A representation of Keynes’s long-term expectation in financial markets

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Marcello Basili
DEPS University of Siena

Alain Chateauneuf
IPAG Business School Paris
and PSE-CES University of Paris-I

Giuliano Antonio Curatola
DEPS University of Siena

Giuseppe Scianna
DIISM University of Siena

Abstract

This paper advances an intuitive representation of Keynes's notion of long-term expectation. We introduce the epsilon-contamination approach and represent the conventional judgment by the Steiner point of agents' common probability set. We anticipate a change in conventional judgment by updating the Steiner point.

Keywords: Keynes, long-term expectation, epsilon contamination, uncertainty, multiple priors.

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1 Introduction

Evaluation of an asset depends on expectations of prospective yields but this long-term expectation, as Keynes claims, is based "partly on existing facts which we can assume to be known more or less for certain, and partially on future events which can only be forecasted with more or less confidence" (p.133). Crucially relevant facts at the base of individual expectation are often *very uncertain*, even if Keynes makes clear that "by very uncertain I do not mean the same thing as improbable" and in so doing he establishes a direct relation between the notion of confidence in the General Theory and the weight of arguments in the Treatise on Probability (Note 1, p. 133). Keynes thinks that professional investors and speculators in the stock exchange are forced to predict the mass psychology of the market, that is to inform and foresee "changes in the conventional basis of valuation a short time ahead of the general public" (Keynes 1936, p. 134). Keynes observes that "knowledge of the factors which will govern the yield of an investment some years hence is very slight and often negligible" (p. 134). Different from heroic times, when, according to Keynes, investment "was partly a lottery, though with the ultimate result largely governed by whether the abilities and character of the managers were above or below the average" (p. 134), if the separation between ownership and management prevails, then "certain classes of investment are governed by the average expectation of those who deal on the Stock Exchange as revealed in the price of shares, rather than by the genuine expectations of the professional entrepreneur" (Keynes 1936, p. 136).

Keynes condenses the process that induces to anticipate the change of convention in the famous metaphor of financial markets as a newspaper beauty contest[1]. Keynes maintains that an investor does not have to anticipate what will be the fundamental value of a firm in the future, but rather should estimate other investors' valuation. The individual assessed value is different from "the outcome of a weighted average of quantitative benefits multiplied by quantitative probabilities" (p. 145). In fact, as Keynes argues, to make an investment decision, "we are assuming, in effect, that the existing market valuation, however arrived at, is uniquely correct in relation to our existing knowledge of the facts which will influence the yield of the investment, and that it will only change in proportion to changes in this knowledge; though, philosophically speaking it cannot be uniquely correct, since our existing knowledge does not provide

[1] Beauty contest explains the activity of professional investors that are forced to anticipate the change of conventional valuation by the following methaphor: "Professional investment may be likened to those newspaper competitions in which the competitors have to pick out the six prettiest faces from a hundred photographs, the prize being awarded to the competitor whose choice most nearly corresponds to the average preferences of the competitors as a whole; so that each competitor has to pick, not those faces which he himself finds prettiest, but those which he thinks likeliest to catch the fancy of the other competitors, all of whom are looking at the problem from the same point of view. It is not a case of choosing those which, to the best of one's judgment, are really the prettiest, nor even those which average opinion genuinely thinks the prettiest. We have reached the third degree where we devote our intelligences to anticipating what average opinion expects the average opinion to be. And there are some, I believe, who practise the fourth, fifth and higher degrees." (1936, p. 140)

a sufficient basis for a calculated mathematical expectation. In point of fact, all sorts of considerations enter into the market valuation and are in no way relevant to the prospective yield" (p. 137).

