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TOWARDS A PROBABILITY THEORY FOR PRODUCT LOGIC: STATES, INTEGRAL REPRESENTATION AND REASONING

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Abstract. The aim of this paper is to extend probability theory from the classical to the product t-norm fuzzy logic setting. More precisely, we axiomatize a generalized notion of finitely additive probability for product logic formulas, called state, and show that every state is the Lebesgue integral with respect to a unique regular Borel probability measure. Furthermore, the relation between states and measures is shown to be one-one. In addition, we study geometrical properties of the convex set of states and show that extremal states, i.e., the extremal points of the state space, are the same as the truth-value assignments of the logic. Finally, we axiomatize a two-tiered modal logic for probabilistic reasoning on product logic events and prove soundness and completeness with respect to probabilistic spaces, where the algebra is a free product algebra and the measure is a state in the above sense.

Keywords. Probability theory; nonclassical events; free product algebras; states; Riesz representation theorem; regular Borel measures; two-tiered modal logics.

1. Introduction

In his monograph [21], Hájek established the theoretical ground for a wide family of fuzzy (thus, many-valued) logics which, since then, has been significantly developed and further generalized, giving rise to a discipline that has been named Mathematical Fuzzy logic (MFL). Hájek's approach consists in fixing the real unit interval as standard domain to evaluate atomic formulas, while the evaluation of compound sentences only depends on the chosen operation which provides the semantics for the so called *strong conjunction* connective. His general approach to fuzzy logics is grounded on the observation that, if strong conjunction is interpreted by a continuous t-norm [22], then any other connective of a logic has a natural standard interpretation.

Among continuous t-norms, the so called Łukasiewicz, Gödel and product t-norms play a fundamental role. Indeed, Mostert-Shields' theorem [22] shows that a t-norm is continuous if and only if it can be built from the previous three ones by the construction of ordinal sum. In other words, a t-norm is continuous if and only if it is an ordinal sum of Łukasiewicz, Gödel and product t-norms. These three operations determine three different algebraizable propositional logics (bringing the same names as their associated t-norms), whose equivalent algebraic semantics are the varieties of MV, Gödel and product algebras respectively.

The first generalization of probability theory to the nonclassical settings of t-norm based fuzzy logics in Hájek sense, is due to Mundici who, in 1995, introduced

the notion of *state* for the class of MV-algebras —the algebraic counterpart Łukasiewicz logic— with the aim of capturing the notion of average degree of truth of a proposition, [28].

In that paper, states are functions mapping an MV-algebra to the real unit interval $[0, 1]$, satisfying a normalization condition and the finite additivity law. Such functions suitably generalize the classical notion of finitely additive probability measures on Boolean algebras, in addition to corresponding to convex combinations of valuations of Łukasiewicz propositional logic. However, states and probability measures were previously studied in [9] (see also [10, 30]) on Łukasiewicz tribes (σ-complete MV-algebras of fuzzy sets) as well as on other t-norm based tribes with continuous operations.

MV-algebraic states have been deeply studied in recent years, as they enjoy several important properties and characterizations (see [18] for a survey). One of the most important results in that framework is Kroupa-Panti theorem [29, §10], a representation result showing that every state of an MV-algebra is the Lebesgue integral with respect to a regular Borel probability measure. Moreover, the correspondence between states and regular Borel probability measures is one-one.

Many attempts of defining suitable notions of state in different structures have been made (see for instance [18, §8] for a short survey). In particular, in [5], the authors provide a definition of state for the Lindenbaum algebra of Gödel logic that corresponds to the integration of the n-place truth-functions corresponding to Gödel formulas, with respect to Borel probability measures on the real unit cube $[0, 1]^n$. Moreover, such states are shown to correspond to convex combinations of finitely many truth-value assignments. Similar results have been obtained for the case Gödel logic expanded with Baaz-Monteiro operator Δ [1], and for the case of Nilpotent Minimum logic [4].

The aims of this contribution are the following: (1) we will introduce and study states for product logic –the remaining fundamental many-valued logic for which such a notion is still lacking– (2) we will prove that our axiomatization results in characterizing Lebesgue integrals of truth-functions of product logic formulas with respect to regular Borel probability measures, and (3) following similar lines to those of [14, 20], we will axiomatize a modal expansion of Łukasiewicz logic for probabilistic reasoning on events described by formulas of product logic. In more detail, we show that states of the Lindenbaum algebra of product logic over n variables, i.e. the free n-generated product algebra, correspond, one-one, to regular Borel probability measures on $[0, 1]^n$.[1] Moreover, and quite surprisingly since in the axiomatization of states the product t-norm operation is only indirectly involved via a condition concerning double negation, we prove that every state belongs to the convex closure of product logic valuations. Finally, these results will allow us to introduce a suitable class of probabilistic-like models with respect to which the modal logic we will introduce in Section 6 turns out to be sound and complete.

The paper is structured as follows. After this introduction, in Section 2 we will recall the functional representation of the free n-generated product algebra $\mathcal{F}_{\mathbb{P}}(n)$, as presented in [3] (see also [11]). We will easily prove that such functions, although they are not continuous, are indeed Borel measurable. In particular, from

[1]Note that, unlike Kroupa-Panti theorem, we do not deal with states of *any* product algebra but of finitely-generated free product algebras.

that functional representation of product logic functions, it follows that the domain $[0,1]^n$ of each such a function can be partitioned in locally compact and Hausdorff subsets of $[0,1]^n$, named G_ε (with ε varying in a certain set Σ, depending on the atoms of the boolean skeleton of $\mathcal{F}_{\mathbb{P}}(n)$). More precisely, each G_ε is an F_σ set since, in fact, it is a countable union of a family $\{G_\varepsilon^q\}_{q\in(0,1]\cap\mathbb{Q}}$ of nested compact subsets of $[0,1]^n$, and hence it is a σ-locally compact set (see [33, §1.11]). Over each G_ε, the function is actually continuous. Moreover, any continuous function with domain one of the compact sets G_ε^q can be uniformly approximated by linear combinations of the functions of $\mathcal{F}_{\mathbb{P}}(n)$ restricted to such subsets.

In Section 3, we will axiomatize our notion of state of $\mathcal{F}_{\mathbb{P}}(n)$, and show its properties together with some examples. In particular, we will investigate states of the 1-generated free product algebra, and see how this analysis reflects into its spectral space.

In Section 4 we will prove our main result, that is to say, for every state s of $\mathcal{F}_{\mathbb{P}}(n)$ there is a unique Borel probability measure μ on $[0,1]^n$ such that s is the Lebesgue integral with respect to μ, and viceversa, every such an integral operator is a state in our sense. In Section 5, we shall prove that the state space of $\mathcal{F}_{\mathbb{P}}(n)$ is convex and closed. Thus, via Krein-Milman theorem (see for instance [19]) every state is a convex combination of extremal ones. We will hence characterize the extremal states, proving that they coincide with the homomorphisms of $\mathcal{F}_{\mathbb{P}}(n)$ into $[0,1]$, that is to say, product logic valuations. Thus, the state space results to be generated by the truth-value assignments of the logic.

Finally, Section 6 is devoted to presenting a logic for probabilistic reasoning on many-valued events represented by formulas of product logic. For that formalism we will provide an (infinitary) axiomatization which is sound and complete with respect to a probabilistic-like semantics given by states of free product algebras.

For the sake of readability we moved some technical proofs in an appendix at the end of the paper.

2. Product algebras and product functions

In this section we are going to recall some basic facts and preliminary notions about product algebras. In particular we will focus on free, finitely generated, product algebras and their functional representation, mainly reporting results from [3]. We assume the reader to be familiar with standard notions of universal algebra and algebraic semantics for many-valued logics. For otherwise we point them to the standard monographs [8] and [21, 12] respectively. To start with, let us recall that a BL-algebra [21] is a bounded, integral and commutative residuated lattice $\mathbf{A} = (A, \odot, \to, \wedge, \vee, 0, 1)$ which satisfies the following equations:

$$(x \to y) \vee (y \to x) = 1 \text{ (prelinearity)},$$
$$x \odot (x \to y) = x \wedge y \text{ (divisibility)}.$$

In what follows we shall adopt the following abbreviations: $\neg x := x \to 0$, for every $n \in \mathbb{N}$, $x^n := x \odot \ldots \odot x$ (n-times).

A BL-algebra $\mathbf{A}$ is a *product algebra* if it further satisfies

$$x \wedge \neg x = 0 \text{ and } \neg\neg x \to ((y \odot x \to z \odot x) \to (y \to z)) = 1.$$

Product algebras form a variety which is denoted by $\mathbb{P}$.

Example 2.1. *(1) Any Boolean algebra is a product algebra. Furthermore for every product algebra* $\mathbf{A}$*, the biggest Boolean subalgebra of* $\mathbf{A}$*,* $\mathscr{B}(\mathbf{A})$ *has universe*

$\{x \in A \mid \neg\neg x = x\}$ *(cf.* [27, Theorem 3.1(1)]*). The Boolean algebra* $\mathscr{B}(\mathbf{A})$ *is called the* Boolean skeleton *of* $\mathbf{A}$*.*

(2) Endow the real unit interval $[0,1]$ *with operations defined in the following manner:* $x \odot y = x \cdot y$ *(the usual product),* $x \to y = 1$ *if* $x \leq y$ *and* $x \to y = y/x$ *otherwise,* $x \wedge y = \min(x,y)$*,* $x \vee y = \max(x,y)$*. Thus* $[0,1]_\Pi = ([0,1], \odot, \to, \wedge, \vee, 0, 1)$ *is a product algebra, known as the* standard *product algebra. Any non Boolean product algebra, such as* $[0,1]_\Pi$*, is generic for* $\mathbb{P}$*, i.e.,* $\mathbb{P}$ *is generated as a variety by* $[0,1]_\Pi$ [21, Corollary 4.1.11].

For every $n \in \mathbb{N}$, let $\mathcal{F}_\mathbb{P}(n)$ be the free product algebra over n free generators. That is, since $[0,1]_\Pi$ is generic for $\mathbb{P}$, $\mathcal{F}_\mathbb{P}(n)$ is isomorphic to the product subalgebra of $[0,1]^{[0,1]^n}$ generated by the projection maps (see [3]). Thus, every element of $\mathcal{F}_\mathbb{P}(n)$ can be regarded as a function $[0,1]^n \to [0,1]$ that we shall call a *product function*.

We are going to recall the description of $\mathcal{F}_\mathbb{P}(n)$ as presented in [3] of which we will also adopt the notation for the sake of uniformity. It is known that $\mathscr{B}(\mathcal{F}_\mathbb{P}(n))$, the Boolean skeleton of $\mathcal{F}_\mathbb{P}(n)$, coincides with the free Boolean algebra over n generators. In particular, $\mathscr{B}(\mathcal{F}_\mathbb{P}(n))$ is finite and hence atomic. Thus, we can safely identify the set of atoms of $\mathscr{B}(\mathcal{F}_\mathbb{P}(n))$, say $at(n)$, with the set $\Sigma = \{1,2\}^n$ of strings $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$ of length n over the binary set $\{1,2\}$ and adopt the same notation of [3] without danger of confusion:

$$at(n) = \{p_\epsilon \in \mathcal{F}_\mathbb{B}(n) \mid \epsilon \in \Sigma\},$$

that is to say, for every $\epsilon = \langle \epsilon_1, \ldots, \epsilon_n \rangle \in \Sigma$, $p_\epsilon = \bigwedge_{i=1}^n \neg^{\epsilon_i} x_i$, where $x_1, \ldots, x_n$ are the free generators of $\mathcal{F}_\mathbb{P}(n)$, while $\neg^1 = \neg$ and $\neg^2 = \neg\neg$. Thus, for every $\epsilon = \langle \epsilon_1, \ldots, \epsilon_n \rangle$, we define

$$G_\epsilon = \{\langle t_1, \ldots, t_n \rangle \in [0,1]^n \mid t_i > 0 \text{ if } \epsilon_i = 2 \text{ and } t_i = 0 \text{ if } \epsilon_i = 1\}.$$

The set $\{G_\epsilon \mid \epsilon \in \Sigma\}$ is then a partition of $[0,1]^n$ (cf. [3]).

For instance, for $n = 2$, we have $p_{(1,1)} = \neg x_1 \wedge \neg x_2$, $p_{(1,2)} = \neg x_1 \wedge \neg\neg x_2$, $p_{(2,1)} = \neg\neg x_1 \wedge \neg x_2$, $p_{(2,2)} = \neg\neg x_1 \wedge \neg\neg x_2$, while the following Figure 1 shows how $[0,1]^2$ is partitioned by $G_{(1,1)}$, $G_{(1,2)}$, $G_{(2,1)}$ and $G_{(2,2)}$.

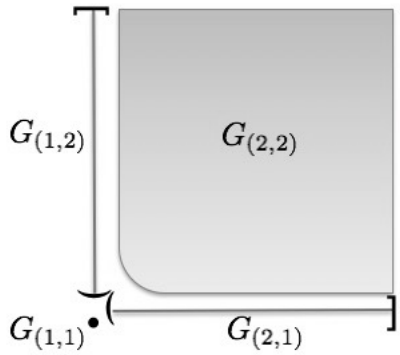


FIGURE 1. The partition of $[0,1]^2$ into $G_{(1,1)}$, $G_{(1,2)}$, $G_{(2,1)}$ and $G_{(2,2)}$.

In what follows, for every function $f : [0,1]^n \to [0,1]$ and for every $\epsilon \in \Sigma$, we will denote by f_ϵ the restriction of f to G_ϵ, i.e. $f_\epsilon = f_{\restriction G_\epsilon}$.

