 \forall t \geq 0\} . \end{aligned}$$

Recall the objective functional

$$J(w, \bar{\mathbf{x}}; \pi) := \mathbb{E} \left(\int_0^{+\infty} e^{-(\rho+\delta)t} \left(\frac{c(t)^{1-\gamma}}{1-\gamma} + \delta \frac{(kB(t))^{1-\gamma}}{1-\gamma} \right) dt \right) ; \quad (15)$$

the part of the integrand in the definition of J that depends on the controls is the *utility function*

$$U(\pi) = \frac{c^{1-\gamma}}{1-\gamma} + \delta \frac{(kB)^{1-\gamma}}{1-\gamma} .$$

Here we write for simplicity J and U as functions of π although they actually only depend on the control triple through c and B and not through θ .

Our goal is to solve the following:

Problem 4.3. *Under Assumptions 2.1, 2.2, 2.5 and 2.7 and for given fixed $(w, \bar{\mathbf{x}}) \in \mathcal{H}$ as in (57), find $\tilde{\pi} \in \Pi(w, \bar{\mathbf{x}})$ such that*

$$J(w, \bar{\mathbf{x}}; \tilde{\pi}) = \max_{\pi \in \Pi(w, \bar{\mathbf{x}})} J(w, \bar{\mathbf{x}}; \pi) .$$

The following result can be proved with a straightforward adaptation of the proof of [10, Proposition 3.1].

Proposition 4.4. *The adjoint operator of A is the operator $A^*: \mathcal{D}(A^*) \subset \mathcal{M}_2^2 \rightarrow \mathcal{M}_2^2$ defined as*

$$\begin{aligned} \mathcal{D}(A^*) &:= \{(\mathbf{y}_0, \mathbf{y}_1) : \mathbf{y}_1 \in W^{1,2}([-d, 0]; \mathbb{R}^2), \mathbf{y}_1(-d) = 0\} , \\ A^*(\mathbf{y}_0, \mathbf{y}_1) &= \left(C^\top \mathbf{y}_0 + \mathbf{y}_1(0), \mathbf{y}_0 \phi - \frac{d}{ds} \mathbf{y}_1 \right) \end{aligned} \quad (58) \quad \text{Astar}$$

where C^\top is the transpose of the matrix C .

4.2 The HJB equation

We now introduce the Hamiltonian for our control problem: formally we expect it to be the function

$$\tilde{\mathbb{H}}: (\mathbb{R} \times D(A)) \times \mathcal{H} \times \mathcal{N}_{\text{sym}} \rightarrow \mathbb{R} \cup \{\pm\infty\}$$

given by

$$\begin{aligned} \tilde{\mathbb{H}}((w, \bar{\mathbf{x}}), (u, \bar{\mathbf{p}}), Q) \\ = \sup_{\pi \in \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{R}_+} \left[\langle \mathcal{B}(w, \bar{\mathbf{x}}, \pi), (u, \bar{\mathbf{p}}) \rangle_{\mathcal{H}} + \frac{1}{2} \text{Tr} [Q \mathcal{S}(w, \bar{\mathbf{x}}, \pi) \mathcal{S}(w, \bar{\mathbf{x}}, \pi)^*] + U(\pi) \right] . \end{aligned}$$

It is however convenient (see also Remark 4.6 below) to define it a bit differently. Using the definitions of \mathcal{B} and \mathcal{S} (and in particular the defining property of A^* with respect to the duality product appearing in \mathcal{B}) together with (55) we can write the Hamiltonian in a more explicit way, separating at the same time the part that depends on the controls from the rest; we thus choose as the Hamiltonian for our problem the function

$$\mathbb{H}: \mathcal{H} \times (\mathbb{R} \times D(A^*)) \times \mathcal{N}_{\text{sym}} \rightarrow \mathbb{R} \cup \{\pm\infty\}$$

given by

$$\mathbb{H}((w, \bar{\mathbf{x}}), (u, \bar{\mathbf{p}}), Q) := \mathbb{H}_0((w, \bar{\mathbf{x}}, (u, \bar{\mathbf{p}}), Q_{22})) + \mathbb{H}_{\max}(P_{1,0} \bar{\mathbf{x}}, u, Q_{11}, Q_{12}) ,$$

where

$$\mathbb{H}_0((w, \bar{\mathbf{x}}), (u, \bar{\mathbf{p}}), Q_{22}) = (r + \delta)wu + P_{1,0} \bar{\mathbf{x}}u + \langle \bar{\mathbf{x}}, A^* \bar{\mathbf{p}} \rangle_{\mathcal{M}_2^2} + \frac{1}{2} Q_{22} (P_{1,0} \bar{\mathbf{x}})^2 \sigma_y \cdot \sigma_y$$

$$= (r + \delta)wu + x_0^{(1)}u + \langle \bar{\mathbf{x}}, A^* \bar{\mathbf{p}} \rangle_{\mathcal{M}_2^2} + \frac{1}{2} Q_{22} \left| x_0^{(1)} \sigma_y \right|^2 ,$$

and

$$\mathbb{H}_{\max} \left(x_0^{(1)}, u, Q_{11}, Q_{12} \right) = \sup_{\pi \in \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{R}_+} \mathbb{H}_c(x_0^{(1)}, u, Q_{11}, Q_{12}, \pi) , \quad (59) \quad \boxed{\text{Hmax}}$$

with

$$\begin{aligned} \mathbb{H}_c(x_0^{(1)}, u, Q_{11}, Q_{12}, \pi) &= [\theta \cdot (\mu - r\mathbf{1}) - c - \delta B] u + \frac{1}{2} |\theta^\top \sigma|^2 Q_{11} + \theta^\top \sigma \sigma_y x_0^{(1)} Q_{12} \\ &\quad + \frac{c^{1-\gamma}}{1-\gamma} + \delta \frac{(kB)^{1-\gamma}}{1-\gamma} . \end{aligned}$$

Reordering the terms in the definition of \mathbb{H}_c we can write

$$\begin{aligned} \mathbb{H}_c(x_0^{(1)}, u, Q_{11}, Q_{12}, \pi) &= \frac{c^{1-\gamma}}{1-\gamma} - cu + \delta \frac{(kB)^{1-\gamma}}{1-\gamma} - \delta Bu \\ &\quad + \frac{1}{2} |\theta^\top \sigma|^2 Q_{11} + \theta^\top \sigma \sigma_y x_0^{(1)} Q_{12} + \theta \cdot (\mu - r\mathbf{1})u \end{aligned}$$

from which is apparent that for each $x_0^{(1)} \in \mathbb{R}$ and $Q_{12} \in \mathbb{R}$ there are three possible situations:

case1

(i) if $u > 0$ and $Q_{11} < 0$ the supremum in (59) is achieved at (θ^*, c^*, B^*) , where

$$\theta^* = -(\sigma \sigma^\top)^{-1} \frac{1}{Q_{11}} \left[(\mu - r\mathbf{1})u + \sigma \sigma_y x_0^{(1)} Q_{12} \right] , c^* = u^{-\frac{1}{\gamma}} , B^* = k^{-b} u^{-\frac{1}{\gamma}}$$

with

$$b = 1 - \frac{1}{\gamma}$$

or equivalently

$$b = \frac{1}{\gamma'} \text{ with } \gamma' = \frac{\gamma}{\gamma - 1} ;$$

(ii) if $u < 0$ or $Q_{11} > 0$ then the supremum in (59) is $+\infty$;

(iii) if $u Q_{11} = 0$ the supremum in (59) can be finite or infinite depending on γ and on the sign of the other terms involved.

The Hamilton-Jacobi-Bellman equation associated with Problem 4.3 is the partial differential equation in the unknown $v: \mathcal{H} \rightarrow \mathbb{R}$

$$(\rho + \delta)v(w, \bar{\mathbf{x}}) = \mathbb{H}(w, \bar{\mathbf{x}}, \nabla v(w, \bar{\mathbf{x}}), \nabla^2 v(w, \bar{\mathbf{x}})) . \quad (60) \quad \boxed{\text{HJB}}$$

def_sol

Definition 4.5. A function $v: \mathcal{H}_{++} \rightarrow \mathbb{R}$ is a classical solution of the Hamilton-Jacobi-Bellman equation (60) if it satisfies:

(a) v is continuously Fréchet differentiable in \mathcal{H}_{++} and its four second Fréchet derivatives with respect to the couple $(w, x_0^{(1)})$ exist and are continuous in \mathcal{H}_{++} ;

regularity

(b) $\partial_{\bar{\mathbf{x}}} v(w, \bar{\mathbf{x}})$ belongs to $\mathcal{D}(A^*)$ for every $(w, \bar{\mathbf{x}}) \in \mathcal{H}_{++}$ and $A^* \partial_{\bar{\mathbf{x}}} v$ is continuous in \mathcal{H}_{++} ;

(c) v satisfies (60) for every $(w, \bar{\mathbf{x}}) \in \mathcal{H}_{++}$.

rem:H_tilde

Remark 4.6. The difference between $\tilde{\mathbb{H}}$ and \mathbb{H} lies in the term involving A , that appears as $\langle A \bar{\mathbf{x}}, \bar{\mathbf{p}} \rangle$ in the former but as $\langle \bar{\mathbf{x}}, A^* \bar{\mathbf{p}} \rangle$ in the latter. This choice makes \mathbb{H} defined on the whole \mathcal{H}_{++} instead than only on $\mathcal{H}_{++} \cap (\mathbb{R} \times \mathcal{D}(A))$, at the price of requiring further regularity of the solution, as specified in Definition 4.5- (b). This will not constitute a problem as we are going to find an explicit solution that satisfies the required properties.

