



On secant defective varieties, in particular of dimension 4

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Abstract

In this paper we prove some general results on secant defective varieties. Then we focus on the 4-dimensional case and we give the full classification of secant defective 4-folds. This paper has been inspired by classical work by G. Scorza.

Keywords Secant varieties · Defective varieties

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Introduction

Let $X \subset \mathbb{P}^r$ be an n -dimensional variety, also called an n -fold (in this paper by *variety* we mean an irreducible, projective, reduced scheme). The variety X is said to be *secant defective*, or simply defective, if its *secant variety* $S(X)$ has dimension $s(X)$ smaller than the expected dimension, which is $\sigma(X) = \min\{r, 2n + 1\}$. The study and classification of secant defective varieties is a classical subject which goes back to the Italian school of algebraic geometry and received in modern times a lot of attention by various authors, too many to be mentioned here. Curves are never defective. The classification of defective surfaces is classical (for a modern account, see [7]): a surface is defective if and only if it is either a cone or the Veronese surface of conics in \mathbb{P}^5 . The classification of secant defective threefolds goes back to G. Scorza in [27], and a modern proof has been given in [9] (see also [10]). The case of 4-folds has been again considered by Scorza in the memoir [28], in which he outlines the classification of secant defective 4-folds, with the use of beautiful geometric ideas. The present paper is devoted to the full classification of secant defective 4-folds, filling

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up Scorza's gaps and simplifying, in several points, his arguments, though often we follow his line of thinking. Our tools, as for Scorza, consist in delicate projective geometric analyses of the various cases which defective 4-folds fall in. These cases correspond to the different values that certain invariants of the 4-fold X acquire. The main invariants in question are basically $f(X) = 9 - s(X)$ and $\gamma(X)$, which is the dimension of the contact locus with X of a general bitangent space to X , and one has $3 \geq \gamma(X) \geq f(X) \geq 1$. The difference $4 - \gamma(X)$ is dubbed by Scorza the *species* of X and we adopt here his terminology. Scorza's arguments are often longwinded and, in some parts, not completely clear also due to a different and ancient terminology. Moreover, he regularly uses results, whose proofs available at that time are considered nowadays unsatisfactory (e.g. the formulas appearing in §8.2 have been recently revised in [23] while Scorza quotes [5]). It would be too long to precisely indicate here all the obscure parts in [28] and all the missing cases in the classification. Scorza himself states in the Introduction of [28] that his analysis is not complete for one class of fourfolds (those of second species). Our treatment is based, among other things, on deep consequences of Terracini Lemma not available to Scorza, who indeed considered 4-folds (or 3-folds) for which two general tangent spaces intersect although he suspected that this property is indeed equivalent to secant defectiveness for $r \geq 2n + 1$, see the Introduction of [31]. Some relations between the invariants of secant defective varieties defined in the paper have not been considered by Scorza and have been applied quite recently also elsewhere to obtain new contributions in the area, see for example [7, 8, 11, 26, 34].

The outcome of our analysis can be summarized in the following classification theorem:

Theorem 1 *Let $X \subset \mathbb{P}^r$ be an irreducible, projective, secant defective variety of dimension 4. Then, modulo projective equivalence, the following cases occur:*

- (i) X is a cone;
- (ii) X sits in a 5 or 6-dimensional cone over a curve;
- (iii) X sits in a 5-dimensional cone over a surface;
- (iv) X is the Segre variety $\text{Seg}(2, 2) = \mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$;
- (v) X is a scroll in 3-spaces as in Example 4.0.1 below;
- (vi) X is a scroll in 3-spaces as in Example 4.0.2 below;
- (vii) $r = 9$ and X is a scroll in 3-spaces as in Example 4.0.3 below;
- (viii) X is an internal projection of the Veronese 4-fold $V_{2,4}$ of quadrics in \mathbb{P}^{14} from finitely many points with the property that it contains at most finitely many lines through its general point;
- (ix) $r = 11$ and X is the projection of $V_{2,4}$ from the plane spanned by a conic on it;
- (x) $r = 9$ and X is the projection of $V_{2,4}$ from a 4-space spanned by a rational normal quartic curve on it;
- (xi) $r = 10$ and X is a hyperplane section of the Segre variety $\text{Seg}(2, 3) = \mathbb{P}^2 \times \mathbb{P}^3 \subset \mathbb{P}^{11}$;
- (xii) $r = 9$ and either X sits in a cone with vertex a line over a hyperplane section of $\text{Seg}(2, 2)$ in \mathbb{P}^7 or X sits in a cone with vertex a point over the Segre variety $\text{Seg}(2, 2) \subset \mathbb{P}^8$;
- (xiii) $9 \leq r \leq 11$ and X sits in a cone with vertex a line over the projection in \mathbb{P}^{r-2} of the Veronese 3-fold $V_{3,2}$;
- (xiv) $r = 9$ and X sits in a 6-dimensional cone over the Veronese surface $V_{2,2}$ in \mathbb{P}^5 ;
- (xv) $r = 9$ and X sits in a cone with vertex a line over a (defective) 3-fold in \mathbb{P}^7 sitting in a cone with vertex a line over the Veronese surface $V_{2,2}$;
- (xvi) X is swept out by a 3-dimensional family \mathcal{R} of lines and it is singular along a linear space Π of dimension ε , with $2 \leq \varepsilon \leq 3$, which is intersected in one point by the general line in \mathcal{R} , and X projects from Π to a 3-dimensional variety $Y \subset \mathbb{P}^{r-\varepsilon-1}$

(with general fibres unions of lines of \mathcal{R}), which contains a 4-dimensional family \mathcal{C} of (generically irreducible) conics such that there is a conic in \mathcal{C} passing through two general points of Y and the counterimage of the general conic of \mathcal{C} via the projection from Π is a non-developable scroll spanning a 4-space with a line directrix¹ sitting in Π ;

- (xvii) X is swept out by a 4-dimensional family \mathcal{S} of surfaces spanning a 4-space, such that two general surfaces in \mathcal{S} intersect at a point. In this case the general surface in \mathcal{S} is rational and X itself is rational;
- (xviii) the general projection of X in \mathbb{P}^9 sits in a 6-dimensional cone with vertex a 3-space over the Veronese surface $V_{2,2}$.

If X is smooth, only the following cases may occur: (iv), (viii), (ix), (x), (xi), (xii) and X sits in a cone with vertex a point over the Segre variety $\text{Seg}(2, 2) \subset \mathbb{P}^8$, and possibly case (xvii) if it exists.

Theorem 1 summarizes the results contained in Proposition 2.9.5 and Theorems 3.0.1, 4.0.4, 5.5.2, 7.4.1, 7.5.2, 7.6.1, 8.2.3, 8.3.1. All the cases listed in Theorem 1 do occur, except, maybe, case (xvii) for which we do not have any example. We point out that starting from dimension four there exist smooth secant defective n -folds which are of general type and have unbounded degree [for example those of the second type in case (xii)]. These phenomena do not appear in dimension two or three, where all the smooth secant defective varieties are rational and belong to a small finite list.

The paper is organized as follows. In §1 we collect some preliminaries and notation. In §2 we introduce some of the main invariants of a defective variety and we recall some important facts and notions, like Terracini’s Lemma, the second fundamental form of a variety, etc. In §3 we prove the classification Theorem 3.0.1 for n -dimensional varieties with very low dimensional secant variety. Section 4 is devoted to the classification of defective 4-folds which are scrolls in 3-spaces. Sections 5, 6 are devoted to the classification of n -folds, in particular 4-folds, of the top species. In §7 we classify 4-folds of the second species and in §8 we classify 4-folds of the first species.

As far as possible, we kept the discussion in the maximal generality, referring to an n -fold and not simply to a 4-fold.

1 Notation

1.1. In this paper we work over the complex field \mathbb{C} .

If X is a scheme of finite type, by $\dim(X)$ we mean the maximum of the dimensions of its irreducible components. We will denote by \mathcal{O}_X the structure sheaf of X . We will denote by $\text{Reg}(X)$ the Zariski open subset of smooth points of X and by $\text{Sing}(X)$ the closed subset of X formed by its singular points. If $x \in X$ we denote by $T_x(X)$ the Zariski tangent space to x at X .

Let now $X \subseteq \mathbb{P}^r$ be an irreducible projective scheme. We will denote by $\text{deg}(X)$ the degree of X and by \mathcal{I}_X the ideal sheaf of X in \mathbb{P}^r .

If $x \in \text{Reg}(X)$ we denote by $T_{X,x} \subseteq \mathbb{P}^r$ the embedded tangent space to X at x , which is a linear subspace of \mathbb{P}^r of dimension equal to $\dim(X)$. If either $P \subseteq T_{X,x}$ is a linear subspace passing through x or $T_{X,x} \subseteq P$, then P is called a tangent subspace to X at x . If $p_0, \dots, p_m \in \text{Reg}(X)$, then we denote by T_{X,p_0,\dots,p_m} the join of T_{X,p_i} with $1 \leq i \leq m$.

¹ A directrix of a scroll surface S is an irreducible curve on S that intersects the general line of the ruling of S in one single variable point.

Let $X \subseteq \mathbb{P}^r$ be a variety. For h a positive integer, an h -secant line to X is any line ℓ not contained in X such that the intersection scheme of X with ℓ has length at least h . An h -secant line is called *proper* if it cuts X at exactly h distinct points.

Let $X \subseteq \mathbb{P}^r$ be a variety. We will denote by $\text{Sym}^2(X)$ the 2-fold symmetric product of X , i.e. the variety of all unordered pairs $[x, y]$, with $x, y \in X$.

1.2. Let $v_{d,r} : \mathbb{P}^r \rightarrow \mathbb{P}^{N(d,r)}$, with $N(d, r) = \binom{d+r}{r} - 1$, be the d -th Veronese map of \mathbb{P}^r . Its image is denoted by $V_{d,r}$ and is called the (d, r) -Veronese variety.

Let r_1, r_2 be non-negative integers. One has the Segre map $s_{r_1,r_2} : \mathbb{P}^{r_1} \times \mathbb{P}^{r_2} \rightarrow \mathbb{P}^{r_1 r_2 + r_1 + r_2}$, whose image is denoted by $\text{Seg}(r_1, r_2)$ and called the (r_1, r_2) -Segre variety.

Let n, r be two non-negative integers, with $r \geq n$. Then $\mathbb{G}(n, r)$ denotes the Grassmann variety of type (n, r) . A point of $\mathbb{G}(n, r)$ represents a linear subspace of dimension n in \mathbb{P}^r , which we also call a n -(sub)space of \mathbb{P}^r .

If $Z \subseteq \mathbb{P}^r$, the span of Z is denoted by $\langle Z \rangle$.

A 2-dimensional irreducible, projective subvariety S of $\mathbb{G}(1, n)$ is called a congruence of lines in \mathbb{P}^n . In particular, for $n = 3$, a congruence S of lines in \mathbb{P}^3 is called of type (a, b) if, given a general point $x \in \mathbb{P}^3$ and a general plane $P \subset \mathbb{P}^3$, there are exactly a [resp. b] lines of the congruence passing through x [resp. contained in P]. Congruences of type $(1, 1)$ consist of the set of lines intersecting two given skew lines that can be also infinitely near (this immediately follows, e.g., from [3, Thm. 2.1]). Recall that an infinitely near line s' to a line s in \mathbb{P}^3 is a projectivity $\phi : s \rightarrow s^*$, where s^* is the pencil of planes containing s . The congruence of lines intersecting s and s' is the set of all lines passing through a (variable) point P of s and lying in the plane $\phi(P)$.

1.3. If $X \subseteq \mathbb{P}^r$ is a variety and P is a linear space of dimension k in \mathbb{P}^r , we can consider the projection $\pi : X \dashrightarrow \mathbb{P}^{r-k-1}$ of X from P . Recall that the projection is a finite morphism onto its image if $P \cap X = \emptyset$. In this case we will say that the projection is *external*. If $P \subseteq X$ we say that the projection is *internal*.

The variety $X \subseteq \mathbb{P}^r$ is said to be *non-degenerate* if $\langle X \rangle = \mathbb{P}^r$.

If $X \subseteq \mathbb{P}^r$ is a variety, an *extension* of X is a variety $X' \subseteq \mathbb{P}^{r+k}$, with $k \geq 1$, such that X is the scheme theoretical intersection of X' with a \mathbb{P}^r . If X' is a cone over X , the extension is said to be *trivial*. If there is a non-trivial extension of X , then X is said to be *extendable*.

1.4. If X is a variety, it is well known the notion of linearly equivalent divisors on X and the relation of this notion with line bundles and linear systems.

Let X be a variety and L a line bundle on X . Then we can consider the *complete linear system* $|L|$ determined by L , as well as linear subsystems of it. If \mathcal{L} is a linear system, we can consider its *base locus* $\text{Bsl}(\mathcal{L})$, that is the scheme consisting of the intersection of all divisors in \mathcal{L} . The divisorial part of $\text{Bsl}(\mathcal{L})$ is called the *divisorial base locus* of \mathcal{L} . The *movable part* of \mathcal{L} is the linear system consisting of all divisors in \mathcal{L} minus the divisorial base locus.

If $X \subseteq \mathbb{P}^r$ is a variety, we can consider the *hyperplane line bundle* $\mathcal{O}_X(1)$ of X . The *hyperplane linear system* \mathcal{H}_X (or simply \mathcal{H} if there is no ambiguity) of X is the subsystem of $|\mathcal{O}_X(1)|$ which is cut out on X by the hyperplanes of \mathbb{P}^r . The variety $X \subseteq \mathbb{P}^r$ is said to be *linearly normal* if there is no variety $X' \subseteq \mathbb{P}^{r+1}$ such that X is isomorphic to X' via an external projection from a point. Equivalently, $X \subseteq \mathbb{P}^r$ is linearly normal if $h^0(X, \mathcal{O}_X(1)) = r + 1$, i.e., if $\mathcal{H}_X = |\mathcal{O}_X(1)|$.

A linear system \mathcal{L} of dimension r determines a rational map $\phi_{\mathcal{L}} : X \dashrightarrow \mathbb{P}^r$. The linear system \mathcal{L} is said to be *birational* if $\phi_{\mathcal{L}}$ is birational onto its image.

If $\mathcal{L}_1, \mathcal{L}_2$ are linear systems on the variety X , then we can consider their *minimal sum* $\mathcal{L}_1 \vee \mathcal{L}_2$, which consists of the minimum linear system which contains all divisors of the form $D_1 + D_2$ with $D_i \in \mathcal{L}_i$ with $i = 1, 2$.

If \mathcal{L} is a linear system on the variety X and Z is a subscheme of X , we denote by $\mathcal{L}(-Z)$ the linear subsystem of \mathcal{L} consisting of all divisors of \mathcal{L} containing the scheme Z . In particular, if $x \in X$ is a point, then $\mathcal{L}(-x)$ consists of all divisors in \mathcal{L} containing x , whereas we denote by $\mathcal{L}(-2x)$ the set of all divisors in \mathcal{L} containing the scheme defined by \mathfrak{m}_x^2 , where \mathfrak{m}_x is the maximal ideal corresponding to the point x . In particular, if $X \subseteq \mathbb{P}^r$ and $x \in X$, then $\mathcal{H}_X(-x)$ is cut out on X by all hyperplanes passing through x , whereas, if $x \in \text{Reg}(X)$, then $\mathcal{H}_X(-2x)$ is cut out on X by all hyperplanes containing $T_{X,x}$, and it is called the linear system of hyperplanes *tangent* at x .

1.5. Let $X \subseteq \mathbb{P}^r$. A *family* \mathcal{F} of subschemes of X is for us a closed subscheme of the Hilbert scheme of X . The family is said to be *filling* X if, given a general point $x \in X$, there is a scheme of \mathcal{F} , i.e., parameterized by a point of \mathcal{F} , passing through x .

2 Secant varieties and contact loci

2.1. Let $X \subseteq \mathbb{P}^r$ be a non-degenerate irreducible projective variety of dimension n . We will denote by $S(X)$ its *secant variety*, namely the Zariski closure in \mathbb{P}^r of the union of all *proper secant lines* to X , i.e. lines $\langle p_0, p_1 \rangle$ with p_0, p_1 distinct points in X . One has $X = S(X)$ if and only if $X = \mathbb{P}^r$.

One can consider the *abstract secant variety* S_X of X , i.e. $S_X \subseteq \text{Sym}^2(X) \times \mathbb{P}^r$ is the Zariski closure of the set of all pairs $([p_0, p_1], x)$ such that $p_0, p_1 \in X$ are distinct points and $x \in \langle p_0, p_1 \rangle$. One has the surjective morphism $p_X : S_X \rightarrow S(X) \subseteq \mathbb{P}^r$, given by the restriction of the projection to the second factor. Hence:

$$s(X) := \dim(S(X)) \leq \sigma(X) := \min \{r, \dim(S_X)\} = \min \{r, 2n + 1\}. \tag{1}$$

The integer $\sigma(X)$ is called the *expected dimension* of $S(X)$. One says that X is *1-defective*, or simply *defective*, when strict inequality holds in (1). One defines

$$\delta(X) := \sigma(X) - s(X)$$

to be the *defect* of X . A slightly different concept of defect, is the one of *fibre defect* defined as

$$f(X) := 2n + 1 - s(X)$$

which is the dimension of the general fibre of p_X , i.e., the dimension of the family of secant lines to X passing through the general point of $S(X)$.

Remark 2.1.1 One has

$$f(X) = \delta(X) \quad \text{if } r \geq 2n + 1$$

whereas, if $r \leq 2n + 1$, one has

$$f(X) = \delta(X) + 2n + 1 - r. \tag{2}$$

In any case $f(X) \geq \delta(X)$ and $f(X) = 0$ implies non-defectiveness.

2.2. Let X, Y be reduced projective schemes in \mathbb{P}^r . The *join* $J(X, Y)$ of X and Y is the closure in \mathbb{P}^r of the set

$$\{x \in \mathbb{P}^r : x \in \langle p, q \rangle \text{ with } p \in X, q \in Y, p \neq q\}.$$

If X is a variety, then $S(X) = J(X, X)$. If X is a reducible projective scheme, then $J(X, X)$ is in general reducible and not pure. It is convenient for us to denote this by $S(X)$

and call it the *secant variety* of X . The reduced schemes X we will be considering next, even if reducible, will have the property that $S(X)$ is pure. For them all the considerations in §2.1 can be repeated verbatim.

2.3 There are several notions of *tangential variety* to a given variety X (see [34] or [18, Lecture 15], as general references). We will adopt the following.

Let $X \subset \mathbb{P}^r$ be an irreducible variety of dimension n . Consider the rational map

$$s : X \times X \dashrightarrow \mathbb{G}(1, r)$$

which sends the point (p_0, p_1) off the diagonal Δ to the line $\langle p_0, p_1 \rangle$. Resolve the indeterminacies of s obtaining a morphism \tilde{s} defined on some blow-up \mathcal{X} of $X \times X$.

Definition 2.3.1 A *tangent line* to X is any line corresponding to images via the map \tilde{s} of points of \mathcal{X} mapping to Δ .

We will denote by $\text{Tan}(X)$ the locus of points of \mathbb{P}^r lying on some tangent line to X . This is the *tangential variety* to X . It does not depend on the blow-up \mathcal{X} of $X \times X$.

Remark 2.3.2 Tangent lines are all limits of secants $\langle p_0, p_1 \rangle$ with $p_0, p_1 \in X$ distinct points, when p_0, p_1 come together. Therefore $\text{Tan}(X) \subseteq S(X)$. Moreover $\text{Tan}(X)$ is a closed subvariety of \mathbb{P}^r , containing the variety

$$T(X) := \overline{\bigcup_{x \in \text{Reg}(X)} T_{X,x}}$$

It is a well known consequence of Fulton–Hansen connectedness theorem (see [16]) that if $X \subset \mathbb{P}^r$, with $r \geq 2n + 1$, is smooth, then either X is not defective (in which case $\dim(\text{Tan}(X)) = 2n$) or $S(X) = \text{Tan}(X)$.

Note that $\text{Tan}(X) = T(X)$ if X is smooth.

2.4. Terracini’s Lemma describes the tangent space to a secant variety at a general point of it (see [31] or, for modern versions, [1, 7, 12, 34]). In the case of interest for us, we may state it as follows.

Theorem 2.4.1 (Terracini’s Lemma) *Let $X \subset \mathbb{P}^r$ be a projective variety. If $p_0, p_1 \in X$ are general points and $x \in \langle p_0, p_1 \rangle$ is a general point of $S(X)$, then*

$$T_{S(X),x} = \langle T_{X,p_0}, T_{X,p_1} \rangle = T_{X,p_0,p_1}.$$

Remark 2.4.2 Let us keep the notation of Terracini’s Lemma. Then

$$f(X) = 2n + 1 - s(X) = \dim(T_{X,p_0} \cap T_{X,p_1}) + 1.$$

Lemma 2.4.3 *Let $X \subseteq \mathbb{P}^r$ be a non-degenerate variety of dimension $n \geq 2$ and let Y be a general hyperplane section of X . Then*

$$f(Y) = \max\{0, f(X) - 1\}.$$

Proof The assertion is trivial if $X = \mathbb{P}^r$, so we may assume that X is strictly contained in \mathbb{P}^r and it is not a linear subspace because it is non-degenerate.

Let Y be the section of X with a general hyperplane H and let p_0, p_1 be general points of Y . Then p_0, p_1 are also general points of X . One has $T_{Y,p_i} = H \cap T_{X,p_i}$ for $i = 0, 1$.

If $T_{X,p_0} \cap T_{X,p_1} = \emptyset$, so that $f(X) = 0$, then also $T_{Y,p_0} \cap T_{Y,p_1} = \emptyset$ and $f(Y) = 0$, and the assertion holds.

Suppose now $T_{X,p_0} \cap T_{X,p_1} \neq \emptyset$. Note that neither p_0 nor p_1 lie in $T_{X,p_0} \cap T_{X,p_1}$, otherwise the general tangent space to X would contain the general point of X and therefore X would coincide with its general tangent space, so it would be a linear space, a contradiction. Then $T_{X,p_0} \cap T_{X,p_1}$ is skew with the line $\langle p_0, p_1 \rangle$, because p_0, p_1 are not in $T_{X,p_0} \cap T_{X,p_1}$ and if a point $p \in \langle p_0, p_1 \rangle$ different from p_0, p_1 lies in $T_{X,p_0} \cap T_{X,p_1}$, then the whole line $\langle p_0, p_1 \rangle = \langle p, p_1 \rangle = \langle p_0, p \rangle$ would be contained in $T_{X,p_0} \cap T_{X,p_1}$, a contradiction. Then by the generality of the hyperplane H , $H \cap T_{X,p_0} \cap T_{X,p_1} = T_{Y,p_0} \cap T_{Y,p_1}$ has dimension $\dim(T_{X,p_0} \cap T_{X,p_1}) - 1$ and the assertion follows. \square

2.5. Given an irreducible projective variety $X \subseteq \mathbb{P}^r$ of dimension n , the *Gauss map* of X is the rational map

$$g_X : X \dashrightarrow \mathbb{G}(n, r)$$

defined at the smooth points of X by mapping $x \in \text{Reg}(X)$ to the tangent space $T_{X,x}$. It is well known that, if $x \in X$ is a general point, then the closure of the fibre of g_X through x is a linear subspace $\Gamma_{X,x}$ of \mathbb{P}^r (see [34, Thm. I.2.3(c)]), which we denote by Γ_x if there is no danger of confusion.

Definition 2.5.1 In the above setting, Γ_x is called the *general Gauss fibre* of X and its dimension is called the *tangential defect* of X , denoted by $t(X)$. We will set $\theta(X) = t(S(X))$.

Note that, if X is smooth, then $t(X) = 0$ (see [34, Corollary I.2.8]).

Remark 2.5.2 In the above setting, there is a dense open set $U \subset \text{Reg}(X)$, which via $g_{X|U}$ is fibred over $g_X(U)$ in open subsets of projective spaces of the same dimension $t(X)$. This implies that if $Y = X \cap H$ is the section of X with a general hyperplane and $x \in Y$, then $\Gamma_{X,x} \cap H = \Gamma_{Y,x}$. Hence

$$t(Y) = \max\{0, t(X) - 1\}.$$

2.6. Given $x \in S(X)$ a general point, i.e. $x \in \langle p_0, p_1 \rangle$ is a general point with $p_0, p_1 \in X$ general points, consider the Zariski closure of the set

$$\{p \in \text{Reg}(X) : T_{X,p} \subseteq T_{S(X),x} = T_{X,p_0,p_1}\}.$$

Note that this set depends only on p_0, p_1 and not on $x \in \langle p_0, p_1 \rangle$. We will denote by Γ_{X,p_0,p_1} the union of all irreducible components of this locus containing p_0 or p_1 . Since p_0, p_1 are general points, we can vary them and exchange them by monodromy and this monodromy action maps Γ_{X,p_0,p_1} to itself. As in [8, Proposition 2.2], this implies that:

Proposition 2.6.1 Γ_{X,p_0,p_1} is smooth at p_0 and p_1 . Hence Γ_{X,p_0,p_1} is either irreducible, or it consists of only two irreducible components, one passing through p_0 (and not through p_1) and the other passing through p_1 (and not through p_0), both with the same dimension.

Proof Let us prove the first assertion. Suppose, by contradiction, that Γ_{X,p_0,p_1} is singular at p_0 (and therefore, by monodromy considerations, also at p_1). In particular Γ_{X,p_0,p_1} has positive dimension at p_0 . Let Γ be an irreducible component of Γ_{X,p_0,p_1} of positive dimension containing p_0 and let p be a general point of Γ . We have that $T_{X,p} \subseteq T_{X,p_0,p_1}$, hence $T_{X,p,p_1} \subseteq T_{X,p_0,p_1}$. Since p is also a general point of X (because it specializes to p_0 that is a general point of X), we have that $\dim(T_{X,p,p_1}) = \dim(T_{X,p_0,p_1})$, hence $T_{X,p,p_1} = T_{X,p_0,p_1}$. As p is a smooth point of Γ and as Γ_{X,p,p_1} has to be singular at p , this implies that Γ is not the

only component of Γ_{X,p,p_1} passing through p . Since p is general in Γ , it immediately follows that Γ_{X,p_0,p_1} contains a subvariety of dimension greater than the dimension of Γ , which contains Γ . This contradicts the fact that Γ is a component of Γ_{X,p_0,p_1} . This contradiction proves the assertion.

The fact that Γ_{X,p_0,p_1} is either irreducible, or it consists of only two irreducible components, one passing through p_0 (and not through p_1) and the other passing through p_1 (and not through p_0) is a trivial consequence of the smoothness of Γ_{X,p_0,p_1} at p_0 and p_1 .

Finally, if Γ_{X,p_0,p_1} is reducible, the fact that its two components have the same dimension follows from the fact that they can be exchanged by monodromy. \square

Let $\gamma(X)$ be the dimension of Γ_{X,p_0,p_1} . By the generality of p_0, p_1 , it does not depend on p_0, p_1 . Note that $\Gamma_{X,p_i} \subseteq \Gamma_{X,p_0,p_1}$ for $i = 0, 1$. We set

$$\Pi_{X,p_0,p_1} = \langle \Gamma_{X,p_0,p_1} \rangle.$$

Of course Π_{X,p_0,p_1} contains $\langle p_0, p_1 \rangle$.

If there is no danger of confusion, we write Γ_{p_0,p_1} and Π_{p_0,p_1} instead of Γ_{X,p_0,p_1} and Π_{X,p_0,p_1} .

Definition 2.6.2 In the above setting, we will call Γ_{X,p_0,p_1} the *tangential contact locus* of X at p_0, p_1 . We will call $\gamma(X)$ the *contact defect* of X .