Definition 2.2. *Let $n \in \mathbb{N}$ and $\mathscr{P}(n)$ be the set of functions $f : [0,1]^n \to [0,1]$ such that, for every $\epsilon \in \Sigma$, either $f_\epsilon = 0$ or, if $f_\epsilon > 0$ pointwise, it is continuous and piecewise monomial. The pointwise application of the operations $\odot$, $\to$, $\wedge$ and $\vee$, together with the functions constantly 0 and 1 make $\mathscr{P}(n)$ into a product algebra that we still denote by $\mathscr{P}(n)$ without danger of confusion.*

The functional representation theorem for free, finitely generated, product algebras then reads as follows:

Theorem 2.3 ([3])**.** *For every $n \in \mathbb{N}$, $\mathcal{F}_{\mathbb{P}}(n)$ is isomorphic to $\mathscr{P}(n)$. Thus a function $f : [0,1]^n \to [0,1]$ is a product function iff f is such that, for every $\epsilon \in \Sigma$, either $f_\epsilon = 0$ or $f_\epsilon > 0$ and it is continuous and piecewise monomial.*

In the rest of this section we shall provide preparatory results about the sets G_ϵ.

Lemma 2.4. *For every $\epsilon \in \Sigma$, G_ϵ is a Borel subset of $[0,1]^n$, locally compact and Hausdorff.*

Proof. First of all, by definition,

$$G_\epsilon = \prod_{i=1}^{n} A_i,$$

where $A_i = (0,1]$ if $\epsilon_i = 2$ and $A_i = \{0\}$ if $\epsilon_i = 1$. Hence, G_ϵ can also be expressed as the following countable union of closed sets in the product topology of $[0,1]^n$,

$$G_\epsilon = \bigcup_{q \in \mathbb{Q} \cap (0,1]} \prod_{i=1}^{n} B_i^q, \tag{1}$$

where $B_i^q = [q,1]$ if $\epsilon_i = 2$ and $B_i^q = \{0\}$ if $\epsilon_i = 1$. Therefore G_ϵ is a Borel subset of $[0,1]^n$. It also easily follows that each G_ϵ is locally compact and Hausdorff. □

In the proof of Lemma 2.4 above, we showed that every G_ϵ is a countable union of compact subsets of $[0,1]^n$, through (1). For the sake of a later use and a lighter notation, let us introduce the following.

Notation 1. *For every $\epsilon \in \Sigma$ and for every $q \in (0,1] \cap \mathbb{Q}$,*

$$G_\epsilon^q = \prod_{i=1}^{n} B_i^q.$$

Remark 2.5. *For every $\epsilon \in \Sigma$, the set $\{G_\epsilon^q \mid q \in (0,1] \cap \mathbb{Q}\}$ is a countable family of compact subsets of G_ϵ and*

$$G_\epsilon = \bigcup_{q \in (0,1] \cap \mathbb{Q}} G_\epsilon^q.$$

Therefore, each G_ϵ is σ-locally compact and Hausdorff (see [35] for further details about σ-compact spaces). Moreover, for each $q_1, q_2 \in (0,1] \cap \mathbb{Q}$, $G_\epsilon^{q_1} \subset G_\epsilon^{q_2}$ iff $q_1 > q_2$.

Given the previous result, we can easily prove what follows.

Theorem 2.6. *Every $f \in \mathscr{P}(n)$ is measurable.*

Proof. We can write each $f \in \mathscr{P}(n)$ as $f = \bigvee_\epsilon (f \wedge p_\epsilon)$, where the restriction of each $f \wedge p_\epsilon$ to G_ϵ is either 0 or is a piecewise monomial function. By Lemma 2.4, each G_ϵ is a Borel set. Thus, each $f \wedge p_\epsilon$ is continuous on a Borel set, and 0 outside, hence it is Borel measurable. The supremum of measurable functions is measurable, thus the claim follows. □

Now, we need to introduce some more notation. For every $n \in \mathbb{N}$ and for every $\epsilon \in \Sigma$, let:

(1) $\mathscr{P}_\epsilon(n) = \{f_\epsilon : G_\epsilon \to [0,1] \mid f \in \mathscr{P}(n)\}$,
(2) $\mathscr{L}_\epsilon(n) = \{g : G_\epsilon \to \mathbb{R} \mid (\exists \lambda_1, \ldots, \lambda_k \in \mathbb{R}\backslash\{0\}) \& (\exists f_{1,\epsilon}, \ldots, f_{k,\epsilon} \in \mathscr{P}_\epsilon(n))$ such that $g = \sum_{i=1}^k \lambda_i f_{i,\epsilon}\}$.

In other words, $\mathscr{P}_\epsilon(n)$ is obtained by restricting each function of $\mathscr{P}(n)$ to G_ϵ, while $\mathscr{L}_\epsilon(n)$ is in fact the *linear span* of $\mathscr{P}_\epsilon(n)$ with nonzero coefficients.

Proposition 2.7. *For every $\epsilon \in \Sigma$ and every $g_\epsilon \in \mathscr{L}_\epsilon(n)$, either $g_\epsilon = 0$ or g_ϵ is a piecewise polynomial function. Moreover, in the latter case, g_ϵ is represented in a unique way as a linear combination of pairwise distinct $f_{1,\epsilon}, \ldots, f_{k,\epsilon} \in \mathscr{P}_\epsilon(n)$ with non-zero coefficients.*

Proof. Assume $g_\epsilon \neq 0$. Then there is a $k \in \mathbb{N}$, $f_{1,\epsilon}, \ldots, f_{k,\epsilon} \in \mathscr{P}_\epsilon(n)$ and $\lambda_1, \ldots, \lambda_k \in \mathbb{R} \setminus \{0\}$ such that $g_\epsilon = \sum_i \lambda_i \cdot f_{i,\epsilon}$. Each $f_{i,\epsilon}$ is piecewise monomial, meaning that there is a partition $\mathcal{P}_i = \{P_{1,i}, \ldots, P_{m_i,i}\}$ of G_ϵ such that the restricted function $f_{i,\epsilon}{\restriction}_{P_{j,i}}$ is monomial. Let $\{Q_1, \ldots, Q_m\}$ be the refined partition of G_ϵ obtained by taking all possible non-void intersections of elements in the $\mathcal{P}_i$'s. Obviously, $f_{i,\epsilon}{\restriction}_{Q_j}$ is monomial for all Q_j. Moreover,

$$g_\epsilon{\restriction}_{Q_j} = \left(\sum_i \lambda_i f_{i,\epsilon}\right){\restriction}_{Q_j} = \sum_i \lambda_i (f_{i,\epsilon}{\restriction}_{Q_j}),$$

whence $g_\epsilon{\restriction}_{Q_j}$ is polynomial. Thus, the claim follows since a finite intersection of semialgebraic sets is semialgebraic, where a semialgebraic set is a set defined by Boolean combination of equalities and inequalities of real polynomials, and hence each Q_j is semialgebraic. This shows that g_ϵ is piecewise polynomial. As to prove that g_ϵ is uniquely determined, assume by way of contradiction that there are two different sets $\{f_{1,\epsilon}, \ldots, f_{k,\epsilon}\}$ and $\{f'_{1,\epsilon}, \ldots, f'_{k',\epsilon}\}$ such that $g_\epsilon = \sum_i \lambda_i \cdot f_{i,\epsilon} = \sum_j \lambda'_j \cdot f'_{j,\epsilon}$. Let $P_1, \ldots, P_m$ and $P'_1, \ldots, P'_{m'}$ be the semialgebraic sets on which, respectively, the $f_{i,\epsilon}$'s and the $f'_{j,\epsilon}$'s are monomial. Let Y be a semialgebraic set contained in $P_i \cap P'_j$ for some i, j. Thus g_ϵ restricted to Y is polynomial, but g_ϵ is not uniquely displayed on Y, since $g_{\epsilon\restriction_Y} = \sum_i \lambda_i \cdot f_{i,\epsilon\restriction_Y} = \sum_j \lambda'_j \cdot f'_{j,\epsilon\restriction_Y}$. But this is contradictory, since polynomial functions have a unique representation on semialgebraic sets. Thus, on each intersection $P_i \cap P'_j$, g_ϵ has a unique representation, and thus it is uniquely determined. □

For every $n \in \mathbb{N}$, every $\epsilon \in \Sigma$ and every $q \in (0,1] \cap \mathbb{Q}$, we denote by $\mathscr{L}^q_\epsilon(n)$ the set of functions obtained by restricting those in $\mathscr{L}_\epsilon(n)$ to G^q_ϵ. Further, for every subset X of $[0,1]^n$, we denote by $\mathscr{C}(X)$ the set of all continuos functions from X to $\mathbb{R}$.

Proposition 2.8. *For every ϵ, for every $q \in (0,1] \cap \mathbb{Q}$ and for every $c \in \mathscr{C}(G^q_\epsilon)$, there is a sequence $g_1, g_2, \ldots \in \mathscr{L}^q_\epsilon(n)$ such that $g_i \leq c$ for every i, and $\{g_i\}$ uniformly converges to c.*

Proof. Since every c is continuous and defined on a compact set, we can get the claim by the Stone-Weierstrass theorem [13, §VIII] if we show that $\mathscr{L}^q_\epsilon(n)$ is a subalgebra of $\mathscr{C}(G^q_\epsilon)$, i.e. it is a vector subspace of $\mathscr{C}(G^q_\epsilon)$ that is closed under multiplication of functions, $\mathscr{L}^q_\epsilon(n)$ contains a non-zero constant function and it separates the points. The first two claims are trivial. Thus, let us show that $\mathscr{L}^q_\epsilon(n)$ separates the points, i.e., for every $x, y \in G^q_\epsilon$, if $x \neq y$, then $g(x) \neq g(y)$ for some $g \in \mathscr{L}^q_\epsilon(n)$. Indeed, each monomial function m defined on a subset of $[0,1]^n$ is strictly increasing and hence $m(x) \neq m(y)$ if $x \neq y$, whence the claim is settled. □

3. States of free product algebras

Let us start introducing the main notion of our investigation, namely, states of free finitely generated product algebras.

Definition 3.1. *A* state of $\mathcal{F}_{\mathbb{P}}(n)$ *is a map* $s : \mathcal{F}_{\mathbb{P}}(n) \to [0,1]$ *satisfying the following conditions:*

S1. $s(1) = 1$ *and* $s(0) = 0$,
S2. $s(f \wedge g) + s(f \vee g) = s(f) + s(g)$,
S3. *If* $f \leq g$, *then* $s(f) \leq s(g)$,
S4. *If* $f \neq 0$, *then* $s(f) = 0$ *implies* $s(\neg\neg f) = 0$.

The following proposition shows some basic facts about states of free product algebras. Their proofs are straightforward and hence omitted.

Proposition 3.2. *For any state* $s : \mathcal{F}_{\mathbb{P}}(n) \to [0,1]$ *the following hold:*

(i) s *restricted to* $\mathscr{B}(\mathcal{F}_{\mathbb{P}}(n))$ *is a finitely additive probability measure;*
(ii) *if* $f \wedge g = 0$, $s(f \vee g) = s(f) + s(g)$. *Thus,* $s(f \vee \neg f) = s(f) + s(\neg f)$*;*
(iii) *if* $f \vee g = 1$, $s(f \wedge g) = s(f) + s(g) - 1$. *Thus,* $s(f \leftrightarrow g) = s(f \to g) + s(g \to f) - 1$*;*
(iv) $s(\neg f) + s(\neg\neg f) = 1$.

Remark 3.3. *It is worth pointing out that states of a free product algebra are lattice valuations (axioms S1–S3) as introduced by Birkhoff in* [7]. *However, if we compare Definition 3.1 with the axiomatization of states of an MV-algebra* [28, 18]*, it is clear that, while for the MV-case the monoidal operations are directly involved in the axiomatization of states, in our case the unique axiom that involves the multiplicative connectives of product logic is S4.*

In the following Proposition 3.4 we will prove that S4 can be equivalently substituted by the condition

(S4') *For every* $\epsilon \in \Sigma$ *and* $f \in \mathcal{F}_{\mathbb{P}}(n)$, *if* $f \wedge p_\epsilon \neq 0$, *then* $s(f \wedge p_\epsilon) = 0$ *implies* $s(p_\epsilon) = 0$,

which involves the atoms of $\mathscr{B}(\mathcal{F}_{\mathbb{P}}(n))$ *and does not make use of the negation connective* $\neg$. *It is also worth noticing that the condition (S4') quite closely resembles the condition (C4) of* [5] *where the authors axiomatized the integral on functions of free Gödel algebras* $\mathcal{F}_{\mathbb{G}}(n)$. *To be more precise, the condition (C4) (see* [5, §2.2]*) says the following: for every* $x, y, z \in \mathcal{F}_{\mathbb{G}}(n)$ *which are either join-irreducible or* 0, *if* $x < y < z$ *and* $s(x) = s(y)$, *then* $s(y) = s(z)$. *Turning back to (S4'), if we take* $0 < f \wedge p_\epsilon < p_\epsilon$, *then we get something similar to (C4). Indeed, if* $0 = s(0) = s(f \wedge p_\epsilon)$, *we have that* $s(p_\epsilon) = 0$ *as well, whence* $s(f \wedge p_\epsilon) = s(p_\epsilon)$.

Proposition 3.4. *The axiom* S4 *is equivalent to*

S4'. *For every* $\epsilon \in \Sigma$, *if* $f \wedge p_\epsilon \neq 0$, *then* $s(f \wedge p_\epsilon) = 0$ *implies* $s(p_\epsilon) = 0$.