If a solution v to (60) satisfies $\nabla v_1 = \partial_w v > 0$ and $\nabla^2 v_{11} = \partial_{ww}^2 v < 0$ uniformly in $(w, \bar{\mathbf{x}})$, then we fall in case (i) above and, plugging θ^*, c^*, B^* in the definition of \mathbb{H} , we find the equation for v to take the form

$$\begin{aligned} (\rho + \delta)v(w, \bar{\mathbf{x}}) = & (r + \delta)w\partial_w v(w, \bar{\mathbf{x}}) + x_0^{(1)}\partial_w v(w, \bar{\mathbf{x}}) - \frac{1}{b}\partial_w(w, \bar{\mathbf{x}})^b (1 + \delta k^{-b}) \\ & + \langle \bar{\mathbf{x}}, A^* \partial_{\bar{\mathbf{x}}} v(w, \bar{\mathbf{x}}) \rangle_{\mathcal{M}_2^2} + \frac{1}{2} \left| x_0^{(1)} \sigma_y \right|^2 \partial_{x_0^{(1)} x_0^{(1)}}^2 v(w, \bar{\mathbf{x}}) \\ & - \frac{1}{2} \frac{1}{\partial_{ww}^2 v(w, \bar{\mathbf{x}})} \left[(\mu - r\mathbf{1})\partial_w v(w, \bar{\mathbf{x}}) + \sigma \sigma_y x_0^{(1)} \partial_{wx_0^{(1)}}^2 v(w, \bar{\mathbf{x}}) \right] \cdot \\ & \cdot (\sigma \sigma^\top)^{-1} \left[(\mu - r\mathbf{1})\partial_w v(w, \bar{\mathbf{x}}) + \sigma \sigma_y x_0^{(1)} \partial_{wx_0^{(1)}}^2 v(w, \bar{\mathbf{x}}) \right] . \end{aligned} \quad (61) \quad \text{HJB_short}$$

Set now

$$\nu = \frac{\gamma}{\rho + \delta - (1 - \gamma) \left(r + \delta + \frac{|\kappa|^2}{2\gamma} \right)}, \quad f_\infty = (1 + \delta k^{-b})\nu \quad (62) \quad \text{defnufinfty}$$

and define, for every $(w, \bar{\mathbf{x}}) \in \mathcal{H}_{++}$,

$$\tilde{v}(w, \bar{\mathbf{x}}) := \frac{f_\infty^\gamma}{1 - \gamma} \Gamma_\infty^{1-\gamma}(w, \bar{\mathbf{x}}), \quad (63) \quad \text{def_v_tilde}$$

where Γ_∞ is defined in (44a).

thm_sol1

Theorem 4.7. *The function \tilde{v} is a classical solution of the Hamilton-Jacobi-Bellman equation (60).*

To prove the theorem we need a brief result that we state separately for later reference.

lemma_Dast

Lemma 4.8. *The element*

$$\bar{\mathbf{i}}_\infty = \begin{pmatrix} (g_\infty, h_\infty) \\ -i_\infty(g_\infty, h_\infty) \end{pmatrix}$$

belongs to $\mathcal{D}(A^)$.*

Proof. Being the integral of an L^2 function, h_∞ is differentiable almost everywhere in $[-d, 0]$ and

$$h'_\infty(s) = -(r + \delta)h_\infty(s) + g_\infty \phi(s)$$

almost everywhere. Set for brevity

$$\beta = K_1(r + \delta) + \int_{-d}^0 e^{(r+\delta)s} \phi(s) ds = r + \delta - \mu_y - \epsilon + \sigma_y \cdot \kappa ;$$

then

$$\beta g_\infty - h_\infty(0) = 1$$

and therefore h_∞ satisfies the differential equation

$$\begin{cases} h' = g_\infty \phi - (r + \delta)h \\ h(0) = \beta g_\infty - 1 . \end{cases} \quad (64) \quad \text{eq_diff}$$

Since $1 \geq e^{-2(r+\delta)(s-\tau)} > 0$ on $-d \leq \tau \leq s \leq 0$ and ϕ is an L^2 function, it is easy to check that h_∞ is in L^2 as well; this implies that actually $h_\infty \in W^{1,2}(-d, 0 : \mathbb{R})$ and since obviously $h_\infty(-d) = 0$ the claim is proved. \square

Proof of Theorem 4.7. Recall that \mathcal{H}_{++} is by definition the set where Γ_∞ is strictly positive. Thanks to the linearity of Γ_∞ the function \tilde{v} is twice continuously Fréchet differentiable in all variables. The derivatives that appear in the Hamiltonian are easily computed:

$$\begin{aligned}\partial_w \tilde{v}(w, \bar{\mathbf{x}}) &= f_\infty^\gamma \Gamma_\infty^{-\gamma}(w, \bar{\mathbf{x}}), \\ \partial_{\bar{\mathbf{x}}} \tilde{v}(w, \bar{\mathbf{x}}) &= f_\infty^\gamma \Gamma_\infty^{-\gamma}(w, \bar{\mathbf{x}}) \bar{\mathbf{I}}_\infty, \\ \partial_{ww}^2 \tilde{v}(w, \bar{\mathbf{x}}) &= -\gamma f_\infty^\gamma \Gamma_\infty^{-(1+\gamma)}(w, \bar{\mathbf{x}}), \\ \partial_{wx_0^{(1)}}^2 \tilde{v}(w, \bar{\mathbf{x}}) &= -\gamma f_\infty^\gamma \Gamma_\infty^{-(1+\gamma)}(w, \bar{\mathbf{x}}) g_\infty, \\ \partial_{x_0^{(1)} x_0^{(1)}}^2 \tilde{v}(w, \bar{\mathbf{x}}) &= -\gamma f_\infty^\gamma \Gamma_\infty^{-(1+\gamma)}(w, \bar{\mathbf{x}}) g_\infty^2.\end{aligned}$$

Therefore thanks to Lemma 4.8 also requirement (b) in the definition of solution is satisfied. It remains to check that \tilde{v} satisfies (60).

Since $f_\infty > 0$, by definition of \mathcal{H}_{++} we have $\partial_w \tilde{v} > 0$ and $\partial_{ww}^2 \tilde{v} < 0$ on \mathcal{H}_{++} , therefore we can consider the simplified form (61) for the Hamilton-Jacobi-Bellman equation.

Let us now look at the various pieces appearing in (61). We have, by simple computations,

$$\begin{aligned}(\rho + \delta)w \partial_w v(w, \bar{\mathbf{x}}) + x_0^{(1)} \partial_w v(w, \bar{\mathbf{x}}) - \frac{1}{b} \partial_w(w, \bar{\mathbf{x}})^b (1 + \delta k^{-b}) \\ = f_\infty^\gamma \Gamma_\infty^{-\gamma}(w, \bar{\mathbf{x}}) \left[(r + d)w + x_0^{(1)} - \frac{\gamma}{\gamma - 1} f_\infty^{-1} \Gamma_\infty(w, \bar{\mathbf{x}}) \left(1 + \delta k^{\frac{1-\gamma}{\gamma}} \right) \right], \\ \frac{1}{2} \left| x_0^{(1)} \sigma_y \right|^2 \partial_{x_0^{(1)} x_0^{(1)}}^2 v(w, \bar{\mathbf{x}}) = -f_\infty^\gamma \Gamma_\infty^{-\gamma}(w, \bar{\mathbf{x}}) \frac{1}{2} \left| x_0^{(1)} \right|^2 |\sigma_y|^2 \gamma g_\infty^2 \Gamma_\infty^{-1}(w, \bar{\mathbf{x}}), \\ -\frac{1}{2} \frac{1}{\partial_{ww}^2 v(w, \bar{\mathbf{x}})} \left[(\mu - r\mathbf{1}) \partial_w v(w, \bar{\mathbf{x}}) + \sigma \sigma_y x_0^{(1)} \partial_{wx_0^{(1)}}^2 v(w, \bar{\mathbf{x}}) \right] \\ \cdot (\sigma \sigma^\top)^{-1} \left[(\mu - r\mathbf{1}) \partial_w v(w, \bar{\mathbf{x}}) + \sigma \sigma_y x_0^{(1)} \partial_{wx_0^{(1)}}^2 v(w, \bar{\mathbf{x}}) \right] \\ = f_\infty^\gamma \Gamma_\infty^{-\gamma}(w, \bar{\mathbf{x}}) \frac{1}{2\gamma} \Gamma_\infty(w, \bar{\mathbf{x}}) \left[|\kappa|^2 - 2\gamma x_0^{(1)} g_\infty \kappa \cdot \sigma_y \Gamma_\infty^{-1}(w, \bar{\mathbf{x}}) + \left| x_0^{(1)} \right|^2 |\sigma_y|^2 \gamma^2 g_\infty^2 \Gamma_\infty^{-2}(w, \bar{\mathbf{x}}) \right]\end{aligned}$$

and finally, using (58) and (64),

$$(A^* \partial_{\bar{\mathbf{x}}} \tilde{v}(w, \bar{\mathbf{x}}))_0 = f_\infty^\gamma \Gamma_\infty^{-\gamma}(w, \bar{\mathbf{x}}) \left(g_\infty (r + \delta + \sigma_y \cdot \kappa) - 1 \right. \\ \left. i_\infty - \epsilon g_\infty - i_\infty g_\infty (r + \delta - \epsilon + \sigma_y \cdot \kappa) \right)$$

and

$$(A^* \partial_{\bar{\mathbf{x}}} \tilde{v}(w, \bar{\mathbf{x}}))_1 = f_\infty^\gamma \Gamma_\infty^{-\gamma}(w, \bar{\mathbf{x}}) (r + \delta) \begin{pmatrix} h_\infty \\ -i_\infty h_\infty \end{pmatrix},$$

hence

$$\begin{aligned}\langle \bar{\mathbf{x}}, A^* \partial_{\bar{\mathbf{x}}} v(w, \bar{\mathbf{x}}) \rangle_{\mathcal{M}_2^2} &= f_\infty^\gamma \Gamma_\infty^{-\gamma}(w, \bar{\mathbf{x}}) x_0^{(1)} (g_\infty (r + \delta + \sigma_y \cdot \kappa) - 1) \\ &\quad + f_\infty^\gamma \Gamma_\infty^{-\gamma}(w, \bar{\mathbf{x}}) x_0^{(2)} (i_\infty - \epsilon g_\infty - i_\infty g_\infty (r + \delta - \epsilon + \sigma_y \cdot \kappa)) \\ &\quad + f_\infty^\gamma \Gamma_\infty^{-\gamma}(w, \bar{\mathbf{x}}) (r + \delta) \langle x_1^{(1)}, h_\infty \rangle - (r + \delta) i_\infty \langle x_1^{(2)}, h_\infty \rangle.\end{aligned}\tag{65}$$