Proposition 2.6.3 *Let X be a projective defective variety of dimension n , let $Y = X \cap H$ with H a general hyperplane and let p_0, p_1 be general points of Y . Then*

$$\Gamma_{X,p_0,p_1} \cap H \subseteq \Gamma_{Y,p_0,p_1}.$$

In particular, $\gamma(Y) \geq \gamma(X) - 1$ and if the equality holds then $\Gamma_{X,p_0,p_1} \cap H = \Gamma_{Y,p_0,p_1}$.

Proof First we prove that $T_{Y,p_0,p_1} = T_{X,p_0,p_1} \cap H$. It is clear that $T_{Y,p_0,p_1} \subseteq T_{X,p_0,p_1} \cap H$, so it suffices to prove that the two spaces have the same dimension. Since $T_{Y,p_i} = H \cap T_{X,p_i}$ for $i = 0, 1$, we have that $T_{Y,p_0} \cap T_{Y,p_1} = H \cap T_{X,p_0} \cap T_{X,p_1}$ (see the proof of Lemma 2.4.3). Then

$$\begin{aligned} \dim(T_{X,p_0,p_1} \cap H) &= 2n - \dim(T_{X,p_0} \cap T_{X,p_1}) - 1 = \\ &= 2n - (\dim(T_{Y,p_0} \cap T_{Y,p_1}) + 1) - 1 = 2(n - 1) - \dim(T_{Y,p_0} \cap T_{Y,p_1}) = \dim(T_{Y,p_0,p_1}), \end{aligned}$$

as wanted.

Now, let $x \in \Gamma_{X,p_0,p_1} \cap H$ be a general point. Then $T_{X,x} \subseteq T_{X,p_0,p_1}$, hence

$$T_{Y,x} \subseteq T_{X,x} \cap H \subseteq T_{X,p_0,p_1} \cap H = T_{Y,p_0,p_1},$$

proving the first assertion.

In particular, the unique component of Γ_{Y,p_0,p_1} passing through $p_i, i = 0, 1$, contains the intersection of H with the unique component of Γ_{X,p_0,p_1} through p_i . So, if $\gamma(Y) = \gamma(X) - 1$, then necessarily $\Gamma_{X,p_0,p_1} \cap H = \Gamma_{Y,p_0,p_1}$. \square

The following result is contained in [11, Proposition 3.9].

Proposition 2.6.4 *Let $X \subset \mathbb{P}^r$ be a non-degenerate variety such that $s(X) < r$. Let $p_0, p_1 \in X$ be general points. Then:*

- (i) *let q_0, q_1 be general points on Γ_{p_0,p_1} , such that q_i specializes to p_i , for $i = 0, 1$ (i.e., q_i is a general point of a component of Γ_{p_0,p_1} containing p_i , for $i = 0, 1$). Then $\Gamma_{p_0,p_1} = \Gamma_{q_0,q_1}$;*

- (ii) $f(\Gamma_{p_0,p_1}) = f(X)$;
- (iii) $\Pi_{p_0,p_1} = S(\Gamma_{p_0,p_1})$ equals the general Gauss fibre $\Gamma_{S(X),x}$ of $S(X)$ (in particular $S(\Gamma_{p_0,p_1})$ is a linear space), whereas $\Gamma_{p_0,p_1} \neq \Pi_{p_0,p_1}$;
- (iv) $\theta(X) = \dim(\Pi_{p_0,p_1}) = 2\gamma(X) + 1 - f(X)$.

Remark 2.6.5 As for point (ii) of Proposition 2.6.4, if Γ_{p_0,p_1} is irreducible, it is clear what $f(\Gamma_{p_0,p_1})$ means. If Γ_{p_0,p_1} is reducible in two components (one containing p_0 and the other containing p_1), then

$$f(\Gamma_{p_0,p_1}) = \dim(T_{\Gamma_{p_0,p_1},q_0} \cap T_{\Gamma_{p_0,p_1},q_1}) + 1,$$

where $q_i \in \Gamma_{p_0,p_1}$ is a point specializing to p_i , for $0 \leq i \leq 1$ (see Remark 2.4.2).

2.7. Let $X \subset \mathbb{P}^r$ be a non-degenerate variety of dimension n and let $p \in X$ be a general point. Consider the projection of X with centre $T_{X,p}$ to \mathbb{P}^{r-n-1} . We call this a *general tangential projection* of X , and we denote it by $\tau_{X,p}$, or by τ_p , or simply by τ . We denote by X_p , or by X_1 , its image, and by n_1 its dimension. As a consequence of Terracini’s Lemma, we have

$$n_1 = n - f(X). \tag{3}$$

Definition 2.7.1 Let $p_0, p_1 \in X$ be general points. Let Ψ_{X,p_0,p_1} be the component of the fibre of τ_{X,p_1} containing p_0 . It is called the *projection contact locus* of X at p_0, p_1 and, by (3), its dimension is $f(X)$.

If there is no danger of confusion, we will write Ψ_{p_0,p_1} rather than Ψ_{X,p_0,p_1} .

Remark 2.7.2 One has that Γ_{p_0,p_1} contains Ψ_{p_0,p_1} . Indeed, if $p \in \Psi_{X,p_0,p_1}$ is a general point (in particular if $p = p_0$), then the projection of $T_{X,p}$ from T_{X,p_1} is the tangent space to X_1 at the point $\tau(p) = \tau(p_0)$. In particular $T_{X,p} \subset T_{X,p_0,p_1}$. Hence

$$\gamma(X) \geq f(X). \tag{4}$$

Actually one has $\gamma(X) = t(X_1) + f(X)$. Indeed, the component of Γ_{p_0,p_1} passing through p_1 is the counterimage, via the tangential projection τ_{p_0} , of the Gauss fibre $\Gamma_{X_{p_0},\tau_{p_0}(p_1)}$. Therefore $\gamma(X) = f(X)$ holds if and only if the Gauss map of X_1 is generically finite to its image, which is equivalent to say that it is birational to its image (see [11, Remark 3.6]). Thus, if $\gamma(X) = f(X)$, then Ψ_{p_0,p_1} is the irreducible component of Γ_{p_0,p_1} containing p_0 .

If X is defective, then $f(X) > 0$, so that $\gamma(X)$ is also positive and X_1 is strictly contained in \mathbb{P}^{r-n-1} . Otherwise $n_1 = r - n - 1$ by (3) yields $f(X) = 2n + 1 - r$ which, together with (2), implies $\delta(X) = 0$, a contradiction.

By Lemma 2.4.3, if $Y = X \cap H$ with H a general hyperplane, then

$$f(Y) = \max\{0, f(X) - 1\}.$$

This implies that, if p_0, p_1 are general points of Y , then $\Psi_{Y,p_0,p_1} = H \cap \Psi_{X,p_0,p_1}$.

Lemma 2.7.3 Let $X \subset \mathbb{P}^r$ be a variety with $f(X) \geq 1$. Let $p \in X$ be a general point and let Y by a general hyperplane section of X passing through p . Then $X_p = Y_p$.

Proof We can assume that Y is a general hyperplane section of X and that p is also a general point of Y . Via the tangential projection τ_p we clearly have that $Y_p \subseteq X_p$, and X_p, Y_p are both irreducible. On the other hand $\dim(X_p) = \dim(X) - f(X) = \dim(Y) - f(Y) = \dim(Y_p)$, so $X_p = Y_p$ as wanted. □

Remark 2.7.4 The above lemma can be proved under milder hypotheses. For example, it suffices that X is homogeneous, Y is any hyperplane section and $f(Y) = f(X) - 1$. Then if p is a general point of Y one still has $Y_p = \tau_{Y,p}(Y) \subseteq \tau_{X,p}(X)$. Moreover the tangential projection $\tau_{X,p}$ behaves as if p were a general point of X by the homogeneity hypothesis. Hence $Y_1 \subseteq X_1$. Finally $\dim(X_1) = \dim(Y_1)$ still holds and we are done.

2.8. Other relevant items related to the secant variety $S(X)$ are the so called *entry loci*.

Definition 2.8.1 Let $x \in S(X)$ be a point. We define the *entry locus* E_x of x with respect to X as the closure of the set

$$\{z \in X : \text{there is } y \in X \text{ with } y \neq z \text{ and } x \in \langle y, z \rangle\}.$$

A useful information about the entry loci is provided by the following result (see [11, Proposition 3.13] or [20, Proposition 2.2]).

Proposition 2.8.2 Let $X \subset \mathbb{P}^r$ be a variety with $s(X) < r$ and let $x \in S(X)$ be a general point. Then E_x is pure of dimension $f(X)$.

Remark 2.8.3 Terracini’s Lemma implies that the entry locus E_x is contained in the tangential contact locus Γ_{p_0,p_1} for all distinct pairs of points $p_0, p_1 \in X$ such that $x \in \langle p_0, p_1 \rangle$. Therefore again $f(X) \leq \gamma(X)$.

2.9. Let $X \subset \mathbb{P}^r$ be a variety. Let $x \in X$ be a general point and let H be a general hyperplane containing $T_{X,x}$. This is a *general tangent hyperplane* to X . Consider the Zariski closure $\Gamma_{X,x}(H)$ of the set

$$\{p \in \text{Reg}(X) : T_{X,p} \subseteq H\}.$$

By [34, Thm. I.2.3, (c)] applied with $m = r - 1$, when H is general then $\Gamma_{X,x}(H)$ is a linear subspace of \mathbb{P}^r . We call it the *general tangent hyperplane contact locus* of X at x and we will denote by $d(X)$ its dimension, which does not depend on x and H . If there is no danger of confusion, we may write $\Gamma_x(H)$ rather than $\Gamma_{X,x}(H)$. Of course $\Gamma_x(H)$ contains Γ_x , thus

$$d(X) \geq t(X).$$

Moreover the *dual variety* X^* of X has dimension

$$\dim(X^*) = r - 1 - d(X)$$

and therefore $d(X)$ is called the *dual defect* of X .

Similarly, let $p_0, p_1 \in X$ be general points, and let H be a general hyperplane containing T_{X,p_0,p_1} , i.e. a *general bitangent hyperplane* to X . Consider the Zariski closure of the set

$$\{p \in \text{Reg}(X) : T_{X,p} \subseteq H\}.$$

Definition 2.9.1 The union $\Gamma_{X,p_0,p_1}(H)$ of all irreducible components of the aforementioned locus containing either p_0 or p_1 is called the *general bitangent hyperplane contact locus* of X at p_0, p_1 . We will denote by $\epsilon(X)$ its dimension, which, by genericity, does not depend on p_0, p_1 and H . We will set

$$\Pi_{X,p_0,p_1}(H) := \langle \Gamma_{X,p_0,p_1}(H) \rangle.$$

We will write $\Gamma_{p_0, p_1}(H)$ and $\Pi_{p_0, p_1}(H)$ instead of $\Gamma_{X, p_0, p_1}(H)$ and $\Pi_{X, p_0, p_1}(H)$ if there is no danger of confusion.

Of course $\Gamma_{p_0, p_1}(H)$ contains Γ_{p_0, p_1} , and therefore

$$\epsilon(X) \geq \gamma(X). \tag{5}$$

Remark 2.9.2 By (4) and (5) we see that

$$\epsilon(X) \geq \gamma(X) \geq f(X).$$

More precisely, looking at a general tangential projection, one has $d(X_1) = \epsilon(X) - f(X) > 0$. Indeed, let H be a general bitangent hyperplane to $X \subset \mathbb{P}^r$ of dimension n , at two general points p_0, p_1 . Let H' be the hyperplane in \mathbb{P}^{r-n-1} image of H via τ_{p_0} , that can be considered as a general tangent hyperplane to X_{p_0} at $\tau_{p_0}(p_1)$. Then $\Gamma_{X, p_0, p_1}(H)$ is the preimage via τ_{p_0} of $\Gamma_{X_{p_0}, \tau_{p_0}(p_1)}(H')$, hence $\epsilon(X) = d(X_{p_0}) + f(X)$ as wanted. Hence we have that $\epsilon(X) > f(X)$ if and only if $d(X_1) = \epsilon(X) - f(X) > 0$, i.e. if and only if X_1 is dual defective.

Recall the following result which parallels Proposition 2.6.4 (see [7, Theorem 1.1] and [8, Theorem 2.4]):

Theorem 2.9.3 *Let $X \subset \mathbb{P}^r$ be a defective variety. Let $p_0, p_1 \in X$ be general points and let H be a general hyperplane containing T_{X, p_0, p_1} . One has:*

- (i) $\Gamma_{p_0, p_1}(H)$ is smooth at p_0, p_1 ; moreover it is either irreducible or it consists of two irreducible components of the same dimension $\epsilon(X)$ and p_0, p_1 can be considered as general points each on one of these components (in accordance with Proposition 2.6.4(i));
- (ii) $f(\Gamma_{p_0, p_1}(H)) = f(X)$;
- (iii) $\Pi_{p_0, p_1}(H) = S(\Gamma_{p_0, p_1}(H))$, whereas $\Gamma_{p_0, p_1}(H) \neq \Pi_{p_0, p_1}$;
- (iv) $\dim(\Pi_{p_0, p_1}(H)) \leq 2\epsilon(X) + 1 - f(X)$.

Remark 2.9.4 As for the meaning of $f(\Gamma_{p_0, p_1}(H))$ in point (ii) of Theorem 2.9.3, we adopt the same convention as in Remark 2.6.5.

Moreover we have the following:

Proposition 2.9.5 *Let $X \subset \mathbb{P}^r$ be a defective variety of dimension n . Then:*

- (i) $r \geq n+3$ and $f(X) \leq n-1$. Moreover if $f(X) = n-1$ (which is the case if $r = n+3$), then X is either a cone over a curve or a cone over the Veronese surface $V_{2,2}$ in \mathbb{P}^5 ;
- (ii) if $f(X) = \gamma(X) := m$, then for $p_0, p_1 \in X$ general, Γ_{p_0, p_1} is a quadric of rank at least 2 in \mathbb{P}^{m+1} . Furthermore Ψ_{p_0, p_1} is an irreducible component of Γ_{p_0, p_1} , hence either it coincides with Γ_{p_0, p_1} or it is a linear subspace of dimension m ;
- (iii) in particular if $f(X) = \epsilon(X) := m$, then for general choices of p_0, p_1 and H containing T_{X, p_0, p_1} , one has $\Gamma_{p_0, p_1}(H) = \Gamma_{p_0, p_1}$ and (ii) holds.

Proof Since X_1 is non-degenerate and strictly contained in \mathbb{P}^{r-n-1} (see Remark 2.7.2), we have $r - n - 1 \geq 2$, proving the first assertion in (i). Similarly $n_1 = n - f(X) \geq 1$, proving the second assertion in (i). Suppose that $r = n + 3$. Since $n \geq 2$, because curves are never defective, we have $r = n + 3 \leq 2n + 1$. Hence, by (2), we have $f(X) = \delta(X) + n - 2 \geq n - 1$, so $f(X) = n - 1$ as stated. The rest of (i) follows from [9, Theorem 3.5].

Let us prove (ii). By Proposition 2.6.4, (iv), we have that Γ_{p_0,p_1} is a hypersurface in Π_{p_0,p_1} , which has dimension $m + 1$. It has to be a quadric otherwise the general secant to X would be a trisecant. By the Trisecant Lemma (see [26, Proposition 1.4.3]), this cannot happen unless $r = n + 1$, which is not the case. Then Γ_{p_0,p_1} is a quadric but it cannot be a linear space, hence Γ_{p_0,p_1} has rank at least 2. The rest of assertion (ii) is clear.

Part (iii) follows from (ii) and Remark 2.9.2. □

Varieties X for which part (ii) of Proposition 2.9.5 holds, i.e., for $p_0, p_1 \in X$ general Γ_{p_0,p_1} is a quadric of rank at least 2, are *locally quadratic entry locus (LQEL)* varieties in the sense of [25] (see [26] for the theory of LQEL varieties and applications). If $\gamma(X) = f(X) = 1$, we will say that they present the *locally conic entry locus (LCEL)* case.

According to Scorza’s terminology (see [28]), we will say that a defective variety $X \subset \mathbb{P}^r$ of dimension n is of the *species k* if $k = n - \gamma(X)$. The *top species* for varieties of dimension n is $n - 1$. Such a variety X has $\gamma(X) = f(X) = 1$, hence it presents the LCEL case. Note that varieties X of dimension n of a given species k could be further classified according to the value of the invariant $\epsilon(X) \geq n - k = \gamma(X)$. There are however some constraints, as the following lemma indicates.

Lemma 2.9.6 *Let $X \subset \mathbb{P}^r$ be a non-degenerate variety of dimension n . Then $\epsilon(X) = n - 1$ if and only if $\gamma(X) = n - 1$.*

Proof If $\gamma(X) = n - 1$, then $\epsilon(X) \geq n - 1$ by (5), and, by Definition 2.9.1, it cannot be $\epsilon(X) = n$ so $\epsilon(X) = n - 1$.

As for the other implication, assume $\epsilon(X) = n - 1$. If $f(X) = n - 1$ then also $\gamma(X) = n - 1$ and we are done. If $f(X) < n - 1$, then the image X_1 of a general tangential projection of X has a curve as a dual (see Remark 2.9.2). Then X_1 , which is dual to a curve, is a hypersurface, and $t(X_1) = n_1 - 1 = n - f(X) - 1$. By Remark 2.7.2 we have $\gamma(X) = t(X_1) + f(X) = n - 1$, as wanted. □

2.10. Let $X \subset \mathbb{P}^r$ be a non-degenerate variety. Let x be a smooth point of X . One can consider the *second fundamental form* $\Pi_{X,x}$ of X at x (see [17, §1(b)] as a general reference or [26, Section 2.3.2] for an introduction in the same spirit of our approach). This is a linear system of quadrics in $\mathbb{P}_{X,x} := \mathbb{P}(T_x(X))$. Recall that $T_x(X)$ is the *Zariski tangent space* to x at X , hence $\mathbb{P}_{X,x}$, the projective space associated to $T_x(X)$, is a projective space of dimension $n - 1$.

Consider the general tangential projection $\tau_x : X \dashrightarrow X_1$. This is not defined at x . Let $\tilde{X} = \text{Bl}_x X$ be the blow-up of X at x , let $\pi_x : \tilde{X} \rightarrow X$ be the induced map and consider the exceptional divisor $E := E_x$, which may be identified with $\mathbb{P}_{X,x}$. We have therefore the rational map

$$\tilde{\tau}_x = \tau_x \circ \pi_x : \tilde{X} \dashrightarrow X_1 \subseteq \mathbb{P}^{r-n-1},$$

whose restriction to E is defined by a linear system of quadrics, which is precisely $\Pi_{X,x}$. We will denote by $B_{X,x}$ its base locus scheme, which consists of the points of E_x corresponding to directions of lines having intersection multiplicity larger than 2 with X at x (see [17]). Hence it contains the subscheme $L_{X,x}$ of E , whose points correspond to tangent directions of the lines on X passing through x .

We may write Π_x, L_x, B_x rather than $\Pi_{X,x}, L_{X,x}, B_{X,x}$ etc., if there is no danger of confusion. We will set

$$b(X) = \dim(B_x) \quad \text{and} \quad \ell(X) = \dim(L_x).$$

Note that if $\ell(X) \geq 0$, then $\ell(X) + n - 1$ is the dimension of the union of the components of the Hilbert scheme of lines contained in X and filling X .

It is useful to collect here the main results about the second fundamental form that we will use later. First of all, it is a classical result by Terracini in [32] (see [17, (5.6), (5.7)] and also [14, Theorem 3.3.1] for a modern treatment) that

$$\dim(T(X)) = n + 1 + \dim(\tilde{\tau}_x(E)).$$

Therefore, if X is smooth and defective, then by Remarks 2.1.1 and 2.3.2 one has

$$\dim(\tilde{\tau}_x(E)) = n - f(X) = \dim(X_1). \tag{6}$$

Thus $\tilde{\tau}_{x|E}$ is dominant onto X_1 , with general fibres of dimension $f(X) - 1$. Since X_1 is non-degenerate in \mathbb{P}^{r-n-1} , one has

$$\dim(\Pi_{X,x}) = r - n - 1. \tag{7}$$

As the following result shows, (7) holds, to a certain extent, even relaxing the smoothness assumption on X .

Lemma 2.10.1 *Let $X \subset \mathbb{P}^r$ be a non-degenerate, defective variety of dimension n . Let $x \in X$ be a general point and assume that the closure of the general fibre of the tangential projection τ_x is irreducible and contains x . Then $\tilde{\tau}_{x|E}$ is dominant onto X_1 , with general fibres of dimension $f(X) - 1$. Hence (6) and (7) hold. Furthermore*

$$r \leq \frac{n(n+3)}{2}. \tag{8}$$

Proof The assumption implies that $\tilde{\tau}_{x|E} : E \dashrightarrow X_1$ is dominant. Since $\tilde{\tau}_{x|E}$ is defined by a linear system of quadrics, one has

$$r - n - 1 \leq \frac{(n-1)(n+2)}{2}$$

and (8) follows. The rest of the assertion is clear. □

3 The codimension 4 case

Since defective surfaces and threefolds are classified (see [7, 9, 10, 27]), we may always assume $n \geq 4$. Let $X \subset \mathbb{P}^r$ be an irreducible, non-degenerate, defective variety of dimension $n \geq 4$. In Proposition 2.9.5 part (i) we proved that $r \geq n + 3$ and that equality holds if and only if X is a cone over a curve or over the Veronese surface $V_{2,2} \subset \mathbb{P}^5$. The next theorem classifies the cases for which $r \geq n + 4$ and $f(X) = n - 2$ and, in particular, those for which the equality $r = n + 4$ holds.

Theorem 3.0.1 *Let $X \subset \mathbb{P}^r$ be a non-degenerate, defective variety of dimension $n \geq 4$, which is not a cone over a curve or over the Veronese surface $V_{2,2}$. Then $r \geq n + 4$ and if $f(X) = n - 2$, which happens in particular if $r = n + 4$, then X is of one of the following types:*

- (i) X is a cone over a non-defective surface;
- (ii) X sits in a $(n + 1)$ -dimensional cone over a curve;
- (iii) X sits in a $(n + 1)$ -dimensional cone over the Veronese surface $V_{2,2}$ in \mathbb{P}^5 ;

- (iv) X is a cone over the Veronese threefold $V_{3,2}$ in \mathbb{P}^9 or over a projection of $V_{3,2}$ in \mathbb{P}^8 or \mathbb{P}^7 ;
- (v) X is either a cone over the Segre variety $\text{Seg}(2, 2) \subset \mathbb{P}^8$ with vertex of dimension $n - 5$, or a cone over a hyperplane section of $\text{Seg}(2, 2)$.

Remark 3.0.2 Suppose X is as in one of the cases listed in Theorem 3.0.1.

If X is either a cone over a curve, or it is a cone over a Veronese surface $V_{2,2}$, then X is defective, with $f(X) = n - 1$ (see Proposition 2.9.5).

Otherwise, we claim that X is defective with $f(X) = n - 2$. This is immediate in cases (i), (iv), (v), since X is a cone.

Suppose we are in case (ii) so that X lies in a cone V over a curve C with vertex a $(n - 1)$ -space Π . We may assume that C sits in a complementary space to Π . Let $x \in X$ be a general point. Let $\pi : V \dashrightarrow C$ be the projection of V to C from Π and let $y = \pi(x)$. The subspace $T_x := \Pi \cap T_{X,x}$ has dimension $n - 2$ and

$$T_{X,x} = \langle T_x, \ell \rangle. \tag{9}$$

with ℓ a suitable tangent line to X through x that is mapped isomorphically via π to $T_{C,y}$ (ℓ exists because $\pi|_X$ is dominant) so that $\ell \cap \Pi = \emptyset$. Then

$$T_{V,x} = \langle \Pi, \ell \rangle. \tag{10}$$

The linear space T_x cannot stay fixed as x varies on X (otherwise X would be a cone over C). If $p_0, p_1 \in X$ are general points, then $\Pi = \langle T_{p_0}, T_{p_1} \rangle$, hence T_{X,p_0,p_1} contains Π , it also contains P_i the n -spaces of the ruling of V through p_i for $i = 0, 1$, and finally T_{V,p_0,p_1} because of (9) and (10). In conclusion $T_{X,p_0,p_1} = T_{V,p_0,p_1}$. Hence

$$\begin{aligned} \dim(T_{X,p_0,p_1}) &= \dim(T_{V,p_0,p_1}) = \dim(T_{V,p_0}) + \dim(T_{V,p_1}) - \dim(\Pi) = \\ &= 2(n + 1) - (n - 1) = n + 3, \end{aligned}$$

which implies $f(X) = 2n + 1 - (n + 3) = n - 2$ (see also Remark 2.4.2).

In case (iii) the argument is analogous to the one in case (ii). Suppose X lies in a cone V over a Veronese surface $S = V_{2,2}$ with vertex a $(n - 2)$ -space Π . We may assume that S sits in a complementary space to Π . Let $x \in X$ be a general point. Let $\pi : V \dashrightarrow S$ be the projection of V to S from Π and let $y = \pi(x)$. The subspace $T_x := \Pi \cap T_{X,x}$ has dimension $n - 3$ and

$$T_{X,x} = \langle T_x, \alpha \rangle \tag{11}$$

with α a suitable tangent plane to X through x that maps via π to $T_{S,y}$. Then

$$T_{V,x} = \langle \Pi, \alpha \rangle. \tag{12}$$

As above, T_x cannot stay fixed as x varies on X , otherwise X would be a cone over S . Thus, for $p_0, p_1 \in X$ general, $\Pi = \langle T_{p_0}, T_{p_1} \rangle$, T_{X,p_0,p_1} contains Π and P_i , the $(n - 1)$ -spaces of the ruling of V through p_i for $i = 0, 1$, and finally T_{V,p_0,p_1} because of (11) and (12). In conclusion $T_{X,p_0,p_1} = T_{V,p_0,p_1}$, yielding $\dim(T_{X,p_0,p_1}) = n + 3$ and $f(X) = n - 2$.

Proof of the Theorem 3.0.1 By hypothesis and by Proposition 2.9.5 part (i), we have $r \geq n + 4$ and $f(X) \leq n - 2$. If $r = n + 4$, then $f(X) = n - 2$. Indeed, if $r = n + 4$, then $r < 2n + 1$ because $n \geq 4$, so that by (2) we have $f(X) = \delta(X) + n - 3 \geq n - 2$, proving the claim.

Next, if $f(X) = n - 2$, then $s(X) = 2n + 1 - f(X) = n + 3$. So we may project generically to \mathbb{P}^{n+4} getting an image X' of X . Note that the projection $X \rightarrow X'$ is finite and bijective.

Claim 3.0.3 If X' is as in one of the cases (i)-(v), then also X is as in the corresponding case (i)-(v).

Proof of the Claim 3.0.3 The assertion is trivial if X' is as in (i), (iv) or (v).

Let us assume that X' is as in (ii), so that X' lies in a $(n + 1)$ -dimensional cone V' over a curve with vertex a $(n - 1)$ -space Π . Then X' is swept out by a 1-dimensional family \mathcal{F}' of varieties of dimension $n - 1$, each lying in a n -space, i.e., a space of a ruling of V' .

Suppose first that the general variety in \mathcal{F}' spans a n -space. Since the projection $X \rightarrow X'$ is finite and birational, then X is also swept out by the $(n - 1)$ -dimensional varieties of a family \mathcal{F} , which are in one-to-one correspondence with the varieties in \mathcal{F}' . Since the projection from X to X' is generic, also the projection of the general member of \mathcal{F} to the general member of \mathcal{F}' is finite and bijective, and they both span a n -space. These n -spaces, as well as their projections to \mathbb{P}^{n+4} , meet along a $(n - 1)$ -space, hence they form a cone V over a curve, in which X sits. So X is as in (ii).