Proof. ($S4 \Rightarrow S4'$). If $f \wedge p_\epsilon \neq 0$, then $s(f \wedge p_\epsilon) = 0$ implies, by $S4$, $s(\neg\neg(f \wedge p_\epsilon)) = 0$ as well. But $\neg\neg(f \wedge p_\epsilon) = \neg\neg f \wedge \neg\neg p_\epsilon = \neg\neg f \wedge p_\epsilon$. Since $f \wedge p_\epsilon \neq 0$, it turns out that $f_\epsilon \neq 0$, and by Theorem 2.3, $f_\epsilon > 0$ on G_ϵ, and hence $\neg\neg f_\epsilon = 1$, thus $\neg\neg f \wedge p_\epsilon = p_\epsilon$.

($S4' \Rightarrow S4$). Let $f \neq 0$. Since

$$f = \bigvee_{\epsilon \in \Sigma} f \wedge p_\epsilon, \text{ then } s(f) = \sum_{\epsilon \in \Sigma} s(f \wedge p_\epsilon).$$

If $s(f) = 0$ then we have $s(f \wedge p_\epsilon) = 0$ for each ϵ. Thus, by $S4'$, if $f \wedge p_\epsilon \neq 0$ we have $0 = s(p_\epsilon) = s(\neg\neg f \wedge p_\epsilon)$; otherwise, if $f \wedge p_\epsilon = 0$, then $\neg\neg f \wedge p_\epsilon = 0$ as well. Therefore, $s(\neg\neg f) = \sum_\epsilon s(\neg\neg f \wedge p_\epsilon) = 0$. □

In the next subsection we will investigate, as an example, the states of the free 1-generated product algebra $\mathcal{F}_{\mathbb{P}}(1)$ with the aim of exhibiting a first representation for these functional in terms of measures on the dual side.

3.1. **States of the free 1-generated product algebra.** The free 1-generated product algebra $\mathcal{F}_{\mathbb{P}}(1) \subset [0,1]^{[0,1]}$ consists of one-variable functions f of the form $f(x) = t(x)$, where the term $t(x)$ can be either 1 (the constant function equal to 1), 0 (the constant function equal to 0), x, $\neg x$, $\neg\neg x$, or it belongs to the following set:

$$\{x^n \mid n \in \mathbb{N}\} \cup \{x^n \vee \neg x \mid n \in \mathbb{N}\}.$$

The lattice structure of $\mathcal{F}_{\mathbb{P}}(1)$ is depicted in Figure 2.

As we recalled in Section 2, the Boolean skeleton $\mathscr{B}(\mathcal{F}_{\mathbb{P}}(1))$ of $\mathcal{F}_{\mathbb{P}}(1)$ coincides with the free Boolean algebra over 1 generator. Thus, in this case, $\Sigma = \{\epsilon_1, \epsilon_2\}$ and the two atoms of $\mathscr{B}(\mathcal{F}_{\mathbb{P}}(1))$ will be denoted by p_1 and p_2. Therefore, identifying terms with functions, the elements of $\mathscr{B}(\mathcal{F}_{\mathbb{P}}(1))$ are 1, 0, $\neg x = p_1$ and $\neg\neg x = p_2$ and the partition $\{G_1, G_2\}$ of $[0,1]$ is given by $G_1 = \{0\}$ and $G_2 = (0,1]$.

Then, as it is easy to check, any map $s : \mathcal{F}_{\mathbb{P}}(1) \to [0,1]$ satisfying the following conditions is a state:

- $s(1) = 1$, $s(0) = 0$,
- $s(\neg x) + s(\neg\neg x) = 1$,
- either $s(\neg\neg x) = s(x) = s(x^n) = 0$ for all n, or all of them are positive,
- $s(x^n) \leq s(x^m)$ whenever $n \geq m$,
- $s(x^n \vee \neg x) = s(x^n) + s(\neg x)$.

For every $y \in \mathcal{F}_{\mathbb{P}}(1)$, let $\langle y \rangle$ denote the principal lattice filter generated by y. The spectrum, denoted by $\mathfrak{P}$, of prime *lattice* filters of $\mathcal{F}_{\mathbb{P}}(1)$, ordered by reverse inclusion, is as in Figure 3.

Notice that $\mathfrak{P}$ is partially ordered as follows: $\langle \neg x \rangle$ is incompatible with any other element of $\mathfrak{P}$; $\langle \neg\neg x \rangle \geq_{\mathfrak{P}} \langle x \rangle \geq_{\mathfrak{P}} \langle x^2 \rangle \geq_{\mathfrak{P}} \langle x^3 \rangle \geq_{\mathfrak{P}} \ldots$. Priestley duality for bounded distributive lattices [32], provides us with a lattice isomorphism $R_{(\cdot)}$ between the lattice subreduct of $\mathcal{F}_{\mathbb{P}}(1)$ and the lattice of those *downsets* of $\mathfrak{P}$, which are clopen with respect to the usual spectral topology. However, since every downset of $\mathfrak{P}$ is clopen, $R_{(\cdot)}$ is onto the whole lattice of downsets of $\mathfrak{P}$. For every $x \in \mathcal{F}_{\mathbb{P}}(1)$, it is:

$$R_x = \{\langle y \rangle \in \mathfrak{P} \mid x \in \langle y \rangle\}.$$

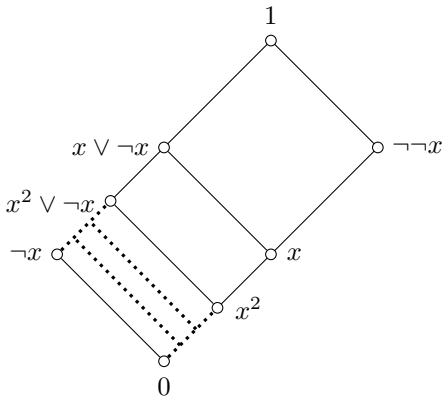


FIGURE 2. The lattice of the free product algebra with one generator $\mathcal{F}_{\mathbb{P}}(1)$

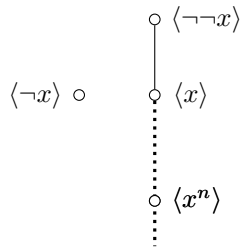


FIGURE 3. The spectral space $\mathfrak{P}$ of the lattice subreduct of the free product algebra with one generator $\mathcal{F}_{\mathbb{P}}(1)$

In particular, we have that for instance, $R_{\neg\neg x} = \{\langle y\rangle \in \mathfrak{P} \mid \langle y\rangle \leq_{\mathfrak{P}} \langle\neg\neg x\rangle\}$ and $R_{\neg x\vee x^n} = \{\langle y_1\rangle \in \mathfrak{P} \mid \langle y_1\rangle \leq_{\mathfrak{P}} \langle\neg x\rangle\} \cup \{\langle y_2\rangle \in \mathfrak{P} \mid \langle y_2\rangle \leq_{\mathfrak{P}} \langle x^n\rangle\}$. (Fig. 4 provides examples aimed at clarifying this correspondence). The following fact clearly holds:

Fact 1. *For every $z \in \mathcal{F}_{\mathbb{P}}(1)$, if $z = z_1 \vee z_2$, then $R_z = R_{z_1} \cup R_{z_2}$ and $R_{z_1\wedge z_2} = R_{z_1} \cap R_{z_2}$.*

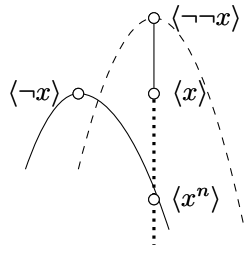


FIGURE 4. This figure shows the downsets $R_{\neg\neg x}$ (dashed parabola) and $R_{\neg x\vee x^n}$ (continuous parabola)

With such a representation in mind, let s be a state of $\mathcal{F}_{\mathbb{P}}(1)$ and let us define a $[0,1]$-valued function d_s on $\mathfrak{P}$ in the following way:

i. $d_s(\langle\neg x\rangle) = s(\neg x)$,
ii. $d_s(\langle\neg\neg x\rangle) = s(\neg\neg x) - s(x)$,
iii. for every $n \in \mathbb{N}$, $d_s(\langle x^n\rangle) = s(x^n) - s(x^{n+1})$.

First of all, let us show that d_s is a *(discrete) probability distribution* on $\mathfrak{P}$, indeed:

$$\begin{array}{rcl}
\sum_{\langle y\rangle\in\mathfrak{P}} d_s(\langle y\rangle) & = & d_s(\langle\neg x\rangle) + d_s(\langle\neg\neg x\rangle) + d_s(\langle x\rangle) + \sum_{n>1} d_s(\langle x^n\rangle) \\
& = & s(\neg x) + s(\neg\neg x) - s(x) + s(x) - s(x^2) + \sum_{n>1} d_s(\langle x^n\rangle) \\
& = & s(\neg x) + s(\neg\neg x) - s(x^2) + s(x^2) - s(x^3) + \sum_{n>2} d_s(\langle x^n\rangle) \\
& = & s(\neg x) + s(\neg\neg x) - s(x^3) + \sum_{n>2} d_s(\langle x^n\rangle) \\
& = & \dots \\
& = & s(\neg x) + s(\neg\neg x) \\
& = & s(\neg x \vee \neg\neg x) \\
& = & s(1) \\
& = & 1.
\end{array}$$

Hence, every state of $\mathcal{F}_{\mathbb{P}}(1)$ determines a distribution function d_s on $\mathfrak{P}$. Moreover, notice that the condition (S4) of Definition 3.1 forces d_s to satisfy the following further condition:

(D) If $d_s(\langle y\rangle) = 0$, then $d_s(\langle y'\rangle) = 0$ for every $\langle y'\rangle \geq_{\mathfrak{P}} \langle y\rangle$.

Conversely, let $d : \mathfrak{P} \to [0,1]$ be a distribution satisfying (D) and define $s_d : \mathcal{F}_{\mathbb{P}}(1) \to [0,1]$ by the following stipulation: for every $z \in \mathcal{F}_{\mathbb{P}}(1)$,

$$s_d(z) = \sum_{\langle y\rangle\in R_z} d(\langle y\rangle). \tag{2}$$

Let us show that s_d is a state of $\mathcal{F}_{\mathbb{P}}(1)$. Obviously $s_d(1) = 1$ and $s_d(0) = 0$. As to prove additivity, let z_1, z_2 in $\mathcal{F}_{\mathbb{P}}(1)$. From Fact 1, $R_{z_1\vee z_2} = R_{z_1} \cup R_{z_2}$ and $R_{z_1\wedge z_2} = R_{z_1} \cap R_{z_2}$. Thus,

$$\begin{array}{rcl}
s_d(z_1 \vee z_2) & = & \sum_{\langle y\rangle\in R_{z_1\vee z_2}} d(\langle y\rangle) \\
& = & \sum_{\langle y\rangle\in R_{z_1}\cup R_{z_2}} d(\langle y\rangle) \\
& = & \sum_{\langle y\rangle\in R_{z_1}} d(\langle y\rangle) + \sum_{\langle y'\rangle\in R_{z_2}} d(\langle y'\rangle) - \sum_{\langle y''\rangle\in R_{z_1}\cap R_{z_2}} d(\langle y''\rangle) \\
& = & \sum_{\langle y\rangle\in R_{z_1}} d(\langle y\rangle) + \sum_{\langle y'\rangle\in R_{z_2}} d(\langle y'\rangle) - \sum_{\langle y''\rangle\in R_{z_1\wedge z_2}} d(\langle y''\rangle) \\
& = & s_d(z_1) + s_d(z_2) - s_d(z_1 \wedge z_2).
\end{array}$$

The monotonicity of s_d can be proved in a similar manner observing that $z_1 \leq z_2$ iff $R_{z_1} \subseteq R_{z_2}$.

Let us finally prove that (S4) is satisfied. The two atoms of $\mathscr{B}(\mathcal{F}_{\mathbb{P}}(1))$ are $\neg x$ and $\neg\neg x$ and for every $y \in \mathcal{F}_{\mathbb{P}}(1)$, either $y \wedge \neg x = \neg x$ if $y = \neg x$, or $y \wedge \neg x = 0$ and in this case (S4) is trivially satisfied. As for $\neg\neg x$, let $y \in \mathcal{F}_{\mathbb{P}}(1)$ such that $y \wedge \neg\neg x \neq 0$. Then, as it is evident from Figure 2 (and skipping the trivial cases of $y = 1$ and $y = \neg\neg x$) either $y = x^n$ for $n \geq 1$, or $y = \neg x \vee x^n$, for $n \geq 1$. In both cases of $y = x^n$ or $y = \neg x \vee x^n$, $\neg\neg x \wedge y = x^n$. Thus, if $s_d(y \wedge \neg\neg x) = s(x^n) = \sum_{\langle t\rangle\in R_{x^n}} d(\langle t\rangle) = 0$, $d(\langle t\rangle) = 0$ for all $\langle t\rangle \in R_{x^n}$. Notice that for any other $\langle a\rangle \in R_{\neg\neg x}$, one has $\langle a\rangle \geq_{\mathfrak{P}} \langle t\rangle$ for each $\langle t\rangle \in R_{x^n}$, whence by (D), $d(\langle a\rangle) = 0$ ensuring that $s_d(\neg\neg x) = \sum_{\langle a\rangle\in R_{\neg\neg x}} d(\langle a\rangle) = 0$.

Thus, the following holds.