Astar_1

Plugging now everything into (61) and multiplying both sides by $f_\infty^{-\gamma} \Gamma_\infty^\gamma(w, \bar{\mathbf{x}})$ (which is a positive quantity on \mathcal{H}_{++} by Assumption 2.7) we find

$$\begin{aligned}\frac{\rho + \delta}{1 - \gamma} \Gamma_\infty(w, \bar{\mathbf{x}}) &= (r + \delta)w - \frac{\gamma}{\gamma - 1} f_\infty^{-1} \Gamma_\infty(w, \bar{\mathbf{x}}) \left(1 + \delta k^{\frac{1-\gamma}{\gamma}} \right) + \frac{1}{2\gamma} |\kappa|^2 \Gamma_\infty(w, \bar{\mathbf{x}}) + (r + \delta) x_0^{(1)} g_\infty \\ &\quad + x_0^{(2)} i_\infty - x_0^{(2)} \epsilon g_\infty - x_0^{(2)} i_\infty g_\infty (r + \delta) + x_0^{(2)} i_\infty g_\infty (\epsilon - \sigma_y \cdot \kappa)\end{aligned}$$

$$\begin{aligned}
& + (r + \delta) \langle x_1^{(1)}, h_\infty \rangle - (r + \delta) i_\infty \langle x_1^{(2)}, h_\infty \rangle \\
& = (r + \delta) \Gamma_\infty(w, \bar{\mathbf{x}}) - \frac{\gamma}{\gamma - 1} f_\infty^{-1} \Gamma_\infty(w, \bar{\mathbf{x}}) \left(1 + \delta k^{\frac{1-\gamma}{\gamma}} \right) + \frac{1}{2\gamma} |\kappa|^2 \Gamma_\infty(w, \bar{\mathbf{x}}) \\
& + x_0^{(2)} (i_\infty - \epsilon g_\infty + i_\infty g_\infty (\epsilon - \sigma_y \cdot \kappa))
\end{aligned}$$

but

$$\begin{aligned}
i_\infty - \epsilon g_\infty + i_\infty g_\infty (\epsilon - \sigma_y \cdot \kappa) & = \frac{\epsilon}{K_2} - \frac{\epsilon}{K_1} + \frac{\epsilon}{K_1 K_2} (\epsilon - \sigma_y \cdot \kappa) \\
& = \frac{\epsilon}{K_1 K_2} (K_1 - K_2 + \epsilon - \sigma_y \cdot \kappa) = 0
\end{aligned} \tag{66} \quad \boxed{\text{comb}_0}$$

therefore, dividing by the positive quantity $\Gamma_\infty(w, \bar{\mathbf{x}})$ we obtain eventually

$$\frac{\rho + \delta}{1 - \gamma} = (r + \delta) - \frac{\gamma}{\gamma - 1} f_\infty^{-1} \left(1 + \delta k^{\frac{1-\gamma}{\gamma}} \right) + \frac{1}{2\gamma} |\kappa|^2$$

and this last equality is easily shown to hold true by the definition of f_∞ . \square

5 Solution of the general problem

5.1 The admissible paths at the boundary

Fix $(w, \bar{\mathbf{x}}) \in \mathcal{H}_+$ and $\pi \in \Pi(w, \bar{\mathbf{x}})$ and let $\mathcal{X}(\cdot; \pi) = (W(\cdot; \pi), \bar{\mathbf{Y}}(\cdot))$ be the corresponding solution of (56). Applying the Ito formula proved in [20, Proposition 1.165] to the process $\langle \bar{\mathbf{I}}_\infty, \bar{\mathbf{Y}} \rangle_{\mathcal{M}_2^2}$ and using (65), (66) and (17) we obtain

$$\begin{aligned}
d\langle \bar{\mathbf{I}}_\infty, \bar{\mathbf{Y}}(t) \rangle_{\mathcal{M}_2^2} & = \langle A^* \bar{\mathbf{I}}_\infty, \bar{\mathbf{Y}}(t) \rangle_{\mathcal{M}_2^2} dt + \langle \bar{\mathbf{I}}_\infty, F(\bar{\mathbf{Y}}(t)) \cdot dZ(t) \rangle_{\mathcal{M}_2^2} \\
& = y(t) (g_\infty(r + \delta + \sigma_y \cdot \kappa) - 1) dt + e(t) (i_\infty - \epsilon g_\infty - i_\infty g_\infty(r + \delta - \epsilon + \sigma_y \cdot \kappa)) dt \\
& + (r + \delta) \langle y(t + \cdot), h_\infty \rangle dt - (r + \delta) i_\infty \langle e(t + \cdot), h_\infty \rangle dt + g_\infty y(t) \sigma_y \cdot dZ(t);
\end{aligned} \tag{67} \quad \boxed{\text{ito}_1\text{-infty}}$$

therefore setting

$$\bar{\Gamma}_\infty(t) := \Gamma_\infty(W(t; \pi), \bar{\mathbf{Y}}(t)) \tag{68} \quad \boxed{\text{def_Gamma_proc}}$$

we have

$$\begin{aligned}
d\bar{\Gamma}_\infty(t) & = y(t) (g_\infty(r + \delta + \sigma_y \cdot \kappa) - 1) dt + e(t) (i_\infty - \epsilon g_\infty - i_\infty g_\infty(r + \delta - \epsilon + \sigma_y \cdot \kappa)) dt \\
& + (r + \delta) \langle y(t + \cdot), h_\infty \rangle dt - (r + \delta) i_\infty \langle e(t + \cdot), h_\infty \rangle dt + g_\infty y(t) \sigma_y \cdot dZ(t) \\
& + (r + \delta) W(t) + (\theta(t) \cdot (\mu - r\mathbf{1}) - c(t) - \delta B(t)) dt + y(t) dt + \theta(t) \cdot \sigma dZ(t) \\
& = (r + \delta) \bar{\Gamma}_\infty(t) dt - (c(t) + \delta B(t)) dt + (g_\infty y(t) \sigma_y + \sigma^\top \theta(t)) \cdot (\kappa dt + dZ(t)) .
\end{aligned} \tag{69} \quad \boxed{\text{dGamma}}$$

In what follows we will denote by τ_+ the first exit time of $\mathcal{X}(\cdot; \pi)$ from \mathcal{H}_{++} :

$$\tau_+ = \inf \left\{ t \geq 0 : \mathcal{X} \in \mathcal{H}_{++}^c \right\} = \inf \{ t \geq 0 : \mathcal{X} \in \partial \mathcal{H}_+ \} = \inf \{ t \geq 0 : \bar{\Gamma}_\infty(t) = 0 \}. \tag{70} \quad \boxed{\text{eq:tau}_+}$$

We can then prove the following result on the behavior of the process $\bar{\Gamma}_\infty$ when it hits the boundary of \mathcal{H}_+ . The proof is postponed to the Appendix.

$\boxed{\text{prop_tau}}$

Proposition 5.1. *Let $(w, \bar{\mathbf{x}}) \in \mathcal{H}_+$;*

(i) *if $\Gamma_\infty(w, \bar{\mathbf{x}}) = \bar{\Gamma}_\infty(0) = 0$ then \mathbb{P} -a.s. $\bar{\Gamma}_\infty(t) = 0$ for every $t > 0$ and*

$$c(t, \omega) = B(t, \omega) = 0, \quad g_\infty y(t, \omega) \sigma_y + \sigma^\top \theta(t, \omega) = 0$$

dt \otimes \mathbb{P} -a.e. on $[0, +\infty) \times \Omega$;

(ii) if $\Gamma_\infty(w, \bar{\mathbf{x}}) > 0$ (i.e. $\bar{\Gamma}_\infty(0) \in \mathcal{H}_{++}$) then \mathbb{P} -a.s. for every $t \geq 0$

$$\mathbf{1}_{(\tau_+, +\infty)}(t) \bar{\Gamma}_\infty(t) = 0$$

and

$$\mathbf{1}_{(\tau_+, +\infty)}(t) c(t, \omega) = \mathbf{1}_{(\tau_+, +\infty)}(t) B(t, \omega) = 0, \quad \mathbf{1}_{(\tau_+, +\infty)}(t) (g_\infty y(t, \omega) \sigma_y + \sigma^\top \theta(t, \omega)) = 0$$

$dt \otimes \mathbb{P}$ -a.e. on $[0, +\infty) \times \Omega$.

5.2 Fundamental identity

In this subsection we assume $\gamma \in (0, 1)$; first we state a key lemma (proved in the Appendix) to deal with the infinite horizon nature of the problem.

lemma_mean_zero

Lemma 5.2. Assume $(w, \bar{\mathbf{x}}) \in \mathcal{H}_{++}$ and $\pi \in \Pi(w, \bar{\mathbf{x}})$. Let $\gamma \in (0, 1)$. Then for every $T > 0$

$$\mathbb{E} \left[e^{(\gamma-1) \left(r + \delta + \frac{|\kappa|^2}{2\gamma} \right) (T \wedge \tau_+)} \tilde{v}(\mathcal{X}(T \wedge \tau_+; \pi)) \right] \leq \tilde{v}(w, \bar{\mathbf{x}}).$$

Moreover

$$\lim_{T \rightarrow +\infty} \mathbb{E} \left[e^{-(\rho+\delta)(T \wedge \tau_+)} \tilde{v}(\mathcal{X}(T \wedge \tau_+; \pi)) \right] = 0.$$

The key step to the main result of our paper is provided by the following result.

prop:fundamental

Proposition 5.3. Assume $(w, \bar{\mathbf{x}}) \in \mathcal{H}_{++}$ and $\pi \in \Pi(w, \bar{\mathbf{x}})$. Then

$$\begin{aligned} \tilde{v}(w, \bar{\mathbf{x}}) &= J(w, \bar{\mathbf{x}}, \pi) \\ &+ \mathbb{E} \int_0^{\tau_+} e^{-(\rho+\delta)s} \left\{ \mathbb{H}_{max} \left(y(s), \partial_w \tilde{v}(\mathcal{X}(s; \pi)), \partial_{ww} \tilde{v}(\mathcal{X}(s; \pi)), \partial_{wx_0^{(1)}} \tilde{v}(\mathcal{X}(s; \pi)) \right) \right. \\ &\quad \left. - \mathbb{H}_c \left(y(s), \partial_w \tilde{v}(\mathcal{X}(s; \pi)), \partial_{ww} \tilde{v}(\mathcal{X}(s; \pi)), \partial_{wx_0^{(1)}} \tilde{v}(\mathcal{X}(s; \pi)), \pi \right) \right\} ds. \end{aligned} \quad (71) \quad \text{eq:vtilde_J}$$

Identity (71) is often called the *fundamental identity*.