We define a general functional to represent Keynes's long-term expectation and following a recent paper (Basili and Chateauneuf, 2016) we also set up the way to represent how a speculator anticipates changes of conventional judgment. Section 2 defines long-term expectation in an epsilon contamination approach incorporating the decision maker's attitude about insufficient and vague information. Section 3 sets up an aggregation scheme of opinions expressed through different probability distributions. Facing the set of all probability distributions attached by agents to possible events, the speculator is assumed to consider the weighted probability distribution of agents'cores, that is the weighted probability distribution of the intersection of all the investors' probability distributions. Such a weighted probability distribution is the Steiner point of the convex capacity that emerges from the aggregation of agents' opinions that represents the conventional judgment. Gajdos et al. (2008) show that in the case of a finite state space, the Steiner point always exists and can be valued through the Shapley value. On the contrary, in the case of an infinite countable state space, since the Steiner point is defined with respect to the outer angle or curvature, the Steiner point has no continuous extension to all convex bodies in infinite dimensional Hilbert space (i.e. Vitale 1985, p. 247). Section 3 approximates the Steiner point at the limit. The idea, that to the best of our knowledge has never been considered before, is very intuitive and straightforward: it is assumed that each agent has an interval of probabilities on each state and that intervals are distributed as a Fisher-Tippet distribution, that is a general distribution for extremes that includes Weibull, Gumbel, and Frechet distributions. By attaching an extreme distribution to intervals, convergence holds: the more extreme are events, the lower are the probabilities and the closer is the interval. Section 4 defines the professional investor long-term expectation that is the result of 'the average expectation of those who deal on the Stock Exchange as revealed in the price of shares" and the competence "to anticipate what average opinion expects the average opinion to be" (Keynes 1936, p. 139). Section 5 points out comprehensive examples of how agents form long-term expectations and speculators anticipate the change of conventional judgements, by updating the Steiner point w.r.t. a real event, Section 6 concludes.

2 Uncertainty, multiple-priors and epsilon-contamination approach

In The General Theory Keynes clarifies that "the state of long-term expectation, upon which our decisions are based, does not solely depend, on the most probable forecast we can make. It also depends on the confidence with which we make this forecast - on how highly we rate the likelihood of our best forecast turning out quite wrong" (p. 133). In this perspective of uncertainty[2], we shall assume that each investor does not have a unique prior on states of the World, but rather a finite set of probability distributions (multiple priors), none of which is considered sufficiently reliable. To represent the individual state of confidence[3], that depends on "the actual observation of the markets and business psychology" (p. 134), we assume that each agent's preferences can be represented by the epsilon contamination ($\varepsilon-$ *contamination*, henceforth) of some probability measure[4]:Eichberger and Kelsey, 1999; Nishimura and Ozaki, 2002; Asano, 2008; Gajdos et al., 2008; Kopylov, 2009; Cerreia et al., 2013.

2.1 Framework

Let $\Omega = \{\omega_1, \omega_2 ..., \omega_n\}$ be the set of states of the World, $\mathcal{P}(\Omega)$ the sigma-algebra of all the subsets of Ω and P the set of probability measures, such that $P = \{p : p \text{ is a probability measure on } \Omega\}$.

Let $E = \{s_1, ...s_k, ..\}$ be a finite or countable set of agents. Suppose every agent has got an opinion, formally an opinion of an agent s_i is a convex set C_i, contained in P. Under no-arbitrage condition, in frictionless and complete financial market, the price of any asset (or security), that is a tradable financial instrument that has a positive or negative cash flow of money, is given by its (discounted) expected value with respect to a unique risk neutral probability, or by a linear pricing rule. If there are incompleteness or trade frictions but arbitrage-free condition holds, an asset price can be evaluated by the Choquet integral with respect to a non-additive probability (capacity) of its payoffs. If the capacity is concave, then the pricing rule is sublinear.

Formally, let $\Omega = \{\omega_1, \dots \omega_n\}$ be a non-empty finite sets of states of the world and let $\mathcal{P}(\Omega)$ be the $\sigma-$algebra of all the events (the power set). A function $\nu : \mathcal{P}(\Omega) \to [0, 1]$ is a capacity if)i $\nu(\emptyset) = 0$; ii) $\nu(\Omega) = 1$; iii) for all $A, B \in \mathcal{P}(\Omega)$ such that $B \subset A$, $\nu(B) \leq \nu(A)$.

A capacity ν is said to be concave (sub-additive as well) if iv) for all $A, B \in \mathcal{P}(\Omega)$ we have $\nu(A \cup B) \leq \nu(A) + \nu(B) - \nu(A \cap B)$.