Proposition 3.5. *There is a one-one correspondence between the set of states of $\mathcal{F}_{\mathbb{P}}(1)$ and the set of distribution functions on $\mathfrak{P}$ that satisfy (D).*

Proof. From what we showed above, we can define a map that associates a distribution d_s to each state s. Thus, it is sufficient to prove that the map is injective, since surjectivity is obvious from (2). To this end, let $s_1 \neq s_2$ be two states of $\mathcal{F}_{\mathbb{P}}(1)$. Thus, there is a $y \in \mathcal{F}_{\mathbb{P}}(1)$ such that $s_1(y) \neq s_2(y)$. Now, if y is one among $\{\neg x, \neg\neg x\} \cup \{x^n \mid n \in \mathbb{N}\}$, it is clear that $d_{s_1}(\langle y\rangle) \neq d_{s_2}(\langle y\rangle)$. If y is of the kind $x^n \vee \neg x$, for $n \in \mathbb{N}$, since $\neg x \wedge x^n = 0$, using (S2) we obtain that $s_1(\neg x) + s_1(x^n) \neq s_2(\neg x) + s_2(x^n)$. Therefore, either $s_1(\neg x) \neq s_2(\neg x)$, and thus $d_{s_1}(\langle \neg x\rangle) \neq d_{s_2}(\langle \neg x\rangle)$, or $s_1(x^n) \neq s_2(x^n)$, and thus $d_{s_1}(\langle x^n\rangle) \neq d_{s_2}(\langle x^n\rangle)$, which settles the proof. □

4. Integral representation

As we recalled in Section 2, product functions are not continuous, thus, unlike the case of (free) MV-algebras, an integral representation for states cannot be obtained by directly applying Riesz representation theorem for linear and monotone functionals.[2] However, the *finite* partition $\{G_\epsilon \mid \epsilon \in \Sigma\}$ of $[0,1]^n$ is made of σ-locally compact sets (Remark 2.5) upon which the restriction f_ϵ of each product function f is continuous. In this setting, we will suitably extend states to real-valued, positive, monotone and linear operators acting on all continuous functions on the restricted compact domain. Only then we will in position to apply Reisz representation theorem to obtain Borel measures over each G^q_ε, in such a way that the Lebesgue integral with respect to these measures will act exactly like our properly restricted functionals. Finally, we will suitably extend the measures obtained by Riesz theorem first to measures on every G_ε, and secondly to a measure μ on the Borel subsets of real unit cube $[0,1]^n$. We will hence prove that the Lebesgue integral with respect to μ behaves like the state s over the functions of $\mathcal{F}_{\mathbb{P}}(n)$.

Thanks to Proposition 2.7, given a state s of $\mathcal{F}_{\mathbb{P}}(n)$, for every $\epsilon \in \Sigma$ we can define a map $\tau_\epsilon : \mathscr{L}_\epsilon(n) \to \mathbb{R}$ in the following way. Indeed, by Proposition 2.7, every $g \in \mathscr{L}_\epsilon(n) \setminus \{0\}$ is uniquely represented as a linear combination $\sum_{i=1}^k \lambda_i \cdot f_{i,\epsilon}$ for (uniquely determined) non-zero parameters $\lambda_1, \ldots, \lambda_k$ and distinct $f_{1,\epsilon}, \ldots, f_{k,\epsilon} \in \mathscr{P}_\epsilon(n)$. Thus, we can properly define:

$$\tau_\epsilon(g) = \tau_\epsilon\left(\sum_{i=1}^k \lambda_i \cdot f_{i,\epsilon}\right) = \sum_{\{i \mid s(f_{i,\epsilon} \wedge p_\epsilon) > 0\}} \lambda_i \cdot \frac{s(f_{i,\epsilon} \wedge p_\epsilon)}{s(p_\epsilon)}. \tag{3}$$

Notice that if, for some i, $s(f_{i,\epsilon} \wedge p_\epsilon) = 0$, (S4') ensures $s(p_\epsilon) = 0$ and hence $s(f_{j,\epsilon} \wedge p_\epsilon) = 0$ as well for any other $j \neq i$, and in such a case, (3) yields $\tau_\epsilon(g) = 0$ with the proviso that the empty sum is taken to be 0.

The definition of τ_ϵ is completed by putting $\tau_\epsilon(0) = 0$.

Proposition 4.1. *For every state s of $\mathcal{F}_{\mathbb{P}}(n)$ and for every ϵ, τ_ϵ is a linear and monotone map.*

[2]Recall that Riesz theorem says that, if X is a locally compact Hausdorff space, then for any positive linear functional I on the space $\mathscr{C}_C(X)$ of continuous functions with compact support on X, there is a unique regular Borel measure μ on X such that $I(f) = \int f \mathrm{d}\mu$, see e.g. [33, Theorem 2.14].

Proof. Linearity follows by the very definition of τ_ϵ. As for the monotonicity of τ_ϵ let $0 < g \leq g'$ with $g = \sum_{i=1}^{k} \lambda_i \cdot f_{i,\epsilon}$ and $g' = \sum_{i=1}^{k'} \lambda'_i \cdot f'_{i,\epsilon}$ as given by Proposition 2.7. Then, $\tau_\epsilon(g) > 0$ implies $\tau_\epsilon(g') \geq \tau_\epsilon(g)$ by definition and the monotonicity of s. On the other hand, if $\tau_\epsilon(g) = 0$, then $s(f_{i,\epsilon} \wedge p_\epsilon) = 0$ for every i whence, by (S4'), $s(p_\epsilon) = 0$. Therefore, $s(f'_{i,\epsilon} \wedge p_\epsilon) = 0$ by monotonicity of s thus, $\tau_\epsilon(g') = 0$. □

Now, for every $q \in (0,1] \cap \mathbb{Q}$, let $\tau^q_\epsilon : \mathscr{L}^q_\epsilon(n) \to \mathbb{R}$ be defined as follows: for every $g \in \mathscr{L}^q_\epsilon(n)$,

$$\tau^q_\epsilon(g) = \inf\{\tau_\epsilon(g') \mid g' \in \mathscr{L}_\epsilon(n),\ g'_{\restriction G^q_\epsilon} = g\}.$$

Proposition 4.2. *τ^q_ϵ is a linear and monotone functional over $\mathscr{L}^q_\epsilon(n)$. Moreover, if $q_2 \leq q_1$, $g \in \mathscr{L}^{q_1}_\epsilon(n)$, and $g' \in \mathscr{L}^{q_2}_\epsilon(n)$ extends g, then $\tau^{q_1}_\epsilon(g) \leq \tau^{q_2}_\epsilon(g')$.*

Proof. See Appendix A. □

Now, we want to extend τ^q_ϵ to a linear and monotone functional on the set $\mathscr{C}(G^q_\epsilon)$ of real-valued continuous functions over G^q_ϵ. For every $c \in \mathscr{C}(G^q_\epsilon)$, let $Seq(c)$ be the set of countable increasing sequences $\overline{g} = \{g_i\}_{i\in\mathbb{N}}$ of elements in $\mathscr{L}^q_\epsilon(n)$ uniformly converging to c, in symbols, $\overline{g} \nearrow c$.

Thus, for every $c \in \mathscr{C}(G^q_\epsilon)$ and for every $\overline{g} \in Seq(c)$ we first define

$$\sigma_{\overline{g}}(c) = \bigvee_{i\in\mathbb{N}} \tau^q_\epsilon(g_i),$$

and finally we put

$$\sigma^q_\epsilon(c) = \bigvee_{\overline{g}\in Seq(c)} \sigma_{\overline{g}}(c). \tag{4}$$

Lemma 4.3. *For every $\epsilon \in \Sigma$ and every $q \in (0,1] \cap \mathbb{Q}$, σ^q_ϵ is a positive, monotone and linear functional. Moreover σ^q_ϵ extends τ^q_ϵ on $\mathscr{L}^q_\epsilon(n)$.*

Proof. See Appendix A. □

The previous Lemma 4.3 has the following immediate consequence.

Theorem 4.4. *For every $\epsilon \in \Sigma$ and every rational q, there is a unique regular Borel measure μ^q_ϵ such that, for any $c \in \mathscr{C}(G^q_\epsilon)$,*

$$\sigma^q_\epsilon(c) = \int_{G^q_\epsilon} c \,\mathrm{d}\mu^q_\epsilon.$$

In particular, for all $g \in \mathscr{L}^q_\epsilon(n)$,

$$\tau^q_\epsilon(g) = \int_{G^q_\epsilon} g \,\mathrm{d}\mu^q_\epsilon.$$

Proof. From Lemma 4.3, σ^q_ϵ is a (positive) linear functional over $\mathscr{C}(G^q_\epsilon)$, with G^q_ϵ being compact, thus the the first part of the claim follows from Riesz representation theorem [33, Theorem 2.14]. The last part of the claim, finally follows from the last part of Lemma 4.3. □

With respect to the notation used in the previous Theorem 4.4, the following lemma holds.

Lemma 4.5. *For every Borel subset B of G_ϵ and for every $q \in (0,1] \cap \mathbb{Q}$, if $q' \leq q$, $\mu^q_\epsilon(B \cap G^q_\epsilon) = \mu^{q'}_\epsilon(B \cap G^q_\epsilon)$.*

Proof. Let us write $Z = B \cap G_\epsilon^q$. First of all, notice that $\mu_\epsilon^q(Z) = \inf\{\sigma_\epsilon^q(c) \mid c \in \mathscr{C}(G_\epsilon^q),\ c \geq \chi_Z\}$ and $\mu_\epsilon^{q'}(Z) = \inf\{\sigma_\epsilon^{q'}(c') \mid c' \in \mathscr{C}(G_\epsilon^{q'}),\ c' \geq \chi_Z\}$ (where χ_Z denotes the characteristic function of Z).

Since for each $r \in (0,1] \cap \mathbb{Q}$, $\mathscr{L}_\epsilon^r(n)$ is dense in $\mathscr{C}(G_\epsilon^r)$, we can safely write, for q^* either being q or q',

$$\mu_\epsilon^{q^*}(Z) = \inf\{\sigma_\epsilon^{q^*}(g) \mid g \in \mathscr{L}_\epsilon^{q^*}(n),\ g \geq \chi_Z\},$$

whence, from Lemma 4.3,

$$\mu_\epsilon^{q^*}(Z) = \inf\{\tau_\epsilon^{q^*}(g) \mid g \in \mathscr{L}_\epsilon^{q^*}(n),\ g \geq \chi_Z\}.$$

Let us now define

$$\Delta = \{g \in \mathscr{L}_\epsilon^q(n),\ g \geq \chi_Z\} \text{ and } \Delta' = \{g' \in \mathscr{L}_\epsilon^{q'}(n),\ g' \geq \chi_Z\}$$

and

$$I = \{f \in \mathscr{L}_\epsilon(n) \mid f_{\restriction G_\epsilon^q} \in \Delta\} \text{ and } I' = \{h \in \mathscr{L}_\epsilon(n) \mid h_{\restriction G_\epsilon^{q'}} \in \Delta'\}.$$

Clearly $I = I'$. Indeed, if $f \in I$, then $f \in \Delta(\epsilon)$, whence $f_{\restriction G_\epsilon^{q'}} \in \mathscr{L}_\epsilon^{q'}(n)$ and $f_{\restriction G_\epsilon^{q'}} \geq \chi_Z$ because $q' \leq q$, whence $Z = B \cap G_\epsilon^q \subseteq B \cap G_\epsilon^{q'}$. Conversely, if $h \in I'$, $h_{\restriction G_\epsilon^q} \geq \chi_Z$ again because $q' \leq q$. Then, for every $g \in \Delta$, there is a $g' \in \Delta'$ such that $\tau_\epsilon^q(g) = \tau_\epsilon^{q'}(g')$ and vice versa. Thus, by the very definition of τ_ϵ^q, the claim is settled. □

Now, recalling Remark 2.5, for every ϵ and for $q_1 \geq q_2$, we have $G_\epsilon^{q_1} \subseteq G_\epsilon^{q_2}$. Thus, the following is an immediate consequence of the above result.

Corollary 4.6. *If B is a Borel subset of G_ϵ^q for some q, then for all $q' \leq q$, $\mu_\epsilon^q(B) = \mu_\epsilon^{q'}(B)$.*

We can now establish an integral representation for the linear and monotone functionals τ_ϵ on $\mathscr{L}_\epsilon(n)$.

Lemma 4.7. *For every $\epsilon \in \Sigma$, there is a Borel probability measure μ_ϵ on the Borel subsets of G_ϵ such that, for every $g \in \mathscr{L}_\epsilon(n)$,*

$$\tau_\epsilon(g) = \int_{G_\epsilon} g \, \mathrm{d}\mu_\epsilon. \tag{5}$$

Proof. See Appendix A. □

Finally, based on the previous results, next theorem provides an integral representation for states of product logic functions.

Theorem 4.8 (Integral representation)**.** *For every state s of $\mathcal{F}_{\mathbb{P}}(n)$ there is a unique regular Borel probability measure μ such that*

$$s(f) = \int_{[0,1]^n} f \, \mathrm{d}\mu.$$

Proof. For every $f \in \mathcal{F}_{\mathbb{P}}(n)$ and for every $\epsilon \in \Sigma$,

$$f = \bigvee_{\epsilon \in \Sigma} (f_\epsilon \wedge p_\epsilon) = \sum_{\epsilon \in \Sigma} (f_\epsilon \wedge p_\epsilon).$$

Moreover, for distinct ϵ_1, ϵ_2, $(f_{\epsilon_1} \wedge p_{\epsilon_1}) \wedge (f_{\epsilon_2} \wedge p_{\epsilon_2}) = 0$ (since $G_{\epsilon_1} \cap G_{\epsilon_2} = \emptyset$), whence $s((f_{\epsilon_1} \wedge p_{\epsilon_1}) \wedge (f_{\epsilon_2} \wedge p_{\epsilon_2})) = 0$. Thus, by axiom S2

$$s(f) = s\left(\bigvee_{\epsilon\in\Sigma} (f_\epsilon \wedge p_\epsilon)\right) = \sum_{\epsilon\in\Sigma} s(f_\epsilon \wedge p_\epsilon). \tag{6}$$

Now, from the definition of τ_ϵ, Proposition 4.1 and Lemma 4.7 it follows that

$$s(f_\epsilon \wedge p_\epsilon) = s(p_\epsilon) \cdot \tau_\epsilon(f_\epsilon) = s(p_\epsilon) \cdot \int_{G_\epsilon} f_\epsilon \, \mathrm{d}\mu_\epsilon \tag{7}$$

for a Borel measure μ_ϵ on the Borel subsets of G_ϵ.