Proof. Set

$$\tau_N := \inf \left\{ t \geq 0 : \bar{\Gamma}_\infty(t) \leq \frac{1}{N} \right\}. \quad (72) \quad \text{tau_N}$$

and choose N large enough so that $\tau_N > 0$ almost surely. Its formula applied on $[0, \tau_N]$ to $e^{-(\rho+\delta)s} f_\infty^\gamma \bar{\Gamma}_\infty^{1-\gamma}(s)$ yields, by (69),

$$\begin{aligned} \frac{1}{1-\gamma} d \left(e^{-(\rho+\delta)s} f_\infty^\gamma \bar{\Gamma}_\infty^{1-\gamma}(s) \right) &= e^{-(\rho+\delta)s} \left\{ -(\rho+\delta) f_\infty^\gamma \bar{\Gamma}_\infty^{1-\gamma}(s) ds + f_\infty^\gamma (1-\gamma) \bar{\Gamma}_\infty^{-\gamma}(s) d\bar{\Gamma}_\infty(s) \right. \\ &\quad \left. - \frac{1}{2} \gamma (1-\gamma) f_\infty^\gamma \bar{\Gamma}_\infty(s)^{-\gamma-1} d[\bar{\Gamma}_\infty](s) \right\} \\ &= e^{-(\rho+\delta)s} \left\{ -(\rho+\delta) \tilde{v}(\mathcal{X}(s; \pi)) ds + \partial_w \tilde{v}(\mathcal{X}(s; \pi)) \left[(r+\delta) \bar{\Gamma}_\infty(s) + g_\infty y(s) \sigma_y \cdot \kappa \right] ds \right. \\ &\quad + \frac{1}{2} \partial_{x_0^{(1)} x_0^{(1)}}^2 |g_\infty y(s) \sigma_y|^2 ds + f_\infty^\gamma \bar{\Gamma}_\infty^{-\gamma}(s) (g_\infty y(s) \sigma_y + \sigma^\top \theta(s)) \cdot dZ(s) \\ &\quad \left. + \partial_w \tilde{v}(\mathcal{X}(s; \pi)) \left[-c(s) - \delta B(s) + \theta(s) \cdot \sigma \kappa - \frac{1}{2} \gamma \bar{\Gamma}_\infty^{-1}(s) (\theta(s)^\top \sigma \sigma^\top \theta(s) + 2g_\infty y(s) \sigma_y \cdot \sigma^\top \theta(s)) \right] ds \right\}. \end{aligned}$$

By the definition of $\bar{\Gamma}_\infty$ as in (44a) and (68) the previous expression is equal to

$$e^{-(\rho+\delta)s} \left\{ -(\rho+\delta) \tilde{v}(\mathcal{X}(s; \pi)) ds + \partial_w \tilde{v}(\mathcal{X}(s; \pi)) \left[(r+\delta) W(s; \pi) + (r+\delta) \langle \bar{\mathbf{I}}_\infty, \bar{\mathbf{Y}}(s) \rangle_{\mathcal{M}_2^2} \right] ds \right\}$$

$$\begin{aligned}
& + \partial_w \tilde{v}(\mathcal{X}(s; \pi)) [-y(s) + y(s) + g_\infty y(s) \sigma_y \cdot \kappa] ds \\
& + \frac{1}{2} \partial_{x_0^{(1)} x_0^{(1)}}^2 \tilde{v}(\mathcal{X}(s; \pi)) |y(s) \sigma_y|^2 ds + f_\infty^\gamma \bar{\Gamma}_\infty^{-\gamma}(s) (g_\infty y(s) \sigma_y + \sigma^\top \theta(s)) \cdot dZ(s) \\
& + \partial_w \tilde{v}(\mathcal{X}(s; \pi)) \left[-c(s) - \delta B(s) + \theta(s) \cdot \sigma \kappa - \frac{1}{2} \gamma \bar{\Gamma}_\infty^{-1}(s) (\theta(s)^\top \sigma \sigma^\top \theta(s) + 2g_\infty y(s) \sigma_y \cdot \sigma^\top \theta(s)) \right] ds \Big\},
\end{aligned}$$

which in turn, thanks to (17), (65) and (66), equals

$$\begin{aligned}
& e^{-(\rho+\delta)s} \left\{ -(\rho + \delta) \tilde{v}(\mathcal{X}(s; \pi)) ds + (r + \delta) W(s; \pi) \partial_w \tilde{v}(\mathcal{X}(s; \pi)) ds + \right. \\
& + \langle \bar{\mathbf{Y}}, A^* \partial_{\bar{\mathbf{x}}} \tilde{v}(\mathcal{X}(s; \pi)) \rangle ds + y(s) \partial_w \tilde{v}(\mathcal{X}(s; \pi)) ds \\
& + \frac{1}{2} \partial_{x_0^{(1)} x_0^{(1)}}^2 \tilde{v}(\mathcal{X}(s; \pi)) |y(s) \sigma_y|^2 ds + f_\infty^\gamma \bar{\Gamma}_\infty^{-\gamma}(s) (g_\infty y(s) \sigma_y + \sigma^\top \theta(s)) \cdot dZ(s) \\
& + \partial_w \tilde{v}(\mathcal{X}(s; \pi)) [-c(s) - \delta B(s) + \theta(s) \cdot (\mu - r \mathbf{1})] + \partial_{wx^{(1)}}^2 \tilde{v}(\mathcal{X}(s; \pi)) y(s) \theta(s) \cdot \sigma \sigma_y ds \\
& \left. + \frac{1}{2} \partial_{ww}^2 \tilde{v}(\mathcal{X}(s; \pi)) \theta(s)^\top \sigma \sigma^\top \theta(s) ds \right\},
\end{aligned}$$

and this last term, by the definitions of \mathbb{H}_0 , \mathbb{H}_{\max} and \mathbb{H}_c , together with the derivatives of \tilde{v} as computed in the proof of Theorem 4.7, is equal to

$$\begin{aligned}
& e^{-(\rho+\delta)s} \left\{ -\mathbb{H}_{\max} \left(y(s), \partial_w \tilde{v}(\mathcal{X}(s; \pi)), \partial_{ww} \tilde{v}(\mathcal{X}(s; \pi)), \partial_{wx_0^{(1)}} \tilde{v}(\mathcal{X}(s; \pi)) \right) ds \right. \\
& + \mathbb{H}_c \left(y(s), \partial_w \tilde{v}(\mathcal{X}(s; \pi)), \partial_{ww} \tilde{v}(\mathcal{X}(s; \pi)), \partial_{wx_0^{(1)}} \tilde{v}(\mathcal{X}(s; \pi)), \pi \right) ds \\
& + \left[\frac{c(s)^{1-\gamma}}{1-\gamma} + \delta \frac{(kB)^{1-\gamma}}{1-\gamma} \right] ds + f_\infty^\gamma \bar{\Gamma}_\infty^{-\gamma}(s) (g_\infty y(s) \sigma_y + \sigma^\top \theta(s)) \cdot dZ(s) \Big\} \\
& = e^{-(\rho+\delta)s} \left\{ -\mathbb{H}_{\max} \left(y(s), \partial_w \tilde{v}(\mathcal{X}(s; \pi)), \partial_{ww} \tilde{v}(\mathcal{X}(s; \pi)), \partial_{wx_0^{(1)}} \tilde{v}(\mathcal{X}(s; \pi)) \right) ds \right. \\
& + \mathbb{H}_c \left(y(s), \partial_w \tilde{v}(\mathcal{X}(s; \pi)), \partial_{ww} \tilde{v}(\mathcal{X}(s; \pi)), \partial_{wx_0^{(1)}} \tilde{v}(\mathcal{X}(s; \pi)), \pi \right) ds \\
& + \left[\frac{c(s)^{1-\gamma}}{1-\gamma} + \delta \frac{(kB)^{1-\gamma}}{1-\gamma} \right] ds + \mathcal{M}(s) \cdot dZ(s) \Big\}
\end{aligned}$$

where we have set

$$\mathcal{M}(s) = f_\infty^\gamma \bar{\Gamma}_\infty^{-\gamma}(s) (g_\infty y(s) \sigma_y + \sigma^\top \theta(s)).$$

For $T \geq 0$ we now integrate on $[0, T \wedge \tau_N]$ and take expectation. The stochastic integral $\int_0^t \mathcal{M}(s) \cdot dZ(s)$ is a martingale. In fact, it is easy to verify that the stochastic integral is a local martingale w.r.t. the sequence of stopping times τ_N as defined in (72). We find

$$\begin{aligned}
& \mathbb{E} \left[e^{-(\rho+\delta)(T \wedge \tau_N)} \tilde{v}(\mathcal{X}(T \wedge \tau_N; \pi)) \right] - \tilde{v}(w, \bar{\mathbf{x}}) \\
& = -\mathbb{E} \int_0^{T \wedge \tau_N} e^{-(\rho+\delta)s} \left\{ \mathbb{H}_{\max} \left(y(s), \partial_w \tilde{v}(\mathcal{X}(s; \pi)), \partial_{ww} \tilde{v}(\mathcal{X}(s; \pi)), \partial_{wx_0^{(1)}} \tilde{v}(\mathcal{X}(s; \pi)) \right) \right. \\
& \quad - \mathbb{H}_c \left(y(s), \partial_w \tilde{v}(\mathcal{X}(s; \pi)), \partial_{ww} \tilde{v}(\mathcal{X}(s; \pi)), \partial_{wx_0^{(1)}} \tilde{v}(\mathcal{X}(s; \pi)), \pi \right) \Big\} ds \\
& \quad - \mathbb{E} \int_0^{T \wedge \tau_N} e^{-(\rho+\delta)s} \left[\frac{c(s)^{1-\gamma}}{1-\gamma} + \delta \frac{(kB)^{1-\gamma}}{1-\gamma} \right] ds.
\end{aligned}$$