Suppose next that the general variety in \mathcal{F}' does not span a n -space. This means that the general variety in \mathcal{F}' is a $(n - 1)$ -space \mathfrak{P} that intersects the vertex Π of V' along a $(n - 2)$ -space \mathfrak{P}' , and the same happens for the general member of \mathcal{F} . Let $\mathfrak{P}'_1, \mathfrak{P}'_2$ two general members of \mathcal{F}' . If $\mathfrak{P}'_1 = \mathfrak{P}'_2$, then all the $(n - 1)$ -spaces of \mathcal{F}' contain the same $(n - 2)$ -space of Π . This implies that X' is a cone over a curve, and then also X is a cone over a curve, a contradiction. So if $\mathfrak{P}'_1, \mathfrak{P}'_2$ are two general members of \mathcal{F}' , then $\mathfrak{P}'_1 \neq \mathfrak{P}'_2$. Thus $\mathfrak{P}'_1, \mathfrak{P}'_2$ intersect along a $(n - 3)$ -space. Since X' is a general projection of X , then also the members of \mathcal{F} intersect along a $(n - 3)$ -space. Then by [10, Lemma 4.1], one of the following cases occur:

- (a) the span of the linear spaces in \mathcal{F} is a k -space with $k \leq n + 2$;
- (b) all the linear spaces in \mathcal{F} contain the same $(n - 3)$ -space;
- (c) the general member of \mathcal{F} cuts a fixed $(n - 1)$ -space along a $(n - 2)$ -space.

Case (a) cannot happen, because then X would span at most a $(n+2)$ -space, a contradiction. In case (b), X would be a cone, as well as X' , a contradiction again. So only case (c) happens, which implies that X also lies on a cone V over a curve, which projects to V' .

The case in which X is as in (iii) is similar, hence we will be brief. Let us assume that X' is as in (iii), so that X' lies in a cone V' over a Veronese surface. Then X' is swept out by a 2-dimensional family \mathcal{F}' of varieties of dimension $n - 2$, each lying in a $(n - 1)$ -space, i.e. a space of the ruling of V' .

Suppose that the general variety in \mathcal{F}' spans a $(n - 1)$ -space. Then X is also swept out by the $(n - 2)$ -dimensional varieties of a family \mathcal{F} , whose generic projections are the varieties in \mathcal{F}' . So also the general variety of \mathcal{F} spans a $(n - 1)$ -space. These $(n - 1)$ -spaces, as well as their projections to \mathbb{P}^{n+4} , meet along a $(n - 2)$ -space, hence they form a cone V over a surface S , in which X sits. Since the cone V projects generically to the cone V' that is a cone over a Veronese surface, we deduce that also S is a Veronese surface, hence in fact $r = n + 4$, $X = X'$ and we are in case (iii).

Suppose next that the general variety in \mathcal{F}' does not span a $(n - 1)$ -space. Then the general variety in \mathcal{F}' is a $(n - 2)$ -space, and the same happens for the general member of \mathcal{F} . As above, one sees that two general members of \mathcal{F}' intersect along a $(n - 4)$ -space. Since X' is a general projection of X , then also the members of \mathcal{F} intersect along a $(n - 4)$ -space. Then by Lemma 4.1 in [10], one of the following cases occur:

- (a') the span of the linear spaces in \mathcal{F} is a k -space with $k \leq n + 1$;
- (b') all the linear spaces in \mathcal{F} contain the same $(n - 4)$ -space;
- (c') the general member of \mathcal{F} cuts a fixed $(n - 2)$ -space along a $(n - 3)$ -space.

As in the previous argument, cases (a') and (b') cannot happen. So only case (c') happens, which implies that X also lies on a cone V over a surface S , and we conclude as above that S is a Veronese surface, $r = n + 4$, $X = X'$ and we are in case (iii). \square

By Claim 3.0.3, by substituting X with X' , we may assume $r = n + 4$.

Let $Y = X \cap P$ be the section of dimension 3 of X with a general 7-dimensional linear subspace P . By Lemma 2.4.3, one has $f(Y) = 1$, so Y is a defective threefold which is non-degenerate in \mathbb{P}^7 , hence Y is in the list of Theorem 1.1 of [9].

Suppose first Y is a cone with vertex a point v over a surface S . Since $f(Y) = 1$, this implies that S is not defective (see Remark 2.4.2). Then we claim we are in case (i). In fact, if X has degree d , also Y has degree d and the vertex v of Y is a point of Y of multiplicity d . Since Y is a general 3-fold section of X , we see that v is also a point of multiplicity d for X , and therefore X has a locus Π of points of multiplicity d which is cut out by a general \mathbb{P}^7 in a point, so Π is a linear space of dimension $n - 3$, and X is a cone with vertex Π over the surface S , that is not defective.

Suppose Y is not a cone and sits in a 4-dimensional cone over a curve with vertex a plane Π . In this case we have $\gamma(Y) = 2$. Indeed, if p_0, p_1 are general points in Y , then Γ_{Y,p_0,p_1} consists of a pair of irreducible surfaces, each contained in one of the 3-spaces $\langle \Pi, p_0 \rangle$ and $\langle \Pi, p_1 \rangle$, and together spanning $\Pi_{Y,p_0,p_1} = \langle \Pi, p_0, p_1 \rangle$ which has dimension 4 (as dictated by Proposition 2.6.4, (iv)).

Let $x \in Y$ be a general point, which is also a general point of X . Consider the tangential projection $\tau_{X,x}$. The image X_1 of X has dimension $n - f(X) = 2$. The restriction of $\tau_{X,x}$ to Y coincides with $\tau_{Y,x}$, whose image Y_1 has also dimension $3 - f(Y) = 2$. This implies that $X_1 = Y_1$. On the other hand it is immediate to see that Y_1 is a cone over a curve. Then $t(X_1) = t(Y_1) = 1$. By Remark 2.7.2, we have $\gamma(X) = t(X_1) + f(X) = n - 1$.

Fix again two general points p_0, p_1 in Y , which are also general points on X . By iterated applications of Proposition 2.6.3, we have that $\Gamma_{Y,p_0,p_1} = \Gamma_{X,p_0,p_1} \cap P$. By the above description of Γ_{Y,p_0,p_1} (that implies that Γ_{Y,p_0,p_1} is reducible in two components containing p_0 and p_1) and by iterated applications of Proposition 2.6.3 that imply that $\Gamma_{X,p_0,p_1} \cap P \subseteq \Gamma_{Y,p_0,p_1}$, Γ_{X,p_0,p_1} is the union of a pair of irreducible Weil divisors on X , each passing through one of the points p_0, p_1 , each spanning at most a n -space, and together spanning Π_{X,p_0,p_1} which has dimension $n + 1$.

Suppose first the components of Γ_{Y,p_0,p_1} both span a 3-space, so that the components of Γ_{X,p_0,p_1} both span a n -space. Then X is swept out by a 1-dimensional family \mathcal{Z} of irreducible Weil divisors such that the general Z in \mathcal{Z} spans a n -space and two general such n -spaces span a $(n + 1)$ -space. Since X is non-degenerate in \mathbb{P}^{n+4} , this implies that all these n -spaces contain the same $(n - 1)$ -dimensional subspace Π .² By projecting X from Π we obtain a curve, so we are in case (ii).

Suppose next that the components of Γ_{Y,p_0,p_1} do not span a 3-space, so they are planes. These planes cannot pass through the same line, because then Y would be a cone over a curve against the fact that $f(Y) = 1$. So these planes pairwise meet at a point. Then X is swept out by a 1-dimensional family \mathcal{Z} such that the general Z in \mathcal{Z} is a $(n - 1)$ -space and two general such $(n - 1)$ -spaces span a $(n + 1)$ -space, hence they meet along a $(n - 3)$ -space. Then the same argument, based on Lemma 4.1 in [10] we made in Claim 3.0.3, implies that: either X is a cone with vertex a $(n - 3)$ -space, hence Y is also a cone, a contradiction; or X sits in a cone with vertex a $(n - 1)$ -space over a curve, and we are in case (ii) again.

² Here (and later) we use the following trivial fact: let \mathcal{S} be a set of n -spaces in \mathbb{P}^r such that any two spaces in \mathcal{S} intersect along a $(n - 1)$ -space (equivalently, they span a $(n + 1)$ -space). Then either all spaces in \mathcal{S} contain the same $(n - 1)$ -space or they are all contained in the same $(n + 1)$ -space.

Suppose now that Y is not a cone and is contained in a 4-dimensional cone V over $V_{2,2}$ with vertex a line L . Then we claim we are in case (iii). Again we have $\gamma(Y) = 2$. Indeed, if p_0, p_1 are general points in Y , then Γ_{Y,p_0,p_1} varies in a 2-dimensional family \mathcal{F} of irreducible surfaces spanning linear spaces of dimension 4 and contained in 3-dimensional quadric cones with vertex L over a conic of $V_{2,2}$. Precisely, if $f : Y \dashrightarrow V_{2,2}$ is the projection from the line L , then Γ_{Y,p_0,p_1} is contained in the 3-dimensional rank 3 quadric cone with vertex L over the unique conic of $V_{2,2}$ passing through the points $f(p_0)$ and $f(p_1)$. Let $S = Y \cap \Pi$ be the intersection of Y with a general hyperplane Π of $P \cong \mathbb{P}^7$, which spans the linear space Π of dimension 6 and sits in the cone over $V_{2,2}$ with vertex $v = \Pi \cap L$. Then S has a 2-dimensional family \mathcal{C} of irreducible curves spanning linear spaces of dimension 3 and contained in quadric cones with vertex v over a conic of $V_{2,2}$: they are the hyperplane sections of the surfaces in \mathcal{F} .

Claim 3.0.4 Let S be the surface as above. Then there is no irreducible curve Γ in Π containing v and such that the projection of S from a general point of Γ is a Veronese surface $V_{2,2}$.

Proof of the Claim 3.0.4 We argue by contradiction and assume there is such a curve Γ . Then for a general point $x \in \Gamma$, the surface S sits in the cone over $V_{2,2}$ with vertex x , and therefore it contains a family \mathcal{C}_x of curves spanning a linear space of dimension 3 and contained in a quadric cone with vertex x over a conic of $V_{2,2}$.

The family \mathcal{C}_x cannot be independent on x . Suppose in fact that $\mathcal{C}_x = \mathcal{C}$ is constant. Then if C is a general curve in \mathcal{C} , it sits in infinitely many quadric cones with vertices moving on Γ . Then C would have either degree 3 or 4. However, if $\text{deg}(C) = 4$ then C is the complete intersection of two quadrics in \mathbb{P}^3 , and it cannot lie on infinitely many cones, so this case is not possible. If $\text{deg}(C) = 3$, then C could lie on infinitely many quadric cones, but their vertices should move on C . This implies that $\Gamma = C$, which is impossible, since C is the general curve in \mathcal{C} .

Since the family \mathcal{C}_x depends on x , this implies that S should have a 3-dimensional family \mathcal{D} (described by all families \mathcal{C}_x when x moves on Γ) of irreducible curves spanning a linear space of dimension 3. Let then $p \in S$ be a general point, and consider the projection in \mathbb{P}^5 of S from p , whose image, birational to S by the Trisecant Lemma (see [26, Proposition 1.4.3]), we denote by S' . Then S' has a family of dimension 2 of irreducible plane curves, the images of the curves of \mathcal{D} passing through p . By the Trisecant Lemma the curves in question are conics and S' , spanning a \mathbb{P}^5 , is defective since it contains an irreducible conic passing through two general points of it. Then S' is the Veronese surface $V_{2,2}$, but this is a contradiction, because S' should also contain the line corresponding to the exceptional divisor of the blow-up at p . This contradiction proves the Claim. \square

Next we argue by induction on the dimension n . If $n = 4$, let \mathcal{P} a general pencil of hyperplanes in \mathbb{P}^8 . Fix general $P \in \mathcal{P}$, which cuts Y on X lying on a cone over $V_{2,2}$ with vertex a line L . If P' is another general element in \mathcal{P} it cuts X along Y' lying on a cone over $V_{2,2}$ with vertex a line L' . The surface section S of X with $P \cap P' := \Pi$ (which depends only on \mathcal{P}) a priori sits inside Π in two cones over $V_{2,2}$, one with vertex $v = L \cap \Pi$ and another with vertex $v' = L' \cap \Pi$. However, when P' moves in \mathcal{P} , by Claim 3.0.4 the point v' cannot move, so that $v = v'$ and therefore $L \cap L' = v$. This implies that the lines which are vertices of the 4-dimensional cones over $V_{2,2}$ in which the hyperplane sections of X sit, pairwise intersect each other, and this intersection is not a fixed point because the hyperplanes of \mathbb{P}^8 have no fixed point. Then all these lines sit in a plane, which is the vertex of a cone over $V_{2,2}$ containing X , so that we are in case (iii).

Suppose next we are in dimension $n > 4$ and let $Z = X \cap H$ be the intersection of X with a general hyperplane H . Then Z is irreducible and non-degenerate. Since X is not a

cone over a curve and neither a cone over $V_{2,2}$, then Z cannot be a cone over a curve, neither it can be a cone over $V_{2,2}$. The general threefold section Y of Z , which is also a general threefold section of X , lies in a 4-dimensional cone V over $V_{2,2}$. In particular, two general tangent spaces to V meet along a plane, hence two general tangent spaces to Y are contained in the same \mathbb{P}^6 . Thus Y is defective, so also Z is defective. It follows that Z satisfies all the assumption that have been assumed and discussed for X up to this point. By induction Z is in case (iii), hence it sits in a n -dimensional cone over $V_{2,2}$ with vertex a linear space T of dimension $n - 3$. If p_0, p_1 are general points in Z , then Γ_{Z,p_0,p_1} varies in a 2-dimensional family \mathfrak{F} of irreducible varieties of dimension $n - 2$ spanning linear spaces of dimension n and contained in quadric cones of dimension $n - 1$ and rank 3 with vertex T over a conic of $V_{2,2}$.

Let $\mathfrak{Y} = Z \cap \Lambda$ be the intersection of Z with a general subspace Λ of H of dimension $n + 2$. So that \mathfrak{Y} is contained in the cone over $V_{2,2}$ with vertex $W = \Lambda \cap T$ of dimension $n - 4$. Then \mathfrak{Y} has a 2-dimensional family \mathfrak{C} of irreducible varieties of dimension $n - 3$ spanning linear spaces of dimension $n - 1$ and contained in rank 3 quadric cones of dimension $n - 2$ with vertex W over a conic of $V_{2,2}$: they are the hyperplane sections of the varieties in \mathfrak{F} .

Claim 3.0.5 Let \mathfrak{Y} be the variety as above. Then there is no irreducible 1-dimensional family \mathfrak{G} of subspaces of dimension $n - 4$ in Λ , containing W such that the projection of V from a general subspace of \mathfrak{G} is a cone over the Veronese surface $V_{2,2}$.

Proof of the Claim 3.0.5 The proof is similar to the one of Claim 3.0.4, so we will be brief.

We argue by contradiction and assume there is such a 1-dimensional family \mathfrak{G} . Then for a general space $\Xi \in \mathfrak{G}$, \mathfrak{Y} sits in the cone over $V_{2,2}$ with vertex Ξ , and therefore it contains a family \mathfrak{C}_Ξ of irreducible varieties of dimension $n - 3$ spanning linear spaces of dimension $n - 1$ and contained in quadric cones of dimension $n - 2$ with vertex W over a conic of $V_{2,2}$.

As in the proof of Claim 3.0.4, one proves that \mathfrak{C}_Ξ is not constant. This implies that \mathfrak{Y} contains a 3-dimensional family \mathfrak{D} of varieties of dimension $n - 3$ spanning linear spaces of dimension $n - 1$. Let $p \in \mathfrak{Y}$ be a general point and consider the projection in \mathbb{P}^{n+1} of \mathfrak{Y} from p , whose image, birational to \mathfrak{Y} by the Trisecant Lemma, we denote by \mathfrak{Y}' . Then \mathfrak{Y}' has a family of dimension 2 of varieties of dimension $n - 3$ spanning linear spaces of dimension $n - 2$, i.e. the images of the varieties in \mathfrak{D} containing p . The general surface section S' of \mathfrak{Y}' spans a \mathbb{P}^5 and it has a 2-dimensional family of irreducible plane curves, so it is the Veronese surface $V_{2,2}$. It is well known that $V_{2,2}$ is not extendable (see [26, Corollary 2.4.5]), so \mathfrak{Y}' is a cone over $V_{2,2}$ with vertex a linear space of dimension $n - 5$. This is however impossible because \mathfrak{Y}' should contain the linear space of dimension $n - 3$ which is the image of the exceptional divisor corresponding to p , whereas the cone over $V_{2,2}$ with vertex a linear space of dimension $n - 5$ contains no such subspace. This contradiction proves the Claim. \square

At this point we can argue exactly as in the case $n = 4$ to see that the spaces T of dimension $n - 3$ vertices of the cones over $V_{2,2}$ containing the hyperplane sections Z of X , pairwise intersect each other in a variable space of dimension $n - 4$, hence they lie in a space of dimension $n - 2$, which is the vertex of a $(n + 1)$ -dimensional cone over $V_{2,2}$ containing X , so we are in case (iii).

Suppose that Y is a projection in \mathbb{P}^7 of the Veronese threefold $V_{3,2}$ and not a hyperplane section of the Segre variety $\text{Seg}(2, 2)$. Note that $\gamma(Y) = 1$ since the tangential contact locus to Y at two general points p_0, p_1 is the unique conic in $V_{3,2}$ passing through p_0, p_1 .

By taking into account Example 2.5 of [9], we may assume that Y is not the projection of $V_{3,2}$ from a secant (not tangent) line to it. We have to prove that we are in case (iv), and it suffices to do it when X has dimension 4. Indeed, if $n > 4$ then by arguing by induction we have that the general hyperplane section of X is a cone and therefore X itself is a cone.

We have $f(Y) = 1$ and $f(X) = 2$. Then $\gamma(X) \geq f(X) = 2$ (see Remark 2.9.2). On the other hand, by Proposition 2.6.3, we have $\gamma(X) \leq \gamma(Y) + 1$, and since $\gamma(Y) = 1$ we get $\gamma(X) \leq 2$ so $\gamma(X) = 2$. Then for a general choice of $p_0, p_1 \in X$, one has that Γ_{p_0, p_1} is a 2-dimensional quadric (see Proposition 2.9.5). Thus X is swept out by a family \mathcal{Q} of 2-dimensional quadrics such that, given two general points $p_0, p_1 \in X$, there is a quadric Q of \mathcal{Q} containing p_0, p_1 . Note that the general quadric Q is irreducible, since so is the conic passing through two general points of Y . Thus $\Gamma_{p_0, p_1} = \Psi_{p_0, p_1}$ because $\Psi_{p_0, p_1} \subseteq \Gamma_{p_0, p_1}$ (see Remark 2.7.2), Γ_{p_0, p_1} is irreducible, and both Ψ_{p_0, p_1} and Γ_{p_0, p_1} have the same dimension $f(X)$ (see Definition 2.7.1 and Proposition 2.6.4, (ii)).

Let x be a general point of X and consider the general tangential projection τ_x of X and its image X_1 , which is a non-degenerate surface in \mathbb{P}^3 . Given a general quadric Q in \mathcal{Q} , let us denote by Q_1 its image via τ_x . There are three possibilities:

- (a) Q_1 is a linear subspace;
- (b) Q_1 is a 2-dimensional quadric;
- (c) Q_1 is a conic.

Case (a) is not possible, because it would imply that the line joining two general points of X_1 lies in X_1 , so X_1 itself would be a linear space, a contradiction. Also case (b) is not possible. Indeed, this would imply that $X_1 = Q_1$ is a quadric. Since $\gamma(X) = f(X) = 2$, then Q_1 has to be a smooth quadric (see Remark 2.7.2). On the other hand X_1 equals Y_1 by Lemma 2.7.3. The surface Y_1 is in turn a projection to \mathbb{P}^3 of the Veronese surface $V_{2,2}$, which is the image of the tangential projection of the Veronese threefold $V_{3,2}$. The only way in which a projection of the Veronese surface $V_{2,2}$ can be a smooth quadric is if it is the projection from a secant (not tangent) line of $V_{2,2}$. But then Y would be the projection of $V_{3,2}$ from a secant (not tangent) line to it, contrary to our assumptions. Thus we are left with case (c), in which Q_1 is irreducible, otherwise Q itself would be reducible, a contradiction. So Q is a cone with vertex a point v lying in $T_{X,x}$. But then X is a cone, since its general tangent space contains v . Thus we have proved we are in case (iv).

Suppose finally that Y is a hyperplane section of the Segre variety $\text{Seg}(2, 2)$, and let H be the hyperplane spanned by Y in the \mathbb{P}^8 where $\text{Seg}(2, 2)$ sits. Recall that $\text{Seg}(2, 2)$ contains a 4-dimensional family \mathcal{F} of rank 4 quadric surfaces parameterized by $(\mathbb{P}^2)^* \times (\mathbb{P}^2)^*$: the general element $Q_{L,N}$ of \mathcal{F} is the image of $L \times N$, with L, N lines in the two factors.

Claim 3.0.6 In the above setting, H is not tangent to all quadrics in \mathcal{F} .

Proof of Claim 3.0.6 Suppose by contradiction H is tangent to all quadrics in \mathcal{F} . Then for general lines L, N of \mathbb{P}^2 , H would cut $Q_{L,N}$ in a pair of lines of the type $L \times \{x\}, \{y\} \times N$, with $x \in N, y \in L$. Fix L and let N vary. Then for a general $N \in (\mathbb{P}^2)^*$, H contains a unique line of the form $L \times \{x\}$, with $x \in N$. Let R_L be the set of all these points $x \in \mathbb{P}^2$ when N varies. By the above considerations, R_L has to intersect the general line N in one point, so R_L is itself a line. So we have a rational map

$$p : L \in (\mathbb{P}^2)^* \dashrightarrow R_L \in (\mathbb{P}^2)^*.$$

Now we have three possibilities:

- (1) the image of p is the whole of $(\mathbb{P}^2)^*$;
- (2) the image of p is a curve \mathcal{C} in $(\mathbb{P}^2)^*$;
- (3) p is constant.

In case (1), H would contain the whole of $\text{Seg}(2, 2)$, which is a contradiction. Indeed, let $x, y \in \mathbb{P}^2$ be general points. Consider the pencil \mathcal{P}_x of lines containing x , whose image via

p is a curve \mathcal{C}_x in $(\mathbb{P}^2)^*$. Hence we find a line in \mathcal{C}_x containing y , so H contains the point $(x, y) \in \text{Seg}(2, 2)$.

One finds the same contradiction also in case (2). Indeed, let again $x, y \in \mathbb{P}^2$ be general points. It is clear that the image of the pencil \mathcal{P}_x of lines containing x via p coincides with \mathcal{C} . So the same argument as above implies that H contains the point $(x, y) \in \text{Seg}(2, 2)$.

In case (3), H would contain the image of $\mathbb{P}^2 \times \mathbb{P}^1$ and, by symmetry, also of $\mathbb{P}^1 \times \mathbb{P}^2$, hence Y would be reducible, again a contradiction. This proves the claim. \square

We prove now that we are in case (v), and we may assume that X has dimension 4 (recall that $\text{Seg}(2, 2)$ is not extendable, see e.g. [26, Corollary 2.4.5]). Then X sits in \mathbb{P}^8 and, having Y as hyperplane section, it has degree 6. We have $f(X) = 2$ and $f(Y) = 1$. Moreover the general tangential projection Y_1 is equal to the general tangential projection of $\text{Seg}(2, 2)$, that is a smooth quadric in \mathbb{P}^3 (see [26, Corollary 5.4.2]). So by Remark 2.7.2, $\gamma(Y) = f(Y) = 1$. By Proposition 2.6.3 we have $\gamma(X) \leq 2$ and therefore $\gamma(X) = 2$ by (4). So, by Proposition 2.9.5(ii), X is swept out by a 4-dimensional family \mathcal{Q} of 2-dimensional quadrics such that, given two general points $p_0, p_1 \in X$, there is a unique quadric Q of \mathcal{Q} containing p_0, p_1 . The general quadric Q is irreducible, since so is the conic passing through two general points of Y , because of Claim 3.0.6.

Let x be a general point of Y , hence of X , and consider the general tangential projection τ_x of X and X_1 its image. This coincides with the image of a general tangential projection Y_1 of Y through x by Lemma 2.7.3. On the other hand, by Remark 2.7.4, this coincides with the image of a tangential projection of $\text{Seg}(2, 2)$, which is a smooth quadric in \mathbb{P}^3 (see [26, Corollary 5.4.2]). Hence X_1 is a smooth quadric surface. For a general quadric Q in \mathcal{Q} , let Q_1 be its image via τ_x . A priori, for Q_1 one of the possibilities (a), (b), (c) in p. 19 may happen. Again, case (a) is not possible, because it would imply that the line joining two general points of X_1 lies in X_1 , so X_1 would be a linear space, a contradiction. In case (c) the conic Q_1 is irreducible, otherwise Q would be reducible, a contradiction. So Q is a cone with vertex a point v lying in $T_{X,x}$. Since the image of Q is a conic under a general tangential projection of X , then the general tangent space contains the vertex v of Q , and this implies that X is a cone, a contradiction. So let us assume we are in case (b) in which Q_1 is an irreducible quadric, thus it coincides with X_1 , and therefore it is smooth. Moreover, since the general fibres of τ_x are quadrics in \mathcal{Q} , we conclude that two general quadrics in \mathcal{Q} intersect transversally at a point.

Let Q be the general quadric in \mathcal{Q} and let Π be its span. Let $\pi : X \dashrightarrow \mathbb{P}^4$ be the projection of X from Π .

Claim 3.0.7 The map $\pi : X \dashrightarrow \mathbb{P}^4$ is birational.

Proof of Claim 3.0.7 One argues as in [11, Claim 8.13]. We reproduce here shortly the argument for the reader’s convenience.

Let X' be the image of X via π . The restriction of π to a general quadric Q in \mathcal{Q} is birational to its image, which is a plane of \mathbb{P}^4 contained in X' . This yields that X' is a subspace of \mathbb{P}^4 and, being non-degenerate, then $X' = \mathbb{P}^4$.

Assume by contradiction that π is not birational. Then, if $x \in X$ is a general point, there is a point $y \in X$, distinct from x , such that $\pi(x) = \pi(y)$. There is some quadric $Q' \in \mathcal{Q}$ containing x and y . Then Q' , as the general $Q \in \mathcal{Q}$, has non-empty intersection with Π . Since the fibre of x in π is 0-dimensional, also the fibre of x in $\pi|_{Q'}$ is 0-dimensional. This implies that $\langle Q' \rangle \cap \Pi$ is only one point $z \in Q'$, and that the line $\langle x, z \rangle$ is not contained in Q' , so it intersects Q' only at x and z , contradicting $\pi(x) = \pi(y)$. \square

The restriction of π to the general hyperplane section Y of X is the projection of Y to \mathbb{P}^4 from the plane spanned by a conic of Y .

Claim 3.0.8 The image of Y via $\pi|_Y$ is a quadric of \mathbb{P}^4 .

Proof Let Ω be a 2-dimensional quadric in $\text{Seg}(2, 2)$ and consider the projection $g : \text{Seg}(2, 2) \dashrightarrow \mathbb{P}^4$ from the 3-space $\langle \Omega \rangle$. By the same argument as in Claim 3.0.7, we see that g is birational and maps a general quadric in $\text{Seg}(2, 2)$ to a plane. This yields that $g^{-1} : \mathbb{P}^4 \dashrightarrow \text{Seg}(2, 2)$ is defined by the linear system of quadrics containing two skew lines a, b in \mathbb{P}^4 and the planes images of the quadrics on $\text{Seg}(2, 2)$ are the planes intersecting in one point each of the lines a, b . Then, via g , the hyperplane section Y of $\text{Seg}(2, 2)$ is mapped to a quadric (containing a and b), and at the same time it is projected to this quadric from the plane of a conic contained in it. \square

Hence the birational map $\pi^{-1} : \mathbb{P}^4 \dashrightarrow X$ is defined by a linear system \mathcal{L} of quadrics.