Let hence define μ on the Borel subsets of $[0,1]^n$ by the following stipulation: for every X Borel subset of $[0,1]^n$,

$$\mu(X) = \sum_{\epsilon\in\Sigma} s(p_\epsilon) \cdot \mu_\epsilon(X \cap G_\epsilon).$$

Since $\sum_{\epsilon\in\Sigma} s(p_\epsilon) = s(\bigvee_{\epsilon\in\Sigma} p_\epsilon) = s(\top) = 1$, μ is a convex combination of the μ_ϵ's. Moreover μ is defined for every X since G_ϵ is a Borel subset of $[0,1]^n$ (recall Lemma 2.4), whence $G_\epsilon \cap X$ is Borel as well. Thus, from (6) and (7),

$$\begin{aligned} s(f) &= \sum_{\epsilon\in\Sigma} s(p_\epsilon) \cdot \tau_\epsilon(f_\epsilon) \\ &= \sum_{\epsilon\in\Sigma} \left(s(p_\epsilon) \cdot \int_{G_\epsilon} f_\epsilon \, \mathrm{d}\mu_\epsilon \right) \\ &= \int_{\bigcup_{\epsilon\in\Sigma} G_\epsilon} \sum_{\epsilon\in\Sigma} f_\epsilon \, \mathrm{d}\left(s(p_\epsilon) \cdot \mu_\epsilon\right) \\ &= \int_{[0,1]^n} f \, \mathrm{d}\mu. \end{aligned}$$

It is left to show that μ is unique. Suppose, by way of contradiction, that for a state s there are two distinct regular Borel measures μ_1 and μ_2 such that, for every $f \in \mathcal{F}_{\mathbb{P}}(n)$, $s(f) = \int_{[0,1]^n} f \, \mathrm{d}\mu_1 = \int_{[0,1]^n} f \, \mathrm{d}\mu_2$.

If $\mu_1 \neq \mu_2$, then there must exist a $c \in \mathscr{C}([0,1]^n)$ such that $\int_{[0,1]^n} c \, \mathrm{d}\mu_1 \neq \int_{[0,1]^n} c \, \mathrm{d}\mu_2$. Since $[0,1]^n = \bigcup_\epsilon G_\epsilon$, there is an ϵ such that $\int_{G_\epsilon} c \, \mathrm{d}\mu_1 \neq \int_{G_\epsilon} c \, \mathrm{d}\mu_2$. Now, since $G_\epsilon = \bigcup_{q>0} G_\epsilon^q$, we have $\lim_{q\to 0} \int_{G_\epsilon^q} c \, \mathrm{d}\mu_1 \neq \lim_{q\to 0} \int_{G_\epsilon^q} c \, \mathrm{d}\mu_2$.

Without loss of generality, assume $\lim_{q\to 0} \int_{G_\epsilon^q} c \, \mathrm{d}\mu_1 < \lim_{q\to 0} \int_{G_\epsilon^q} c \, \mathrm{d}\mu_2$. Hence, there is q such that for every q', $\int_{G_\epsilon^{q'}} c \, \mathrm{d}\mu_1 < \int_{G_\epsilon^q} c \, \mathrm{d}\mu_2$. In particular, $\int_{G_\epsilon^q} c \, \mathrm{d}\mu_1 < \int_{G_\epsilon^q} c \, \mathrm{d}\mu_2$.

Now, over G_ϵ^q, c is the limit of an increasing sequence $\{g_k\}_{k\in\mathbb{N}} \subseteq \mathscr{L}_\epsilon^q(n)$, and by the continuity of the integral,

$$\sup_k \int_{G_\epsilon^q} g_k \, \mathrm{d}\mu_1 < \sup_k \int_{G_\epsilon^q} g_k \, \mathrm{d}\mu_2$$

So there is k such that, for every k',

$$\int_{G_\epsilon^q} g_{k'} \, \mathrm{d}\mu_1 < \int_{G_\epsilon^q} g_k \, \mathrm{d}\mu_2$$

in particular,

$$\int_{G_\epsilon^q} g_k \, \mathrm{d}\mu_1 < \int_{G_\epsilon^q} g_k \, \mathrm{d}\mu_2.$$

But g_k is the restriction of a function $g \in \mathscr{L}_\epsilon(n)$ on G_ϵ^q and hence g is of the form $\sum_i \lambda_i f_i$, with $f_i \in \mathscr{P}_\epsilon(n)$. Thus, we can apply Theorem 4.4 and, since the τ_ϵ^q's are uniquely determined from the state s, we get that $\tau_\epsilon^q(g_k) = \int_{G_\epsilon^q} g_k \, \mathrm{d}\mu_1$ and $\tau_\epsilon^q(g_k) = \int_{G_\epsilon^q} g_k \, \mathrm{d}\mu_2$, whence:

$$\tau_\epsilon^q(g_k) < \tau_\epsilon^q(g_k),$$

which is a contradiction. □

We shall now see that the converse also holds.

Theorem 4.9. *For every regular Borel probability measure $\mu : \mathcal{B}([0,1]^n) \to [0,1]$, the function $s : \mathcal{F}_\mathbb{P}(n) \to [0,1]$ defined as*

$$s(f) = \int_{[0,1]^n} f \, \mathrm{d}\mu.$$

is a state of $\mathcal{F}_\mathbb{P}(n)$.

Proof. First we observe that for each $f \in \mathcal{F}_\mathbb{P}(n)$, $\int_{[0,1]^n} f \, \mathrm{d}\mu \in [0,1]$, since μ is normalized to 1 and the functions of the free product algebra take values in $[0,1]$. In order to prove that s is a state, we need to show that the integral of product functions satisfy the properties S1-S4:

(S1) $\int_{[0,1]^n} \mathbf{0} \, \mathrm{d}\mu = 0$ and $\int_{[0,1]^n} \mathbf{1} \, \mathrm{d}\mu = 1$, where $\mathbf{0}$ and $\mathbf{1}$ are respectively the functions constantly equal to 0 and 1.

(S2) $\int_{[0,1]^n} (f \wedge g) \, \mathrm{d}\mu + \int_{[0,1]^n} (f \vee g) \, \mathrm{d}\mu = \int_{[0,1]^n} f \, \mathrm{d}\mu + \int_{[0,1]^n} g \, \mathrm{d}\mu$, for each $f, g \in \mathcal{F}_\mathbb{P}(n)$.

(S3) If $f, g \in \mathcal{F}_\mathbb{P}(n)$ are such that $f \leq g$, then $\int_{[0,1]^n} f \, \mathrm{d}\mu \leq \int_{[0,1]^n} g \, \mathrm{d}\mu$.

(S4) For every $\epsilon \in \Sigma$, for any $f \in \mathcal{F}_\mathbb{P}(n)$ such that $f \wedge p_\epsilon$ is non-zero, $\int_{[0,1]^n} f \wedge p_\epsilon \, \mathrm{d}\mu = 0$ implies $\int_{[0,1]^n} p_\epsilon \, \mathrm{d}\mu = 0$.

Properties (S1) and (S3) are well-known properties of the integral with respect to probability measures. About property (S2), it is not difficult to realize that, since the operations are defined pointwise, it holds that $f + g = \min(f, g) + \max(f, g)$, which settles the proof. In order to prove (S4), we shall observe that

$$\int_{[0,1]^n} (f \wedge p_\epsilon) \, \mathrm{d}\mu = \int_{[0,1]^n \setminus G_\epsilon} (f \wedge p_\epsilon) \, \mathrm{d}\mu + \int_{G_\epsilon} (f \wedge p_\epsilon) \, \mathrm{d}\mu.$$

The first integral is 0 since the function $f \wedge p_\epsilon$ is 0 outside G_ϵ. Thus, if $\int_{[0,1]^n} (f \wedge p_\epsilon) \, \mathrm{d}\mu = 0$ the second one must be 0 as well, and since $f \wedge p_\epsilon$ is strictly positive over G_ϵ (if it is 0 in one point, it is 0 in the whole G_ϵ, [3, Lemma 3.2.3]) then it must be $\mu(G_\epsilon) = 0$, whence $\int_{[0,1]^n} p_\epsilon \, \mathrm{d}\mu = 0$. □

Therefore, our main result can be stated in the following concise way.

Corollary 4.10. *For every $n \in \mathbb{N}$, and for every map $s : \mathcal{F}_\mathbb{P}(n) \to [0,1]$ the following are equivalent:*

(1) *s is a state,*

(2) *there is a unique regular Borel measure μ such that, for every $f \in \mathcal{F}_\mathbb{P}(n)$,*

$$s(f) = \int_{[0,1]^n} f \, \mathrm{d}\mu.$$

5. The state space and its extremal points

In this section we shall prove that states of $\mathcal{F}_{\mathbb{P}}(n)$ are actually convex combinations of product logic valuations. The idea is to show first that the state space is convex and compact and hence, by Krein-Milman theorem[3], every state is in the closure of convex combination of extremal. Second, we show that the extremal states coincide with product logic valuations, i.e. homomorphisms of $\mathcal{F}_{\mathbb{P}}(n)$ into $[0,1]_{\Pi}$.

Let n be any positive integer. Let us denote by $\mathcal{H}(n)$ the set of homomorphisms of $\mathcal{F}_{\mathbb{P}}(n)$ to the product algebra $[0,1]_{\Pi}$; $\mathcal{S}(n)$ stands for the set of states of $\mathcal{F}_{\mathbb{P}}(n)$; $\mathcal{M}(n)$ denotes the set of regular Borel probability measures on $\mathbf{B}([0,1]^n)$, the σ-algebra of Borel subsets of $[0,1]^n$.

Proposition 5.1. *For every $n \in \mathbb{N}$, there is a bijection between $\mathcal{H}(n)$ and $[0,1]^n$.*

Proof. Let $\varphi : [0,1]^n \to [0,1]^{\mathcal{F}_{\mathbb{P}}(n)}$ be the map that associates to every $x \in [0,1]^n$ the function $\varphi_x : f \mapsto f(x)$, for every $f \in \mathcal{F}_{\mathbb{P}}(n)$. Clearly, φ_x is a homomorphism, and it is easy to see that if $x_1 \neq x_2$ then $\varphi_{x_1} \neq \varphi_{x_2}$. Moreover, every homomorphism h is such that $h = \varphi_x$, for some $x \in [0,1]^n$. Indeed, let $x = (h(\pi_1), \ldots, h(\pi_n))$, where π_i denotes the i-th projection. Moreover, for every $f \in \mathcal{F}_{\mathbb{P}}(n)$ there is a term t_f such that $f = t_f[\pi_1, \ldots, \pi_n]$. Thus $h(f) = h(t_f[\pi_1, \ldots, \pi_n]) = t_f[h(\pi_1), \ldots, h(\pi_n)] = f(x) = \varphi_x(f)$. □

It is quite obvious that $\mathcal{S}(n)$ and $\mathcal{M}(n)$ are convex subsets of $[0,1]^{\mathcal{F}_{\mathbb{P}}(n)}$ and $[0,1]^{\mathbf{B}([0,1]^n)}$ respectively. Furthermore, $\mathcal{M}(n)$ is clearly compact with respect to the subspace product topology. As for $\mathcal{S}(n)$, let us prove that it is closed, whence compact.

Proposition 5.2. *$\mathcal{S}(n)$ is closed in the Tychonoff cube $[0,1]^{\mathcal{F}_{\mathbb{P}}(n)}$.*

Proof. See Appendix A. □

By Proposition 5.2 above and Krein-Milman theorem, $\mathcal{S}(n)$ and $\mathcal{M}(n)$ are generated by their extremal points. It is well-known that the extremal points of $\mathcal{M}(n)$ are Dirac measures, i.e. those maps $\delta_x : \mathbf{B}([0,1]^n) \to \{0,1\}$, for each $x \in [0,1]^n$, such that $\delta_x(B) = 1$ iff $x \in B$ and $\delta_x(B) = 0$ otherwise.

Let us consider the map

$$\delta : \mathcal{S}(n) \to \mathcal{M}(n)$$

which associates, to each state $s \in \mathcal{S}(n)$ the unique regular Borel measure $\mu \in \mathcal{M}(n)$ provided by Theorem 4.8, such that for every $f \in \mathcal{F}_{\mathbb{P}}(n)$, $s(f) = \int_{[0,1]^n} f \, \mathrm{d}\mu$. The following holds:

Proposition 5.3. *For every $n \in \mathbb{N}$, the map $\delta : \mathcal{S}(n) \to \mathcal{M}(n)$ defined as above is bijective and affine.*

Proof. Injectivity follows from Theorem 4.8, and surjectivity from Theorem 4.9. In order to prove that δ is affine, let us suppose that $s = \lambda s_1 + (1-\lambda)s_2$, with

[3] Recall that Krein-Milman theorem says that if X is a locally convex topological vector space and K is a compact convex subset of X, then K coincides with the closure of the convex hull of its extreme points, see e.g. [19, Theorem 5.17].