Taking now the limit $N \rightarrow +\infty$ we can use the theorem on the first expectation and the monotone convergence theorem to the terms on the right hand side (as the integrands are nonnegative almost surely) to obtain

$$\tilde{v}(w, \bar{\mathbf{x}}) = \mathbb{E} \int_0^{T \wedge \tau_+} e^{-(\rho+\delta)s} \left[\frac{c(s)^{1-\gamma}}{1-\gamma} + \delta \frac{(kB)^{1-\gamma}}{1-\gamma} \right] ds$$

$$\begin{aligned}
& + \mathbb{E} \int_0^{T \wedge \tau_+} e^{-(\rho+\delta)s} \left\{ \mathbb{H}_{\max} \left(y(s), \partial_w \tilde{v}(\mathcal{X}(s; \pi)), \partial_{ww} \tilde{v}(\mathcal{X}(s; \pi)), \partial_{wx_0^{(1)}} \tilde{v}(\mathcal{X}(s; \pi)) \right) \right. \\
& \left. - \mathbb{H}_c \left(y(s), \partial_w \tilde{v}(\mathcal{X}(s; \pi)), \partial_{ww} \tilde{v}(\mathcal{X}(s; \pi)), \partial_{wx_0^{(1)}} \tilde{v}(\mathcal{X}(s; \pi)), \pi \right) \right\} ds \\
& + \mathbb{E} \left[e^{-(\rho+\delta)(T \wedge \tau_+)} \tilde{v}(\mathcal{X}(T \wedge \tau_N; \pi)) \right].
\end{aligned}$$

We finally take the limit $T \rightarrow +\infty$ and use the monotone convergence Theorem and Lemma 5.2 to obtain

$$\begin{aligned}
\tilde{v}(w, \bar{\mathbf{x}}) &= \mathbb{E} \int_0^{\tau_+} e^{-(\rho+\delta)s} \left[\frac{c(s)^{1-\gamma}}{1-\gamma} + \delta \frac{(kB)^{1-\gamma}}{1-\gamma} \right] ds \\
&+ \mathbb{E} \int_0^{\tau_+} e^{-(\rho+\delta)s} \left\{ \mathbb{H}_{\max} \left(y(s), \partial_w \tilde{v}(\mathcal{X}(s; \pi)), \partial_{ww} \tilde{v}(\mathcal{X}(s; \pi)), \partial_{wx_0^{(1)}} \tilde{v}(\mathcal{X}(s; \pi)) \right) \right. \\
&\left. - \mathbb{H}_c \left(y(s), \partial_w \tilde{v}(\mathcal{X}(s; \pi)), \partial_{ww} \tilde{v}(\mathcal{X}(s; \pi)), \partial_{wx_0^{(1)}} \tilde{v}(\mathcal{X}(s; \pi)), \pi \right) \right\} ds,
\end{aligned}$$

where the right hand side is finite because the left hand side is. To conclude just notice that by definition of τ_+ and Proposition 5.1 we have

$$\mathbb{E} \int_0^{\tau_+} e^{-(\rho+\delta)s} \left[\frac{c(s)^{1-\gamma}}{1-\gamma} + \delta \frac{(kB)^{1-\gamma}}{1-\gamma} \right] ds = J(w, \bar{\mathbf{x}}, \pi).$$

□

We introduce the *value function* $V : \mathcal{H} \rightarrow \bar{\mathbb{R}}$ defined as

$$V(w, \bar{\mathbf{x}}) := \sup_{\pi \in \Pi(w, \bar{\mathbf{x}})} J(w, \bar{\mathbf{x}}; \pi);$$

note that we allow at this point V to take the values $+\infty$ or $-\infty$.

coro:V_finite

Corollary 5.4. *The value function is finite on \mathcal{H}_+ and $V(w, \bar{\mathbf{x}}) \leq \tilde{v}(w, \bar{\mathbf{x}})$ for every $(w, \bar{\mathbf{x}}) \in \mathcal{H}_+$.*

Proof. By definition of \mathbb{H}_{\max} we have that the integrand in (71) is always nonnegative, therefore $\tilde{v}(w, \bar{\mathbf{x}}) \geq J(w, \bar{\mathbf{x}}, \pi)$ for every $(w, \bar{\mathbf{x}})$ and every $\pi \in \Pi(w, \bar{\mathbf{x}})$, so that the claim follows from the definition of the value function. □

m:newuniqueness

Remark 5.5. *Observe that, from the proof the above Proposition 5.3, we easily obtain that the fundamental identity (71) holds when, in place of \tilde{v} , we put any classical solution v of the Hamilton-Jacobi-Bellman equation (60) which satisfies (τ being the first exit time from \mathcal{H}_{++}),*

$$\lim_{T \rightarrow +\infty} \mathbb{E} \left[e^{-(\rho+\delta)(T \wedge \tau_+)} v(W_\pi(T \wedge \tau_+), X(T \wedge \tau)) \right] = 0. \tag{73}$$

eq:trasvnew

Hence the same observation made in [10, Remark 4.13] still hold in this case.

We actually aim to show that $V = \tilde{v}$ on \mathcal{H}_+ . In doing so we will also provide optimal feedback strategies.

STRATEGY_INF_RET

Definition 5.6. *Fix $(w, \bar{\mathbf{x}}) \in \mathcal{H}_+$. A strategy $\tilde{\pi} := (\tilde{c}, \tilde{B}, \tilde{\theta})$ is called an optimal strategy if $\tilde{\pi} \in \Pi(w, \bar{\mathbf{x}})$ and*

$$V(w, \bar{\mathbf{x}}) = J(w, \bar{\mathbf{x}}, \tilde{\pi}), \tag{74}$$

that is, the supremum in Problem 4.3 is achieved at $\tilde{\pi}$.

Definition 5.7. We say that a function $(\boldsymbol{\theta}, \mathbf{c}, \mathbf{B}) : \mathcal{H}_+ \longrightarrow \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{R}_+$ is an optimal feedback map if for every $(w, \bar{\mathbf{x}}) \in \mathcal{H}_+$ the closed loop equation

$$\begin{cases} dW(t) = [(r + \delta)W(t) + \boldsymbol{\theta}(W(t), \bar{\mathbf{Y}}(t)) \cdot (\mu - r\mathbf{1}) + P_{1,0}\bar{\mathbf{Y}}(t) - \mathbf{c}(W(t), \bar{\mathbf{Y}}(t)) \\ \quad - \delta \mathbf{B}(W(t), \bar{\mathbf{Y}}(t))] dt + \boldsymbol{\theta}(W(t), \bar{\mathbf{Y}}(t)) \cdot \sigma dZ(t), \\ d\bar{\mathbf{Y}}(t) = A\bar{\mathbf{Y}}(t)dt + F(\bar{\mathbf{Y}}(t)) dZ(t), \\ (W(0), \bar{\mathbf{Y}}(0)) = (w, \bar{\mathbf{x}}) \end{cases}$$

has a unique solution $(W^*, \bar{\mathbf{Y}}) =: \mathcal{X}^*$, and the associated control strategy $(\tilde{c}, \tilde{B}, \tilde{\theta})$

$$\begin{cases} \tilde{c}(t) &:= \mathbf{c}(W^*(t), \bar{\mathbf{Y}}(t)), \\ \tilde{B}(t) &:= \mathbf{B}(W^*(t), \bar{\mathbf{Y}}(t)), \\ \tilde{\theta}(t) &:= \boldsymbol{\theta}(W^*(t), \bar{\mathbf{Y}}(t)) \end{cases} \quad (75)$$

is an optimal strategy.

In the Hamilton-Jacobi-Bellman equation (60), the role of the variables u, Q_{11}, Q_{12} in the definition of \mathbb{H} is played by, respectively, $\partial_w v(w, \bar{\mathbf{x}}), \partial_{ww}^2 v(w, \bar{\mathbf{x}}), \partial_{wx_0}^2 v(w, \bar{\mathbf{x}})$, where v is the unknown. Thus, recalling what we called case (i) after we introduced the Hamiltonian, it makes sense to define the maps

$$\begin{cases} \mathbf{c}_f(w, \bar{\mathbf{x}}) := f_\infty^{-1} \Gamma_\infty(w, \bar{\mathbf{x}}), \\ \mathbf{B}_f(w, \bar{\mathbf{x}}) := k^{-b} f_\infty^{-1} \Gamma_\infty(w, \bar{\mathbf{x}}), \\ \boldsymbol{\theta}_f(w, \bar{\mathbf{x}}) := (\sigma \sigma^\top)^{-1} (\mu - r\mathbf{1}) \frac{\Gamma_\infty(w, \bar{\mathbf{x}})}{\gamma} - (\sigma \sigma^\top)^{-1} \sigma \sigma_y g_\infty x_0^{(1)} \\ \quad = \frac{1}{\gamma} \Gamma_\infty(w, \bar{\mathbf{x}}) (\sigma^\top)^{-1} \kappa - g_\infty x_0^{(1)} (\sigma^\top)^{-1} \sigma_y. \end{cases} \quad (76) \quad \text{EQ_DEF_FEEDBACK_MAP}$$

We want to prove that this is an optimal feedback map.