Since the general hyperplane through Π is mapped to the general hyperplane of \mathbb{P}^4 , the hyperplanes of \mathbb{P}^4 , plus a fixed hyperplane P' , form a sublinear system of \mathcal{L} . This implies that $2P' \in \mathcal{L}$, hence the base locus B of \mathcal{L} sits in P' . Moreover P' has to be the image of the exceptional divisor of the blow-up of X along Q , and therefore P' is the image of the hyperplane P tangent to X along Q . Note that P cuts X along a threefold Z of degree 6 singular along Q .

The general quadric of Q maps via π to a plane which intersects P' along a line. The planes thus obtained vary in a 4-dimensional family \mathcal{Q}' with the property that given two general points x, y of \mathbb{P}^4 there is a unique plane of the family \mathcal{Q}' containing x, y . The line $\langle x, y \rangle$ cuts P' in a point z . There is then a unique secant line ℓ to the base locus B of \mathcal{L} passing through z (because of the uniqueness of the quadric in Q containing two general points of X), so that the unique plane of \mathcal{Q}' through x, y is $\langle x, y, \ell \rangle$. This implies that the intersection lines of the planes in \mathcal{Q}' with P' vary in a congruence \mathcal{C} of secants to B such that given a general point $z \in P'$, there is a unique line in \mathcal{C} containing z .

We claim that \mathcal{C} is a congruence of type $(1, 1)$. Let in fact S be a general plane in P' . We have to show that in S there is a unique line of the congruence \mathcal{C} . Note that S is the projection via π of a general subspace T of dimension 6 contained in P and containing Π . The intersection of T with X is a surface (of degree 6) consisting of Q counted twice, plus another quadric Q' . The lines of \mathcal{C} in S are just the images of Q' via the projection from Π . This implies that Q and Q' intersect along a line and the projection of Q' from Π is the unique line of the congruence \mathcal{C} sitting on S . This proves our claim, hence \mathcal{C} is given by all lines in P' which meet two skew lines L_1, L_2 of P' that, a priori, can be also infinitely near (see §1.2) which form the base locus B of \mathcal{L} .

In conclusion \mathcal{L} is the 8-dimensional linear system of quadrics in \mathbb{P}^4 passing through two skew lines. The skew lines cannot be infinitely near because, as we said, the image X_1 of a general tangential projection of X is a smooth quadric, whereas, if the two lines were infinitely near, it would be a cone. Finally the image of \mathbb{P}^4 via this linear system of quadrics is $\text{Seg}(2, 2)$ (see Claim 3.0.8). \square

Remark 3.0.9 In the hypotheses of Theorem 3.0.1, if in addition we assume that X is smooth, then $X = \text{Seg}(2, 2)$.

Indeed, since $s(X) = 2n + 1 - f(X) = n + 3$, if X is smooth, it can be isomorphically projected to a \mathbb{P}^{n+3} and Zak's theorem on linear normality (see Chapter I, Corollary 2.11 of [34]) implies $3n \leq 2(n + 2)$ and hence $n = 4$. Then the generic projection of X to \mathbb{P}^8 is a Severi variety (see again Definition 1.2 and Theorem 4.7, Chapter IV, [34]; see also [27]), whence the assertion.

4 Scrolls

In this section we will dispose of the classification of defective, non-degenerate 4-folds X in \mathbb{P}^r , which are scrolls in 3-spaces over a curve. The cases $f(X) \geq 2$ (and $r = 8$) have been dealt with in Proposition 2.9.5 and in Theorem 3.0.1. So we will assume here that $f(X) = 1$ and $r \geq 9$.

Before stating and proving the classification theorem, let us construct some examples.

Example 4.0.1 Let us consider a plane Π in \mathbb{P}^r , with $r \geq 9$. Consider an irreducible surface scroll Σ sitting in a $(r - 3)$ -space skew with Π . Let \mathcal{C} be the curve parameterizing the lines of Σ . Let $C \subset \Pi^*$ be an irreducible curve. Suppose there is a rational dominant map $\phi : C \dashrightarrow C$. Then we can consider the 4-fold scroll X which is the Zariski closure of the union of the 3-spaces joining a line of Σ , corresponding to a point c of \mathcal{C} where ϕ is defined, to the line corresponding to the point $\phi(c)$. Assume that $C \subset \Pi^*$ is not a line, so that the 3-spaces of the ruling of X do not pass through the same point, hence X is not a cone. Since the 3-spaces of the ruling of X pairwise meet at a point, then also two general tangent spaces to X meet so that X is defective.

Note that X is singular at all points where two 3-spaces of the ruling meet.

Example 4.0.2 Consider an irreducible 1-dimensional family \mathcal{P} of planes in \mathbb{P}^r , with $r \geq 9$, such that two general planes of the family intersect at a point, and it does not happen that either all the planes of \mathcal{P} pass through the same point or all the planes of \mathcal{P} intersect the same plane along a line. Then by Lemma 4.1 from [9] (Morin's theorem), the planes of \mathcal{P} span at most a 5-space.

Consider an irreducible curve $C \subset \mathbb{P}^r$ which has a dominant rational map $\phi : C \dashrightarrow \mathcal{P}$, and is in a sufficiently general position with respect to the planes of the family \mathcal{P} . Then consider the 4-dimensional scroll X which is the Zariski closure of the union of all 3-spaces joining a point $x \in C$ where ϕ is defined, with the plane corresponding to $\phi(x)$. Then X is not a cone and it is defective. As in Example 4.0.1, X is singular.

Example 4.0.3 Let $S \subset \mathbb{P}^9$ be an irreducible, projective surface, and let \mathcal{C} be an irreducible 1-dimensional family of curves on S not passing through the same point, such that the general curve in \mathcal{C} is irreducible, spans a 3-dimensional space and two general curves of \mathcal{C} intersect at a point. Let X be the 4-fold scroll which is swept out by the 3-dimensional spaces spanned by the curves of \mathcal{C} . Then X is not a cone and it is defective. As in Example 4.0.1, X is singular.

For a specific example of this type, consider the Veronese surface $V_{2,3} \subset \mathbb{P}^9$. It contains the images of the lines of \mathbb{P}^2 which are rational normal cubics, each spanning a 3-dimensional linear space. Let \mathcal{C} be an irreducible 1-dimensional family of these curves corresponding to a family of lines which do not pass through the same point.

Other infinitely many examples are obtained in the following way. Consider the Veronese 3-fold $V_{2,3} \subset \mathbb{P}^9$, which is defective, any two of its tangent spaces meeting at a point. We can view $V_{2,3}$ as the image of the *dual Veronese map* which sends a point H of the dual $(\mathbb{P}^3)^*$ of \mathbb{P}^3 (i.e., H is a plane of \mathbb{P}^3) to the quadric $2H$ in the \mathbb{P}^9 parameterizing the quadrics of \mathbb{P}^3 . Then the tangent space to $V_{2,3}$ at the point $2H$ consists of all the points of \mathbb{P}^9 corresponding to reducible quadrics of the form $H + H'$ with H' varying in $(\mathbb{P}^3)^*$. Consider any irreducible curve $\Gamma \subset (\mathbb{P}^3)^*$ and consider the 4-fold X which is swept out by the tangent spaces to $V_{2,3}$ at the points $2H$ corresponding to points H of Γ . We claim that X is of the type described above. Indeed, consider the surface $S \subset \mathbb{P}^9$, birational to the symmetric product $\Gamma(2)$ of the curve Γ , consisting of all points $H + H'$ with $H, H' \in \Gamma$. Then the tangent space to $V_{2,3}$ at the point $2H$, where $H \in \Gamma$, cuts S in the curve C_H , isomorphic to Γ , described by the

reducible quadrics $H + H'$, with H' varying in Γ . When H varies in Γ the curve C_H varies in a 1-dimensional family \mathcal{C} of curves on S , enjoying the required property.

Theorem 4.0.4 *Let $X \subset \mathbb{P}^r$, with $r \geq 9$, be a non-degenerate, linearly normal, defective 4-fold scroll in 3-spaces over a curve with $f(X) = 1$. Then one of the following cases occurs:*

- (i) X is a cone with vertex a point over a non-defective threefold;
- (ii) X is as in Example 4.0.1;
- (iii) X is as in Example 4.0.2;
- (iv) $r = 9$ and X is as in Example 4.0.3.

In all these cases X is singular.

Proof If X is a cone, the assumption $f(X) = 1$ implies that X has only one point as vertex and that its general hyperplane section is non-defective (see Lemma 2.4.3). Assume next that X is not a cone.

Claim 4.0.5 Two general 3-spaces of the ruling of X meet at a single point.

Proof of the Claim 4.0.5 Let $p_0, p_1 \in X$ be general points, and let P_0, P_1 be the 3-spaces of the ruling of X through p_0, p_1 respectively. Consider the tangential projection $\tau_{p_0} : X \dashrightarrow X_1$. Since X is defective with $f(X) = 1$, then $X_1 \subset \mathbb{P}^{r-5}$ is a non-degenerate 3-fold by (3). Since X is a scroll and X_1 is non-degenerate, the 3-spaces of the ruling of X are mapped by τ_{p_0} to planes, hence P_1 (that is contained in T_{X,p_1}) intersects T_{X,p_0} at a point p which is the unique intersection point of T_{X,p_0} and T_{X,p_1} because $f(X) = 1$ (see Remark 2.4.2). Similarly, P_0 has to intersect T_{X,p_1} at p , hence P_0 and P_1 have to intersect at p , and only at p . □

Let P be a general 3-space of the ruling of X . The intersection point of P with another general 3-space P' of the ruling cannot stay fixed as P' varies, because we are assuming X is not a cone. Then the Zariski closure of the union of these intersections is an irreducible curve C_P in P . Moreover if P and P' are general 3-spaces of the ruling of X , then C_P and $C_{P'}$ intersect at one point, i.e., the intersection point of P and P' .

If C_P is a line, then these lines pairwise meet at a point. Since we are assuming X is not a cone, then they lie in one and the same plane. Then we are in case (ii).

If C_P is not a line, but spans a plane, then these planes pairwise meet at a point but cannot pass through the same point because X is not a cone. So, by applying Lemma 4.1 from [9], either they intersect the same plane along a line, or they span at most a 5-space. In the former case we are again in case (ii) and the curves C_P are lines, contrary to our assumption. So we are in the latter case and in case (iii).

Finally, assume C_P spans P . Then take four general 3-spaces P_i of the ruling, for $1 \leq i \leq 4$. If $p_i = P \cap P_i$, for $1 \leq i \leq 4$, then the points p_1, \dots, p_4 are independent, so they span P . This implies that X is contained in $\langle P_1, P_2, P_3, P_4 \rangle$ which has dimension at most 9. On the other hand, we have $r \geq 9$, hence $r = 9$, and we are in case (iv). □

5 Top species: the irreducible case

5.1. Here we will consider the case of an irreducible, non-degenerate, defective variety $X \subset \mathbb{P}^r$ of dimension $n \geq 4$, with $r \geq 2n + 1$. Then we have $\delta(X) = f(X)$ (see Remark 2.1.1). The case $\delta(X) = n - 2$ is covered by Theorem 3.0.1. So we may assume $1 \leq \delta(X) \leq n - 3$.

In this section we will assume $f(X) = \delta(X) = \gamma(X)$. Hence, for $p_0, p_1 \in X$ general points, one has that $\Gamma_{p_0, p_1} = \Psi_{p_0, p_1}$ equals the general entry locus and it is a quadric of rank at least 2 in $\mathbb{P}^{\delta(X)+1}$ (see Proposition 2.9.5, (ii)). Therefore X is a, perhaps singular, LQEL variety (see §2.9). If $\delta(X) = 1$, then X presents the LCEL case. This happens when X is a variety of the top species.

By taking a general linear section Y of X of dimension $m = n - \delta(X) + 1$ we have $Y \subset \mathbb{P}^s$, with $s = r - \delta(X) + 1 \geq 2m + 1$, and $\delta(Y) = 1$ by Lemma 2.7.3. Hence the varieties X in question are extensions of varieties of the top species, thus presenting the LCEL case. Therefore, as a first instance, we will limit ourselves to consider only the top species case. In this section we will also stick to the *irreducible case*, i.e. the case in which the general entry locus is an irreducible conic.

5.2. From now on in this section we let $X \subset \mathbb{P}^r$ be a non-degenerate, defective variety of dimension $n \geq 4$ with $r \geq 2n + 1$. We will assume that X is of the top species, so that $\delta(X) = f(X) = \gamma(X) = 1$, and presents the irreducible case. By definition of LCEL variety (see p. 11), there is an irreducible family \mathcal{C} of dimension $2n - 2$ of generically irreducible conics such that, given two general points $p_0, p_1 \in X$, there is a unique conic of \mathcal{C} containing p_0, p_1 .

Lemma 5.2.1 *In the above setting, let $x \in X$ be a general point and let t be a general tangent direction to X at x . Then there is a unique irreducible conic of the family \mathcal{C} passing through x and tangent to t .*

Proof Let $X[2]$ be the Hilbert scheme of X parameterizing the 0-dimensional subschemes of X of length 2. Consider the set

$$\Omega = \{(\Gamma, \xi) \in \mathcal{C} \times X[2] : \xi \subset \Gamma\}$$

The second projection $p_2 : \Omega \rightarrow X[2]$ is birational, by definition of LCEL variety, and it is surjective by the projectivity of \mathcal{C} . Suppose, by contradiction, that, given x and t general, there is more than one conic through x tangent to t . Then, by Zariski’s Main Theorem, given a general non-reduced length 2 subscheme ξ of X (which is a smooth point of $X[2]$), there would be a positive dimensional family of conics in \mathcal{C} containing ξ . This would imply that the subset Ω' of Ω consisting of pairs (Γ, ξ) such that ξ is non-reduced has a component of dimension at least $2n$, since the locus of non reduced schemes of length 2 of X has dimension $2n - 1$. The first projection $p_1 : \Omega' \rightarrow \mathcal{C}$ has general fibres of dimension 1, so $p_1(\Omega')$ would have dimension at least $2n - 1$, a contradiction. The general conic in \mathcal{C} passing through x is smooth at x by hypothesis and its tangent direction at x is general by the unicity property proved above. In conclusion, there is a unique irreducible conic in \mathcal{C} passing through a general point x and having a general tangent direction t at x . □

If $x \in X$ is a general point, define the *Scorza map*

$$S_x : X \dashrightarrow T_{X,x}$$

sending a general point $y \in X$ to the point $T_{X,x} \cap T_{X,y}$. If $C_{x,y}$ is the unique conic in X passing through two general points x and y of X , then one has

$$S_x(y) = T_{C_{x,y},x} \cap T_{C_{x,y},y}.$$

Indeed, $T_{C_{x,y},x} \cap T_{C_{x,y},y}$ is a point, contained in $T_{X,x} \cap T_{X,y}$ and therefore coinciding with $T_{X,x} \cap T_{X,y}$, which is also a point. If there is no danger of confusion, we write S instead of S_x .

Proposition 5.2.2 *In the above setting, the Scorza’s map S_x is birational so that X is rational. Moreover its inverse S_x^{-1} is defined by a linear system of hypersurfaces of degree d having a point of multiplicity $d - 2$ at x .*

Proof We prove that S_x is birational by constructing its inverse S_x^{-1} . Indeed, let $y \in T_{X,x}$ be a general point. Let $r = \langle x, y \rangle$. By Lemma 5.2.1, there is a unique irreducible conic $C \in \mathcal{C}$ passing through x and tangent to r at x . Let s be the tangent line to C through y other than r . It is tangent to C at a point z . Then $z = S_x^{-1}(y)$. This proves that S_x is birational.

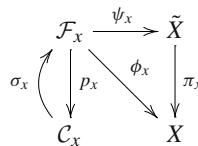
As for the rest of the assertion, it follows from the fact that S_x^{-1} maps a general line through x to a conic on X . □

The main point of our analysis is to study the Scorza map. Specifically, following Scorza’s ideas in [28], we will prove that, under suitable hypotheses, its inverse is defined by a linear system of quadrics.

5.3. We will study now the family of conics \mathcal{C} . In doing so, we will use some ideas contained in [20], where however X is assumed to be smooth.

Let $x \in X$ be a general point. We will denote by $p_x : \mathcal{F}_x \rightarrow \mathcal{C}_x$ the universal family of conics in \mathcal{C} passing through x . One has $\mathcal{F}_x \subseteq \mathcal{C}_x \times X$. Let $\phi_x : \mathcal{F}_x \rightarrow X$ be the projection to the second factor, which is birational. Then \mathcal{F}_x is irreducible, so also \mathcal{C}_x is irreducible with $\dim(\mathcal{C}_x) = n - 1$ and $\dim(\mathcal{F}_x) = n$.

The map p_x has a natural section σ_x mapping a point of \mathcal{C}_x corresponding to a conic C , to the point $x \in C \subset \mathcal{F}_x$. The image \mathcal{E}_x of σ_x is contracted by ϕ_x to the point x . Consider the blow-up $\pi_x : \tilde{X} \rightarrow X$ of X at x . So we have the following commutative diagram



where ψ_x is a birational map whose indeterminacy locus is contained in \mathcal{E}_x . Let S_x be the closed subscheme of \mathcal{C}_x parameterizing those conics in \mathcal{C}_x which are singular at x . We will denote by \mathcal{D}_x the image of S_x via the section σ_x . Then we claim that \mathcal{D}_x is exactly the indeterminacy locus of ψ_x . In fact \mathcal{D}_x is clearly contained in the indeterminacy locus of ψ_x . To prove that ψ_x is defined off \mathcal{D}_x , one first remarks that \mathcal{E}_x is the preimage of x via ϕ_x . Then [19, Lemma 4.3] assures that \mathcal{E}_x is a Cartier divisor at points off \mathcal{D}_x . So we restrict ψ_x to the open subset $U_x = \mathcal{F}_x \setminus \mathcal{D}_x$ and we have that the schematic preimage of x via ϕ_x is a Cartier divisor on U_x . Then we can apply the universal property of the blow-up to conclude that ψ_x is defined on U_x .

Thus ψ_x is defined at the general point of \mathcal{E}_x and its inverse is defined at the general point of the exceptional divisor E_x of the blow-up $\pi_x : \tilde{X} \rightarrow X$. Hence ψ_x induces a birational map $\psi_{x,0} : \mathcal{E}_x \dashrightarrow E_x$ (see also Lemma 5.2.1). The geometric meaning of the map $\psi_{x,0}$ is clear: it associates to a point $(c, x) \in \mathcal{E}_x \setminus \mathcal{D}_x$ the point of E_x corresponding to the direction of the tangent line at x to the conic C corresponding to the point $c \in \mathcal{C}_x$.

Let D_x be the closure in E_x of the points corresponding to all directions at x of irreducible components of conics in S_x . We set

$$\chi(X) = \dim(D_x),$$

which is independent on $x \in X$. Note that $D_x \subseteq L_x$, where L_x was introduced in §2.10, thus $\chi(X) \leq \ell(X)$. Given a point in S_x corresponding to a conic C singular at x the plane Π_C

spanned by C is tangent to X at x and the points corresponding to the directions of the lines in Π_C through x describe a line r_C in E_x . It may either happen that C has rank 2 or rank 1. In the former case we get two distinct points $p, q \in D_x$ and the line r_C joins the points p and q . In the latter case C consists of a line Z counted with multiplicity 2, whose direction corresponds to a unique point $z \in E_x$. The plane Π_C still gives rise to a line r_C in E_x through z . The following lemma shows that something similar to the rank 2 case for C happens in this situation (recall the definitions and notation from §2.10).

Lemma 5.3.1 *In the above setting, i.e., when C consists of a line Z counted with multiplicity 2, the line r_C belongs to the Zariski tangent space to the scheme L_x at the point z . Every point in L_x is a base point for Π_x and all quadrics in Π_x are tangent to r_C at z .*

Proof The plane Π_C is tangent to X at the general point of the line Z . Therefore we have an inclusion of normal sheaves $N_{Z, \Pi_C} \rightarrow N_{Z, X}$, hence an inclusion $N_{Z, \Pi_C}(-x) \rightarrow N_{Z, X}(-x)$, which induces an inclusion $H^0(Z, N_{Z, \Pi_C}(-x)) \subseteq H^0(Z, N_{Z, X}(-x))$. The 2-dimensional space $H^0(Z, N_{Z, \Pi_C}(-x))$ is the tangent space to the pencil of lines through the point x in the plane Π_C , whereas $H^0(Z, N_{Z, X}(-x))$ is the Zariski tangent space to L_x at z . So this proves the first assertion. Every point in L_x is a base point for Π_x (see §2.10). As for the final assertion, consider any quadric Q in Π_x . Then there is a hyperplane section H of X tangent at x , having a double point at x with the projectified tangent cone at x coinciding with Q . The hyperplane cutting out H contains the plane Π_C which is tangent also to H along Z . This implies that the quadric Q has intersection multiplicity at least 2 with r_C at z , as wanted. \square

Let us denote by $J_{X,x}$, or simply by J_x , the Zariski closure of the union of all tangent spaces to L_x in E_x at the points of D_x .

Lemma 5.3.2 *In the above setting, let $D \subseteq E_x \setminus D_x$ be an irreducible complete curve contracted to a point by $\psi_{x,0}$. Then the general point $(c, x) \in D$ corresponds to a rank 2 conic $C = N + M$, where N is a fixed line through x and M describes a cone not containing x , with vertex a point of N .*

If $q \in E_x \setminus D_x$ and if there is an irreducible curve D in $E_x \setminus D_x$ such that $\psi_{x,0}(D) = \{q\}$, then D is not complete and $q \in J(D_x, D_x) \cup J_x$.

Proof We shall adapt to our situation an argument of Kebekus (see [21, Proof of Theorem 3.4]). Let $\psi_{x,0}(D) = p$. Take \tilde{D} , the normalization of D , and consider the scheme Y over \tilde{D} which is gotten by base-change from our family.

Suppose first the general fiber of $\pi_Y : Y \rightarrow \tilde{D}$ is irreducible, so that Y is an irreducible surface. Let $D_0 \subset Y$ be the section of $\pi_Y : Y \rightarrow \tilde{D}$ induced by σ_x . Note that Y is smooth along D_0 by the hypothesis that $D \cap D_x$ is empty. Since the induced morphism contracts D_0 to the point $x \in X$, the restriction of the differential of $\phi_Y : Y \rightarrow X$ induces a morphism $T_{\phi_Y} : N_{D_0, Y} \rightarrow l_p \simeq \mathbb{C}$, where l_p is the line in $T_x(X)$ corresponding to $p \in E_x$. Since $N_{D_0, Y}$ is not trivial ($\phi_Y(D_0) = x$ implies $D_0^2 < 0$ the map T_{ϕ_Y} is not constant, so it has a zero. The corresponding curve of the family is singular at x . This is a contradiction (here we used Kebekus' argument).

Suppose now that every fiber of $\pi_Y : Y \rightarrow \tilde{D}$ is reducible, consisting of two irreducible components each one isomorphically mapping to different lines contained in X . Only one of these two lines, call it N , contains x and has the direction corresponding to p . So N does not move. Let M be the other line, which varies in a 1-dimensional family. By the assumption $D \cap D_x = \emptyset$, the intersection point y of N with M has to stay fixed when M varies. Then Y has an irreducible component mapping to a cone with vertex y not passing through x .

Let $q \in E_x$ be an indeterminacy point of $\psi_{x,0}^{-1}$, and let $D \subseteq \mathcal{E}_x \setminus \mathcal{D}_x$ be an irreducible curve such that $\psi_0(D) = q$. Assume that $q \notin J_x$, in particular $q \notin D_x$. Then the general point $(c, x) \in D$ corresponds to an irreducible conic C containing x , whose tangent line has direction corresponding to q . By Kebekus' argument that we already used above, D cannot be complete, hence there is some point $(c_0, x) \in \overline{D} \cap \mathcal{D}_x$. Let C_0 be the conic corresponding to c_0 , which is singular at x . Then the direction corresponding to q has to belong to the Zariski tangent space to C_0 at x . If C_0 has rank 1, then $q \in J_x$ by Lemma 5.3.1, a contradiction. Then C_0 has rank 2, and, by the above argument, $q \in J(D_x, D_x)$. \square

Next we will restrict our attention to the case $\chi(X) \leq 0$, i.e. $\dim(D_x) \leq 0$.

Lemma 5.3.3 *Suppose $\dim(D_x) \leq 0$. Then $\psi_{x,0}^{-1}$ induces an isomorphism between $E_x \setminus (J(D_x, D_x) \cup J_x)$ and its image, so that for every point $q \in E_x \setminus (J(D_x, D_x) \cup J_x)$ there is a unique conic in \mathcal{C}_x with tangent line at x having direction corresponding to q .*

Proof Let

$$\begin{array}{ccc}
 \tilde{\mathcal{E}}_x & & \\
 \alpha \downarrow & \searrow \tilde{\psi}_{x,0} & \\
 \mathcal{E}_x & \longrightarrow & E_x,
 \end{array}$$

be a resolution of $\psi_{x,0} : \mathcal{E}_x \dashrightarrow E_x$.

Consider a point $q \in E_x \setminus (J(D_x, D_x) \cup J_x)$, and suppose that $\psi_{x,0}^{-1}$ is not defined at q . Then also $\tilde{\psi}_{x,0}^{-1}$ is not defined at q . By Zariski's Main Theorem, there is an irreducible, complete curve $C \subseteq \tilde{\psi}_{x,0}^{-1}(q)$. Let $D = \alpha(C)$, which we may assume to be a curve by the assumption that $\psi_{x,0}^{-1}$ is not defined at q . Clearly D cannot be contained in \mathcal{D}_x , which is a finite set, since by assumption D_x is finite. This gives a contradiction to Lemma 5.3.2.

Thus $\tilde{\psi}_{x,0}^{-1}$ is defined at q , hence also $\psi_{x,0}^{-1}$ is defined at q . If $(c, x) = \psi_{x,0}^{-1}(q)$, then the conic C corresponding to c is smooth at x , with tangent line having the direction corresponding to q . \square

If $\chi(X) \leq 0$ one has

$$\dim(J(D_x, D_x)) \leq 1. \tag{13}$$

We also need an estimate on the dimension of J_x .

Lemma 5.3.4 *If $\chi(X) \leq 0$, one has $\dim(J_x) \leq n - 3$.*

Proof Since $\dim(D_x) \leq 0$, J_x is a union of finitely many linear spaces $T_{L_x,z}$ with $z \in D_x$. We have to prove that none of these spaces has dimension $m \geq n - 2$.

Suppose first there is a point $z \in D_x$ such that $T_{L_x,z}$ has dimension $n - 1$, i.e. it coincides with E_x . Since $T_{L_x,z} \subseteq T_{B_x,z} \subseteq T_{Q,z}$ for all quadrics Q in $\Pi_{X,x}$, we deduce that all quadrics in $\Pi_{X,x}$ are cones whose vertex contains z and hence that the map defined by $\Pi_{X,x}$ on E_x has fibres of positive dimension. Since this map coincides with $\tilde{\tau}_{x|E_x} : E_x \dashrightarrow X_1$ (see Lemma 2.10.1 and remember that $f(X) = 1$ so that the general fibre of τ_x is an irreducible conic), which is generically finite, we get a contradiction.

Next, suppose that there is a point $z \in D_x$ such that $T_{L_x,z}$ is a hyperplane in E_x . Then if a quadric in $\Pi_{X,x}$ has a double point at a general point $y \in E_x$, then it is singular along the whole line $\langle y, z \rangle$. Since the quadrics in $\Pi_{X,x}$ singular at y correspond to hyperplanes tangent to the image of E_x at the image of y , the general Gauss fibre of the image of $\Pi_{X,x}$

has positive dimension. Since the image of E via the map determined by $\Pi_{X,x}$ is X_1 , we find a contradiction because, being $\gamma(X) = f(X)$, the Gauss map of X_1 is generically finite (see Remarks 2.7.2 and 2.9.2). \square

5.4. We go back now to the study of Scorza’s map S_x . If $\chi(X) \leq 0$, by (13) and Lemma 5.3.4, a general line $M \subset E_x = \mathbb{P}^{n-1}$ does not intersect $J(D_x, D_x) \cup J_x$. Moreover, the line M does not intersect the base locus B_x of Π_x because $\dim(B_x) \leq n - 3$. Indeed, the linear system of quadrics Π_x cannot have a fixed component because this would be a hyperplane in E_x . Then the map induced by Π_x would be a linear projection being given by a linear system of hyperplanes and X_1 , coinciding with the image of E_x via a linear projection, would be a linear space. This is not possible, because X_1 is non-degenerate and strictly contained in \mathbb{P}^{r-n-1} (see Remark 2.7.2). In particular, M does not intersect L_x that is contained in B_x .