$\lambda \in [0,1]$. Then we have, for every $f \in \mathcal{F}_{\mathbb{P}}(n)$,

$$\begin{aligned} s(f) &= \lambda s_1(f) + (1-\lambda)s_2(f) \\ &= \lambda \int_{[0,1]^n} f \, \mathrm{d}\mu_1 + (1-\lambda) \int_{[0,1]^n} f \, \mathrm{d}\mu_2 \\ &= \int_{[0,1]^n} f \, \mathrm{d}(\lambda\mu_1) + \int_{[0,1]^n} f \, \mathrm{d}[(1-\lambda)\mu_2] \\ &= \int_{[0,1]^n} f \, \mathrm{d}[(\lambda\mu_1) + (1-\lambda)\mu_2]. \end{aligned}$$

Thus, $\delta(s) = \delta(\lambda s_1 + (1-\lambda)s_2) = \lambda\delta(s_1) + (1-\lambda)\delta(s_2)$, which proves that δ is affine. □

Before showing the main result of this section (Theorem 5.4 below), let us point out an immediate but interesting consequence of Proposition 5.3 above which reveals a remarkable analogy between states of MV-algebras and states of product algebras. Indeed, the Kroupa-Panti theorem shows that for every positive integer n, the state space $\mathcal{S}_{MV}(n)$ of the free MV-algebra over n-free generators is affinely isomorphic to $\mathcal{M}(n)$. Thus, in particular, $\mathcal{S}(n)$ and $\mathcal{S}_{MV}(n)$ are affinely isomorphic via an isomorphism which is defined in the obvious way.

The main result of this section hence reads as follows.

Theorem 5.4. *The following are equivalent for a state $s : \mathcal{F}_{\mathbb{P}}(n) \to [0,1]$*

(1) *s is extremal;*
(2) *$\delta(s)$ is a Dirac measure;*
(3) *$s \in \mathcal{H}(\mathcal{F}_{\mathbb{P}}(n))$.*

Proof. (1) $\Rightarrow$ (2). If s is extremal then its corresponding measure $\delta(s)$ is extremal in the space of Borel probability measures on $[0,1]$ since by Proposition 5.3 δ is affine, whence it preserves extremality. Extremal Borel measures on $[0,1]$ are exactly Dirac measures (see for instance [29, Corollary 10.6]), thus $\delta(s) = \delta_x$ for some $x \in [0,1]^n$.

(2) $\Rightarrow$ (1). If s is such that $\delta(s)$ is a Dirac measure, then it is extremal. Indeed, by way of contradiction, let us suppose that s can be expressed as a convex combination of two states s_1, s_2, that is, $s = \lambda s_1 + (1-\lambda)s_2,\ \lambda \in (0,1)$, but this would mean that $\delta(s) = \delta(\lambda s_1 + (1-\lambda)s_2) = \lambda\delta(s_1) + (1-\lambda)\delta(s_2)$, which contradicts the extremality of $\delta(s)$.

Hence, we proved that (1) $\Leftrightarrow$ (2).

(3) $\Rightarrow$ (2). Follows from Proposition 5.1.

(2) $\Rightarrow$ (3). Let us suppose that $\delta(s)$ is a Dirac measure $\delta(s) = \delta_x$, and let us prove that s is a homomorphism. By Theorem 4.8, for every $f \in \mathcal{F}_{\mathbb{P}}(n)$,

$$s(f) = \int_{[0,1]^n} f \, \mathrm{d}\, \delta_x = f(x),$$

thus clearly s is a homomorphism to $[0,1]$.

Hence we proved (2) $\Leftrightarrow$ (3), which settles the proof. □

Thus, via Krein-Milman theorem, we obtain the following:

Corollary 5.5. *The state space $\mathcal{S}(n)$ is the convex closure of the set of product homomorphisms from $\mathcal{F}_{\mathbb{P}}(n)$ into $[0,1]$.*

Remark 5.6. *For every n, the set of extremal states of the free MV-algebra $\mathcal{F}_{\mathbb{MV}}(n)$, with the topology inherited by restriction from the product space $[0,1]^{\mathcal{F}_{\mathbb{MV}}(n)}$, constitutes a compact Hausdorff space* $\mathrm{ext}(\mathcal{S}_{MV}(n))$ *which is homeomorphic to* $[0,1]^n$ *(see* [28, Theorem 2.5] *and* [29, Corollary 10.6]*). Thus,* $\mathrm{ext}(\mathcal{S}_{MV}(n))$ *is closed. A similar result for extremal product states is false. Indeed, as a consequence of* [25, Theorem 4.6]*, $\mathcal{H}(n) = \mathrm{ext}(\mathcal{S}(n))$ is not closed in the Tychonoff cube $[0,1]^{\mathcal{F}_{\mathbb{P}}(n)}$. Thus, it cannot be homeomorphic to $[0,1]^n$. However, Theorem 5.4 still provides us with a bijection between* $\mathrm{ext}(\mathcal{S}_{MV}(n))$, $\mathrm{ext}(\mathcal{S}(n))$ *and* $[0,1]^n$.

6. A logic to reason about the probability of product logic events

In this section we define a logic to reason about probabilities (in the sense of states) of product logic events. The idea is to follow the same fuzzy logic approach that has been used in the literature to formalise reasoning with different models of uncertainty, like probabilistic [20, 21, 14], possibilistic [15] or evidential models [17].

The logic we will define, $FP(\Pi, \mathrm{Ł}_\Delta)$, is a two-tiered logic: an inner logic to represent the events (which will be product logic), and an outer logic to reason about the probability of the inner logic events. To express the additivity property of states, the outer logic will be a suitable (modal-like) extension of Łukasiewicz logic: for each product logic proposition φ, $P\varphi$ will be an atomic modal formula in the outer logic that will be read as "φ is probable". Note that we will not allow the nesting of the modality P.

In more detail, the language of $FP(\Pi, \mathrm{Ł}_\Delta)$ contains the following sets of formulas:

- Non-modal formulas: built from a countable set of propositional variables using product logic connectives, i.e. propositional product logic formulas.
- Atomic modal formulas: of the form $P\varphi$, where φ is a non-modal formula (of product logic).
- Modal formulas: built from atomic modal formulas using $\mathrm{Ł}_\Delta$ logic[4] connectives.

We will denote by Fm the set of non-modal formulas and by PFm the set of all modal formulas of $FP(\Pi, \mathrm{Ł}_\Delta)$. In the following, by a *modal theory* we will refer to an arbitrary set of modal formulas.

We will provide semantics for $FP(\Pi, \mathrm{Ł}_\Delta)$ based on states on product logic formulas, that is, by mappings $\sigma : Fm \to [0,1]$ satisfying the following conditions:

S1. $\sigma(\top) = 1$ and $\sigma(\bot) = 0$,
S2. $\sigma(\varphi \wedge \psi) + \sigma(\varphi \vee \psi) = \sigma(\varphi) + \sigma(\psi)$,
S3. If $\vdash_\Pi \varphi \to \psi$, then $\sigma(\varphi) \leq \sigma(\psi)$,
S4. If $\nvdash_\Pi \neg\varphi$, then $\sigma(\varphi) = 0$ implies $\sigma(\neg\neg\varphi) = 0$.

Note that, due to S3, logical equivalence is preserved by states on formulas, that is, if φ and ψ are logically equivalent product logic formulas, then necessarily $\sigma(\varphi) = \sigma(\psi)$ for any state σ.

Remark 6.1. *It is worth noticing that, if σ is a state on Fm, its restriction σ^n on FM^n, the set of formulas built from a finite subset of propositional variables $\{p_1, \ldots, p_n\}$, is again a state (in the sense that $\sigma^n : FM^n \to [0,1]$ satisfies all the*

[4] $\mathrm{Ł}_\Delta$ is the expansion of Łukasiewicz logic with the Baaz-Monteiro projection connective Δ, see [21, §2.4].

above properties S1 - S4). Therefore, there is a one-one correspondence between states on formulas of FM^n and states on the free algebra $\mathcal{F}_{\mathbb{P}}(n)$.

Following [26, §7.2], interpretations for $FP(\Pi, \text{Ł}_\Delta)$-formulas will consist of pairs (e, σ), where e is a $[0,1]$-evaluation of propositional variables, that extends to propositional formulas as usual with product logic truth-functions from $[0,1]_\Pi$, and σ is a state on Fm. Every interpretation (e, σ) assigns a truth-value $\|\Phi\|_{e,\sigma} \in [0,1]$ to every $FP(\Pi, \text{Ł}_\Delta)$-formula Φ as follows:

- If $\Phi = \varphi$ is a propositional formula from Fm, $\|\Phi\|_{e,\sigma} = e(\varphi)$.
- if $\Phi = P\varphi$ is an atomic modal formula, $\|\Phi\|_{e,\sigma} = \sigma(\varphi)$.
- If Φ is a propositional combination with Ł_Δ connectives, then $\|\Phi\|_{e,\sigma}$ is computed from its atomic modal subformulas by using their truth-functions from $[0,1]_{\text{Ł}_\Delta}$.

Note that for non-modal formulas $\varphi \in Fm$, $\|\varphi\|_{e,\sigma}$ only depends on e, while for modal formulas $\Phi \in PFm$, $\|\Phi\|_{e,\sigma}$ only depends on the state σ. Therefore, for the sake of a simpler notation, we will also write $\|\varphi\|_e$ and $\|\Phi\|_\sigma$ respectively. Now we define the following notion of logical consequence for $FP(\Pi, \text{Ł}_\Delta)$.

Definition 6.2. *Let $\Gamma \cup \{\Phi\}$ be a (arbitrary) set of $FP(\Pi, \text{Ł}_\Delta)$-formulas. Then define $\Gamma \models_{FP(\Pi, \text{Ł}_\Delta)} \varphi$ if, for any interpretation (e, σ), it holds that if $\|\Psi\|_{e,\sigma} = 1$ for all $\Psi \in \Gamma$, then $\|\Phi\|_{e,\sigma} = 1$.*

As for the axiomatization of $\models_{FP(\Pi, \text{Ł}_\Delta)}$, we need to properly capture properties $S1 - S4$ of states in terms of product logic formulas. Actually $S1, S2$, and $S3$ can be suitably encoded only using the language of Łukasiewicz logic with the following schemes:

P1. $P\top$, $\neg P \neg \bot$,
P2. $P(\varphi \vee \psi) \leftrightarrow P\varphi \oplus (P\psi \ominus P(\varphi \wedge \psi))$,
P3. $P\varphi \to P\psi$, for φ, ψ such that $\vdash_\Pi \varphi \to \psi$.

However, the axiom S4 of Definition 3.1 cannot be written within the language of Łukasiewicz logic, since this logic cannot express that a formula is not totally false. This is the reason for considering Ł_Δ, the expansion of Łukasiewicz logic with the well-known Monteiro-Baaz Δ operator, for the outer logic. Indeed, using the language of Ł_Δ, then S4 can be encoded by the following scheme:

P4. $\Delta(\neg P\varphi) \to \neg P \neg\neg \varphi$, for φ such that $\nvdash_\Pi \neg\varphi$.

As outlined in [26, §7.2] and in the proof of Theorem 6.4 below, the usual technique to prove completeness for a probabilistic modal logic as $FP(\Pi, \text{Ł}_\Delta)$ consists, mainly, in the following steps: (1) translating, at the propositional level of the outer logic (in this case Ł_Δ), all modal axioms and rules; (2) using the completeness of the outer logic with respect to a standard algebra, build a model for the probabilistic modal logic. The typical problem of this strategy is that, as in this specific case, the propositional translation of the modal axioms leads to an infinite theory for Ł_Δ which, however, is not strongly complete with respect to the algebra on $[0,1]$, i.e., if $\Gamma \cup \{\varphi\}$ is an infinite set of propositional formulas of Ł_Δ, it might happen that, although every model of Γ is a model of φ, φ cannot be proved from Γ. Therefore, we will need to equip the outer logic Ł_Δ with the following infinitary rule that makes it strongly complete (see [26] for full details):

$$(IR)\ \frac{\Theta \vee (\Phi \to \Psi^n),\ \text{for each } n \in \mathbb{N}}{\Theta \vee (\neg\Phi \vee \Psi)}$$

For this reason, we will henceforth extend the outer logic Ł_Δ with the previous rule (IR) and we will denote it by Ł^+_Δ.

Definition 6.3. *$FP(\Pi, \text{Ł}_\Delta)$ is the logic, in the language defined above, whose axioms and rules are the following:*

- (Π) *Axioms and rule of product logic for non-modal formulas*
- (Ł_Δ) *Axioms and rules of Ł_Δ for modal formulas*
- (P1) $P\top, \neg P\neg\bot$
- (P2) $P(\varphi \vee \psi) \leftrightarrow P\varphi \oplus (P\psi \ominus P(\varphi \wedge \psi))$
- (P3) $P\varphi \to P\psi$, *for every φ, ψ such that* $\vdash_\Pi \varphi \to \psi$
- (P4) $\Delta(\neg P\varphi) \to \neg P\neg\varphi$, *for every φ such that* $\nvdash_\Pi \neg\varphi$
- (IR) *from* $\{\Theta \vee (\Phi \to \Psi^n) \mid n \in \mathbb{N}\}$ *infer* $\Theta \vee (\neg\Phi \vee \Psi)$, *for* $\Theta, \Phi, \Psi \in PFm$

Note that since we have two arrows $\to$ in $FP(\Pi, \text{Ł}_\Delta)$, the inner one (from product logic Π) and the outer one (from the logic Ł_Δ), and two Modus Ponens rules, one for each arrow. Moreover, in the outer logic we have the necessitation rule for Δ for modal formulas.

Notice that in $FP(\Pi, \text{Ł}_\Delta)$ the presence of the infinitary rule (IR) requires to slightly change the notion of *proof* in such a way ensuring that if, for every $n \in \mathbb{N}$, we have a proof of $\Theta \vee (\Phi \to \Psi^n)$ from the same set of premises Γ, then we also have a proof of $\Theta \vee (\neg\Phi \vee \Psi)$ from Γ as well.[5]

In the following we restrict ourselves to prove completeness for deductions from modal theories.