For given $(w, \bar{\mathbf{x}}) \in \mathcal{H}_+$, denote with $W_f^*(t)$ the unique solution of the associated closed loop equation

$$\begin{cases} dW(t) = [(r + \delta)W(t) + \boldsymbol{\theta}_f(W(t), \bar{\mathbf{x}}(t)) \cdot (\mu - r\mathbf{1}) + P_{1,0}\bar{\mathbf{Y}}(t) - \mathbf{c}_f(W(t), \bar{\mathbf{x}}(t)) \\ \quad - \delta \mathbf{B}_f(W(t), \bar{\mathbf{x}}(t))] dt + \boldsymbol{\theta}_f(t) \cdot \sigma dZ(t), \\ d\bar{\mathbf{Y}}(t) = A\bar{\mathbf{Y}}(t)dt + F(\bar{\mathbf{Y}}(t)) dZ(t), \\ (W(0), \bar{\mathbf{Y}}(0)) = (w, \bar{\mathbf{x}}) \end{cases} \quad (77) \quad \text{CLOSED_LOOP_EQUATION}$$

and set

$$\Gamma_\infty^*(t) = \Gamma_\infty(W_f^*(t), \bar{\mathbf{Y}}(t)) = W_f^*(t) + \langle \bar{\mathbf{I}}_\infty, \bar{\mathbf{Y}}(t) \rangle_{\mathcal{M}_2^2}. \quad (78) \quad \text{DEF_GAMMA_INFTY_STRAT}$$

The control strategy associated with (76) is then

$$\begin{cases} \tilde{c}_f(t) := \mathbf{c}_f(W_f^*(t), \bar{\mathbf{Y}}(t)) = f_\infty^{-1} \Gamma_\infty^*(t), \\ \tilde{B}_f(t) := \mathbf{B}_f(W_f^*(t), \bar{\mathbf{Y}}(t)) = k^{-b} f_\infty^{-1} \Gamma_\infty^*(t), \\ \tilde{\theta}_f(t) := \boldsymbol{\theta}_f(W_f^*(t), \bar{\mathbf{Y}}(t)) = \frac{\Gamma_\infty^*(t)}{\gamma} (\sigma^\top)^{-1} \kappa - g_\infty P_{1,0} \bar{\mathbf{Y}}(t) (\sigma^\top)^{-1} \sigma_y. \end{cases} \quad (79) \quad \text{EQ_FEEDBACK_STRATEGY}$$

We first show that this strategy is admissible.

Lemma 5.8. Let $(w, \bar{\mathbf{x}}) \in \mathcal{H}_+$. The process Γ_∞^* defined in (78) is a stochastic exponential satisfying equation

$$d\Gamma_\infty^*(t) = \Gamma_\infty^*(t) \left(r + \delta + \frac{1}{\gamma} |\kappa|^2 - f_\infty^{-1} (1 + \delta k^{-b}) \right) dt + \frac{\Gamma_\infty^*(t)}{\gamma} \kappa \cdot dZ(t). \quad (80) \quad \text{DYN_H^*_INFTY}$$

Proof. Substituting (79) into the first equation of (77) we get

$$dW_f^*(t) = \left\{ W_f^*(t)(r + \delta) + \Gamma_\infty^*(t) \left[\frac{|\kappa|^2}{\gamma} - f_\infty^{-1}(1 + \delta k^{-b}) \right] + P_{1,0} \bar{\mathbf{Y}}(t) - g_\infty P_{1,0} \bar{\mathbf{Y}}(t) \sigma_y \cdot \kappa \right\} dt \quad (81)$$

$$+ \left\{ \frac{\Gamma_\infty^*(t)}{\gamma} \kappa - g_\infty P_{1,0} \bar{\mathbf{Y}}(t) \sigma_y \right\} \cdot dZ(t). \quad (82)$$

Since

$$d\Gamma_\infty^*(t) = dW_f^*(t) + d\langle \bar{\mathbf{I}}_\infty, \bar{\mathbf{Y}}(t) \rangle_{\mathcal{M}_2^2},$$

using (67) and proceeding as in the computation leading to (69) we find the claim. \square

We now state and prove our main result in the case $\gamma \in (0, 1)$.

THEOREM_INF_RET

Theorem 5.9. *We have $V = \tilde{v}$ in \mathcal{H}_+ . Moreover the function $(\mathbf{c}_f, \mathbf{B}_f, \boldsymbol{\theta}_f)$ defined in (76) is an optimal feedback map. Finally, for every $(w, \bar{\mathbf{x}}) \in \mathcal{H}_+$ the strategy $\tilde{\pi}_f := (\tilde{c}_f, \tilde{B}_f, \tilde{\theta}_f)$ is the unique optimal strategy.*

Proof. We first take $(w, \bar{\mathbf{x}}) \in \partial\mathcal{H}_+ = \{\Gamma_\infty = 0\}$. We thus have, by equation (80), that almost surely $\Gamma_\infty^*(t) = 0$ for every $t \geq 0$. This in turn implies, by (79), that

$$\tilde{c}_f \equiv 0, \quad \tilde{B}_f \equiv 0, \quad \tilde{\theta}_f \equiv -g_\infty P_{1,0} \bar{\mathbf{Y}}(t) (\sigma^\top)^{-1} \sigma_y.$$

It follows from Proposition 5.1 that this is the only admissible strategy, therefore it must be optimal.

Now we consider $(w, \bar{\mathbf{x}}) \in \mathcal{H}_{++}$. First we observe that $\tilde{\pi}_f$ is an admissible strategy; indeed by Lemma 5.8 $\Gamma_\infty^*(\cdot)$ is a stochastic exponential, therefore almost surely strictly positive for any strictly positive initial condition $\Gamma_\infty^*(0) = \Gamma_\infty(w, x)$, and this implies that the constraint (43) is always satisfied, so that the strategy is admissible provided that \tilde{c}_f and \tilde{B}_f are nonnegative. This last fact follows however immediately from (79).

Concerning optimality we observe that, as recalled above, the feedback map is obtained taking the maximum points of the Hamiltonian given in (76) and substituting the derivatives $\partial_w \tilde{v}, \partial_{ww}^2 \tilde{v}, \partial_{wx_0^{(1)}} \tilde{v}$ in place of u, Q_{11}, Q_{12} , respectively. This implies that, substituting $\tilde{\pi}_f$ in the fundamental identity (71), we obtain

$$\tilde{v}(w, \bar{\mathbf{x}}) = J(w, \bar{\mathbf{x}}; \tilde{\pi}_f).$$

Hence, by Corollary 5.4 and the definition of the value function,

$$V(w, \bar{\mathbf{x}}) \leq \tilde{v}(w, \bar{\mathbf{x}}) = J(w, \bar{\mathbf{x}}; \tilde{\pi}_f) \leq V(w, \bar{\mathbf{x}}),$$

which gives $V(w, \bar{\mathbf{x}}) = J(w, \bar{\mathbf{x}}; \tilde{\pi}_f)$, namely, optimality of $\tilde{\pi}_f$.

We now prove uniqueness. When $(w, \bar{\mathbf{x}}) \in \partial\mathcal{H}_+$ the claim again easily follows from Proposition 5.1. When instead $(w, \bar{\mathbf{x}}) \in \mathcal{H}_{++}$, uniqueness follows from the fundamental identity (71). Indeed, since $\tilde{v} = V$, if a given strategy π is optimal at $(w, \bar{\mathbf{x}}) \in \mathcal{H}_{++}$ it must satisfy $\tilde{v}(w, \bar{\mathbf{x}}) = J(w, \bar{\mathbf{x}}; \pi)$, which implies, substituting in (71), that the integral in (71) is zero. This implies that, on $[0, \tau_+]$ we have $\pi = \tilde{\pi}_f$, $dt \otimes \mathbb{P}$ -almost everywhere. This is enough for uniqueness as, for $t > \tau_+$, we still must have $\pi = \tilde{\pi}_f$ $dt \otimes \mathbb{P}$ -almost everywhere, due to Proposition 5.1. \square

rm:gamma>1

Remark 5.10. *The result analogous to Theorem 5.9 for the case $\gamma > 1$ can be obtained using the same approach proposed in [10, Subsection 4.6]. We do not present this here for brevity.*

6 Back to the original problem

sec:back

We now explain what our main result (Theorem 5.9) on the general Problem 4.3 says on our original problem of Section 2.

First of all, from Remark 2.4 and Proposition 4.1 we observe that our original problem can be seen as a “subproblem” of the general problem in the sense that, when the initial conditions for y and e are as in (13) the optimal strategies of our general problem is also the optimal strategies of the original problems with the initial data of y in (13).

From Theorem 5.9 and Remark 5.10 we get the following result.

th:mainlast

Theorem 6.1. *Consider our original problem of Section 2 with initial datum $(w, x) \in \mathbb{R} \times \mathcal{M}_2$. The initial total capital is $\Gamma_\infty(w, (x, x))$ and is given by*

$$\Gamma_\infty(w, x, x) = w + g_\infty(1 - i_\infty) \left[x_0 + \int_{-d}^0 G(s)x_1(s) ds \right], \quad (83)$$

eq:GammaInfyLast

where (g_∞, i_∞, G) are given in (39)-(40) (recall that, by (40) we have $h_\infty(\cdot) = g_\infty G(\cdot)$). Hence the value function V of our original problem of Section 2 is given by

$$V(w, x_0, x_1) = \frac{f_\infty^\gamma \left(w + g_\infty(1 - i_\infty) \left[x_0 + \int_{-d}^0 G(s)x_1(s) ds \right] \right)^{1-\gamma}}{1 - \gamma}, \quad (84)$$

eq:vflast

where f_∞ is defined in (62). Moreover for every $(w, x) \in \mathbb{R} \times \mathcal{M}_2$ there exists a unique optimal strategy $\pi^* = (c^*, B^*, \theta^*) \in \Pi$ starting at (w, x) . Such strategy can be represented as follows. Denote total wealth by

$$\Gamma_\infty^*(t) := W^*(t) + g_\infty(y(t) - i_\infty \mathbb{E}[y(t)]) + g_\infty \int_{-d}^0 G(s) (y(t+s) - i_\infty \mathbb{E}[y(t+s)]) ds, \quad (85)$$

Gamma-infy2

where $W^*(\cdot)$ is the solution of equation (77) with initial datum w and control π^* , whereas $y(\cdot)$ is the solution of (10b) with datum $x = (x_0, x_1) \in \mathcal{M}_2$. Then, Γ_∞^* has dynamics

$$d\Gamma_\infty^*(t) = \Gamma_\infty^*(t) \left(r + \delta + \frac{|\kappa|^2}{\gamma} - f_\infty^{-1}(1 + \delta k^{-b}) \right) dt + \frac{\Gamma_\infty^*(t)}{\gamma} \kappa \cdot dZ(t), \quad (86)$$

DYN_GAMMA*_PB1

(here f_∞ is defined in (62), and κ in (17)) and the optimal strategy triplet $\pi^* = (c^*, B^*, \theta^*)$ for our original problem of Section 2 is given by

$$\begin{aligned} c^*(t) &:= f_\infty^{-1} \Gamma_\infty^*(t), \\ B^*(t) &:= k^{-b} f_\infty^{-1} \Gamma_\infty^*(t), \\ \theta^*(t) &:= \frac{\Gamma_\infty^*(t)}{\gamma} (\sigma^\top)^{-1} \kappa - g_\infty y(t) (\sigma^\top)^{-1} \sigma_y. \end{aligned} \quad (87)$$

OPTIMAL_STRATEGIES

This explicit result provides a strong basis to analyze the practical consequences of the model and fit them with real data. This is out of the scope of the present paper and is the subject of our current research. Nevertheless we think it is interesting to give some ideas on the comparison between the optimal solution in this case and the one of [10] (where the path-dependency term ϕ is present but the mean reversion speed ϵ disappears) and even [19] (where both path-dependency and mean reversion disappear).