The Choquet integral (Chochet, 1954) of a function $f : \Omega \to \mathbf{R}$ with respect to a capacity ν is defined as $\int f dv := \int_0^{+\infty} \nu(\{\omega \in \Omega | f(\omega) \geq t\})dt + \int_{-\infty}^0 [\nu(\{\omega \in \Omega | f(\omega) \geq t\}) - 1]dt$. Finally, given a capacity ν we define its core as $\text{core}(\nu) = \{p \mid \text{ p is a probability measure on } \mathcal{P}(\Omega), p(A) \geq \nu(A) \text{ for all } A \in \mathcal{P}(\Omega)\}$.

[2]Because of continuity with Keynes terminology we call *uncertainty* what in current decision theory is named *ambiguity*.

[3]Under uncertainty, individual state of confidence represents reliability of the probability distribution. In literature reliability of a probability distribution can be represented by: a capacity (Choquet expected theory), a distorted probability (Cumulative Prospect Theory, Prospect Theory, Rank Dependent Theory), a belief function or a fuzzy measure and a similarity function (Case-based Theory).

[4]The $\varepsilon-$ *contamination* emerges as a robust Bayesian method to quantify, in terms of a class of possible distributions, how partial and incomplete is the subjective information encompassed in a single prior distribution. In fact, "quantification of prior beliefs can never be done without error, and hence that one is left at the end of the elicitation process with a set Γ of prior distributions which reflect true prior belief; i.e., π_T is an unknown element of Γ" (Berger 1984, p. 73). Details are in Moreno and Cano (1991).

The core represents the set of all the agent's opinions coherent with the market information[5]. Following Chateauneuf et al (Theorem 1.1, 1996), Jouini (2000), Jouini and Kallal (2001), Castagnoli et al. (2002), Araujo et al. (2012; 2018), for any asset $f \in F$ there exists a financial pricing rule $D : R^{\Omega} \rightarrow R$, that is a function over future payoffs contingenty to state space $\Omega = \{\omega_1, \omega_2 ..., \omega_n\}$. Such a pricing rule D is *subadditive, arbitrage free, positive homogeneous, monotonic and constant additive*[6]. Araujo et al. (2012) point out (Theorem 2) that for a given pricing rule $D : R^{\Omega} \rightarrow R$, there exists a unique closed and convex set $K \subset P$ of probability measures, where at least one element is strictly positive, such that for any asset f: $D(f) = \max_{k \in K} E_k(f)$, where $E_k(.)$ is the standard expectation with respect to k.

2.2 Individual long-term expectation

The $\varepsilon-$contamination approch allows to consider the agent's asset evaluation as the combination of $\overline{D}$, the asset price observed in the market and the confidence in his most reliable forecast. Because of uncertainty every agent forms his long-term expectation by distorting asset price with his confidence and combining it with his own most reliable, or any other motivated probability distribution such as the probability distribution that induces minimum expected utility that solves the Ellsberg Paradox, evaluation of that asset.

Then agent's long-term expectation can be summarized by the following criterion

Criterion 1 *Agent's long term expectation can formally be defined by*

$$\gamma_i(f) = [\epsilon \overline{D}(f) + (1 - \epsilon) D_i(f)] \tag{2.1}$$

where $p_i \in C_i \subset P$, $\epsilon \in [0, 1]$ *and* $D_i(f)$ *is the expectation of* f *with respect to* p_i.

Agent's long-term expectation reveals that he is $\varepsilon \times 100\%$ confident that the uncertainty he faces is summarized by the market price, but at the same time, he is aware that with $(1 - \varepsilon) \times 100\%$ uncertainty could be better represented by another probability distribution in the set C_i of all reasonable evaluations. In sum, the $\varepsilon-$ *contamination* interpretation of agent's long-term expectation allows describing imprecision of knowledge and behavioral effects of its awareness.

3 Main motivation

As noted before, Keynes assumes a convention influences investment decisions and such a general evaluation is "the outcome of the mass psychology of a

[5]Chateauneuf et al (1996) first studied and characterized the sub-additive Choquet pricing rule and showed that if the non-additive probability is a concave capacity, the set of the agent's probability distributions is unique and coincides with the core.