Hence, by Lemma 5.3.3, $L = \psi_{x,0}^{-1}(M)$ is a smooth irreducible rational curve such that for every point $(c, x) \in L$ the conic C corresponding to c is smooth. By taking the universal family over L we obtain a smooth surface S with a morphism $\beta : S \rightarrow L = \mathbb{P}^1$, such that every fiber F of β is irreducible and maps to a conic in X through x via the tautological morphism $\phi_S : S \rightarrow X$, which is clearly birational onto its image $Y = \phi_S(S)$. The surface S is isomorphic to a surface \mathbb{F}_e , with $e \geq 0$. The section E_0 with $E_0^2 = -e$ is contracted to x on Y , hence $e \geq 1$.

The next lemma is the main tool for the study of Scorza’s map.

Lemma 5.4.1 *In the above setting, suppose again $\chi(X) \leq 0$. Then Y is a Veronese surface $V_{2,2}$.*

Proof First of all we claim that $T_{X,x} \cap Y = \{x\}$. Indeed, suppose $z \in Y$ is different from x . Then there is a smooth conic C on Y containing x and z . The line $\langle x, z \rangle$ intersects transversally C at x . If $z \in T_{X,x}$, then the plane $\langle C \rangle$ would be contained in $T_{X,x}$ because it contains two distinct tangent lines to X at x , namely $\langle x, z \rangle$ and the tangent line to C at x . Moreover the tangent line to C at x has intersection multiplicity larger than 2 with X at x (because it is the limit of lines tangent at x and secant to X at a general point of C) and therefore the point in E_x corresponding to its direction sits in the base locus B_x of Π_x , a contradiction.

Consider now the tangential projection $\tau_x : X \dashrightarrow X_1$. The restriction of τ_x to Y is thus defined on $Y \setminus \{x\}$ and $\tau_x(Y)$, which is a curve because the irreducible conics through x contained in Y are contracted to points by τ_x , coincides with $\tilde{\tau}_x(M)$, which, by the genericity of M , is a conic Γ . This yields $\dim(\langle Y \rangle) \geq 5$, because Y is projected from the span of its tangent cone at x , which is at least a plane.

Let $E_1 \subset S$ be the general curve in the linear system $|E_0 + eF|$, where F is a fibre of the ruling of $S \cong \mathbb{F}_e$ and E_0 is the negative section. Then E_1 does not intersect E_0 , therefore $\phi_S(E_1)$ does not pass through x so that it is projected via τ_x onto Γ . Moreover the restriction of τ_x to $\phi_S(E_1)$ is birational, because the pull-back to S of the fibres of the restriction of τ_x to Y are the curves in $|F|$ and $F \cdot E_1 = 1$. This yields $\deg(\phi_S(E_1)) = 2$. Let H be a general divisor in the linear system $|\phi_S^*(\mathcal{O}_Y(1))|$. Then $H \equiv 2E_0 + bF$ (remember that the curves in $|F|$ are mapped to conics on Y , so $H \cdot F = 2$). Since $0 = H \cdot E_0 = -2e + b$, we have $H \equiv 2(E_0 + eF)$. Moreover, as we saw, $2 = H \cdot E_1 = 2E_1^2 = 2e$, hence $e = 1$. This implies the assertion because $\dim(\langle Y \rangle) \geq 5$. \square

We can now prove our main result about Scorza’s map which can be seen as a strong improvement of Lemma 2.10.1.

Theorem 5.4.2 *Let $X \subset \mathbb{P}^r$, with $r \geq 2n + 1$, be a non-degenerate, defective variety, of dimension $n \geq 4$. Assume that X is of the top species, presents the irreducible case and*

$\chi(X) \leq 0$. Then S_x^{-1} is defined by a linear system of quadrics, $r \leq \frac{n(n+3)}{2}$ and X is a projection to \mathbb{P}^r of the Veronese variety $V_{2,n}$.

Proof By Lemma 5.4.1, the restriction of S_x^{-1} to a general plane Π through x maps Π to a Veronese surface $V_{2,2}$. This implies that S_x^{-1} is defined by a linear system of quadrics (see Proposition 5.2.2), whence the claim follows. \square

5.5. The previous results suffice for the classification of defective 4-folds of the top species presenting the irreducible case. Recall the definition of $b(X)$ from §2.10.

Proposition 5.5.1 *Let $X \subset \mathbb{P}^r$, with $r \geq 9$, be a non-degenerate, defective variety, of dimension 4. Assume that X is of the top species and presents the irreducible case. Then $b(X) \leq 1$ and if the equality holds then, for a general point $x \in X$, one has:*

- (i) either B_x is a smooth conic and $r = 9$;
- (ii) or B_x is a line and $r \leq 11$;
- (iii) or B_x is the union of a line L and of a point $p \notin L$, and $r \leq 10$.

In cases (ii) and (iii), one has $\epsilon(X) \geq 2$.

Proof By hypothesis, $\dim(X_1) = 3$ and the general fiber of the general tangential projection $\tau_x : X \dashrightarrow X_1 \subset \mathbb{P}^{r-5}$ is an irreducible conic. Then $\tilde{\tau}_x(E_x) = X_1 \subset \mathbb{P}^{r-5}$ (see Lemma 2.10.1) so that $\dim(\Pi_x) = r - 5 \geq 4$. Therefore the positive dimensional irreducible components of $B_x \subset E_x = \mathbb{P}^3$, which are one dimensional because Π_x has no fixed components (see §5.4), are necessarily lines or conics. Indeed, an irreducible non-degenerate curve $A \subset \mathbb{P}^3$ has degree at least three and $h^0(\mathcal{I}_A(2)) \leq 3$ (see also [7, Theorem 6.1]). If the irreducible component is not a line, then it is a plane curve defined by quadric surfaces in \mathbb{P}^3 and hence a conic.

If B_x contains a conic C , then $\Pi_x \subseteq |H^0(\mathcal{I}_{C,E_x}(2))| = \mathbb{P}^4$ together with $\dim(\Pi_x) = r - 5 \geq 4$ implies that equality holds and hence that $B_x = C$. Therefore X_1 is a quadric hypersurface in \mathbb{P}^4 , being the image of the second fundamental form of X at x by Lemma 2.10.1. Since X is of the top species, i.e., $\gamma(X) = f(X) = 1$, then X_1 is smooth (recall that $\gamma(X) = t(X_1) + f(X)$, see Remark 2.7.2) and hence C is smooth.

If B_x contains a line L , then it cannot contain any other line. Indeed, as we have shown above, B_x cannot contain any other line intersecting L because in this case it would be a reducible conic C and X_1 would be an irreducible quadric cone, yielding $\gamma(X) = f(X) + 1 > f(X)$ contrary to our assumption. Since the linear system of quadrics containing two skew lines has only dimension 3, then B_x cannot contain any line skew with L . Suppose there is a point $p \in L$ such that all quadrics in Π_x are tangent to a given plane Π at p . Then if $q \in E_x$ is a general point, the quadrics in Π_x singular at q are singular at p (the tangent space at p contains $\langle \Pi, q \rangle = E_x$) and hence they must be singular along the whole line $\langle p, q \rangle$. This would imply that X_1 has general Gauss fiber of positive dimension (by the argument used in the second part of the proof of Lemma 5.3.4), against the hypothesis that X is of the top species, i.e., $\gamma(X) = 1$. If there are two distinct points $p, q \in B_x$ off L , again X_1 would be a quadric cone in \mathbb{P}^4 , which leads again to a contradiction. In fact, the line $L' = \langle p, q \rangle$ has to be skew with L , otherwise it is contained in B_x , a contradiction. Moreover the map determined by the linear system of quadrics Π_x , contracts L' to a point $v \in X_1 \subset \mathbb{P}^4$, and the projection of X_1 from v to \mathbb{P}^3 is the image of $E_x = \mathbb{P}^3$ via the linear system of quadrics containing L and L' , that is well known to be a smooth quadric in \mathbb{P}^3 . This proves that only cases (i), (ii), (iii) are possible. In case (i), one has $\dim(\Pi_x) = 4$ and $r = 9$. In case (ii), one has $\dim(\Pi_x) \leq 6$ and $r \leq 11$. In case (iii), one has $\dim(\Pi_x) \leq 5$ and $r \leq 10$.

In cases (ii), (iii), X_1 turns out to be a scroll in planes, and therefore $d(X_1) > 0$ (see §2.9). This proves the final assertion (see Remark 2.9.2). \square

We can now prove the classification theorem for defective 4-folds of the top species presenting the irreducible case. It suffices to assume the 4-fold to be linearly normal.

Theorem 5.5.2 *Let $X \subset \mathbb{P}^r$, with $r \geq 9$, be a non-degenerate, linearly normal, defective variety, of dimension 4. Assume that X is of the top species and presents the irreducible case. Then X is one of the following:*

- (i) *an internal projection of the Veronese 4-fold $V_{2,4} \subset \mathbb{P}^{14}$ from finitely many points with the property that $\ell(X) \leq 0$;*
- (ii) *the projection of $V_{2,4} \subset \mathbb{P}^{14}$ from the plane spanned by a conic on it;*
- (iii) *the projection of $V_{2,4} \subset \mathbb{P}^{14}$ from a 4-space spanned by a rational normal quartic curve on it;*
- (iv) *a hyperplane section of $\text{Seg}(2, 3) \subset \mathbb{P}^{11}$.*

Proof If $\ell(X) \leq 0$, then also $\chi(X) \leq 0$. Then, as we saw in Theorem 5.4.2, S_x^{-1} is defined by a linear system Λ of quadrics in \mathbb{P}^4 and the base locus B of Λ is clearly finite, because $\ell(X) = 0$. Thus we are in case (i).

Assume that $\ell(X) > 0$. Since $L_x \subseteq B_x$, we have $b(X) \geq \ell(X)$, Proposition 5.5.1 yields $\ell(X) = 1$.

Suppose S_x^{-1} is still defined by a linear system Λ of quadrics (with a base locus B), which is the case if $\chi(X) \leq 0$ (see Theorem 5.4.2). If $z \in \mathbb{P}^4$ is a general point, mapped by S_x^{-1} to a general point $y \in X$, then Π_y is projectively equivalent to the projection in \mathbb{P}^3 from z of the linear system of cones in Λ with vertex in z . By Proposition 5.5.1 and since $\ell(X) = 1$, one has $b(X) = 1$ and:

- (a) either B is a smooth conic;
- (b) or B is a line;
- (c) or B is the union of a line L and of a point $p \notin L$.

Cases (a), (b) lead to cases (iii) and (ii) respectively because $S_x^{-1} : \mathbb{P}^4 \dashrightarrow X \subset \mathbb{P}^r$ is given by the complete linear system of quadrics vanishing on a conic in \mathbb{P}^4 in case (a), respectively on a line in \mathbb{P}^4 in case (b) (the linear systems are complete because X is linearly normal). In case (c), X is the image of \mathbb{P}^4 by the complete linear system of quadrics vanishing on a line and at a point so that X is an internal projection of the 4-fold appearing in case (ii) and it sits in \mathbb{P}^{10} (again the linear systems are complete because X is linearly normal). It is easy to compute $\deg(X) = 10$. Indeed, this degree is the number of intersection points of four general quadrics in \mathbb{P}^4 containing a line L and a point $p \notin L$, off p and L . This is the same as computing the self intersection minus 1 of the linear system cut out on a smooth Del Pezzo surface S of degree 4 in \mathbb{P}^4 (intersection of two general quadrics in \mathbb{P}^4 through L) by the quadrics containing the line L of S : one easily checks that this self-intersection is 11. On the other hand X also sits on $\text{Seg}(2, 3)$, since Λ is the sum of two linear systems of hyperplanes: Λ_1 given by the hyperplanes containing the line L ; Λ_2 given by the hyperplanes containing the point p , with dimension 2 and 3 respectively. Indeed, Λ_1 yields a rational map $\phi_1 : \mathbb{P}^4 \dashrightarrow \mathbb{P}^2$ while Λ_2 defines a rational map $\phi_2 : \mathbb{P}^4 \dashrightarrow \mathbb{P}^3$. Hence $\phi_1 \times \phi_2 : \mathbb{P}^4 \dashrightarrow \mathbb{P}^2 \times \mathbb{P}^3 \subset \mathbb{P}^{11}$ is given by a linear system of quadrics vanishing on L and p , that is in Λ , proving the claim. Hence X is contained in a hyperplane section of $\text{Seg}(2, 3)$. As we saw, $\deg(X) = 10$. Since also $\deg(\text{Seg}(2, 3)) = 10$, we see that X coincides with a hyperplane section of $\text{Seg}(2, 3)$, so we are in case (iv).

Assume now that S_x^{-1} is not defined by a linear system of quadrics, so that $\chi(X) = \ell(X) = 1$ and $b(X) = 1$.

To treat this case we need a preliminary remark. Since $\ell(X) = 1$, there is an irreducible component \mathcal{R} of the Hilbert scheme of lines in X of dimension 4, thus, by [24, Theorem 2], we have the following possibilities:

- (1) either X is swept out by a 1-dimensional family \mathcal{Q} of 3-dimensional quadrics generically of rank 5;
- (2) or X is swept out by a 2-dimensional family \mathcal{P} of planes;

and the lines of the family \mathcal{R} are the lines contained in the quadrics of \mathcal{Q} in case (1), the lines contained in the planes of \mathcal{P} in case (2).

By Proposition 5.5.1, in case (1) we have only one quadric Q of the family \mathcal{Q} passing through the general point $x \in X$ (because the lines contained in such a Q through x are parametrized by a conic $C_x \subseteq L_x \subseteq B_x$) and in case (2) we have only one plane P of the family \mathcal{P} passing through x (because the lines contained in such a P through x are parametrized by a line $\ell_x \subseteq L_x \subseteq B_x$).

Let us now set $B := B_x$. Then by Proposition 5.5.1 we have for B the three cases (a), (b) and (c).

Case (a) corresponds to case (1) above, i.e. $X \subset \mathbb{P}^9$ is ruled by a 1-dimensional family \mathcal{Q} of generically smooth 3-dimensional quadrics. Let $\psi : \tilde{X} \rightarrow X$ be the tautological morphism associated to the universal family $\alpha : \tilde{X} \rightarrow \mathcal{Q}$. Then ψ is birational since, as we saw, there is only one quadric of the family \mathcal{Q} passing through the general point $x \in X$. Let $x \in X$ be a general point, which can be seen as a general point of \tilde{X} . Recall that X possesses a family \mathcal{C} of (generically irreducible) conics such that given two general points of X there is a conic of \mathcal{C} containing them. Consider the strict transform \tilde{C}_x on \tilde{X} of the family \mathcal{C}_x of conics of \mathcal{C} through x . Let $D \subset \tilde{X}$ be a Cartier divisor which is the pull-back via α of a divisor in a base point free linear system on \mathcal{Q} . If C corresponds to the general point of \tilde{C}_x , then $C \cdot D > 0$. On the other hand the proper transform C' on \tilde{X} of a reducible conic of \mathcal{C} with a double point at $x \in X$ is not contained in the exceptional locus of ψ by the generality of x and it is contained in the unique element Q_x of $\tilde{\mathcal{Q}}$ (that is the family of fibres of α) passing through x (because Q_x contains all the lines in X passing through x). Hence $C' \cdot D = 0$. Therefore these reducible conics are not contained in \mathcal{C}_x , contrary to the assumption $\chi(X) = 1$. Case (a) is thus excluded.

Cases (b) and (c) correspond to case (2) above, i.e., X is ruled by a 2-dimensional family \mathcal{P} of planes. In case (b) one comes to a contradiction with an argument similar to the one we made in case (a). Indeed, let $\psi : \tilde{X} \rightarrow X$ be the birational tautological morphism over the universal family $\alpha : \tilde{X} \rightarrow \mathcal{P}$. The proper transform C' on \tilde{X} of a reducible conic with a double point at the general point x lies in the plane of the family through x . Let $D \subset \tilde{X}$ be a Cartier divisor which is the pull-back via α of a divisor in a base point free linear system on \mathcal{P} . Then $D \cdot C' = 0$, while $C \cdot D > 0$ for the strict transform of a general conic in \mathcal{C}_x through x , a contradiction.

In case (c) we have $X \subset \mathbb{P}^{10}$ and the image of the general tangential projection τ_x is $X_1 = \text{Seg}(1, 2) \subset \mathbb{P}^5$, since it is the image of $E_x = \mathbb{P}^3$ via the second fundamental form Π_x , that is the linear system of quadrics with base locus a line and a point. Thus we have two rational maps $\phi_1 : X \dashrightarrow \mathbb{P}^1$ and $\phi_2 : X \dashrightarrow \mathbb{P}^2$. Set $\mathcal{L}_1 = \phi_1^*(|H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))|)$ [resp. $\mathcal{L}_2 = \phi_2^*(|H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))|)$]. Then $\mathcal{L}_1 \vee \mathcal{L}_2$ coincides $\tau_x^*(|H^0(\text{Seg}(1, 2), (1))|)$, which is $\mathcal{H}(-2x)$ (the linear system of hyperplanes with a double point at x) and both linear systems \mathcal{L}_1 and \mathcal{L}_2 have a base point at x . Indeed L_1 [resp. L_2] induces on E_x the linear system of planes passing through the line L [resp. containing the point p] and therefore x must be a base

point for L_1 [resp. L_2]. By moving x we see that \mathcal{L}_1 and \mathcal{L}_2 vary in continuous families, and therefore in larger linear systems, since X is rational (see Proposition 5.2.2). If \mathcal{N}_1 and \mathcal{N}_2 are the complete linear systems containing \mathcal{L}_1 and \mathcal{L}_2 , we have $\dim(\mathcal{N}_1) = \dim(\mathcal{L}_1) + 1 = 2$ and $\dim(\mathcal{N}_2) = \dim(\mathcal{L}_2) + 1 = 3$, because \mathcal{L}_1 and \mathcal{L}_2 have the base point x and, by their very definition, are complete under this constraint, because their minimal sum gives the rational map τ_x from X to $\text{Seg}(1, 2)$ which is linearly normal. Finally, since the minimal sum $\mathcal{L}_1 \vee \mathcal{L}_2$ coincides with $\mathcal{H}(-2x)$, we have $\mathcal{N}_1 \vee \mathcal{N}_2 = \mathcal{H}$. By using \mathcal{N}_1 and \mathcal{N}_2 we define a morphism $\phi : X \rightarrow \text{Seg}(2, 3)$, which is an embedding because $\mathcal{N}_1 \vee \mathcal{N}_2 = \mathcal{H}$. Then X is realized as a hyperplane section of $\text{Seg}(2, 3) \subset \mathbb{P}^{11}$ because $\dim(\mathcal{H}) = 10$. In conclusion, we are in case (iv). \square

Remark 5.5.3 Let X be as in Theorem 5.5.2 and assume in addition that X is smooth. Then, either we are in cases (ii), (iii) or (iv), or we are in case (i) and either X is the Veronese variety $V_{2,4}$ in \mathbb{P}^{14} , or X is the projection to \mathbb{P}^{13} of $V_{2,4}$ from a point on it. Indeed, any internal projection of $V_{2,4}$ from more than one point is singular because this projection contracts the unique conic through two (distinct or infinitely near) points to a singular point of the image.

6 Top species: the reducible case

6.1. Next we continue the analysis of varieties $X \subset \mathbb{P}^r$ of dimension $n \geq 4$, with $r \geq 2n + 1$, of the top species, but we turn to the reducible case in which the general entry locus is a reducible conic. Remember that by definition of top species we have $\gamma(X) = 1$, which implies $f(X) = 1$ (and $\delta(X) = 1$ because $r \geq 2n + 1$).

Let p_0, p_1 be general points of X . Then Γ_{p_0, p_1} consists of two incident lines R_0, R_1 , with $p_i \in R_i$ and $p_i \notin R_{1-i}$, $i = 0, 1$. In this case $\Psi_{p_0, p_1} = R_0$ (see Definition 2.7.1 and Proposition 2.9.5). This implies, by monodromy, that R_0, R_1 move in one and the same irreducible family \mathcal{R} of lines filling X .

Lemma 6.1.1 *In the above setting, either X is a cone with vertex a point over a non-defective variety of dimension $n - 1$ or $\dim(\mathcal{R}) \geq n$.*

Proof Since there is some line in \mathcal{R} through the general point of X , one has $\dim(\mathcal{R}) \geq n - 1$. Suppose the equality holds. Then that there is a finite number t of lines of \mathcal{R} passing through the general point of X , and, by the irreducibility of \mathcal{R} , they are exchanged by monodromy.

Let p_0, p_1 be general points of X and let, as above, $\Gamma_{p_0, p_1} = R_0 \cup R_1$ be the unique tangential contact locus containing p_0, p_1 , with $p_i \in R_i$ and $p_i \notin R_{1-i}$, $i = 0, 1$. Since, given p_1 , the line $R_0 \in \mathcal{R}$ through p_0 is uniquely determined, this is the unique line in \mathcal{R} containing p_0 and $t = 1$.

If we let p_0 stay fixed and move p_1 , the line R_0 stays also fixed, whereas R_1 , the unique line in \mathcal{R} through p_1 , always intersects R_0 . This proves that the lines in \mathcal{R} pairwise meet at a point, and therefore they all pass through a fixed point, thus X is a cone. \square

The previous lemma suggests that our analysis would take advantage from some classification result for varieties of dimension n with a family of lines of dimension at least n . Unfortunately such a classification is missing in general, but it is luckily available for $n = 4$ (see [24]). Therefore we turn our attention to the 4-dimensional case.

Theorem 6.1.2 *Let $X \subset \mathbb{P}^r$, $r \geq 9$, be a non-degenerate, linearly normal, defective variety of dimension 4 of the top species, not a cone, presenting the reducible case. Then X is a scroll over a curve and therefore one of the cases listed in Theorem 4.0.4 occurs.*

Proof By Lemma 6.1.1, X is covered by lines moving in a family \mathcal{R} of dimension at least 4. Let \mathcal{S} be an irreducible component of the Hilbert scheme of lines contained in X and containing \mathcal{R} . If $\dim(\mathcal{S}) \geq 6$, then through two general points of X there would pass a line contained in X and $X \subset \mathbb{P}^r$ would be a linear subspace. Hence $\dim(\mathcal{S}) \leq 5$ and if the equality holds, then X is swept out by a 1-dimensional family \mathcal{F} of 3-spaces by a classical result of B. Segre (see [29] or the modern reference [24, Theorem 1]). Thus, if $\dim(\mathcal{S}) = 5$, then X is a scroll and we are done.

Suppose now that $\mathcal{S} = \mathcal{R}$ has dimension 4. By [24, Theorem 2], as in the proof of Theorem 5.5.2, from which we keep the notation, we have the following alternatives (1) and (2):

- (1) either X is swept out by a 1-dimensional family \mathcal{Q} of 3-dimensional quadrics generically of rank 5;
- (2) or X is swept out by a 2-dimensional family \mathcal{P} of planes;

Suppose we are in case (1). Let $x, y \in X$ be general points and let Q_x, Q_y be 3-dimensional quadrics in \mathcal{Q} containing r_x and r_y respectively. Let p be the intersection point of r_x and r_y , and let $Z_{x,y}$ be the union of the irreducible components of $Q_x \cap Q_y$ containing p , that has pure dimension 2 by Krull Theorem. We claim that $Z_{x,y}$ is an irreducible quadric surface. Indeed, $Z_{x,y}$ is a surface whose linear span has dimension at most 3 (the linear spans of Q_x and Q_y are distinct by genericity) and it is defined by quadrics. Moreover $Z_{x,y}$ is irreducible because Q_x, Q_y have rank 5 and cannot contain planes.

Consider now the 1-dimensional family \mathcal{K} of 4-spaces spanned by the quadrics of \mathcal{Q} . By keeping, in the above argument, x fixed and letting y vary, we see that two general 4-spaces in \mathcal{K} intersect in a 3-space. Hence the 4-spaces in \mathcal{K} all contain the same 3-space Π . Thus X is contained in a 5-dimensional cone over a curve C with vertex Π and $S(X)$ would be the cone over $S(C)$ from Π , yielding $s(X) = 7$, which contradicts $\delta(X) = 1$.

Suppose we are in case (2). Then, with the same argument as above, we see that the planes in \mathcal{P} pairwise meet. By [9, Lemma 4.1] and taking into account that X is of the top species and is not a cone, we see that X should be contained in a 5-dimensional cone with vertex a plane Π over a non-defective surface Y and $S(X)$ would be the cone with vertex Π over $S(Y)$. Let π be the projection of X to Y from Π . Given two general points $p_0, p_1 \in X$ mapping via π to general points $q_0, q_1 \in Y$, then Γ_{X,p_0,p_1} is the pull-back via π of $\Gamma_{Y,q_0,q_1} = \{q_0, q_1\}$, which contradicts $\gamma(X) = 1$. □

7 Second species

7.1. From now on we will concentrate on the 4-fold case. Let $X \subset \mathbb{P}^r, r \geq 9$, be a non-degenerate, defective, projective variety of dimension 4. By Proposition 2.9.5 and Theorem 3.0.1 we may assume $f(X) = \delta(X) = 1$. By Theorem 4.0.4 we may also assume X is neither a cone nor a scroll. It suffices to assume the 4-fold to be linearly normal.

We will consider here the case in which X is of the second species, i.e. $\gamma(X) = 2$. Therefore, given $p_0, p_1 \in X$ general points, the tangential contact locus Γ_{p_0,p_1} is a surface, spanning a 4-space Π_{p_0,p_1} by Proposition 2.6.4, part (v). Again we have two cases: the *irreducible* and the *reducible* case, according to the possibilities that Γ_{p_0,p_1} is irreducible or it consists of two irreducible components each passing through one of the two points p_0, p_1 .

7.2. We first examine the irreducible case. In this case the tangential contact surfaces move in a 4-dimensional irreducible family \mathcal{S} , such that given two general points $p_0, p_1 \in X$ there is a unique irreducible surface $\Gamma = \Gamma_{p_0,p_1}$ in \mathcal{S} containing p_0 and p_1 (see §2.6). A priori the

surfaces in \mathcal{S} may have a base locus scheme B which is the largest subscheme B of X such that B is contained in all surfaces in \mathcal{S} .

There are two subcases to be considered:

- (i) two general surfaces in \mathcal{S} have no isolated point in common off the base locus scheme B ;
- (ii) two general surfaces in \mathcal{S} have some isolated point in common off the base locus scheme B .

Let $x \in X$ be a general point. Then we have a subfamily \mathcal{S}_x of surfaces in \mathcal{S} passing through x . Each irreducible component of \mathcal{S}_x has dimension 2. Actually \mathcal{S}_x is irreducible, because otherwise if $y \in X$ is another general point, there would be more than one surface in \mathcal{S} containing x and y .

If we are in case (i), two general surfaces in \mathcal{S}_x have a positive dimensional intersection off B , hence they intersect off B in a curve C passing through x . Since the surfaces in \mathcal{S}_x span in general a 4-space which varies with the surface, there are three subcases to be considered:

- (i1) the curve C spans a 3-space;
- (i2) the curve C spans a plane;
- (i3) the curve C is a line.

7.3. In this section we discuss the subcase (i1). We prove that:

Lemma 7.3.1 *The subcase (i1) does not occur.*

Proof In subcase (i1) the spans of two general surfaces in \mathcal{S}_x intersect along a 3-space containing the point x . Since they fill up X they cannot lie in a 5-space, so they pass through the same 3-space P_x containing all curves which are the intersections of two general surfaces in \mathcal{S}_x .