First of all we observe that, due to the linear character of the infinite dimensional Merton's model, the dynamics of the optimal total capital Γ_∞^* is the same as in [10] and [19] as it does not depend on ϵ , nor on ϕ . However, its initial value is different since here $\Gamma_\infty^*(0) = \Gamma_\infty(w, x, x)$ which is now different as it contains the part coming from the mean reverting term. Moreover, as in [10] and [19], the dynamics of the optimal consumption and bequest are constant fractions (independent of the mean reversion speed) of the optimal total capital Γ_∞^* . Hence, the differences with the solutions of [10] and [19], concerning the optimal total capital, the value function, the

optimal consumption and the optimal bequest, come only from the initial total capital $\Gamma_\infty(w, x, x)$ defined in (83).

From (83) we immediately see that the dependence of $\Gamma_\infty(w, x, x)$ on ϵ arises only through the term $g_\infty(1 - i_\infty)$. Moreover from (39) we easily see (with little computations on derivatives) that g_∞ and i_∞ are both increasing in ϵ while $g_\infty(1 - i_\infty)$ is decreasing. Hence, the presence of a mean reverting term with $\epsilon < 0$ (respectively $\epsilon > 0$) makes such variables to increase (decrease) their value. This fact substantially says that the presence of an aggregation force among incomes increases the total capital, the consumption and the welfare of the population. The opposite happens when there is a divergence force of the incomes in the population. See on this, e.g., [48] and [41]. Note that the optimal feedback rule chosen by the agent in consumption and bequest does not change with ϵ .

On the other hand, things are different if one looks at the optimal trading strategy. This is the sum of a constant fraction of the optimal total capital Γ_∞^* , which behaves as explained above, and of a so-called negative hedging demand term. Here such term is $-g_\infty y(t)(\sigma^\top)^{-1}\sigma_y$, so, here, differently from the feedback rules in c and B , the feedback rule of this part of θ depends on ϵ through the presence of g_∞ . This is increasing in ϵ (see (39)). Hence, when $\epsilon < 0$ (resp. $\epsilon > 0$) the hedging demand rule is a smaller (resp. bigger) fraction of y than the one in the case $\epsilon = 0$. This can be interpreted as the fact that the aggregation force of the incomes decreases the need of the hedging demand by the agent. Viceversa for a dispersion force.

Finally, since we have now two interacting forces (persistence and benchmark) in our model it would be interesting to understand which one is dominant and when. However this seems to us a nontrivial question which needs a much deeper analysis, also based on numerical approximations and real data. We think that the answer to this research question, together with a deeper financial analysis of our model deserves a stand alone paper that we leave for the near future.

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Appendix

Proof of Proposition 3.3. (i) Consider the equivalent formulation of system (24) given by (29) with $t_0 = 0$ and introduce the 2×2 -matrix-valued finite measure on $[-d, 0]$

$$a(d\lambda) = C_0 \delta_0(d\lambda) + \text{Id}_{2 \times 2} \phi(\lambda) d\lambda. \quad (88)$$

The operator A_0 can be then written in the form

$$A_0(\mathbf{x}_0, \mathbf{x}_1) = \left(\int_{-d}^0 \mathbf{x}_1(\lambda) a(d\lambda), \frac{d\mathbf{x}_1}{ds} \right), \quad (89)$$

eq_gen

therefore it generates a strongly continuous semigroup by [16, Proposition A.27].

- (ii) The compactness property of $S(t)$ for t big enough is proven for example in [27, Chapter 7, Lemma 1.2].
- (iii) Existence and uniqueness of a weak solution given by (33) for deterministic $\bar{\mathbf{m}}$ is classical (see [16, Proposition A.5] and the related references therein); the fact that it is actually a strong solution if $\bar{\mathbf{m}} \in \mathcal{D}(A_0)$ is proved for example in [16, Proposition A.7]; the generalization to random $\bar{\mathbf{m}}$ is immediate. Property (34) then follows from uniqueness of the solution.
- (iv) If $\mathbf{n}(t_0; \cdot)$ is the unique solution to (29) then the \mathcal{M}_2^2 -valued process $(\mathbf{n}(t_0; t), \mathbf{n}(t_0; t + \cdot))_{t \geq t_0}$ solves (32) by [3, Part II, Chapter 4, Theorem 4.3]. Since also the latter has a unique solution, its first component must in fact be the solution to (29).
- (v) This is an immediate consequence of (iv).

□

Proof of Lemma 3.5. If $\lambda \in \mathbb{R} \cap R(A_0)$ then both $K_1(\lambda)$ and $K_2(\lambda)$ are nonzero, by Lemma 3.4. To compute $R(\lambda, A_0)$, we will consider for a fixed $((\frac{m_0}{e_0}), (\frac{m_1}{e_1})) \in \mathcal{M}_2^2$ the equation

$$(\lambda - A_0) \left(\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}, \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} \right) = \left(\begin{pmatrix} m_0 \\ e_0 \end{pmatrix}, \begin{pmatrix} m_1 \\ e_1 \end{pmatrix} \right), \quad (90)$$

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in the unknown $((\frac{u_0}{v_0}), (\frac{u_1}{v_1})) \in \mathcal{D}(A_0)$, that by definition of A_0 is equivalent to

$$\left\{ \begin{array}{l} \lambda u_0 - (\epsilon + \mu_y - \sigma_y \cdot \kappa) u_0 + \epsilon v_0 - \int_{-d}^0 u_1(\tau) \phi(\tau) d\tau = m_0 \\ \lambda v_0 - \mu_y v_0 - \int_{-d}^0 v_1(\tau) \phi(\tau) d\tau = e_0 \\ \lambda u_1 - \frac{du_1}{ds} = m_1 \\ \lambda v_1 - \frac{dv_1}{ds} = e_1. \end{array} \right. \quad (91) \quad \boxed{\text{system3}}$$

Then

$$u_1(s) = e^{\lambda s} u_0 + \int_s^0 e^{-\lambda(s_1-s)} m_1(s_1) ds_1, \quad s \in [-d, 0],$$

and

$$v_1(s) = e^{\lambda s} v_0 + \int_s^0 e^{-\lambda(s_1-s)} e_1(s_1) ds_1, \quad s \in [-d, 0].$$

Therefore v_0 is determined by the equation

$$(\lambda - \mu_y) v_0 = \left[e_0 + \int_{-d}^0 \left(e^{\lambda \tau} v_0 + \int_{\tau}^0 e^{-\lambda(s-\tau)} e_1(s) ds \right) \phi(\tau) d\tau \right]$$

yielding

$$K_2(\lambda) v_0 = e_0 + \int_{-d}^0 \int_{-d}^s e^{-\lambda(s-\tau)} \phi(\tau) d\tau e_1(s) ds.$$

If we now substitute the expressions for v_0 and u_1 in the first expression of (91), we obtain (35).

□

Proof of Proposition 5.1. Under the probability measure $\tilde{\mathbb{P}}_T$ (defined in (21)), the process $\bar{\Gamma}_\infty$ satisfies, on $[0, T]$,

$$d\bar{\Gamma}_\infty(t) = [(r + \delta)\bar{\Gamma}_\infty(t) - c(t) - \delta B(t)] dt + [\sigma^\top \theta(t) + g_\infty P_{1,0} \bar{\mathbf{Y}}(t) \sigma_y] \cdot d\tilde{Z}(t).$$

Thus we obtain, under $\tilde{\mathbb{P}}_T$, for every $0 \leq t \leq T$,

$$\bar{\Gamma}_\infty(t) = e^{(r+\delta)t} \left[\bar{\Gamma}_\infty(0) - \int_0^t e^{-(r+\delta)s} (c(s) + \delta B(s)) ds + \int_0^t e^{-(r+\delta)s} [\sigma^\top \theta(s) + g_\infty P_{1,0} \bar{\mathbf{x}}(s) \sigma_y] \cdot d\tilde{Z}(s) \right]. \quad (92) \quad \boxed{\text{eq:Gammatau}}$$

Setting $\bar{\Gamma}_\infty^0(t) := e^{-(r+\delta)t} \bar{\Gamma}_\infty(t)$ the above (92) is rewritten as

$$\bar{\Gamma}_\infty^0(t) = \bar{\Gamma}_\infty^0(0) - \int_0^t e^{-(r+\delta)s} (c(s) + \delta B(s)) ds + \int_0^t e^{-(r+\delta)s} [\sigma^\top \theta(s) + g_\infty P_{1,0} \bar{\mathbf{x}}(s) \sigma_y] \cdot d\tilde{Z}(s), \quad (93) \quad \boxed{\text{eq:Gamma0tau}}$$

which implies that the process $\bar{\Gamma}_\infty^0(t)$ is a supermartingale under $\tilde{\mathbb{P}}_T$ on $[0, T]$. By the optional sampling theorem we then have, for every couple of stopping times $0 \leq \tau_1 \leq \tau_2 \leq T$, denoting by $\tilde{\mathbb{E}}_T$ the expectation under $\tilde{\mathbb{P}}_T$,

$$\tilde{\mathbb{E}}_T [\bar{\Gamma}_\infty^0(\tau_2) | \mathcal{F}_{\tau_1}] \leq \bar{\Gamma}_\infty^0(\tau_1), \quad \tilde{\mathbb{P}}_T\text{-a.s.} \quad (94) \quad \boxed{\text{eq:OST}}$$