Theorem 6.4 (Soundness and Completeness)**.** *Let $\Gamma \cup \{\Phi\} \subset PFm$ be an (arbitrary) modal theory. Then, $\Gamma \vdash_{FP(\Pi, \text{Ł}_\Delta)} \Phi$ iff $\Gamma \models_{FP(\Pi, \text{Ł}_\Delta)} \Phi$, that is, iff for every state σ, if $\|\Psi\|_\sigma = 1$ for all $\Psi \in \Gamma$, then $\|\Phi\|_\sigma = 1$ as well.*

Proof. Soundness is easy. As for completeness, we apply the usual technique in this kind of modal-like fuzzy probabilistic logics of inductively defining a translation mapping $(\ )^*$ from the modal language PFm into the propositional Ł^+_Δ-language built from atomic modal formulas $P\varphi$ taken as propositional variables (see [14, 16] and [26, §7.2]).

Accordingly, each proof of Φ from Γ in $FP(\Pi, \text{Ł}_\Delta)$ can be translated into a proof of Φ^* in the logic Ł^+_Δ from the set of (propositional) formulas $\Gamma^* \cup AX^*$, where AX^* is the set of all translated instances of axioms $(S1)$-$(S4)$. And viceversa, every proof of Φ^* from $\Gamma^* \cup AX^*$ in Ł^+_Δ gives rise to a proof of Φ from Γ in $FP(\Pi, \text{Ł}_\Delta)$. In other words, $\Gamma \vdash_{FP(\Pi, \text{Ł}_\Delta)} \Phi$ iff $\Gamma^* \cup AX^* \vdash_{\text{Ł}^+_\Delta} \Phi^*$. Note that, independently of whether Γ is finite or not, AX^* is infinite. But now, since Ł^+_Δ is strongly complete [26, Theorem 4], $\Gamma^* \cup AX^* \vdash_{\text{Ł}^+_\Delta} \Phi^*$ iff $\Gamma^* \cup AX^* \models_{\text{Ł}^+_\Delta} \Phi^*$.

Finally, we can check that Ł^+_Δ-evaluations that are model of AX^* are clearly in one-one correspondence with states on non-modal formulas. Namely, if e is a Ł^+_Δ-evaluation validating the translations of all instances of axioms (P1)–(P4), then the map $\sigma : Fm \to [0, 1]$ defined as $\sigma(\varphi) = e((P\varphi)^*)$ is state on product logic formulas,

[5] Formally, a proof of a formula Φ from Γ is defined as a well-founded tree (i.e. with of possibly infinite width and depth, but with no branches of infinite length) where (i) the root Φ can have an infinite degree, (ii) the leaves are formulas from Γ or instances of the axioms of $FP(\Pi, \text{Ł}_\Delta)$, and (iii) for each node of the tree with a formula Ψ there is an inference rule in $FP(\Pi, \text{Ł}_\Delta)$ deriving Ψ from its predecessors.

in the sense as defined above. In other words, $\Gamma^* \cup AX^* \models_{\mathrm{L}^+_\Delta} \Phi^*$ iff $\Gamma \models_{FP(\Pi,\mathrm{L}_\Delta)} \Phi$. This completes the proof. □

7. Conclusions and future work

In this paper we have defined and studied the notion of state for free product algebras, i.e., the Lindenbaum-Tarski algebras of product logic. We may recall that Product logic is the third formalism which, together with Łukasiewicz and Gödel logics, stands at the ground of all continuous t-norm based logics, since any continuous t-norm can be obtained as an ordinal sum of isomorphic copies of Gödel t-norm (i.e. the minimum t-norm), Łukasiewicz t-norm and product t-norm.

Our main result is a Kroupa-Panti-like representation for states. In other words we have proved that our axiomatization of states captures the Lebesgue integral of product functions with respect to regular Borel probability measures and the relation between states and measures is one-one. That result, besides supporting the appropriateness of our axiomatization, has several interesting consequences as welcome side effects. First of all, when studying the geometric properties of the state space, it allows us to fully characterize extremal states in terms of $[0, 1]$-valued product homomorphisms which, in turn, correspond one-one to Dirac measures on the space of extremal states. Furthermore the integral representation theorem shows that, for every natural number n, the state space of the free n-generated product algebra is affinely isomorphic to the state space of the free n-generated MV-algebra.

In the last section of this paper, in order to point out the close relation between states and probabilistic logic, we introduced a modal-like fuzzy logic obtained by the combination of product logic and a suitable expansion of Łukasiewicz calculus to reason about the probability of product logic events. The resulting logic turned out to be sound and complete with respect to the intended semantics given by states of free product algebras.

The paper leaves several interesting open problems for further research. A first future direction clearly concerns with the generalization to the frame of product logic of a coherence (no-Dutch-book) criterion *à la* de Finetti. In this regard, the non-finiteness of free product algebras and the discontinuity of product implication, makes the problem of generalizing de Finetti's theorem to this setting non-trivial and hence particularly challenging. However, it is worth pointing out that the results contained in Section 5 pave the way for a first step in this direction.

Secondly, as Łukasiewicz, Gödel and product logics are the building blocks of Hájek logic BL, it is reasonable to think that the integral representation theorem for states of free product algebras, together with its analogous results for MV and Gödel algebras, and the remarkable functional representation theorem for free BL-algebras [2], are the necessary ingredients to shed a light on the problem of providing an appropriate axiomatization for states of free BL-algebras.

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Appendix A. Proofs

Proof of Proposition 4.2.

Proof. Let us start showing that τ^q_ϵ is monotone.

Let $g, g' \in \mathscr{L}^q_\epsilon(n)$, with $g \leq g'$. In order to prove the monotonicity of τ^q_ϵ, we will show that for each $h \in \mathscr{L}_\epsilon(n)$ such that $h_{\restriction G^q_\epsilon} = g'$ we can find $k \in \mathscr{L}_\epsilon(n)$ such that

$$k_{\restriction G^q_\epsilon} = g \text{ and } k \leq h. \tag{8}$$

Thus the claim will follow from the definition of τ^q_ϵ and the monotonicity of τ_ϵ. Let hence $h_{\restriction G^q_\epsilon} = g'$ and let $l \in \mathscr{L}_\epsilon(n)$ which extends g. Thus, let $k = h \wedge l$. Clearly $k \in \mathscr{L}_\epsilon(n)$ and (8) holds.

Now, we prove the linearity of τ^q_ϵ. First, we prove that if $h \in \mathscr{L}_\epsilon(n)$ is such that $h_{\restriction G^q_\epsilon} = g + g'$, for some $g, g' \in \mathscr{L}^q_\epsilon(n)$, then there are $z, z' \in \mathscr{L}_\epsilon(n)$ such that $z_{\restriction G^q_\epsilon} = g, z'_{\restriction G^q_\epsilon} = g'$, and $h = z + z'$. Since $g \leq h_{\restriction G^q_\epsilon}$, from the previous point, we know there is a $z \leq h$ and extending g. Thus, let $z' = h - z$. Hence, $\tau^q_\epsilon(g + g') = \inf\{\tau_\epsilon(h) \mid h_{\restriction G^q_\epsilon} = g + g'\} = \inf\{\tau_\epsilon(z + z') \mid z_{\restriction G^q_\epsilon} = g, z'_{\restriction G^q_\epsilon} = g'\} = \inf\{\tau_\epsilon(z) + \tau_\epsilon(z') \mid z_{\restriction G^q_\epsilon} = g, z'_{\restriction G^q_\epsilon} = g'\}$, where the last equality follows by the linearity of τ_ϵ.

Thus, we shall now prove the following: $\inf\{\tau_\epsilon(z) + \tau_\epsilon(z') \mid z_{\restriction G^q_\epsilon} = g, z'_{\restriction G^q_\epsilon} = g'\} = \inf\{\tau_\epsilon(z) \mid z_{\restriction G^q_\epsilon} = g\} + \inf\{\tau_\epsilon(z') \mid z'_{\restriction G^q_\epsilon} = g'\}$. Since one inequality is obviously valid, we are left to prove that

$$\inf\{\tau_\epsilon(z)+\tau_\epsilon(z') \mid z_{\restriction G^q_\epsilon} = g, z'_{\restriction G^q_\epsilon} = g'\} \leq \inf\{\tau_\epsilon(z) \mid z_{\restriction G^q_\epsilon} = g\}+\inf\{\tau_\epsilon(z') \mid z'_{\restriction G^q_\epsilon} = g'\}.$$

As to prove this claim, it suffices to notice that for any z such that $z_{\restriction G^q_\epsilon} = g$ and z' such that $z'_{\restriction G^q_\epsilon} = g'$ it is always possible to find a $\hat{z} = \hat{z} + \hat{z}'$ where $\hat{z} \leq z, \hat{z}' \leq z'$, with $\hat{z}_{\restriction G^q_\epsilon} = g$, $\hat{z}'_{\restriction G^q_\epsilon} = g'$. Thus, being τ_ϵ monotone, $\tau_\epsilon(\hat{z}) + \tau_\epsilon(\hat{z}') \leq \tau_\epsilon(z) + \tau_\epsilon(z')$, and the claim is settled.

In a very similar way, we can show that $\tau^q_\epsilon(\lambda z) = \lambda \tau^q_\epsilon(z)$. Thus, τ^q_ϵ is linear.

Finally, in order to conclude the proof, let $q_2 \leq q_1$ and let $g' \in \mathscr{L}^{q_2}_\epsilon(n)$ extending $g \in \mathscr{L}^{q_1}_\epsilon(n)$. Then, $\tau^{q_1}_\epsilon(g) \leq \tau^{q_2}_\epsilon(g')$ from the very definition of τ^q_ϵ. □

*

Proof of Lemma 4.3.

Proof. The fact that σ^q_ϵ is a positive functional on $\mathscr{C}(G^q_\epsilon)$ follows by the very definition. In order to prove that σ^q_ϵ is monotone, let $c, c' \in \mathscr{C}(G^q_\epsilon)$ and assume $c \leq c'$. The following holds:

Fact 2. *For each sequence $\{g_1, g_2, \ldots\} \nearrow c$, there is a sequence $\{g'_1, g'_2, \ldots\} \nearrow c'$ and and index i_0 such that, for every $i \geq i_0$, $g'_i \geq g_i$.*

Proof. (of Fact 2). Let $\{g_1, g_2, \ldots\} \nearrow c$, and $\{g^*_1, g^*_2, \ldots\} \nearrow c'$. Define, for every i, $g'_i = g_i \vee g^*_i$ and easily check that this settles the claim. □

Thus, we prove that $\sigma^q_\epsilon(c) \leq \sigma^q_\epsilon(c')$. Indeed,

$$\sigma^q_\epsilon(c) = \bigvee_{\overline{g} \in Seq(c)} \sigma_{\overline{g}}(c) = \bigvee_{\overline{g} \in Seq(c)} (\bigvee_{i \in \mathbb{N}} \tau^q_\epsilon(g_i))$$

Fact 2 ensures that, given a $\{g_i\}\nearrow c$, there is a $\{r_i\}\nearrow c'$ and, for every $i \geq i_0$, $g_i \leq r_i$, and hence, since τ_ϵ^q is monotone, $\tau_\epsilon^q(g_i) \leq \tau_\epsilon^q(r_i)$. Whence, for every $\overline{g} \in Seq(c)$ there is $\overline{r} \in Seq(c')$ such that $\sigma_{\overline{g}}(c) \leq \sigma_{\overline{r}}(c')$. Therefore

$$\sigma_\epsilon^q(c) = \bigvee_{\overline{g}\in Seq(c)} \sigma_{\overline{g}}(c) \leq \bigvee_{\overline{r}\in Seq(c')} \sigma_{\overline{r}}(c') = \sigma_\epsilon^q(c')$$

showing that σ_ϵ^q is monotone.

Now, it is left to show that σ_ϵ^q is linear. To this end let us begin with the following claims:

Fact 3. *For every $c, c' \in \mathscr{C}(G_\epsilon^q)$ and for every $\lambda \in \mathbb{R}$, the following hold*

(1) For each $\{t_1, t_2, \ldots\}\nearrow c + c'$, there are $\{a_1, a_2, \ldots\}\nearrow c$ and $\{a'_1, a'_2, \ldots\}\nearrow c'$ such that, for every i, $t_i = a_i + a'_i$.

(2) For each $\{t_1, t_2, \ldots\}\nearrow \lambda c$, there is $\{a_1, a_2, \ldots\}\nearrow c$ such that, for every i, $t_i = \lambda a_i$.

Proof. (of Fact 3). (1) Let $\{t_1, t_2, \ldots\}$ be as given by hypothesis and let $\{a_1, a_2, \ldots\}$ be any sequence converging to c. Then let, for every i, $a'_i = t_i - a_i \in \mathscr{L}_\epsilon^q(n)$. Thus, $\{a'_1, a'_2, \ldots\}\nearrow (c + c') - c$, that is $\{a'_1, a'_2, \ldots\}\nearrow c'$ and this settles the claim.