[6]Details are in Araujo et al. 2012.

large number of ignorant individuals" (1936, p. 138). A way to define the mass psychology is by aggregating agents' opinions, expressed by probability distributions on future states of the World[7]. An aggregation of agents' opinions is that of choosing a particular set of agents E, at most countable, each one giving a range of probability distributions; every such an agent s_i, as Ω is the space of states, has a family of probabilities C_i on it, which he considers reasonable. An adequate way to do all this is, for every such an individual, to associate with him a convex subset C_i of the probabilities on Ω . Let $\mathcal{K}$ be the family of all convex sets in P. An opinion multifunction is every $\mathcal{O} : E \to \mathcal{K}$, $s_i \to C_i$. Finally, given an opinion $\mathcal{O}$, the prevailing opinion $\mathcal{O}_E$ is defined as: $\bigcap_i C_i$. In the finite dimensional case, i.d. if Ω is finite and under the hypothesis that $\bigcap_i C_i$ is not the empty set [8], the idea is that the properly balanced opinion has got to be the Steiner point of $\bigcap_i C_i$: the conventional judgement[9]. If the number of the events is not finite yet countable, some difficulties occur: so in Section 3 we define a suitable aggregation of agents' opinion[10].

3.1 Preliminaries

Let (X, d) be a metric space. In what follows $B(x; r) \subset X$ is the usual ball centered on x and with radius r. If $X = R^d$ with its usual Euclidean norm, we set S^{d-1} the unit hypersphere centered on the origin. If $\mathcal{H}$ is a Hilbert space, we denote with $\langle \cdot, \cdot \rangle$ its standard inner product and $\|\cdot\|$ the induced Hilbert norm.

Definition 1 *Let (X, d) be a metric space. For every couple C_1,C_2 of bounded closed subsets of X we define their Hausdorff distance as:*
$d_{H,X}(C_1, C_2). = \{\inf \rho > 0 : C_2 \subset C_1 + B(0, \rho), C_1 \subset C2 + B(0, \rho)\}.$

Let now X be a Banach space. We denote with $\mathcal{C}(\mathcal{H})$ the family of all the closed sets of $\mathcal{H}$ and let $\mathcal{K}(X)$ be the family of its compact and convex subsets. Let also $\mathcal{C}_F(\mathcal{H})$ be the family of all the finite-dimensional elements of $\mathcal{C}(\mathcal{H})$, that is the family of those contained in some finite-dimensional affine subspace of $\mathcal{H}$ and let $\mathcal{K}_F(\mathcal{H}) := \mathcal{C}_F(\mathcal{H}) \cap \mathcal{K}(X)$ be the set of finite-dimensional compact and convex sets of X. Finally, for a $C \in \mathcal{C}_F(\mathcal{H})$ define $dim(C) := min\{dim(L) : C \subset L, L$ a finite-dimensional affine subspace of $\mathcal{H}\}$.

It is a well-known result (Castaing and Valadier 1977, Theorem II-14, p. 47) that:

[7] Opinion as a distribution is a usual assumption, e.g. de Finetti.

[8] $\cap C_i$ can be considered the agents common information set (opinions), that is their subjectively elaborated and evaluated information about market asset evaluation.

[9] The Steiner point or curvature centroid of smooth convex bodies is additive, uniformly continuous and satisfies an invariance property with respect to isometries.

[10] The Bayesian axiomatic approach to consensus distribution would not appear satisfying, not even in the sophisticated versions (copula models) and elicitation based on behavioral combination methods (e.g., DeGroot and Montera, 1991). If investors' opinions are not all independent and equally likely, each investor has to cope with ambiguity and stochastically dependent evaluations. As a consequence, each investor could calibrate the aggregation of investors' opinions through her confidence or degree of belief by pooling methods based on Dempster's rule of combination or theory of evidence, combination rules based on possibility distributions and fuzzy measures, or aggregation based on multiple priors or capacity.

Proposition 1 $(\mathcal{K}(X), d_{H,X})$ *is a complete metric space.*

Next, we recall the definition of the classical Steiner point for a d-dimensional convex body (see e.g. Schneider 1993). Let $\mathcal{H}$ be a Hilbert space.