We claim that then all surfaces in \mathcal{S}_x must contain the same curve C_x , passing through x , which spans P_x . In fact, let Γ be the general surface in \mathcal{S}_x . If the other surfaces in \mathcal{S}_x cut out on Γ different curves C passing through x , then these curves would, on one side, sweep out Γ , on the other all lie in P_x , hence Γ would lie in P_x which is a 3-space, a contradiction. This proves our claim.

When x moves, the curve C_x moves in a family \mathcal{C} of dimension 3, which fills up X . Let $y \in X$ be another general point. Then the surface Γ of \mathcal{S} which contains x and y , contains also C_x and C_y (actually, for a general point $z \in \Gamma$, it contains C_z), and this implies that P_x and P_y , that are contained in the 4-space spanned by Γ , intersect in a plane. This yields that there is a plane Π such that for $x \in X$ general, the 3-space P_x contains Π . Let $\pi : X \dashrightarrow \mathbb{P}^{r-3}$ be the projection from Π and let Y be the image of X , which has to span the \mathbb{P}^{r-3} . Moreover $\dim(Y) \leq 3$ because for $x \in X$ general the curve C_x is contracted to a point by π . Let $q_0, q_1 \in Y$ be general points and let p_0, p_1 be points in the counterimages of q_0, q_1 respectively. The surface Γ_{p_0, p_1} , which spans a 4-space containing Π , is mapped by π to the line $\langle q_0, q_1 \rangle \subset Y$. This implies that Y is a linear space, a contradiction. \square

7.4. In this section we discuss the subcase (i2). We keep the notation introduced above. We prove that:

Theorem 7.4.1 *In subcase (i2) then either:*

- (a) $r = 9$ and X sits in a cone with vertex a line over a hyperplane section of $\text{Seg}(2, 2)$ in \mathbb{P}^7 ;

(b) or $9 \leq r \leq 11$ and X sits in a cone with vertex a line over a projection in \mathbb{P}^{r-2} of the Veronese 3-fold $V_{2,3}$ in \mathbb{P}^9 from a linear space of dimension $r - 10$, intersecting $V_{2,3}$ in a 0-dimensional scheme of length $11 - r$.

Proof Let $x \in X$ be a general point and let Γ, Γ' be general elements in S_x . Then they intersect in a curve C which spans a plane P and passes through x .

Claim 7.4.2 Keeping Γ fixed and letting Γ' vary, the curve C stays fixed.

Proof of the Claim 7.4.2 Suppose the assertion is false. Consider the family \mathcal{F}_x of plane curves cut out on Γ by the other surfaces of S_x , and recall that S_x has dimension 2.

If \mathcal{F}_x has dimension 2, then, since x is a general point of Γ , the surface Γ possesses a 3-dimensional family \mathcal{F} of plane curves. This implies that Γ is either a plane or spans a 3-space, a contradiction since Γ spans a 4-space. In fact, since \mathcal{F} has dimension 3, two general curves of \mathcal{F} intersect in more than one point (through two general points of Γ there passes a 1-dimensional family of curves in \mathcal{F}). If these intersection points span a plane, then two general curves lie in the same plane and Γ would be the plane containing all the curve in \mathcal{F} . If they span a line, the planes spanning two general curves in \mathcal{F} intersect along a line, so the planes spanning curves in \mathcal{F} pairwise intersect along a moving line, hence they span a 3-space.

Hence \mathcal{F}_x has dimension 1, i.e., a general curve in \mathcal{F}_x belongs to a 1-dimensional family of surfaces in S_x . Then Γ has a 2-dimensional family \mathcal{F}_Γ of plane curves and therefore there is some curve of \mathcal{F}_Γ which passes through two general points of Γ . By moving Γ we see that X possesses an irreducible family \mathcal{F} of plane curves such that there is some curve of \mathcal{F} passing through two general points of X : indeed, if Γ is general in S , then two general points of Γ are also general on X . This yields that \mathcal{F} has dimension at least 6.

Consider now the set I of all pairs (C, Γ) , with C in \mathcal{F} and Γ in S such that $C \subset \Gamma$. Since on the general surface $\Gamma \in S$ there is the general curve $C \in \mathcal{F}$, we can find an irreducible component J of I such that the two projections $\pi_1 : J \rightarrow \mathcal{F}$ and $\pi_2 : J \rightarrow S$ are both surjective. Looking at π_2 , since S has dimension 4 and since for Γ general in S the family \mathcal{F}_Γ has dimension 2, we see that $\dim(J) = 6$. On the other hand, by what we saw above, the general fibre of $\pi_1 : J \rightarrow \mathcal{F}$ has dimension at least 1, which implies that \mathcal{F} has dimension at most 5, a contradiction. □

By Claim 7.4.2, all surfaces in S_x contain a fixed plane curve C passing through x . Let us denote again by \mathcal{F} the family of these curves C , which has the property that, given a general point $x \in X$ there is a unique curve of \mathcal{F} containing x . Hence \mathcal{F} has dimension 3. Moreover the general surface Γ in S contains a 1-dimensional family \mathcal{F}_Γ of curves in \mathcal{F} . We denote by \mathcal{P} the family of planes spanned by the curves in \mathcal{F} .

Let $p_0, p_1 \in X$ be general points. The surface Γ_{p_0, p_1} which contains p_0 and p_1 contains also the curves C_0, C_1 of \mathcal{F} which pass through p_0, p_1 respectively. Hence the planes P_0, P_1 in \mathcal{P} spanned by C_0, C_1 , lying in the 4-space spanned by Γ_{p_0, p_1} , intersect in at least one point.

Claim 7.4.3 Two general planes in \mathcal{P} do not intersect in a single point.

Proof of the Claim 7.4.3 Suppose by contradiction that two general planes in \mathcal{P} intersect in a single point. Then by [9, Lemma 4.1], the only possibilities are:

- (1) there is a point p such that all the planes in \mathcal{P} contain p ;
- (2) there is a plane Π such that all the planes in \mathcal{P} intersect Π along a line.

Assume (1) happens. Let $\pi : X \dashrightarrow \mathbb{P}^{r-1}$ be the projection of X from p . The image Y of X is a 4-fold, otherwise X would be a cone with vertex p , a contradiction. Hence $\pi : X \dashrightarrow Y$ is a generically finite rational map. If Γ is a general surface in \mathcal{S} , then the 4-dimensional span of Γ contains p , so Γ is projected from p to a surface in \mathbb{P}^3 . Hence Y contains a 4-dimensional family \mathcal{Q} of irreducible surfaces in 3-space such that there is a surface Q of \mathcal{Q} through two general points of Y . Hence a general secant line to Y lies in the 3-space generated by Q . By the Trisecant Lemma (see [26, Proposition 1.4.3]), a general secant line intersects Y (and a fortiori Q) transversally only in the two points. This implies that the surfaces in \mathcal{Q} are quadrics and that through the general point of $S(Y)$ there passes at least a 2-dimensional family of secant lines to Y . Then $\gamma(Y) = f(Y) = 2$ and the general contact locus of Y (see Definition 2.6.2) is an irreducible quadric. So we can apply Theorem 3.0.1 to Y . Cases (i) and (ii) of Theorem, 3.0.1 do not apply to Y , because in those cases the contact surfaces are not irreducible. In case (iii) of Theorem 3.0.1 the tangential contact loci are not surfaces, so also this case does not apply. In case (iv) of Theorem 3.0.1, Y is a cone with vertex a point v and the images of the planes of \mathcal{P} via π are the lines generating the cones. This implies that the planes of \mathcal{P} all intersect along the fixed line $\langle p, v \rangle$, contrary to the assumption. Suppose we are in case (v) of Theorem 3.0.1. If Y is a cone over a hyperplane section of $\text{Seg}(2, 2)$, the same argument as above proves that the planes of \mathcal{P} all intersect along a fixed line, a contradiction. It remains to discuss the case in which $Y = \text{Seg}(2, 2)$. Then X sits in a cone with vertex p over $\text{Seg}(2, 2)$. Now the quadrics in \mathcal{Q} are just all the quadrics in $\text{Seg}(2, 2)$, because they form a family of dimension 4. Two general of them pairwise intersect at a single point. On the other hand, if $y \in Y$ is a general point, two general quadrics in \mathcal{Q} containing y intersect along the image via π of a curve in \mathcal{F} , hence they must intersect along a line, a contradiction again.

We claim that also case (2) cannot happen. Indeed, let $\pi : X \dashrightarrow \mathbb{P}^{r-3}$ be the projection of X from the plane Π . Then the general curve of \mathcal{F} is contracted to a point by π , so the image of π is a 3-fold Y . Let $q_0, q_1 \in Y$ be general points and let $p_0, p_1 \in X$ be such that $\pi(p_i) = q_i$, for $i = 1, 2$. We may assume that $p_0, p_1 \in X$ are also general points. Consider the surface Γ_{p_0, p_1} . It contains the curves C_0, C_1 of \mathcal{F} which pass through p_0, p_1 respectively. The planes P_0, P_1 , spanned by C_0, C_1 respectively, cut Π in two distinct lines. This implies that the span of Γ_{p_0, p_1} contains Π , hence the image of Γ_{p_0, p_1} via π is a line, precisely the line joining q_0 and q_1 . This yields that $Y \subset \mathbb{P}^{r-3}$ is a linear subspace because the line through two general points of Y is contained in Y . This contradicts the fact that Y has to span \mathbb{P}^{r-3} because $r - 3 \geq 6 > 3$. □

Finally we have to examine the case in which the planes spanned by the curves of \mathcal{F} meet pairwise along a line, and then they all pass through the same line s . Let $\pi : X \dashrightarrow \mathbb{P}^{r-2}$ be the projection of X from s . The general curve in \mathcal{F} is contracted to a point by π , so the image of π is a 3-fold Y .

Claim 7.4.4 Y is a defective 3-fold with $f(Y) = 1$, possessing a 4-dimensional family \mathcal{C} of generically irreducible conics such that if $q_0, q_1 \in Y$ are general points there is a unique conic in \mathcal{C} containing q_0 and q_1 .

Proof Let $q_0, q_1 \in Y$ be general points and let $p_0, p_1 \in X$ be such that $\pi(p_i) = q_i$, for $i = 0, 1$. Then we may assume that $p_0, p_1 \in X$ are also general points. Consider the surface Γ_{p_0, p_1} spanning a 4-space Π_{p_0, p_1} , which is swept out by a family \mathcal{F}_Γ of plane curves whose planes all contain the line s . Then the projection of Γ_{p_0, p_1} from s is a curve spanning the plane image of Π_{p_0, p_1} and $\gamma(Y) = 1$. Hence Y has a 4-dimensional family \mathcal{C} of generically irreducible plane curves such that if $q_0, q_1 \in Y$ are general points there is a unique curve in

\mathcal{C} containing q_0 and q_1 . By the Trisecant Lemma, this implies that the curves in \mathcal{C} are conics. Hence $1 \leq f(Y) \leq \gamma(Y) = 1$, yielding $f(Y) = 1$. □

In conclusion, in this case X , that is linearly normal, sits in the cone with vertex s over Y , that is also linearly normal and is in the list of defective threefolds in [9, Theorem 1.1]. Actually in that list the only 3-folds Y possessing a 4-dimensional family \mathcal{C} of generically irreducible conics such that if $q_0, q_1 \in Y$ are general points there is a unique conic in \mathcal{C} containing q_0 and q_1 , are the hyperplane section of $\text{Seg}(2, 2)$ in \mathbb{P}^7 and the Veronese 3-fold $V_{2,3}$ and its projections, that must be internal projections because of the linear normality. This leads to cases (a) and (b) of Theorem 7.4.1. □

Remark 7.4.5 If X is a 4-fold as in Theorem 7.4.1, then X is in fact defective, with $\delta(X) = f(X) = 1$ and irreducible tangential contact loci of dimension 2.

First, suppose X sits in a cone with vertex a line s over a hyperplane section Y of $\text{Seg}(2, 2)$ in \mathbb{P}^7 . Hence we have a projection $\pi : X \dashrightarrow Y \subset \mathbb{P}^7$ from s . If $p_0, p_1 \in X$ are general points and $q_i = \pi(p_i)$ with $i = 0, 1$, then from $\langle s, T_{X,p_i} \rangle \cap \mathbb{P}^7 = T_{Y,q_i}$, we get $\langle s, T_{Y,q_0}, T_{Y,p_1} \rangle = \langle T_{X,p_0}, T_{X,p_1} \rangle = T_{X,p_0,p_1}$. Since $T_{Y,q_0,q_1} = \mathbb{P}^6$ (recall that $\gamma(Y) = f(Y) = 1 = \delta(Y)$), we get $T_{X,p_0,p_1} = \mathbb{P}^8$ and $f(X) = 1$. From $T_{X,p_0,p_1} = \langle s, T_{Y,q_0,q_1} \rangle$ we deduce that the general tangential contact locus is the pull back of a general conic in Y via π so that it has dimension 2, yielding $\gamma(X) = 2$ (see Remark 3.0.2 for the relations between the invariant of X and Y).

The other case of Theorem 7.4.1 is similar because a projection $Y \subset \mathbb{P}^{r-2}$ of $V_{2,3} \subset \mathbb{P}^9$ has $\delta(Y) = f(Y) = \gamma(Y) = 1$ (as well as $V_{2,3}$, its projection Y is a LCEL variety).

Remark 7.4.6 Suppose X is a 4-fold as in Theorem 7.4.1. Then we claim $X \subset \mathbb{P}^9$ is never smooth. The 4-fold X projects from the line s onto the 3-fold Y . If X is smooth, then $s \subset X$. Indeed, if this is not the case, $s \cap X = \{p_1, \dots, p_h\}$ with $h \geq 1$. If $m_i = \text{length}_{p_i}(s \cap X) \geq 1$, we can resolve the projection from s by blowing-up a finite number of points above each p_i because the strict transform of s will cut the exceptional divisor in a finite number of points. The last exceptional divisor is a smooth \mathbb{P}^3 , which would be sent by the projection onto a linear space of dimension 3 (the linear system of hyperplanes through s induces on this \mathbb{P}^3 a base point free linear system in $|\mathcal{O}(1)|$ because we have blown-up a point and the strict transform of a general hyperplane through s has now multiplicity one at this point). Hence Y would be a linear space of dimension 3, which is impossible since Y is non-degenerate (see the proof of [15, Lemma 3.19] for a similar argument).

Let us assume that $s \subset X$, let $E \subset \tilde{X}$ be the exceptional divisor of the blowing-up of X along s and let $\tilde{\pi} : \tilde{X} \rightarrow Y$ be the induced morphism. Then E , which is a \mathbb{P}^2 -bundle over s , is mapped onto Y in such a way that each \mathbb{P}^2 in E is a linear space in Y . In particular Y is ruled by planes.

Suppose that $Y \subset \mathbb{P}^7$ is a hyperplane section of $\text{Seg}(2, 2) \subset \mathbb{P}^8$, that is a divisor of type $(1, 1)$ on $\mathbb{P}^2 \times \mathbb{P}^2$. Recall that a plane on $\text{Seg}(2, 2) = \mathbb{P}^2 \times \mathbb{P}^2$ is either of the form $\mathbb{P}^2 \times \{p\}$ or of the form $\{q\} \times \mathbb{P}^2$, with p, q points in \mathbb{P}^2 . Since Y contains infinitely many planes, there would exist a curve $\Gamma \subset \mathbb{P}^2$ such that either $Y = \mathbb{P}^2 \times \Gamma$ or $Y = \Gamma \times \mathbb{P}^2$, contradicting the fact that Y is a divisor of type $(1, 1)$.

Suppose now that Y is a projection of $V_{2,3} \subset \mathbb{P}^9$. If $r = 11$, then $Y = V_{2,3}$ that does not contain planes. If $r = 10$, then Y is the internal projection of $V_{2,3}$ from a point, that contains a unique plane, i.e., the image of the exceptional divisor of the blow-up of $V_{2,3}$ at the centre of projection. If $r = 9$, then Y is the projection of $V_{2,3}$ from a secant or tangent line. This is also easily seen not to contain infinitely many planes, concluding the proof of our claim. Indeed, if the threefold Y contains infinitely many planes, it would be a scroll in planes and its general hyperplane section Z would be a surface scroll. However the general hyperplane

section Z of Y in this case is the image of a smooth quadric surface Q in \mathbb{P}^3 birationally mapped in \mathbb{P}^6 via the linear system $|\mathcal{O}_Q(2) \otimes I_\zeta|$, where ζ is a lenght two 0-dimensional subscheme of Q . Then the irregularity of Z is 0. On the other hand, the general hyperplane section of Z is a smooth elliptic curve, because so is the general curve in $|\mathcal{O}_Q(2) \otimes I_\zeta|$. If Z would be a scroll it would have irregularity 1, a contradiction.

7.5. In this section we discuss the subcase (i3). We keep notation and conventions introduced above, in particular we assume that X is not a cone.

Lemma 7.5.1 *In the subcase (i3), if $x \in X$ is a general point, then all surfaces in \mathcal{S}_x contain the same line s_x passing through x . Hence X contains a 3-dimensional family \mathcal{R} of lines such that there is a unique line in \mathcal{R} passing through the general point of X . Moreover the surfaces Γ in \mathcal{S} are ruled, swept out by a 1-dimensional family of lines in \mathcal{R} .*

Proof Fix Γ a general surface in \mathcal{S}_x . If the other surfaces in \mathcal{S}_x intersect Γ in a moving line, then Γ would possess a positive dimensional family of lines through its general point. Then Γ would be a plane, which is impossible. So the intersection of all surfaces in \mathcal{S}_x is a unique line through x and the assertion follows. □

We will prove the following:

Theorem 7.5.2 *In the subcase (i3) then either:*

- (1) $r = 9$ and X sits in a cone with vertex a line over a hyperplane section of $\text{Seg}(2, 2)$ in \mathbb{P}^7 ;
- (2) or $9 \leq r \leq 11$ and X sits in a cone with vertex a line over a projection in \mathbb{P}^{r-2} of the Veronese 3-fold $V_{2,3}$ in \mathbb{P}^9 from a linear space of dimension $r - 10$, intersecting $V_{2,3}$ in a 0-dimensional scheme of lenght $11 - r$;
- (3) or $r = 9$ and X sits in a cone with vertex a line over a linearly normal (defective) 3-fold in \mathbb{P}^7 sitting in a cone with vertex a line over the Veronese surface $V_{2,2}$, in particular X sits in a 6-dimensional cone with vertex a 3-space over the Veronese surface $V_{2,2}$;
- (4) or X is swept out by a 3-dimensional family \mathcal{R} of lines and it is singular along a linear space Π of dimension ε , with $2 \leq \varepsilon \leq 3$, which is cut out in one point by the general line in \mathcal{R} , and X projects from Π to a 3-dimensional variety $Y \subset \mathbb{P}^{r-\varepsilon-1}$ (with general fibres unions of lines of \mathcal{R}), which contains a 4-dimensional family \mathcal{C} of (generically irreducible) conics such that there is a conic in \mathcal{C} passing through two general points of Y and the counterimage of the general conic of \mathcal{C} via the projection from Π is a non-developable scroll spanning a 4-space with a line directrix sitting in Π . In any event $r \leq 10 + \varepsilon$.

Before proving Theorem 7.5.2, we need a few preliminary lemmas which are due to Scorza (see [28], §11).

First of all we consider $\Sigma \subset \mathbb{P}^r$, with $r \geq 3$, an irreducible, non-degenerate, projective surface, which is a non-developable scroll. Let s be a line of the ruling of Σ such that the general point of s is smooth for Σ and the tangent plane to Σ at the smooth points of s is not constant. We will say that s is a *non-developable* line of the ruling, otherwise, if the tangent plane at the smooth points of s is constant, the line is called *developable*. We recall the following standard fact in basic projective-differential geometry (see, e.g., [30, §1]):

Proposition 7.5.3 (Chasles' Theorem) *In the above setting, if s is a non-developable line of the ruling of Σ , there is a 3-space P_s containing s and there is a projectivity $\omega_s : s \rightarrow s^*$ (where s^* is the pencil of planes containing s in P_s), such that for all points $x \in s$ smooth for Σ , one has $T_{\Sigma,x} = \omega_s(x)$.*

The 3-space P_s is called the *tangent space* to Σ along s . If s is instead a developable line of the ruling of Σ , we denote by P_s the plane tangent to Σ at a general point of s .

One has the possibility of defining a notion of tangent plane to Σ even at points that are not smooth for Σ . We proceed in this way. We consider a pair (s, x) , where s is a line of the ruling whose general point is smooth for Σ and $x \in s$. Then we define the *tangent plane* $T_{\Sigma,s,x}$ to Σ at (s, x) in the following way:

- $T_{\Sigma,s,x} = \omega_s(x)$ if s is a non-developable line of the ruling;
- $T_{\Sigma,s,x} = T_{\Sigma,y}$ where y is a general point of s , if s is a developable line of the ruling.

Sometimes one says that $T_{\Sigma,s,x}$ is a *tangent plane* to Σ at x on s and note that it is a limit of tangent planes at smooth points of Σ on s .

Remark 7.5.4 Let $C \subset \Sigma$ be a directrix of Σ , let $x \in C$ be a general point and let s be a (general) line of the ruling containing x . Then $T_{\Sigma,s,x}$ contains the tangent line to C at x . This follows from the discussion in [30, §1].

Note that given $x \in \Sigma$ there are at most finitely many lines of the ruling of Σ containing x , because Σ is not a cone. So there are at most finitely many tangent planes to Σ at x . Moreover it is clear that if x is smooth for Σ there is a unique tangent plane to Σ at x that coincides with $T_{\Sigma,x}$.

Lemma 7.5.5 *Let $\Sigma \subset \mathbb{P}^4$ be an irreducible, non-degenerate, projective surface, which is a non-developable scroll. Let s be a general line of the ruling of Σ . Then there is a curve \mathcal{C}_s on Σ , union of irreducible components such that some tangent plane to Σ at the general points of each of these components intersect s at a point. More precisely, \mathcal{C}_s consists of an irreducible component \mathcal{C}'_s which is unisecant to the lines of the ruling of Σ , plus finitely many lines of the ruling, whose set we denote by \mathfrak{S}_s . If the curve $\mathcal{C}'_s = \mathcal{C}'$ is independent on s , then Σ has a directrix line R and in this case $\mathcal{C}' = R$.*

Proof Let t be a further general line of the ruling of Σ and let P_t be the tangent 3-space to Σ along t . Then P_t cannot contain s which is a general line of the ruling, because Σ is non-degenerate in \mathbb{P}^4 . Hence P_t cuts s at a point q_t . Then the plane $\langle q_t, t \rangle$ is tangent to Σ at a point $p_s \in t$. The curve \mathcal{C}'_s , which is unisecant to the lines of the ruling, whose general point is p_s , is a component of the desired curve \mathcal{C}_s . The remaining components of \mathcal{C}_s are those non-developable lines u of the ruling such that P_u contains s plus the developable lines u of the ruling such that P_u intersects s , that fill up the set \mathfrak{S}_s .

Suppose now $\mathcal{C}'_s = \mathcal{C}'$ does not depend on s . Keeping the above notation, take t general but fixed and let s vary. Then the point $p := p_s = \mathcal{C}' \cap t$ does not depend on s , and therefore the general line of the ruling intersects in one point the tangent plane $T_{\Sigma,t,p}$, that contains t . Similarly, the general line of the ruling intersects in one point the tangent plane $T_{\Sigma,s,q}$, where $q = \mathcal{C}' \cap s$, that contains s . Note that $T_{\Sigma,t,p}$ and $T_{\Sigma,s,q}$ are distinct (because t and s being general lines of the ruling of Σ are not coplanar), and intersect along a line, precisely the line R joining the two points $T_{\Sigma,t,p} \cap s$ and $T_{\Sigma,s,q} \cap t$. So $\langle T_{\Sigma,t,p}, T_{\Sigma,s,q} \rangle$ is a 3-space. Since all lines of the ruling intersect both $T_{\Sigma,t,p}$ and $T_{\Sigma,s,q}$ and since Σ is non-degenerate in \mathbb{P}^4 , then the general line of the ruling intersects R in a variable point (because Σ is not a cone), i.e., Σ has the line directrix R .

Finally, assume that Σ has a line directrix R . Then $\mathcal{C}' = R$, because if $p \in R$ is a general point, and t a (general) line of the ruling containing p , then $T_{\Sigma,t,p}$ contains R (see Remark 7.5.4), hence it meets the general line of the ruling. □

To state the second lemma, recall that an irreducible, projective surface $S \subset \mathbb{G}(1, r)$ is called a *congruence* of lines in \mathbb{P}^r . Given an irreducible projective curve $C \subset S$ it corresponds

to a scroll in \mathbb{P}^r , and it is well known that the scroll is developable if and only if for the general point $x \in C$ the tangent line to C at x is contained in $\mathbb{G}(1, r)$ (see for example [4, Proposition 12.4.1]).

Lemma 7.5.6 *Let S be a congruence of lines in \mathbb{P}^r , with $r \geq 3$. If all curves contained in S correspond to developable scrolls, then one of the following occurs:*

- (α) *there is a plane $\Pi \subset \mathbb{P}^r$ such that S consists of all the lines in Π ;*
- (β) *there is a point $p \in \mathbb{P}^r$ such that S consists of lines containing p .*

Proof Consider the variety $W \subset \mathbb{P}^r$, $r \geq 3$, described by the lines in S . By projecting W down generically in \mathbb{P}^3 it suffices to prove the assertion for $r = 3$. So we assume $r = 3$, hence $S \subset \mathbb{Q} \subset \mathbb{P}^5$, where $\mathbb{Q} = \mathbb{G}(1, 3)$ is a smooth quadric.

Let $x \in S$ be a general point. Every line passing through x in the tangent plane $T_{S,x}$ is tangent to some curve on S , and therefore this line is contained in \mathbb{Q} . Hence $T_{S,x} \subset \mathbb{Q}$ and therefore $T(S) \subseteq \mathbb{Q}$. This implies that all tangent planes to S in smooth points of S belong to one and the same system of planes of \mathbb{Q} , hence they pairwise intersect at a point, thus either S is defective or it spans a 4-space.

If S is defective, it could either be a cone or the Veronese surface $V_{2,2}$. The latter case is impossible, because $T(V_{2,2})$ is a cubic hypersurface in \mathbb{P}^5 and not a quadric.

Claim 7.5.7 *If the surface S is a cone, then either case (α) or (β) occurs.*

Proof of Claim 7.5.7 We want to show that the lines of S pairwise intersect, because this implies that we are in case (α) or (β). So we argue by contradiction and assume that two general lines in S are skew. Since S is a cone, its vertex corresponds to a line s in \mathbb{P}^3 such that all lines corresponding to the points of S intersect s . Now take a curve C on S such that two general points on C correspond to skew lines. Then C corresponds to a developable surface Σ in \mathbb{P}^3 that is not a cone since it contains skew lines of the ruling. So Σ is the tangent developable surface to a curve Z that is not a plane curve, because two general tangent lines to Z are skew. However, the general tangent line to Z has to intersect s , which is not possible. This contradiction proves the assertion. \square

Suppose now that S spans a 4-space Π , which therefore contains also $T(S)$. This means that the quadric \mathbb{Q}' cut out by Π on \mathbb{Q} is singular, because it contains planes. Hence \mathbb{Q}' is a cone with vertex a point corresponding to a line s , and the lines corresponding to the points of S all intersect the line s . Even in this case, arguing as before, we can conclude that we are either in case (α) or (β). \square

As a consequence we have the:

Lemma 7.5.8 *Let $V \subset \mathbb{G}(1, r)$ be a projective variety of dimension $n \geq 2$ possessing an irreducible family of dimension $2(n - 1)$ of (generically) irreducible curves C such that:*

- (i) *all curves in C correspond to developable scrolls;*
- (ii) *given two general points of V there is a curve in C containing them.*

Then there is a point $p \in \mathbb{P}^r$ such that any point of V corresponds to a line containing p (hence $r \geq n + 1$).