(2) Let $\{t_1, t_2, \ldots\}$ as in the hypothesis and since $\lambda \neq 0$ put, for every i, $a_i = t_i/\lambda$. Thus, $\{a_1, a_2, \ldots\}\nearrow \lambda c/\lambda$, that is, $\{a_1, a_2, \ldots\}\nearrow c$. □

Now we prove that σ_ϵ^q is linear. Let $c, c' \in \mathscr{C}(G_\epsilon^q)$. Then

$$\sigma_\epsilon^q(c + c') = \bigvee_{\overline{t}\in Seq(c+c')} \sigma_{\overline{t}}(c + c').$$

Fact 3(1) shows that, for each $\{t_1, t_2, \ldots\}\nearrow (c + c')$, we can find $\{a_1, a_2, \ldots\}\nearrow c$ and $\{b_1, b_2, \ldots\}\nearrow c'$ such that, for every i, $t_i = a_i + b_i$. Thus, $\sigma_{\overline{t}}(c+c') = \bigvee_{i\in\mathbb{N}} \tau_\epsilon^q(a_i + b_i)$ and since τ_ϵ^q is linear, $\tau_\epsilon^q(a_i + b_i) = \tau_\epsilon^q(a_i) + \tau_\epsilon^q(b_i)$. Thus,

$$\begin{array}{rcl} \sigma_{\overline{t}}(c + c') & = & \bigvee_{i\in\mathbb{N}} \tau_\epsilon^q(a_i) + \tau_\epsilon^q(b_i) \\ & = & \lim_{i\in\mathbb{N}} \tau_\epsilon^q(a_i) + \tau_\epsilon^q(b_i) \\ & = & \lim_{i\in\mathbb{N}} \tau_\epsilon^q(a_i) + \lim_{i\in\mathbb{N}} \tau_\epsilon^q(b_i) \\ & = & \sigma_{\overline{a}}(c) + \sigma_{\overline{b}}(c'), \end{array}$$

where the previous limits exist because every sequence $\{a_i\}$ and $\{a'_i\}$ is bounded by c and c' which are continuous functions and $\{a_i\}$ and $\{a'_i\}$ converge to c and c' on the compact set G_ϵ^q. In a similar way, we can prove that

$$\begin{array}{rcl} \sigma_\epsilon^q(c + c') & = & \bigvee\{\sigma_{\overline{t}}(c + c') \mid \overline{t} \in Seq(c + c')\} \\ & = & \bigvee\{\sigma_{\overline{t}}(c + c') \mid \overline{t} = \overline{a} + \overline{b}, \overline{a} \in Seq(c), \overline{b} \in Seq(c')\} \\ & = & \bigvee\{\sigma_{\overline{a}}(c) + \sigma_{\overline{b}}(c') \mid \overline{a} \in Seq(c), \overline{b} \in Seq(c')\} \\ & = & \bigvee_{\overline{a}\in Seq(c')} \sigma_{\overline{a}}(c) + \bigvee_{\overline{b}\in Seq(c')} \sigma_{\overline{b}}(c') \\ & = & \sigma_\epsilon^q(c) + \sigma_\epsilon^q(c') \end{array}$$

Finally, using a similar argument, but using Fact 3(2), $\sigma_\epsilon^q(\lambda c) = \lambda\sigma_\epsilon^q(c)$ so proving that σ_ϵ^q is linear.

In order to conclude the proof, notice that, for each $g \in \mathscr{L}_\epsilon^q(n)$, the constant sequence $\{g\}$ belongs to $Seq(g)$, and for any other sequence $\overline{t} = \{t_1, t_2 \ldots\}\nearrow g$ we have $t_i \leq g$, whence

$$\sigma_\epsilon^q(g) = \bigvee_{\overline{t}\in Seq(g)} (\bigvee_{i\in\mathbb{N}} \tau_\epsilon^q(t_i)) = \bigvee_{i\in\mathbb{N}} \tau_\epsilon^q(g) = \tau_\epsilon^q(g).$$

□

*

Proof of Lemma 4.7.

Proof. Let, for every $q \in \mathbb{Q}$, μ^q_ϵ be a Borel measure that provides an integral representation of τ^q_ϵ (Corollary 4.4). Let us define for each μ^q_ϵ, the map $\hat{\mu}^q_\epsilon$ over the Borel subset of G_ϵ in the following way:

$$\hat{\mu}^q_\epsilon(B) = \mu^q_\epsilon(B \cap G^q_\epsilon).$$

From Proposition 4.2 the sequence $\{\hat{\mu}^q_\epsilon\}$ is increasing and clearly bounded. Thus, by the Vitali-Hahn-Saks theorem [13, §III.10], it converges to a σ-additive measure μ_ϵ. Further notice that, by Corollary 4.6, for every Borel subset X of G^q_ϵ,

$$\mu_\epsilon(X) = \mu^q_\epsilon(X) = \hat{\mu}^q_\epsilon(X). \tag{9}$$

Now, let us define for each $g \in \mathscr{L}_\epsilon(n)$, the function $g_q : G_\epsilon \to [0,1]$ which equals $g_{\restriction G^q_\epsilon}$ over G^q_ϵ and takes 0 outside. Observe that each g_q is not continuous but it is measurable. Clearly, each sequence $\{g_q\}_{q\in\mathbb{Q}}$ is non-decreasing and it converges pointwise to g: $\lim_q g_q(x) = g(x)$, for every $x \in G_\epsilon$. Then, by Levi's theorem (cf. [23, §30, Theorem 2]),

$$\lim_q \int_{G_\epsilon} g_q \mathrm{d}\mu_\epsilon = \int_{G_\epsilon} g \mathrm{d}\mu_\epsilon.$$

Finally, observe that

$$\tau^q_\epsilon(g_{\restriction G^q_\epsilon}) = \int_{G^q_\epsilon} g_{\restriction G^q_\epsilon} \mathrm{d}\mu^q_\epsilon = \int_{G_\epsilon} g_q \mathrm{d}\hat{\mu}^q_\epsilon,$$

and also, by the definition of τ^q_ϵ and Proposition 4.2, $\lim_q \tau^q_\epsilon(g_{\restriction G^q_\epsilon}) = \tau_\epsilon(g)$. Thus,

$$\tau_\epsilon(h) = \lim_q \tau^q_\epsilon(g_{\restriction G^q_\epsilon}) = \lim_q \int_{G_\epsilon} g_q \mathrm{d}\hat{\mu}^q_\epsilon = \lim_q \int_{G_\epsilon} g_q \mathrm{d}\mu_\epsilon = \int_{G_\epsilon} g \mathrm{d}\mu_\epsilon,$$

where the third equality follows from (9) recalling that $g_q(y) = 0$ for each $y \in G_\epsilon \setminus G^q_\epsilon$. □

*

Proof of Proposition 5.2.

Proof. Let $\{s_i\}_{i\geq 0}$ be a sequence of states of $\mathcal{F}_{\mathbb{P}}(n)$ such that $\lim_{i\in\mathbb{N}} s_i = s$ exists, and let us prove that such s is a state. Condition $S1$ of Definition 3.1 is clearly verified. Let us show that s respects condition $S2$. We need to prove that $s(f\vee g) = s(f) + s(g) - s(f\wedge g)$. Being each s_n a state, we have that:

$$\lim_{i\in\mathbb{N}} s_n(f\vee g) = \lim_{i\in\mathbb{N}}(s_n(f) + s_n(g) - s_n(f\wedge g))$$

and also, it clearly holds that:

$$\lim_{i\in\mathbb{N}}(s_n(f) + s_n(g) - s_n(f\wedge g)) = \lim_{i\in\mathbb{N}} s_n(f) + \lim_{i\in\mathbb{N}} s_n(G) - \lim_{i\in\mathbb{N}} s_n(f\wedge g),$$

thus the claim directly follows. It is easy to prove condition $S3$, since given $f, g \in \mathcal{F}_{\mathbb{P}}(n)$, if $f \leq g$ then $s_n(f) \leq s_n(g)$ for every $n \in \mathbb{N}$. Thus, it follows that:

$$s(f) = \lim_{i\in\mathbb{N}} s_n(f) \leq \lim_{i\in\mathbb{N}} s_n(g) = s(g).$$

Let us finally prove $S4$. Let $f \in \mathcal{F}_{\mathbb{P}}(n), f \neq 0$, such that $s(f) = 0$. We shall prove that $s(\neg\neg f) = 0$. Let $\mathrm{supp}(f) = \{x \in [0,1]^n \mid f(x) > 0\}$. Then $\mathrm{supp}(f)$ is a union of G_ϵ's, whence it is a Borel subset of $[0,1]^n$. This observation, with Corollary 4.10, imply that:

$$s(f) = \lim_i \int_{[0,1]^n} f \, \mathrm{d}\mu_i = \lim_i \int_{\mathrm{supp}(f)} f \, \mathrm{d}\mu_i = 0.$$

Fact 4. *If* $\lim_i \int_{\mathrm{supp}(f)} f \, \mathrm{d}\mu_i = 0$ *then* $\lim_i \mu_i(\mathrm{supp}(f)) = 0$.

Proof. (of Fact 4) As we already noticed, $\mathrm{supp}(f) = \bigcup_{\epsilon \in \Sigma^*} G_\epsilon$, for some $\Sigma^* \subseteq \Sigma$. Thus, if $\lim_{i \in \mathbb{N}} \int_{\mathrm{supp}(f)} f \, \mathrm{d}\mu_i = 0$, and since the G_ϵ's are disjoint, the following holds:

$$\lim_{i \in \mathbb{N}} \int_{\mathrm{supp}(f)} f \, \mathrm{d}\mu_i = \lim_{i \in \mathbb{N}} \int_{\bigcup_{\epsilon \in \Sigma^*} G_\epsilon} f \, \mathrm{d}\mu_i = \lim_{i \in \mathbb{N}} \left(\sum_{\epsilon \in \Sigma^*} \int_{G_\epsilon} f \mathrm{d}\mu_i \right) = \sum_{\epsilon \in \Sigma^*} \left(\lim_{i \in \mathbb{N}} \int_{G_\epsilon} f \mathrm{d}\mu_i \right).$$

Therefore, $\sum_{\epsilon \in \Sigma^*} (\lim_{i \in \mathbb{N}} \int_{G_\epsilon} f \mathrm{d}\mu_i) = 0$, whence $\lim_{i \in \mathbb{N}} \int_{G_\epsilon} f \mathrm{d}\mu_i = 0$ for all $\epsilon \in \Sigma^*$. Now, $G_\epsilon = \bigcup_{q \in \mathbb{Q} \cap (0,1]} G_\epsilon^q$, and hence

$$\int_{G_\epsilon} f \mathrm{d}\mu_i = \sup_{q \in \mathbb{Q} \cap (0,1]} \left(\int_{G_\epsilon^q} f \mathrm{d}\mu_i \right).$$

Therefore:

$$0 = \lim_i \int_{G_e} f \mathrm{d}\mu_i = \lim_i \sup_{q \in \mathbb{Q} \cap (0,1]} \left(\int_{G_e^q} f \mathrm{d}\mu_i \right) \geq \sup_{q \in \mathbb{Q} \cap (0,1]} \lim_i \left(\int_{G_e^q} f \mathrm{d}\mu_i \right).$$

Hence, for all $q \in \mathbb{Q} \cap (0,1]$, it follows that $\lim_i (\int_{G_\epsilon^q} f \mathrm{d}\mu_i) = 0$. But since G_ϵ^q is compact, and f is strictly positive on it, $\lim_i \mu_i(G_e^q) = 0$. Indeed, let $r = \min\{f(x) \mid x \in G_\epsilon^q\}$, thus $\lim_{i \in \mathbb{N}} (\int_{G_\epsilon^q} r \, \mathrm{d}\mu_i) \leq \lim_{i \in \mathbb{N}} (\int_{G_\epsilon^q} f \mathrm{d}\mu_i) = 0$. Thus,

$$0 = \lim_{i \in \mathbb{N}} \left(\int_{G_\epsilon^q} r \, \mathrm{d}\mu_i \right) = r \cdot \lim_{i \in \mathbb{N}} \mu_i(G_e^q),$$

that is, $\lim_{i \in \mathbb{N}} \mu_i(G_e^q) = 0$. Therefore,

$$\lim_{i \in \mathbb{N}} \mu_i \left(G_\epsilon \right) = \lim_{i \in \mathbb{N}} \mu_i \left(\bigcup_{q \in \mathbb{Q} \cap (0,1]} G_\epsilon^q \right) \leq \lim_{i \in \mathbb{N}} \sum_{q \in \mathbb{Q} \cap (0,1]} \mu_i \left(G_\epsilon^q \right) = \sum_{q \in \mathbb{Q} \cap (0,1]} \lim_{i \in \mathbb{N}} \mu_i \left(G_\epsilon^q \right) = 0$$

that is to say, $\lim_{i \in \mathbb{N}} \mu_i(G_\epsilon) = 0$ for all $\epsilon \in \Sigma^*$. Hence, finally,

$$\lim_{i \in \mathbb{N}} \mu_i(\mathrm{supp} f) = \lim_{i \in \mathbb{N}} \sum_{\epsilon \in \Sigma^*} \mu_i(G_\epsilon) = \sum_{\epsilon \in \Sigma^*} \lim_{i \in \mathbb{N}} \mu_i(G_e) = 0.$$

□

Now, since $\mathrm{supp}(f) = \mathrm{supp}(\neg\neg f)$, and $(\neg\neg f)(x) = 1$ for every $x \in \mathrm{supp}(f)$,

$$\begin{aligned} s(\neg\neg f) &= \lim_{i\in\mathbb{N}} s_i(\neg\neg f) \\ &= \lim_{i\in\mathbb{N}} \int_{[0,1]^n} \neg\neg f \, \mathrm{d}\mu_i \\ &= \lim_{i\in\mathbb{N}} \int_{\mathrm{supp}(\neg\neg f)} \neg\neg f \, \mathrm{d}\mu_i \\ &= \lim_{i\in\mathbb{N}} \int_{\mathrm{supp}(f)} \neg\neg f \, \mathrm{d}\mu_i \\ &= \lim_{i\in\mathbb{N}} \int_{\mathrm{supp}(f)} 1 \, \mathrm{d}\mu_i \\ &= \lim_{i\in\mathbb{N}} \mu_i(\mathrm{supp}(f)) \\ &= 0, \end{aligned}$$

where the last equality clearly follows from Fact 4 above. □

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