Definition 2 *Let $C \in \mathcal{K}(\mathcal{H})$ and $C \subset L$ where L is a d-dimensional linear subspace of $\mathcal{H}$. Then its Steiner point $s(C)$ is defined as*

$$s(C) := d \int_{S_{\mathcal{H}} \cap L} u h_C(u) d\sigma(u),$$

where $h_C(u) := \sup \{\langle u, x\rangle : x \in C\}$ is the support function of C, $S_{\mathcal{H}}$ denotes the unit hypersphere in $\mathcal{H}$ centered on the origin and σ is the normalized Lebesgue measure on $S_{\mathcal{H}} \cap L$.

The Steiner point is independent of the choice of the finite-dimensional Euclidean subspace L containing C, so that the previous definition makes sense; it only depends on the inner product.

Let first analyze the case of $\Omega = \{w_1, ..., w_n\}$ a finite set of states of the World; so to set our ideas in a simpler situation. We shall treat the countable case further.

Given the sigma-algebra $\mathcal{P}(\Omega)$ of all the subsets of Ω , we identify isometrically the convex set $P(\Omega) = \{\pi : \pi \text{ is a probability measure on } \Omega\} = \{\pi : \Omega \longrightarrow [0,1] \text{ such that } \sum_{j=1}^{n} \pi(j) = 1\}$, with $[0,1]^n \cap \{(\pi(1), \ldots, \pi(n) \in \mathbf{R}^n | \sum_{j=1}^{n} \pi(j) = 1\}$.

Finally, we consider every euclidean space $\mathbf{R}^n$ naturally continuosly immersed in l^2 or in l^1 by $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, 0, \ldots 0, \ldots)$.

A reasonable way to investigate these events is simply to choose a certain number of agents E to give an opinion, or a range of opinions as follows: every such an agent s_i is asked to give to Ω a probability or, in more uncertain situations, a possible set of probabilities C_i.Thus, with the previous identifications, for every agent s_i, C_i is contained in a linear set of dimension less or equal to $n-1$, and the common opinion is a convex set contained in a linear set of dimension less or equal to $n-1$.

For every agent chain $E = \{s_1, ...s_k, ..\}$ a reasonable way to have an aggregation of agents opinions, as remarked in a recent paper Basili and Chateauneuf (2016) is to choose the Steiner point of the common opinion $\mathcal{O}_E$. In that paper an opinion was chosen this way: for every agent s_i a range of possible values for every adimissable value $\pi(j)$ is chosen such that $a_{ij}(j) \leq \pi(j) \leq b_{ij}$, $1 \leq j \leq n$. So, $C_i = ([a_{i1}, b_{i1}] \times ... \times [a_{in}, b_{in}]) \cap \left\{\sum_{j=1}^{n} \pi(j) = 1\right\}$.

The following inportant result is an Hilbert space adapted situation of classical results (See Shvartsman 2004, Theorem 1.2, with the Lipschitz constant asymptotic evaluation due to Vitale (1985, Appendix)).

Proposition 2 *Let $\mathcal{H}$ be a Hilbert space. Then the mapping $s\colon (\mathcal{K}_F(\mathcal{H}), d_{H,\mathcal{H}}) \to \mathcal{H}$ which associates to every element of $\mathcal{K}_F(\mathcal{H})$ its Steiner point is such that, for* *every $C_1, C_2 \in \mathcal{K}_F(\mathcal{H})$, and setting $d = dim(C_1 \cup C_2)$,*

$$\|s(C_2) - s(C_1)\| \leq l(d) d_{H,\mathcal{H}}(C_2, C_1),$$

where

$$l(d) = \frac{\Gamma(d/2+1)}{\sqrt{\pi}\Gamma(d/2+1/2)} \sim \sqrt{d/2+1},$$

as $d \to +\infty$, with Γ the standard Euler Gamma function.

Furthermore, $l(d)$ is the minimal possible constant fulfilling the previous inequality.

We finally remark that if the opinions are chosen as in Basili and Chateauneuf (2016), then we actually restrict our Steiner selector to the set of compact convex sets of $P(\Omega)$ contained in a linear space whose dimension does not exceeds $n-1$, having thus the possibility of a unique Lipschitz constant. The stability with respect to the Hausdorff metric is at its best in such a situation.