Proof The assumption (ii) implies that given a general point $x \in V$ and a general tangent line t to V at x , there is a curve in C which passes through x , is smooth at x and whose tangent line at x is t . This in turn implies that $T(V) \subseteq \mathbb{G}(1, r)$ and all curves in V correspond to

developable scrolls and the same happens for every surface $S \subset V$. By Lemma 7.5.6 this implies that the points of V correspond to lines which pairwise intersect at a point. They cannot correspond to the lines in a plane, hence they correspond to lines passing through a fixed point. \square

Now we are ready for the:

Proof of Theorem 7.5.2 By Lemma 7.5.1, X contains a 3-dimensional family \mathcal{R} of lines such that there is a unique line in \mathcal{R} passing through the general point of X . Let $p_0, p_1 \in X$ be general points. Let s_0, s_1 be the unique lines in \mathcal{R} containing p_0, p_1 respectively. Then $\Gamma := \Gamma_{p_0, p_1}$ (that moves in the family \mathcal{S} of dimension 4) is a scroll of which s_0, s_1 are general lines of the ruling. Then we may apply Lemma 7.5.8 to \mathcal{R} and conclude that the general surface in \mathcal{S} is not a developable scroll, otherwise X would be a cone.

By Lemma 7.5.5, there is a point $x \in s_1$ such that a tangent plane to Γ at x intersects s_0 . We need to consider two cases:

- (a) for general p_0 , either the point $x \in s_1$ moves (and so is a smooth point of X) or it is fixed but it is still a regular point of X ;
- (b) for general p_0 , the point x is in $\text{Sing}(X)$ (in particular the point x does not move on s_1 when p_0 moves).

We first assume that we are in case (a). In particular, the point x is smooth for X .

Claim 7.5.9 Given a general line s in \mathcal{R} consider the linear space

$$\Pi_s = \left\langle \overline{\bigcup_{y \in s \cap \text{Reg}(X)} T_{X,y}} \right\rangle.$$

Then:

- (1) the general line $s' \in \mathcal{R}$ intersects Π_s ;
- (2) given two general lines $s, s' \in \mathcal{R}$, one has $\dim(\langle \Pi_s, \Pi_{s'} \rangle) \leq 8$.

Proof of the Claim 7.5.9 Given two general lines $s, s' \in \mathcal{R}$, there is a unique surface Γ containing both of them. Moreover, there is a point $x \in s$ such that a tangent plane to Γ at x on s intersects s' . This implies that s' intersects Π_s and it proves (1) by the generality of s and s' .

Both Π_{s_0}, Π_{s_1} are contained in the 8-space T_{X, p_0, p_1} which is tangent to X along Γ , proving (2). \square

Set now

$$\Pi := \bigcap_{s \in \mathcal{R}} \Pi_s$$

and $\varepsilon = \dim(\Pi)$.

Claim 7.5.10 If $s \in \mathcal{R}$ is a general line, then s intersects Π in a point. Hence $\varepsilon \geq 1$.

Proof of the Claim 7.5.10 Let $s, s', t \in \mathcal{R}$ be general lines. Then t has non-empty intersection with both Π_s and $\Pi_{s'}$ by part (1) of Claim 7.5.9. If t intersects Π_s and $\Pi_{s'}$ in distinct points, then t lies in $\langle \Pi_s, \Pi_{s'} \rangle$. This is not possible, because $t \in \mathcal{R}$ is general, and then all of X would lie in $\langle \Pi_s, \Pi_{s'} \rangle$ which has dimension at most 8 by part (2) of Claim 7.5.9. Hence t intersects $\langle \Pi_s, \Pi_{s'} \rangle$ in a point of their intersection. By the genericity of s and s' we see that t intersects Π . It is not possible that Π is a point, because then X would be a cone. Hence $\varepsilon \geq 1$. \square

Consider now the irreducible closed set $\mathfrak{J} \subset \mathcal{S} \times (\mathbb{P}^r)^*$ whose general member is a pair (Γ, H) with $\Gamma \in \mathcal{S}$ general and H a hyperplane tangent to X along Γ . We have the two projections $p_1 : \mathfrak{J} \rightarrow \mathcal{S}$ and $p_2 : \mathfrak{J} \rightarrow (\mathbb{P}^r)^*$. The map p_1 is surjective, with general fibre of dimension $r - 9$, hence $\dim(\mathfrak{J}) = \dim(\mathcal{S}) + (r - 9) = r - 5$. Let \mathfrak{J} be the image of p_2 . One has $\dim(\mathfrak{J}) \geq 1$, because otherwise there would be a unique hyperplane tangent to X along the general surface $\Gamma \in \mathcal{S}$, that is impossible, because X is non-degenerate.

We denote by h the dimension of the general fibre of p_2 so that $\dim(\mathfrak{J}) = r - 5 - h$. If $h \geq 1$ and if $H \in \mathfrak{J}$ is general, then H is tangent to X along a h -dimensional family of surfaces $\Gamma \in \mathcal{S}$, that fills up a 3-dimensional variety X_H .

Claim 7.5.11 If $h \geq 1$ and if $x \in X$ is general, the general 3-fold X_H containing x contains the unique line of \mathcal{R} containing x .

Proof of the Claim 7.5.11 Since $h \geq 1$, the general 3-fold X_H is swept out by surfaces in \mathcal{S} , so there is some surface $\Gamma \in \mathcal{S}$ contained in X_H and containing x . Then Γ contains the unique line in \mathcal{R} containing x as wanted. □

Claim 7.5.12 The map p_2 is generically finite to \mathfrak{J} , i.e., $h = 0$, unless, may be, $r = 9$ and $h = 2$.

Proof of the Claim 7.5.12 We argue by contradiction and assume $h \geq 1$. We have a rational map $\eta : X \dashrightarrow \mathcal{R}$ that sends a general point $x \in X$ to the unique line in \mathcal{R} containing x . Claim 7.5.11 implies that if $h \geq 1$ and H is general in \mathfrak{J} , there is a surface Σ_H contained in \mathcal{R} , such that $X_H = \eta^*(\Sigma_H)$. We can assume Σ_H is smooth, otherwise we work on a desingularization. Given the general surface $\Gamma \in \mathcal{S}$ along which H is tangent to X , there is an irreducible curve $C_{H,\Gamma}$ on Σ_H such that $\Gamma = \eta^*(C_{H,\Gamma})$. The curves of the type $C_{H,\Gamma}$ form a family of dimension h on Σ_H . Since two general surfaces in \mathcal{S} intersect along a line, we see that $C^2_{H,\Gamma} = 1$ on Σ_H . Then, by [6, Prop. 0.18], one has $h \leq 2$. Hence $\dim(\mathfrak{J}) = r - 5 - h \geq r - 7 \geq 2$.

Let now p_0, p_1 be general points of X . We have a family \mathfrak{G} of dimension $r - 7 - h$ (which is positive, unless $r = 9$ and $h = 2$) of $H \in \mathfrak{J}$ such that $p_0, p_1 \in X_H$. For all $H \in \mathfrak{G}$, X_H has to contain the unique surface Γ containing p_0, p_1 and H has to be tangent to X along Γ . Note that \mathfrak{G} is not a linear system, because otherwise it would cut out a linear system on X and the general divisor of this linear system would have singular points filling up X , against Bertini’s theorem. Hence \mathfrak{G} has dimension strictly smaller than the linear system of hyperplanes tangent to X along Γ . Hence we have $\dim(\mathfrak{G}) = r - 7 - h < r - 9$, so $h > 2$ a contradiction. □

Claim 7.5.13 If $h = 0$ then $\varepsilon \leq 3$. If $r = 9$ and $h = 2$, then $\varepsilon \leq 5$.

Proof of the Claim 7.5.13 Assume first $h = 0$. The linear space Π is contained in all hyperplanes of the family \mathfrak{J} . If $h = 0$, we have $\dim(\mathfrak{J}) = r - 5$. Moreover, \mathfrak{J} is not a linear system, because otherwise it would cut out a linear system on X and the general divisor of this linear system would have singular points filling up X , against Bertini’s theorem. Hence \mathfrak{J} has dimension strictly smaller than the family of hyperplanes containing Π , thus $r - 5 < r - (\varepsilon + 1)$, i.e., $\varepsilon \leq 3$.

Consider next the case $r = 9$ and $h = 2$. In this case $\dim(\mathfrak{J}) = 2$. As above, \mathfrak{J} is not a linear system. Hence \mathfrak{J} has dimension strictly smaller than the family of hyperplanes containing Π , thus $2 < 9 - (\varepsilon + 1)$ i.e., $\varepsilon \leq 5$. □

Next we consider the projection $\pi : X \dashrightarrow \mathbb{P}^{r-\varepsilon-1}$ from Π and we denote by Y its image.

Claim 7.5.14 The case $r = 9, h = 2$ and $\varepsilon \geq 4$ does not happen. Hence one always has $\varepsilon \leq 3$.

Proof of the Claim 7.5.14 We have $\dim(\mathfrak{J}) = 2$. Let H be general in \mathfrak{J} and consider again the threefold X_H along which H is tangent to X . By the Infinitesimal Bertini’s Theorem (see [7, Thm. 2.2]), X_H spans a linear space of dimension at most 6 containing Π .

Suppose first $\varepsilon = 4$. If H and H' are general in \mathfrak{J} , they do intersect at some general point of X , because $\dim(\mathfrak{J}) = 2$ and if $x \in X$ is a general point, there is a 1-dimensional family of hyperplanes H in \mathfrak{J} such that $x \in X_H$.

The 3-fold X_H (for $H \in \mathfrak{J}$ general) is mapped by π either to a point p_H or to a line ℓ_H . In either case, both p_H or ℓ_H cannot stay fixed when H varies, because X is non-degenerate in \mathbb{P}^9 . On the other hand, if the image of X_H is a point p_H , since general H and H' in \mathfrak{J} intersect at some general point of X , then $p_H = p_{H'}$, a contradiction. By the same argument, if the image of X_H is a line ℓ_H , then for general H and H' in \mathfrak{J} one has that ℓ_H and $\ell_{H'}$ intersect at general points of both ℓ_H and $\ell_{H'}$. Since Y is swept out by lines of the type ℓ_H , we see that Y should be a plane, and this is a contradiction since Y has to be non degenerate in \mathbb{P}^4 .

The case $\varepsilon = 5$ can be excluded in a similar way. In this case in the map π the 3-fold X_H is mapped to a point p_H that cannot stay fixed when H varies, because X is non-degenerate in \mathbb{P}^9 . On the other hand, since general H and H' in \mathfrak{J} intersect at some general point of X , then $p_H = p_{H'}$, a contradiction. □

So, from now on we may assume that $\varepsilon \leq 3$.

Claim 7.5.15 One has $2 \leq \dim(Y) \leq 3$.

Proof of the Claim 7.5.15 The projection π contracts all lines of \mathcal{R} to points in Y , hence $\dim(Y) \leq 3$. It cannot be the case that $\dim(Y) = 0$ because Y has to span $\mathbb{P}^{r-\varepsilon-1}$ and $r - \varepsilon - 1 \geq 8 - \varepsilon \geq 5$. To finish we have to exclude that $\dim(Y) = 1$. Assume, by contradiction, that this is the case. Then X is swept out by a 1-dimensional family of 3-folds (the fibres of π), each lying in a $\mathbb{P}^{\varepsilon+1}$. This implies $\varepsilon = 3$, otherwise X would be a cone with vertex Π over a curve. Then the general tangent space to X intersects Π in at least a plane, two general tangent spaces to X intersect at least along a line and $\delta(X) \geq 2$ by Terracini Lemma, a contradiction. □

Claim 7.5.16 The case $\dim(Y) = 2$ is not possible.

Proof of the Claim 7.5.16 Assume by contradiction that $\dim(Y) = 2$. Then X sits in the $(\varepsilon + 3)$ -dimensional cone V with vertex Π over Y . Since X is not a cone, we have $\varepsilon \geq 2$.

Let $x \in X$ be general. Then $T_{X,x} \subset T_{V,x}$, and $T_{V,x}$ is a $(\varepsilon + 3)$ -space which contains Π . Since the projection of $T_{X,x}$ from Π is the general tangent space to Y , we see that $T_{X,x}$ intersects Π in a line.

Let s_x be the line which $T_{X,x}$ cuts out on Π . Since X is not a cone, the line s_x moves when x varies and describes a family \mathcal{L} of lines in Π .

We claim that two general lines in \mathcal{L} intersect each other. Otherwise, two general lines in \mathcal{L} are skew, and then $\varepsilon = 3$. Let $p_0, p_1 \in X$ be general points. Then T_{X,p_0,p_1} contains Π , T_{X,p_0} and T_{X,p_1} , hence it coincides with T_{V,p_0,p_1} , which is tangent to X along the union of the two surfaces which are the fibres of π passing through p_0 and p_1 . This is a contradiction, because the surface Γ_{p_0,p_1} is irreducible by hypothesis. We have thus proved that two general lines in \mathcal{L} meet.

Since the lines in \mathcal{L} pairwise meet, then they all belong to the same plane $P \subseteq \Pi$, because they cannot pass through the same point otherwise X would be a cone. Consider the projection $\pi' : X \dashrightarrow \mathbb{P}^{r-3}$ from P . The image of π' is a surface Y' because the general tangent space to X intersects the centre of projection along a line. Moreover X sits in the 5-dimensional cone W over Y' with vertex P . Now we repeat the same argument as above. Namely, let $p_0, p_1 \in X$ be general points. Then T_{X,p_0,p_1} , containing P , T_{X,p_0} and T_{X,p_1} , coincides with T_{W,p_0,p_1} , which is tangent to X along the union of two surfaces, i.e., the fibres of π' passing through p_0 and p_1 , leading again to a contradiction. \square

Hence one has $\dim(Y) = 3$. Note now that, by the proof of (2) of Claim 7.5.9, all tangent 8-spaces to X along a surface Γ in \mathcal{S} contain Π . Under the projection π from Π all these 8-spaces are mapped to a family of dimension 4 ($=\dim(\mathcal{S})$) \mathcal{F} of $(7 - \varepsilon)$ -spaces, which are tangent to Y along the projection of the surfaces in \mathcal{S} from Π . If $q_0, q_1 \in Y$ are general points, then through them there passes a projection of a surface in \mathcal{S} and $T_{Y,q_0,q_1} = \langle T_{Y,q_0}, T_{Y,q_1} \rangle$ is contained in a $(7 - \varepsilon)$ -space in \mathcal{F} . This implies that the two tangent spaces to Y at the general points $q_0, q_1 \in Y$ intersect in dimension λ with

$$6 - \lambda = \dim(T_{Y,q_0}) + \dim(T_{Y,q_1}) - \lambda = \dim(T_{Y,q_0,q_1}) \leq 7 - \varepsilon$$

which implies $\lambda \geq \varepsilon - 1$.

If $\varepsilon = 3$, then two general tangent spaces to $Y \subset \mathbb{P}^{r-4}$ intersect in dimension 2. Then a general curve section of Y has tangent lines which pairwise meet, hence it would be a plane curve, whereas it spans a \mathbb{P}^{r-6} and $r - 6 \geq 3$, a contradiction.

If $\varepsilon = 2$, then two general tangent spaces to $Y \subset \mathbb{P}^{r-3}$ intersect in dimension 1. Then the general hyperplane section of Y , which spans a \mathbb{P}^{r-4} , is a defective surface, hence either it is a cone or a Veronese surface $V_{2,2}$. Hence Y itself is either a cone over a curve or a cone over the Veronese surface $V_{2,2}$. In either case, X would be contained in a cone over a curve or in a cone over a Veronese surface and hence it would be of the first species by Remark 3.0.2, contradiction.

Finally, if $\varepsilon = 1$, then two general tangent spaces to $Y \subset \mathbb{P}^{r-2}$ intersect in dimension 0, i.e., Y is a defective 3-fold and X sits in a cone with vertex a line over Y . By [9, Theorem 1.1], then Y is either a hyperplane section of $\text{Seg}(2, 2)$ (and then $r = 9$), or the Veronese 3-fold $V_{2,3}$ in \mathbb{P}^9 or a projection of it in \mathbb{P}^8 or \mathbb{P}^7 (and then $9 \leq r \leq 11$), or it sits in a 4-dimensional cone over the Veronese surface $V_{2,2}$ (and again $r = 9$) or in a 4-dimensional cone over a curve. However the last case is not possible. Indeed, if Y sits in a 4-dimensional cone over a curve, then the general contact locus of X would be reducible. Hence we are in one of the cases (1), (2), (3) of the statement of Theorem 7.5.2.

Now we consider case (b). So let us assume that $x \in \text{Sing}(X)$. Note that, by Lemma 7.5.5, this is the case only if Γ_{p_0,p_1} has a line directrix which passes through x and, by the generality of s_1 , it consists of singular points of X . We have the:

Claim 7.5.17 Suppose that when p_0 moves, the point x does not move on s_1 and $x \in \text{Sing}(X)$. Then there is a subspace $\Pi \subseteq \text{Sing}(X)$ of dimension ε , with $1 \leq \varepsilon \leq 3$, such that if $\ell \in \mathcal{R}$ is a general line, then ℓ intersects Π in a point. Moreover if Γ is a general surface in \mathcal{S} , then Γ is a (non-developable) scroll with line directrix contained in Π .

Proof of the Claim 7.5.17 In this situation, as p_0, p_1 vary, the point x describes a variety $V \subseteq \text{Sing}(X)$, of dimension ε , with $1 \leq \varepsilon \leq 3$. So V is described by the line directrices of the scrolls Γ_{p_0,p_1} as p_0, p_1 vary. Hence the general line in \mathcal{R} intersects V in its general point because the intersection point is on a general directrix. Let $x, y \in V$ be general points.

Take lines $a, b \in \mathcal{R}$, with $x \in a$ and $y \in b$, which are general lines in \mathcal{R} , and consider the surface Γ in \mathcal{S} containing a, b (we can suppose that a passes through a general point p_0 , that b passes through a general point p_1 and take $\Gamma = \Gamma_{p_0, p_1}$). By construction and by definition of x and y , the line joining x and y is exactly the line directrix of Γ , and it lies on V . In conclusion, V is a linear space Π proving the assertion. \square

Now again we consider the projection $\pi : X \dashrightarrow \mathbb{P}^{r-\varepsilon-1}$ from Π and let Y be its image. Exactly as in Claims 7.5.15 and 7.5.16, we see that Y has to be a 3-fold. If Γ is a general surface in \mathcal{S} , its image via π is a plane curve, because Γ spans a 4-space and we are projecting it from the line directrix which sits in this 4-space. Hence Y contains a family \mathcal{C} of generically irreducible plane curves (the images of the surfaces Γ in \mathcal{S}), such that there is a curve of \mathcal{C} passing through two general points of Y . Then, by the Trisecant Lemma, the curves of \mathcal{C} are conics. The counterimage via π of the general conic in \mathcal{C} is a non-developable scroll with a line directrix contained in Π . Note that either $r - \varepsilon - 1 \leq 6$, or $Y \subset \mathbb{P}^{r-\varepsilon-1}$ is defective. Suppose that $r - \varepsilon - 1 > 6$ and $Y \subset \mathbb{P}^{r-\varepsilon-1}$ is defective. Then, by [9, Theorem 1.1], either $r - \varepsilon - 1 \leq 9$, or Y is a cone, or Y sits in a 4-dimensional cone over a curve. We claim that these two latter cases do not occur. Suppose first that Y is a cone. It could be a cone with vertex a line over a curve C or a cone with vertex a point p over a surface S . In the former case, since Y contains the family \mathcal{C} of conics, then C is a conic, hence Y (that is linearly normal) sits in \mathbb{P}^4 , so that $r - \varepsilon - 1 = 4$, a contradiction, since we are assuming $r - \varepsilon - 1 > 6$. In the latter case, since Y contains the family \mathcal{C} of conics, the surface S has also a family \mathcal{C}' of conics (the projections of the conics in \mathcal{C} from p), such that there is a conic in \mathcal{C}' passing through two general points of S . Then, by the classification of defective surfaces, S is either the Veronese surface $V_{2,2}$ or it is contained in a \mathbb{P}^n , with $n \leq 4$. Hence $r - \varepsilon - 1 \leq 6$, again a contradiction. In conclusion, we have $r - \varepsilon - 1 \leq 9$, i.e., $r \leq 10 + \varepsilon$. If $\varepsilon = 1$ and $r - 2 \geq 7$, again by [9, Theorem 1.1], we are in cases (1), (2) or (3). Otherwise we are in case (4) of the statement of Theorem 7.5.2. \square

Remark 7.5.18 If X is as in cases (1), (2), (3) of Theorem 7.5.2, then X is indeed defective, $f(X) = 1$ and the general contact loci of X are irreducible surfaces. In case (4), which has been overlooked by Scorza in [28, §11], X is still defective with $f(X) = 1$, because X is covered by a 4-dimensional family of scrolls Γ each spanning a 4-space and such that through two general points of X there is a unique scroll Γ containing them. Then the general secant line s to X is also a secant to a scroll Γ and therefore there is a 1-dimensional family of secants to Γ , hence to X , containing the general point of s . It is however not clear if the surfaces Γ are contact loci, neither we have examples of such 4-folds.

We can repeat verbatim the argument of Remark 7.4.6 and conclude that all 4-folds as in Theorem 7.5.2 are singular.

7.6. In this section we discuss case (ii), in which two general surfaces in \mathcal{S} have some isolated point in common off the base locus scheme B . We keep all notation and conventions introduced above, in particular we assume X is not a cone. We will denote by j the dimension of the intersection of the spans of two general surfaces in \mathcal{S} , which are \mathbb{P}^4 . Thus $0 \leq j \leq 3$.

We will prove the:

Theorem 7.6.1 *In case (ii) we have the following possibilities:*

- (*) X sits in a cone with vertex a point over $\text{Seg}(2, 2)$ in \mathbb{P}^8 ;
- (**) $r = 9$ and X sits in a cone with vertex a line over a hyperplane section of $\text{Seg}(2, 2)$ in \mathbb{P}^7 ;

- (***) or $9 \leq r \leq 11$ and X sits in a cone with vertex a line over the Veronese 3-fold $V_{2,3}$ in \mathbb{P}^9 or the projection of it in \mathbb{P}^8 or \mathbb{P}^7 ;
- (***)) $j = 0$, hence given two general surfaces Γ, Γ' in \mathcal{S} , the 4-spaces spanned by them intersect in one point which is also the intersection of Γ and Γ' . In this case the general surface in \mathcal{S} is rational and X itself is rational.

Proof We start with the following:

Claim 7.6.2 There is no subspace Π of \mathbb{P}^r with $\dim(\Pi) = k < r - 1$ such that for the general surface Γ in \mathcal{S} the 4-space spanned by Γ intersects Π in a subspace of dimension 3.

Proof of the Claim 7.6.2 Suppose by contradiction that there is such a subspace Π . Let $x \in X$ be a general point. Then for the general surface Γ in \mathcal{S}_x the span of such a surface is contained in $\langle \Pi, x \rangle$, which has dimension $k + 1 < r$. Hence the whole of X would be contained in $\langle \Pi, x \rangle$, a contradiction because X is non-degenerate in \mathbb{P}^r . □

Claim 7.6.3 There is no plane P such that for the general surface Γ in \mathcal{S} the 4-space spanned by Γ contains P .

Proof of the Claim 7.6.3 Suppose by contradiction that there is such a plane P . Let $\pi : X \dashrightarrow \mathbb{P}^{r-3}$ be the projection from P , and Y the image of π . If Γ is the general surface in \mathcal{S} , then Γ is mapped to a line by π . Thus Y is a 3-fold such that if $x, y \in Y$ are general points, there is a line in Y containing x and y . Hence Y would be a 3-space, a contradiction, because Y has to span the \mathbb{P}^{r-3} and $r - 3 \geq 6$. □

Let first be $j = 3$. Then, since the 4-spaces spanned by the surfaces in \mathcal{S} cannot lie in a 5-space, they have to pass through the same 3-space, and this contradicts Claim 7.6.2.

Let $j = 2$. Consider two general surfaces Γ_1, Γ_2 in \mathcal{S} , and their spans Π_1, Π_2 . Then $\Pi' = \langle \Pi_1, \Pi_2 \rangle$ has dimension 6. Let Γ be another general surface in \mathcal{S} and Π be its span. Of course Π cannot be contained in Π' and by Claim 7.6.2 it cannot intersect it in a 3-space. This implies that Π intersects Π' in the intersection plane of Π_1 and Π_2 , which contradicts Claim 7.6.3.

Let $j = 1$. Let us argue as in the case $j = 2$. Consider two general surfaces Γ_1, Γ_2 in \mathcal{S} , and their spans Π_1, Π_2 . Then $\Pi' = \langle \Pi_1, \Pi_2 \rangle$ has dimension 7. Let Γ be another general surface in \mathcal{S} and Π be its span, which cannot be contained in Π' and by Claim 7.6.2 it cannot intersect it in a 3-space. Then either Π intersects Π' in the intersection line of Π_1 and Π_2 (hence all the spans of the surfaces of \mathcal{S} pass through the same line ℓ), or it intersects Π' in a plane which contains the two distinct intersection lines of Π with Π_1 and Π_2 .

In the former case let $\pi : X \dashrightarrow \mathbb{P}^{r-2}$ be the projection of X from the line ℓ and Y be its image. We claim that Y is a defective 3-fold such that there is an irreducible conic in Y containing two general points of Y . To prove the claim, note first that if Γ is a general surface in \mathcal{S} , then its projection from ℓ is not a plane. Otherwise Y would be swept out by a family of planes such that through two general points of Y there is a plane of the family passing through them. This implies that $Y \subset \mathbb{P}^{r-2}$ would be a linear subspace of dimension at most 4, which is a contradiction because $r - 2 \geq 7$ and Y has to be non-degenerate. Then the general Γ in \mathcal{S} is projected from ℓ to a plane curve, which implies that $\dim(Y) = 3$. Moreover Y is covered by a family of generically irreducible plane curves such that through two general points of Y there is a curve of the family passing through them. The Trisecant Lemma implies that the plane curves are conics and Y is in particular defective, because $r - 2 \geq 7$.

We claim that in this situation we are in cases (***) or (***)). Indeed, let us look at the list of defective 3-folds in [9, Theorem 1.1]. To prove the claim it suffices to prove that:

- Y cannot be a cone;
- Y cannot be contained in a 4-dimensional cone over a curve;
- Y cannot be contained in a 4-dimensional cone over the Veronese surface $V_{2,2}$.

To prove the first and second items, we argue as at the end of the proof of Theorem 7.5.2. Indeed, suppose Y is a cone. If it is a cone with vertex a line over a curve then this curve is a conic, hence Y sits in \mathbb{P}^4 , a contradiction, because Y is linearly normal in \mathbb{P}^{r-2} and $r - 2 \geq 7$. If Y is a cone with vertex a point over a surface S , then S has also a family of conics such that there is a conic passing through two general points of S . Then S is either the Veronese surface $V_{2,2}$ or it sits in a \mathbb{P}^n , with $n \leq 4$. So Y would be contained in a \mathbb{P}^m , with $m \leq 6$, whereas Y is linearly normal in \mathbb{P}^{r-2} and $r - 2 \geq 7$. To prove the last item, we notice that in this case we have $r = 9$. Moreover, by [9, Example 2.4], a threefold Y contained in a 4-dimensional cone over the Veronese surface $V_{2,2}$ has $f(Y) = 1$ and $\gamma(Y) = 2$. Let $p_0, p_1 \in X$ general points and set $q_0 = \pi(p_0), q_1 = \pi(p_1)$. The surface Γ_{X,p_0,p_1} spans a 4-space that contains the line ℓ , hence the hyperplane T_{X,p_0,p_1} in \mathbb{P}^9 also contains ℓ and it is projected from ℓ to the hyperplane T_{Y,q_0,q_1} in \mathbb{P}^7 . Since $\gamma(Y) = 2$, we have that T_{Y,q_0,q_1} is tangent to Y along a surface Γ_{Y,q_0,q_1} , and this would imply that T_{X,p_0,p_1} is tangent to X along $\pi^{-1}(\Gamma_{Y,q_0,q_1})$ that is a 3-fold, a contradiction since $\gamma(X) = 2$.