The admissibility of the strategy π , and the fact that \mathbb{P} and $\tilde{\mathbb{P}}_T$ are equivalent on \mathcal{F}_T , implies that $\bar{\Gamma}_\infty^0(\tau_2) \geq 0$, $\tilde{\mathbb{P}}_T$ -a.s., hence also

$$\tilde{\mathbb{E}}_T [\bar{\Gamma}_\infty^0(\tau_2) | \mathcal{F}_{\tau_1}] \geq 0, \quad \tilde{\mathbb{P}}_T\text{-a.s.}$$

Now let $\tau_1 := \tau_+ \wedge T$ where τ_+ is defined in (70) (which is taken to be identically 0 when $\Gamma_\infty(w, \bar{\mathbf{x}}) = 0$). Then $\bar{\Gamma}_\infty^0(\tau_1) = 0$ on $\{\tau_+ < T\}$, and from (94) we get

$$\mathbf{1}_{\{\tau_+ < T\}} \tilde{\mathbb{E}}_T \left[\bar{\Gamma}_\infty^0(\tau_2) | \mathcal{F}_{\tau_1} \right] = \tilde{\mathbb{E}}_T \left[\bar{\Gamma}_\infty^0(\tau_2) \mathbf{1}_{\{\tau_+ < T\}} | \mathcal{F}_{\tau_1} \right] = 0, \quad \tilde{\mathbb{P}}_{T\text{-a.s.}}$$

and, consequently,

$$\bar{\Gamma}_\infty^0(\tau_2) \mathbf{1}_{\{\tau_+ < T\}} = 0, \quad \tilde{\mathbb{P}}_{T\text{-a.s.}} \quad \text{eq:GammaInd0}$$

We now use (93) to compute $\bar{\Gamma}_\infty^0(\tau_2) - \bar{\Gamma}_\infty^0(\tau_1)$ getting

$$\bar{\Gamma}_\infty^0(\tau_2) - \bar{\Gamma}_\infty^0(\tau_1) = - \int_{\tau_1}^{\tau_2} e^{-(r+\delta)s} (c(s) + \delta B(s)) ds + \int_{\tau_1}^{\tau_2} e^{-(r+\delta)s} [\sigma^\top \theta(s) + g_\infty P_{1,0} \bar{\mathbf{Y}}(s) \sigma_y] \cdot d\tilde{Z}(s). \quad (96) \quad \text{eq:Gamma0taubis}$$

Again using the optional sampling theorem we get

$$\tilde{\mathbb{E}}_T \left[\bar{\Gamma}_\infty^0(\tau_2) | \mathcal{F}_{\tau_1} \right] - \bar{\Gamma}_\infty^0(\tau_1) = - \tilde{\mathbb{E}}_T \left[\int_{\tau_1}^{\tau_2} e^{-(r+\delta)s} (c(s) + \delta B(s)) ds | \mathcal{F}_{\tau_1} \right], \quad \tilde{\mathbb{P}}_{T\text{-a.s.}}$$

Hence, taking $\tau_2 \equiv T$

$$0 \leq \mathbf{1}_{\{\tau_+ < T\}} \tilde{\mathbb{E}}_T \left[\bar{\Gamma}_\infty^0(T) | \mathcal{F}_{\tau_1} \right] = - \tilde{\mathbb{E}}_T \left[\int_0^T \mathbf{1}_{\{\tau_+ < s\}} e^{-(r+\delta)s} (c(s) + \delta B(s)) ds | \mathcal{F}_{\tau_1} \right], \quad \tilde{\mathbb{P}}_{T\text{-a.s.}}$$

which implies

$$\mathbf{1}_{\{\tau_+ < s\}}(\omega) c(s, \omega) = \mathbf{1}_{\{\tau_+ < s\}}(\omega) B(s, \omega) = 0, \quad ds \otimes \tilde{\mathbb{P}}_{T\text{-a.e.}} \text{ in } [0, T] \times \Omega. \quad (97) \quad \text{eq:cB0PT}$$

We now multiply (96) by $\mathbf{1}_{\{\tau_+ < T\}}$ and we use (95) and (97) to get

$$0 = \int_0^{\tau_2} e^{-(r+\delta)s} \mathbf{1}_{\{\tau_+ < s\}} [\sigma^\top \theta(s) + g_\infty P_{1,0} \bar{\mathbf{Y}}(s) \sigma_y] \cdot d\tilde{Z}(s), \quad \tilde{\mathbb{P}}_{T\text{-a.s.}}$$

Since the integral of the right hand side is a martingale the above implies that

$$\mathbf{1}_{\{\tau_+ < s\}} \sigma^\top \theta(s) + g_\infty P_{1,0} \bar{\mathbf{Y}}(s) \sigma_y = 0, \quad ds \otimes \tilde{\mathbb{P}}_{T\text{-a.e.}} \text{ in } [0, T] \times \Omega. \quad (98) \quad \text{thetaOPT}$$

Using (95), (97), (98), the fact that \mathbb{P} and $\tilde{\mathbb{P}}_T$ are equivalent on \mathcal{F}_T and the arbitrariness of T we eventually get the claim. \square

Proof of Lemma 5.2. Since almost surely for $t < \tau_+$ we have $\Gamma_\infty(\mathcal{X}(t; \pi)) > 0$, we can apply the Ito formula to the process

$$e^{(\gamma-1)\left(r+\delta+\frac{|\kappa|^2}{2\gamma}\right)t} f_\infty^\gamma \frac{\bar{\Gamma}_\infty^{1-\gamma}(t)}{1-\gamma} \quad (99) \quad \text{supermartingale}$$

obtaining, by (69),

$$\begin{aligned} d \left[e^{(\gamma-1)\left(r+\delta+\frac{|\kappa|^2}{2\gamma}\right)t} f_\infty^\gamma \frac{\bar{\Gamma}_\infty^{1-\gamma}(t)}{1-\gamma} \right] &= e^{(\gamma-1)\left(r+\delta+\frac{|\kappa|^2}{2\gamma}\right)t} f_\infty^\gamma \left\{ -\bar{\Gamma}_\infty^{-\gamma}(t) (c(t) + \delta B(t)) \right. \\ &\quad \left. - \frac{1}{2\gamma} \bar{\Gamma}_\infty^{-\gamma-1}(t) |\bar{\Gamma}_\infty(t) \kappa - \gamma (\sigma^\top \theta(t) + g_\infty y(t) \sigma_y)|^2 \right\} dt \\ &\quad + e^{(\gamma-1)\left(r+\delta+\frac{|\kappa|^2}{2\gamma}\right)t} f_\infty^\gamma \bar{\Gamma}_\infty^{-\gamma}(t) (\sigma^\top \theta(t) + g_\infty y(t) \sigma_y) \cdot dZ(t). \end{aligned}$$

The drift term in the equation above is negative, because $t < \tau_+$ and both c and B take values in \mathbb{R}_+ , thus the process given by (99) is a local \mathbb{F} -supermartingale up to the exit time τ_+ .

Set now

$$\tau_N := \inf \left\{ t \geq 0 : \bar{\Gamma}_\infty(t) \leq \frac{1}{N} \right\}.$$

Taking N sufficiently large we have that $\tau_N > 0$ almost surely and both the drift and the diffusion coefficients above are integrable, therefore the process

$$e^{(\gamma-1)\left(r+\delta+\frac{|\kappa|^2}{2\gamma}\right)(T\wedge\tau_N)} f_\infty^\gamma \frac{\bar{\Gamma}_\infty^{1-\gamma}(T\wedge\tau_N)}{1-\gamma}$$

is in L^1 , hence

$$\mathbb{E} \left[e^{(\gamma-1)\left(r+\delta+\frac{|\kappa|^2}{2\gamma}\right)(T\wedge\tau_N)} f_\infty^\gamma \frac{\bar{\Gamma}_\infty^{1-\gamma}(T\wedge\tau_N)}{1-\gamma} \right] \leq \frac{f_\infty^\gamma}{1-\gamma} \mathbb{E} [\bar{\Gamma}_\infty^{1-\gamma}(0)] = \tilde{v}(w, \bar{\mathbf{x}}). \quad (100) \quad \boxed{\text{supermartingale_E}}$$

Since $\tau_N \uparrow \tau_+$ as $N \rightarrow +\infty$ and the quantity inside the expectation in the left hand side of (100) is nonnegative, the first claim follows from Fatou's lemma.

To prove the second claim notice first that almost surely $\bar{\Gamma}_\infty(\tau_+) = 0$ because almost surely $t \mapsto \bar{\Gamma}_\infty(t)$ is continuous; this implies

$$\begin{aligned} \mathbb{E} \left[e^{-(\rho+\delta)(T\wedge\tau_+)} \tilde{v}(\mathcal{X}(T\wedge\tau_+; \pi)) \right] &= e^{-(\rho+\delta)T} \mathbb{E} [\mathbf{1}_{(\tau_+, +\infty)}(T) \tilde{v}(\mathcal{X}(T; \pi))] \\ &= e^{-\left(\rho+\delta+(\gamma-1)\left(r+\delta+\frac{|\kappa|^2}{2\gamma}\right)\right)T} \mathbb{E} \left[\mathbf{1}_{(\tau_+, +\infty)}(T) e^{(\gamma-1)\left(r+\delta+\frac{|\kappa|^2}{2\gamma}\right)T} \tilde{v}(\mathcal{X}(T; \pi)) \right] \\ &\leq e^{-\left(\rho+\delta+(\gamma-1)\left(r+\delta+\frac{|\kappa|^2}{2\gamma}\right)\right)T} \tilde{v}(w, \bar{\mathbf{x}}) \end{aligned}$$

and this last quantity converges to 0 as $T \rightarrow +\infty$ thanks to Assumption 2.7. \square