3.2 Steiner point with countable states of the World

Unfortunately, there is no way to define a suitable generalization of the notion of Steiner point to general convex bodies not contained in a finite dimensional subspace of a Banach or even a Hilbert space This is because, e.g. in the Hilbert case, the Lipschitz constant l in Proposition 2 increases as $\sqrt{d}$ when the dimension d increases, not permitting, in general, any approximation argument by means of finite dimensional convex bodies[11]. So, in general, when the set Ω is a countable set there is no way to proceed. As a matter of fact, some reasonable possibility arises when there is a natural way to create an ordering of Ω, when the tail of Ω is considered constituted by extreme events, for example. In situations like this, it usually happens that the way the Lipschitz constant l behaves as d goes to $+\infty$ is compensated the right way by the distribution itself. Crucially, to the best of our knowledge, this is the first approach which allows to approximate the Steiner point at the limit.

To proceed with this way of analysis, let now consider the more general case of a countable set of states of the World $\Omega = \{w_1, ..., w_n, ...\}$ and a finite number of agents $E = \{s_1, ...s_k\}$ (the case of a countable number of agents can be analogously treated, with minor changes in notation and no difference in methods, even if it is not realistic in our economical analysis). This case we shall use the Hilbert space $l_2 = \left\{(x_1, x_2, ...) : \ \sum_{j=1}^{+\infty} x_j^2 < \infty\right\}$ and the Banach space $l_1 = \left\{(x_1, x_2, ...) : \sum_{j=1}^{+\infty} |x_j| < \infty\right\}$.

We recall that l_1 is the dual space of the separable Banach space $c = \{(x_1, x_2, ...) : \lim_{j\to\infty} x_j \text{exists and is finite}\}$. Thus the unit ball of l_1 centered in the null sequence is sequentially weakly star compact. Given the sigma-algebra $\mathcal{P}(\Omega)$ of all the subsets of Ω, we shall identify the set of all the probability measures $P(\Omega)$ on Ω with $\left\{(x_j) : 0 \leq x_j \leq 1, \sum_{j=1}^{+\infty} x_j = 1\right\} \subset l_1(\Omega) \subset l_2(\Omega)$.

[11] See R. A. Vitale (1985) for more discussion and details.

Assume further $\Delta_i^n = [a_{i1}, b_{i1}] \times ... \times [a_{in}, b_{in}]$ where, with a little abuse of notation, we also suppose the possibility that for some i and j, $a_{ij} = b_{ij}$, so that in this case whe set $[a_{ij}, b_{ij}] := \{a_{ij}\} = \{b_{ij}\}$. Set $\Delta^n = \cap_{i=1}^k \Delta_i^n$, which is either the empty set or it is a possibly degenerate n-rectangle $[\alpha_{n1}, \beta_{n1}] \times ... \times [\alpha_{nn}, \beta_{nn}]$, where also in this case with a little abuse of notation, we consider the possibility that, for some j, $\alpha_j = \beta_j$, with $[\alpha_i, \beta_j] := \{\alpha_j\} = \{\beta_j\}$. Finally, consider $O_E^n = \Delta_n \cap P(\Omega)$. Using the previously introduced notation, we now state the following

Hypotheses 1

i) there exists an event j_0 such that for every agent s_i, $a_{ij} = 0$ if $j \geq j_0$; suppone furthermore that $O_E^{j_0} \neq \emptyset$;

ii) there exists a sequence (ζ_i) of non-negative real numbers such that, for every agent i, for all possible event j, $b_{ij} \leq \zeta_i$; furthermore let $\sum_{j=1}^{+\infty} \zeta_j < +\infty$.

iii) Suppose $\sum_{j=1}^{+\infty} \zeta_j l(j) < +\infty$, with $l(j)$ as in Proposition 2.

Note that i) can be interpreted as that there exists some elementary event after which any agent can reasonably give no lower bound for the probabilities: only (possibly very decreasing) upper bounds can be given for all elementary events. Furthermore, to ii) to be fulfilled, one should choose for (ζ_i) a suitable extreme events distribution, for example.