In the case in which a general 4-space Π intersects Π' in a plane which contains the two distinct intersection lines of Π with Π_1 and Π_2 , the intersection lines of the spans of two general surfaces of \mathcal{S} are generically distinct but they pairwise intersect in one point. Then either all these lines lie in the same plane or they pass through the same point p but do not lie in the same plane. The former case is impossible by Claim 7.6.3. In the latter case consider the projection $\pi : X \dashrightarrow \mathbb{P}^{r-1}$ from p and let Y be its image. Since X is not a cone, then Y is a 4-fold. The general surface in \mathcal{S} , which is a tangential contact surface on X , maps to a surface in Y , which is a tangential contact surface spanning a 3-space. Hence Y has an irreducible, 4-dimensional family \mathcal{Q} of generically irreducible tangential contact surfaces spanning a 3-space, such that given two general points of Y there is a surface in \mathcal{Q} containing the two points. By the Trisecant Lemma, the general surfaces in \mathcal{Q} are irreducible quadrics and therefore $f(Y) \geq 2$.

If $f(Y) = 3$, then by Proposition 2.9.5, (i), Y would either be a cone over a curve or a cone over $V_{2,2}$. Both cases are impossible because in the former case Y would not contain the family \mathcal{Q} of contact quadrics, in the latter Y would lie in a 7-space, whereas Y spans a \mathbb{P}^{r-1} and $r - 1 \geq 8$.

So we have that $f(Y) = 2$, and Y is in the list of Theorem 3.0.1. The only cases in this list which are compatible with the present situation are the cases (iv) and (v) of Theorem 3.0.1 (in the other cases the contact loci are not surfaces), and then we are in cases (*), (**) or (***)).

Finally suppose we are in case $j = 0$. Then we are in case (***) and we will prove that the general surface in \mathcal{S} is rational and X itself is rational. To see this, we need the following:

Claim 7.6.4 Let $x \in X$ be a general point and let t be a general tangent direction of X at x . Then there is a unique surface in \mathcal{S}_x tangent to t .

Proof of the Claim 7.6.4 We assume \mathcal{S}_x , that has dimension 2, to be irreducible, otherwise we substitute \mathcal{S}_x with one of its components. Let $\pi_x : \mathcal{F}_x \rightarrow \mathcal{S}_x$ be the universal family over \mathcal{S}_x , that has dimension 4. There is the obvious rational map $f_x : \mathcal{F}_x \dashrightarrow X$ that is birational because through two general points of X there passes a unique surface in \mathcal{S} . The map f_x induces another obvious birational map $\tilde{f}_x : \mathcal{F}_x \dashrightarrow \tilde{X}$, where $\tilde{X} \rightarrow X$ is the blow-up of X at x , with exceptional divisor E . Look at the birational map \tilde{f}_x^{-1} . Its indeterminacy locus on

\tilde{X} has codimension at least 2, so \tilde{f}_x^{-1} is defined at the general point of E . If ξ is a general point of E , that corresponds to a pair (x, t) , with t general tangent direction at x , then $\tilde{f}_x^{-1}(\xi)$ is the unique required surface. \square

Claim 7.6.4 implies that \mathcal{S}_x , which has dimension 2, is unirational, because there is a dominant map of $\mathbb{P}(T_x(X)) \cong \mathbb{P}^3$ to \mathcal{S}_x (recall that $T_x(X)$ is the Zariski tangent space to X at x), so \mathcal{S}_x is rational. Let now Γ be a general surface in \mathcal{S} . We have a natural map $\rho : \mathcal{S}_x \dashrightarrow \Gamma$, which takes the general surface Γ' in \mathcal{S}_x and maps it to the unique intersection point of Γ' and Γ , which is also the unique intersection point of their spans. This map is dominant by the generality of Γ , hence Γ is unirational, so it is rational. Finally let $x, y \in X$ be general points. There is a natural map $\sigma : \mathcal{S}_x \times \mathcal{S}_y \dashrightarrow X$ which maps the pair (Γ, Γ') of general surfaces in $\mathcal{S}_x \times \mathcal{S}_y$ to their unique intersection point. This map is dominant because through a general point $z \in X$ there passes a unique surface in \mathcal{S}_x and a unique surface in \mathcal{S}_y , intersecting at z . Moreover, it is also generically injective, because given $z \in X$ general, there is a unique surface in \mathcal{S}_x [resp. in \mathcal{S}_y] containing z . This proves that X is rational. \square

Remark 7.6 In cases $(*)$, $(**)$, $(***)$ listed in Theorem 7.6.1, it is clear that X is defective and verifies (ii). In case $(****)$ the 4-fold X is still defective, since the surfaces Γ on \mathcal{S} span 4-spaces, hence $f(X) = 1$. However we do not have examples of such 4-folds, hence we do not know if this case really occurs.

Moreover, if X is smooth, as we have seen many times, cases $(**)$ and $(***)$ are not possible.

7.7. Here we discuss the reducible case, in which given two general points $p_0, p_1 \in X$ there is a unique tangential contact surface Γ_{p_0, p_1} containing p_0, p_1 , which consists of two irreducible components each passing through one of the two points p_0, p_1 . We will denote by \mathcal{T} the family of surfaces of X such that any tangential contact surface Γ is the sum of two components each in \mathcal{T} . By the uniqueness of the tangential contact locus, given a general point $x \in X$ there is a unique surface in \mathcal{T} containing x , hence \mathcal{T} is irreducible (see §2.6) and clearly $\dim(\mathcal{T}) = 2$. We will denote by i the dimension of the span of the general surface in \mathcal{T} . Recall that, since we are studying the case $\gamma(X) = 2$ and $f(X) = \delta(X) = 1$, the general tangential contact surface Γ spans a 4-space by Proposition 2.6.4. Hence the union of two general surfaces in \mathcal{T} spans a 4-space. Therefore $2 \leq i \leq 3$. Indeed, clearly $i \geq 2$ and moreover $i \leq 4$ because, as we said, the union of two general surfaces in \mathcal{T} spans a 4-space. Moreover it cannot be the case that $i = 4$, otherwise the span of a general surface in \mathcal{T} would contain any other surface in \mathcal{T} , hence it would contain X .

Proposition 7.7.1 *In the above setting, if X is not a cone, then X sits in a cone with vertex a plane over a surface.*

Proof With the above notation, assume that $i = 3$. Then if S, S' are two general surfaces in \mathcal{T} , one has $\dim(\langle S \rangle \cap \langle S' \rangle) = 2$. Then either the spans of the surfaces in \mathcal{T} lie in a projective space of dimension 3, which is clearly not possible, or there is a plane P which is contained in $\langle S \rangle$ for all S in \mathcal{T} . By projecting from P we see X sits in a cone with vertex the plane P over a surface.

Assume $i = 2$, then \mathcal{T} is a family of planes. These planes pairwise span a 4-space, hence they pairwise intersect at a point. By [9, Lemma 4.1] there is a plane P such that all planes in \mathcal{T} intersect P along a line. By projecting from P we see again that X sits in a cone with vertex the plane P over a surface. \square

Remark 7.7.2 If X is a 4-fold as in Proposition 7.7.1, then X is defective. In fact if P is the plane vertex of the 5-dimensional cone in which X sits, and if $x \in X$ is a general point, then $T_{X,x}$ intersects P along a line, and therefore if $y \in X$ is another general point, $T_{X,x}$ and $T_{X,y}$ do intersect at a point.

A 4-fold X as in Proposition 7.7.1 is never smooth, because it is singular at points of the plane P vertex of the 5-dimensional cone over a surface S in which it sits. To see this we proceed in this way.

First, assume that X does not contain P , so that it intersects P along a curve C . Take a general hyperplane H in \mathbb{P}^r and consider the section Y of X with H , that sits in a cone with vertex the line $s = P \cap H$ over the surface S . Then Y intersects s at the intersection points p_1, \dots, p_h of H with C . We show that Y is singular at one of these points, thus showing, by the genericity of H , that X is singular. To see this, argue as in Remark 7.4.6. Namely, resolve the projection of Y from s by blowing-up a finite number of points above each p_i . If Y is smooth at one of the points p_i , the last exceptional divisor is a \mathbb{P}^2 , which would be sent by the projection onto a plane. Hence S would be a plane, which is impossible since Y is non-degenerate.

Alternatively, X could contain P . In this case, again proceed as in Remark 7.4.6 by blowing up P in X . Then the exceptional divisor E of the blow-up is a \mathbb{P}^2 -bundle over P , and it is mapped under the projection from P onto S in such a way that each \mathbb{P}^2 in E is mapped to a plane. In particular S would be a plane, which leads to the same contradiction as above.

8 First species

8.1. Next we consider an irreducible, non-degenerate 4-fold, not a scroll, $X \subset \mathbb{P}^r$, $r \geq 9$, of the first species, i.e., $\gamma(X) = 3$. Taking into account Proposition 2.9.5 and Theorem 3.0.1 we may assume $f(X) = \delta(X) = 1$.

Given $p_0, p_1 \in X$ general points, the tangent spaces to X at p_0 and p_1 intersect at one point and the tangential contact locus Γ_{p_0,p_1} is a 3-fold, spanning a 6-space Π_{p_0,p_1} (see Proposition 2.6.4). We have two cases: the *irreducible* and the *reducible* case, according to the possibilities that Γ_{p_0,p_1} is irreducible or it consists of two irreducible components each passing through one of the two point p_0, p_1 .

Since all the above properties are preserved by a general projection to \mathbb{P}^9 because $\dim(S(X)) = 8$, we will assume that $r = 9$.

Lemma 8.1.1 *In the above setting we have the following cases:*

- (i) *in the irreducible case the 3-folds Γ_{p_0,p_1} move in a 2-dimensional family which is a linear system;*
- (ii) *in the reducible case, the two components of Γ_{p_0,p_1} move in a 1-dimensional (not necessarily rational) pencil.*

Proof Since we are assuming $r = 9$ and $f(X) = 1$, given $p_0, p_1 \in X$ general points, there is a unique hyperplane bitangent to X at p_0 and p_1 , hence there is a unique 3-fold Γ_{p_0,p_1} as above, which is smooth at p_0 and p_1 . In the irreducible case the family of 3-folds Γ_{p_0,p_1} has dimension two since through a general point there passes a one dimensional family of such 3-folds. It follows from [7, Theorem 5.10] that we are in case (i). Case (ii) follows from the fact that given $x \in X$ general there is a unique component of a contact variety passing through x (hence the family has dimension 1). □

Let $x \in X$ be a general point and consider the tangential projection $\tau_x : X \dashrightarrow X_1 \subset \mathbb{P}^4$. Since X is defective and $f(X) = 1$, then X_1 is a 3-fold.

Lemma 8.1.2 *In the above setting, we have $t(X_1) = d(X_1) = 2$, hence X_1 is swept out by a 1-dimensional family \mathcal{P} of planes along which the tangent space to X_1 is constant, so that the dual variety X_1^* of X_1 is a curve. Each plane of the family \mathcal{P} is the image via τ_x of a 3-fold $\Gamma_{x,y}$.*

Proof The first assertion follows by Remark 2.7.2 and by the fact that for a hypersurface Z in projective space one has $t(Z) = d(Z)$. The second assertion follows from the fact that the general contact locus of X projects onto the general tangential locus of X_1 , which is a plane of the family \mathcal{P} . □

8.2 In this section we examine the irreducible case. We keep all notation and convention we introduced above.

Lemma 8.2.1 *In the above setting, if we are in the irreducible case, then the curve X_1^* is rational.*

Proof The varieties $\Gamma_{x,y}$ passing through x form a rational pencil, by Lemma 8.1.1,(i), so that the family \mathcal{P} is also rational. The rationality of X_1^* follows. □

Next we want to study more closely the general tangential projection $\tau_x : X \dashrightarrow X_1 \subset \mathbb{P}^4$ and the 3-fold X_1 . By Lemma 8.1.2 and by [17, Theorem 2.20], we deduce that X_1 is of one of the following types:

- (i) there is a non-degenerate, irreducible, projective curve $C \subset \mathbb{P}^4$ such that X_1 is the closure of the union of the osculating planes to C at all non-flex points of C ; the tangent space to X_1 along a general osculating plane to C is the corresponding osculating 3-space;
- (ii) X_1 is the cone with vertex a point over a tangent developable scroll Σ in \mathbb{P}^3 ;
- (iii) X_1 is the cone with vertex a line over a plane curve.

Lemma 8.2.2 *In the above setting only case (iii) occurs.*

Proof We follow Scorza’s argument in §6 of [28]. Suppose first we are in case (i). There are formulae (see [5], p. 490, and, for modern versions, [23], [2, p. 39] and [13, Proposition 10.4.13]) that give the degree n_1 of the tangent developable to C , the degree n_2 of X_1 , i.e., the locus of osculating planes to C , the class n_3 of C , i.e., the degree of X_1^* . For any branch ξ of C , let $(\alpha, \alpha_1, \alpha_2)$ be the (truncated) rank sequence of ξ :

- α is the degree of the branch, i.e. the intersection multiplicity of the branch with the general hyperplane through the origin of the branch;
- $\alpha + \alpha_1$ is the intersection multiplicity of ξ with a general hyperplane containing the tangent line to ξ at the origin (the tangent line to a branch singular at the origin is the reduced tangent cone to the branch at the origin);
- $\alpha + \alpha_1 + \alpha_2$ is the intersection multiplicity of ξ with a general tangent hyperplane passing through the osculating plane to the branch.

Note that for a general branch one has $\alpha = \alpha_1 = \alpha_2 = 1$. Let d be the degree of C , while, as we know, its genus is 0. The formulae in question give

$$\begin{aligned}
 n_1 &= 2(d - 1) - \sum (\alpha - 1) \\
 n_2 &= 3(d - 2) - \sum (2\alpha + \alpha_1 - 3) \\
 n_3 &= 4(d - 3) - \sum (3\alpha + 2\alpha_1 + \alpha_2 - 6)
 \end{aligned}
 \tag{14}$$

where the sums are taken over all branches of C .

Consider again the general tangential projection $\tau_x : X \dashrightarrow X_1 \subset \mathbb{P}^4$. Let Y be a general hyperplane section of X passing through x . Note that Y is not defective because $f(X) = \delta(X) = 1$ (recall Lemma 2.4.3). However if p_0, p_1 are general points of Y , the 7-space T_{Y,p_0,p_1} is tangent to Y along a surface Γ_{Y,p_0,p_1} , i.e. it is tangent along the intersection of the 3-fold Γ_{X,p_0,p_1} with the hyperplane spanned by Y .

We can then consider the tangential projection $\tau_{Y,x} : Y \dashrightarrow Y_1 \subset \mathbb{P}^4$. Since Y is not defective, then Y_1 has dimension 3 and therefore $Y_1 = X_1$ (because $f(X) = 1, f(Y) = 0$ and (3) and Lemma 2.7.3 hold). We let s be the degree of the map $\tau_{Y,x}$.

Take a general plane Π in \mathbb{P}^4 which cuts out a plane curve Z of degree n_2 on X_1 . By Bertini's Theorem the curve Z is irreducible. By Lemma 8.2.1 the curve Z is rational. We let D be the normalization of the preimage Z' of Z via the generically finite map $\tau_{Y,x}$, so that D is a smooth irreducible curve.

Note that, since Z is rational, on D there is the linear series g_s^1 consisting of the fibres of the map of D to Z (that is rational). This is the pull-back on D of the movable part of the linear series cut out on Z' by the rational pencil described by the surfaces $\Gamma_{Y,x,y}$ on Y passing through x . The hyperplanes of \mathbb{P}^4 (which come via the tangential projections $\tau_{X,x}$ and $\tau_{Y,x}$ from hyperplanes containing $T_{X,x}$ or $T_{Y,x}$ respectively) cut out on D the pull-back to D of the linear series cut out on Z by the lines of the plane Π , hence the pull-back on D of the linear series cut out on Z' by these hyperplanes has dimension 2 and degree sn_2 , i.e., it is a $g_{sn_2}^2$, which is composite with the above g_s^1 . Hence, a hyperplane passing through $T_{Y,x}$ and tangent to Y along a surface $\Gamma_{Y,x,y}$ cuts out on Z' a divisor whose pull-back on D consists of a divisor of the g_s^1 , counted with multiplicity 2, plus $n_2 - 2$ more divisors of the g_s^1 .

Let $p \in D$ be a general point, which by abusing notation, we may identify with a general point of Z' hence with a general point of X and let q be its image on Z . Note that q is also a general point of X_1 .

The number of hyperplanes passing through $T_{Y,x}$ and p and tangent to Y along surfaces of the type $\Gamma_{Y,x,y}$ which do not pass through p equals the number of divisors of the $g_{sn_2}^2$ which contain the divisor of the g_s^1 containing p and then another divisor of the g_s^1 counted with multiplicity 2. This number coincides with the number of lines of the plane Π passing through q and tangent to Z at a point different from q . By Hurwitz formula, and taking into account that Z is rational, we have that this number is $2(n_2 - 2)$.

On the other hand, we can compute this number in another way. Indeed, this number is nothing but the number of tangent hyperplanes to X_1 passing through q but not tangent to X_1 in q . By duality this equals the number of intersections of a general tangent hyperplane to X_1^* with X_1^* off the contact point, i.e., $n_3 - 2$.

In conclusion we have $n_3 - 2 = 2(n_2 - 2)$, which, by using (14), reads

$$\sum(\alpha - 1) = 2(d - 1) + \sum(\alpha_2 + 5)$$

and implies $n_1 = 2(d - 1) - \sum(\alpha - 1) < 0$, a contradiction. Hence case (i) cannot happen.

Assume next we are in case (ii). Consider the tangent developable scroll $\Sigma \subset \mathbb{P}^3$ and let \mathcal{C} be the rational curve of degree d whose tangent lines fill up Σ . For any branch ξ of \mathcal{C} , let (α, α_1) be the (truncated) rank sequence of ξ , with the same meaning as above. Then the degree of Σ , hence of X_1 , is

$$n_2 = 2(d - 1) - \sum(\alpha - 1)$$

whereas the class of \mathfrak{C} , which coincides with the degree of X_1^* , is

$$n_3 = 3(d - 2) - \sum (2\alpha + \alpha_1 - 3).$$

Then the same argument as above leads to the equality

$$d + \sum (\alpha_1 - 1) = 0$$

which is also impossible. Then also case (ii) cannot happen, and only case (iii) can occur. \square

Theorem 8.2.3 *Let $X \subset \mathbb{P}^9$ be a non-degenerate variety of dimension 4, not a scroll, of the first species, which is not a cone. If X presents the irreducible case, then X sits in a 6-dimensional cone with vertex a 3-space over a Veronese surface $V_{2,2}$ in a \mathbb{P}^5 skew with the vertex.*

Proof By Lemma 8.2.2, given a general point $x \in X$, all hyperplanes containing $T_{X,x}$ and tangent to a 3-fold $\Gamma_{x,y}$ also pass through a fixed \mathbb{P}^6 , i.e., the join of $T_{X,x}$ with the line vertex of X_1 (which is a cone with vertex a line by Lemma 8.2.2). By Lemma 8.2.1, these hyperplanes form, in the dual plane of this \mathbb{P}^6 an irreducible rational curve.

Let us consider the 2-dimensional family \mathcal{K} of hyperplanes tangent to X along a 3-fold of the type Γ_{p_0,p_1} . By duality, this family describes an irreducible surface $S \subset (\mathbb{P}^9)^*$ which possesses a family of dimension at least 2 of irreducible, rational, plane curves.

Claim 8.2.4 The surface S spans a \mathbb{P}^n , with $2 \leq n \leq 5$.

Proof of the Claim 8.2.4 The bound $n \geq 2$ is obvious. Let us prove that $n \leq 5$. By [9, Lemma 4.1], the elements of the family \mathcal{P} of the planes spanned by the plane curves on S either pass through the same point p , or intersect a given plane P along lines, or they span at most a 5-space. In the first case, by projecting S from p we find a surface which possesses a 2-dimensional family of lines, hence it is a plane, thus $n = 3$. In the second case, by projecting S from P , we see that each plane curve of S is contracted to one point and, since these curves intersect each other, S itself is contracted to a point, which again implies $n = 3$. The Claim is thus proved. \square

More precisely:

Claim 8.2.5 One has $n = 5$. Moreover all tangent spaces to X intersect a fixed 3-space Π along a line.

Proof of the Claim 8.2.5 First of all $n = 2$ is impossible, because then the hyperplanes of \mathcal{K} vary in a 2-dimensional linear system cutting on X a linear system of surfaces which are all singular with the singular points filling up X , contrary to Bertini’s theorem.

Assume $n \geq 3$. In this case all the hyperplanes of \mathcal{K} contain one and the same subspace Π of dimension $8 - n$. Let $x \in X$ be a general point. The hyperplanes in \mathcal{K} containing $T_{X,x}$ all contain the same \mathbb{P}^6 (see the beginning of the proof), which contains Π and also $T_{X,x}$. Hence $T_{X,x}$ cuts Π along a subspace of dimension at least $6 - n$.

If $n = 3$, two general tangent spaces to X intersect in at least a line, a contradiction, because $f(X) = 1$. Assume $n = 4$. All tangent spaces to X intersect Π along a plane. Consider the projection $\pi : X \dashrightarrow \mathbb{P}^4$, of X from Π . The image of X is a curve Λ . The general hyperplane of \mathcal{K} maps via π to a general bitangent hyperplane to Λ and the 3-folds Γ_{p_0,p_1} would then be reducible in pairs of fibres of π , a contradiction since we are in the irreducible case.

In conclusion this proves that $n = 5$. Then all the hyperplanes of \mathcal{K} contain one and the same 3-space Π and we can conclude again that all tangent spaces to X intersect Π along a line. □

Consider the projection $\pi : X \dashrightarrow \mathbb{P}^5$ of X from Π . The image of X is a surface S . The general hyperplane of \mathcal{K} maps via π to a general bitangent hyperplane to S , which is also tangent along an irreducible curve, the image of Γ_{p_0,p_1} . To finish our proof we need to prove the:

Claim 8.2.6 One has $S = V_{2,2}$

Proof of the Claim 8.2.6 If $x, y \in S$ are general points, there is a bitangent hyperplane to S at x and y , hence the tangent spaces to S at x and y intersect at a point, thus S is defective with irreducible contact curve. Hence $S = V_{2,2}$ (see [7, Theorem 1.3]). □

This ends our proof. □

Remark 8.2.7 Let $X \subset \mathbb{P}^9$ be a 4-fold as in the statement of Theorem 8.2.3, sitting in a 6-dimensional cone V with vertex a 3-space Π over a Veronese surface $V_{2,2}$. We claim that X is in general defective, of the first species, presenting the irreducible case. Indeed, let $p_0, p_1 \in X$ be general points. The space T_{V,p_0,p_1} has dimension 8 and it contains T_{X,p_0} and T_{X,p_1} , which therefore in general intersect at a point, proving that X is defective with $f(X) = 1$. Moreover V is swept out by a 2-dimensional family of quadric cones of rank 3, which project from Π the conics of the Veronese surface $V_{2,2}$. For each of these quadric cones Q , there is a hyperplane which is tangent to V along Q . In general the intersection of Q with X is an irreducible 3-fold of the type Γ_{p_0,p_1} , proving our claim.

Also in this case X is never smooth.

8.3. In this section we examine the reducible case. We consider irreducible, defective, projective threefolds $X \subset \mathbb{P}^r$, of the first species, with $r \geq 9$, not a scroll, with $f(X) = 1$. By projecting generically to \mathbb{P}^9 we may assume that $r = 9$. If $p_0, p_1 \in X$ are general points, the tangent spaces to X at p_0 and p_1 span a hyperplane which is tangent to X along a contact 3-fold Γ_{p_0,p_1} which is reducible in two irreducible components each containing p_0 and p_1 respectively. By Lemma 8.1.1, these two components vary in a pencil which we denote by \mathcal{G} , and we denote by Φ its general element.

Theorem 8.3.1 *Let $X \subset \mathbb{P}^r$ be a non-degenerate variety of dimension 4, not a scroll, of the first species with $f(X) = 1$, which is not a cone. If X presents the reducible case, then one of the following occurs:*

- (i) X lies in a 6-dimensional cone with vertex a \mathbb{P}^4 over a curve;
- (ii) X lies in a 5-dimensional cone with vertex a plane over a scroll surface.

Proof Let us denote by m the dimension of $\langle \Phi \rangle$, with $\Phi \in \mathcal{G}$ a general element. Since we are assuming X not a scroll, we have $m \geq 4$. On the other hand, if Φ and Φ' are two general elements of \mathcal{G} , then $\langle \Phi, \Phi' \rangle$ has dimension 6 (see Proposition 2.6.4,(iv)). This implies that $m \leq 5$.

If $m = 4$ and if Φ and Φ' are two general elements of \mathcal{G} , then $\langle \Phi \rangle \cap \langle \Phi' \rangle$ has dimension 2. By [9, Lemma 4.1], we have only the following two possibilities:

- (a) there is a fixed plane Π such that, for a general element Φ of \mathcal{G} , one has $\Pi \subset \langle \Phi \rangle$;
- (b) there is a fixed 4-space Π such that, for a general element Φ of \mathcal{G} , one has that $\Pi \cap \langle \Phi \rangle$ has dimension 3.

If (a) holds, we are in case (ii) and if (b) holds we are in case (i).

If $m = 5$ and if Φ and Φ' are two general elements of \mathcal{G} , then $\langle \Phi \rangle \cap \langle \Phi' \rangle$ has dimension 4. This implies that there is a 4-space Π such that if Γ is a general element of \mathcal{G} , one has that $\Pi \subset \langle \Phi \rangle$. Then we are again in case (i). \square

Remark 8.3.2 Note that if (i) of Theorem 8.3.1 occurs, then $X \subset \mathbb{P}^r$ is actually defective and in general it has $f(X) = 1$ and presents the reducible case. Indeed, if $p_0, p_1 \in X$ are general points, and V is the 6-dimensional cone containing X , then T_{V, p_0, p_1} has dimension 8 and it contains T_{X, p_0} and T_{X, p_1} , which therefore in general intersect at a point.

If we are in case (ii) of Theorem 8.3.1, X is still defective with, in general, $f(X) = 1$. In fact, let V be the 5-dimensional cone on which X sits, with vertex the plane Π . Then V is swept out by a 1-dimensional family \mathcal{K} of 4-spaces, i.e., the spans of Π with the lines of the scroll Σ which is the base of the cone. The general such 4-space P intersects X along a 3-fold X_P . If $x \in X$ is a general point and P is the unique element of \mathcal{K} containing x , then $T_{X, x}$ intersects P in the 3-space $T_{X_P, x}$ and therefore it intersects Π along a line. Therefore two general tangent spaces to X intersect in general at a point in Π , proving that X is defective.

It is interesting to note that, in case (ii) of Theorem 8.3.1, X is not in general of the first species. In fact, let $p_0, p_1 \in X$ be general points. Then T_{V, p_0, p_1} has dimension 8 and, containing T_{X, p_0, p_1} , it coincides with it. Now in general the bitangent hyperplane to V at p_0, p_1 is only tangent along the 3-spaces generators of V passing through p_0, p_1 , hence in a general situation it is bitangent to X only along the surfaces in which these generators cut X .

However, it may happen that, in some particular cases, X is of the top species. For example this is the case if the scroll Σ is developable. In this case in fact the tangent space to V is constant along the 4-spaces of the family \mathcal{K} so that in this case a general bitangent hyperplane to X is tangent along two threefolds of the form X_P with $P \in \mathcal{K}$.

In any event, X is never smooth in these cases.

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