Remark 1 *if i) is fulfilled, (O_E^n) is a sequence of strongly compact convex sets of l^2 which, for $n \geq j_o$, is not-decreasing with respect to set inclusion and not identically equal to the empty set; if also ii) is fullfilled, $\beta_n \leq \zeta_n$. Furthermore, $\sum_{j=1}^{+\infty} \beta_j$ is convergent. If also iii) is fullfilled, then $\sum_{j=1}^{+\infty} \beta_j l(j)$ is convergent.*

We next need the following

Lemma 1 *Suppose i) is fulfilled. Then, for $n \geq j_0$, $d_{H,l_2}(O_E^{n+1}, O_E^n) \leq \sqrt{2}\beta_{n+1}$.*

Proof. Let $\bar{\mathbf{x}} = (\bar{x}_1, .., \bar{x}_n, \bar{x}_{n+1}) \in O_E^{n+1}$. Then, setting π_n the projection of the whole l_2 onto R^n, isometrically identified with its subspace having zero components after the n-th, we get: $\pi_n(\bar{\mathbf{x}}) = (\bar{x}_1, .., \bar{x}_n) \in \Delta_n$, so that: $\alpha_j \leq \bar{x}_j \leq \beta_j$, $j \leq n$ and, furtermore, $0 \leq \bar{x}_{n+1} \leq \beta_{n+1}$, which implies $\sum_1^n \bar{x}_j = 1 - \bar{x}_{n+1} \leq 1$. Next, remark that, because $\Delta_n \neq \emptyset$ we have $\sum_1^n \beta_n \geq 1$.

Set $\phi_{\bar{\mathbf{x}}} \colon [\bar{x}_1, \beta_1] \times .. \times [\bar{x}_n, \beta_n] \to R$, $(y_1, .., y_n) \mapsto \sum_1^n y_j$.

By the intermediate value theorem there exists a $\bar{\mathbf{y}}$ such that $\phi_{\bar{\mathbf{x}}}(\bar{\mathbf{y}}) = \sum_1^n \bar{y}_j = 1$. Then $\bar{\mathbf{y}} \in O_E^n$. Notice that, for $j \leq n$, $\bar{y}_j \geq \bar{x}_j$ and $\sum_1^n |\bar{y}_j - \bar{x}_j| = \sum_1^n \bar{y}_j - \bar{x}_j = \bar{x}_{n+1} \leq \beta_{n+1}$. So $\|\bar{\mathbf{y}} - \pi_n(\bar{\mathbf{x}})\| \leq \beta_{n+1}$. Because $\|\pi_n(\bar{\mathbf{x}}) - \bar{\mathbf{x}}\| = \bar{x}_{n+1} \leq \beta_{n+1}$, by the orthogonality between $\bar{\mathbf{y}} - \pi_n(\bar{\mathbf{x}})$ and $\pi_n(\bar{\mathbf{x}}) - \bar{\mathbf{x}}$ and using Pythagoras theorem we get that $\|\bar{\mathbf{y}} - \bar{\mathbf{x}}\| \leq \sqrt{2}\beta_{n+1}$; because $O_E^n \subset O_E^{n+1}$this is enough to get the thesis. ■

Theorem 1 *Suppose Hypotheses 1 are fulfilled. Then*

i) there exists a strongly compact convex set $O \subset l_2$ such that (O_E^n) converges to O in the Hausdorff metric d_{H,l_2};

ii) the sequence $(s(O_E^n))$ of the corresponding Steiner points converges to a point $\bar{s}(O) = (\bar{s}_j)$ strongly in l_2 and weakly star in l_1. In particular $\sum_{j=1}^{+\infty} \bar{s}_j = 1$, with $\bar{s}_j \geq 0$ for all j, so that $\bar{s}(O) \in P(\Omega)$.

Proof. By Lemma 1 and Remark 1 (O_E^n) is a Cauchy sequence in $(\mathcal{K}(l_2), d_{H,l_2})$; so, by Proposition 1, *i)* is proved.

In order to prove *ii)*, notice that by Proposition 2 and Remark 1 $(s(O_E^n))$ is a Cauchy sequence in l_2 such that $\|s(O_E^n)\|_{l_1} = \sum_{j=1}^{+\infty} s(O_E^n)_j = 1$, so it is strongly convergent in l_2 and, because the unit ball in l_1 is sequentially weakly star compact, it is weakly star convergent in l_1; finally, because the sequence (1) which is constantly equal to 1 is in c and every $s(O_E^n)_j \geq 0$, we get that $\sum_{j=1}^{+\infty} \bar{s}(O_E)_j = \|\bar{s}(O_E)\|_{l_1} = 1$. ∎

4 Conventional judgement and confidence in market asset price: a professional investor's behavior

Keynes considers the stock exchange populated by professional investors and speculators who are forced to anticipate the mass psychology of the market. As a consequence, the behavior of professional investors and speculators is the result of two different components: "the average expectation of those who deal on the Stock Exchange as revealed in the price of shares" and the competence "to anticipate what average opinion expects the average opinion to be" (Keynes 1936, p. 140). Then we obtain our primary result:

Criterion 2 *The professional investor's or speculator's long-term expectation $I(f)$ can be formally defined by*

$$I(f) = [\mu\overline{D}(f) + (1-\mu)\overline{s}(N(f))] \tag{4.1}$$

where $\mu \in [0,1]$ is the confidence of a professional investor in the asset price and $\overline{s}(N(f))$ is the expected value with respect to the Steiner point, that is what he considers conventional judgement, times future returns.

Crucially $(1-\mu)$ is the weight attached to what Keynes considers *the average opinion expects the average opinion to be.*

Then (4.1) precisely summarized the competitor's behavior in the newspaper beauty contest suggested by Keynes. Then, the previous expectation is the solution of the problem and accurately explains how skilled Keynesian individuals that are long term investors, or speculators should solve the newspaper beauty contest. Confronting (2.1) and (4.1) it is clear that speculators and professional investors differ from ordinary agents through the ability or superior knowledge in trying to estimate the conventional valuation.

4.1 Updating

When an uncertain event occurs, people may change their long-term expectation. In fact the "conventional valuation which is established as the outcome of the mass psychology of a vast number of ignorant individuals is liable to change violently as the result of a sudden fluctuation of opinion due to factors which do not really make much difference to the prospective yield" (Keynes 1936, p. 138).

The investor has to anticipate this change, but because of dynamic consistency, he can not update the Steiner point only, since it could induce an order that is not coherent with his preference. A very simple way to update multiple priors models[12] is to apply the Bayes rule for each probability distribution (prior-by-prior) in P and then *re-evaluate* the Steiner point. This method can guarantee dynamic consistency but is strenuous. It is possible to reduce the number of the probability distributions that need to be updated to calculate the new Steiner point after a given non-null even Ξ occurred.

Araujo et al (2016) point out how news modify the asset price. Araujo et al. characterize a new approach to updating the pricing rule that satisfies above conditions and the property called Dynamic Consistency to Certainty[13]

It follows that $N^{\Xi}(f)$, i.e. the conditional (f) w.r.t. an event Ξ, is the updated pricing rule such that $p^{\Xi} \in P$, and for any asset f and real number h, $N(f_h^{\Xi}) \geq h$ if and only if $N^{\Xi}(f) \geq h$, that is *if the unconditional price of* f *is at least equal to* h*, then its conditional price must also be at least equal to* h*.*

So doing the investor anticipates the change in the conventional judgment and includes this anticipation, so that

$$I(f)_{\Xi} = [\mu \overline{D}(f) + (1-\mu)\overline{s}(N^{\Xi}(f))], \tag{4.2}$$

where $\mu \in [0, 1]$.

Interesting enough the conditional Steiner point is elicited by the simple full Bayesian updating rule and it represents an appropriate 'averange opinion' that can be considered as a preferred rule with respect to every non bayesian rule unconditionally (de Finetti 1954).

The long-term expectation [4.2] represents the solution of the "battle of wits to anticipate the basis of conventional valuation a few months" (Keynes 1936, p.139). In fact, as Keynes argues, "it happens, however, that the energies and skill of the professional investor and speculator are mainly occupied otherwise. For most of these persons are, in fact, largely concerned, not with making superior long-term forecasts of the probable yield of an investment over its whole life, but with foreseeing changes in the conventional basis of valuation a short time ahead of the general public. They are concerned, not with what an investment

[12] Different solutions are: *rectangularity*, *menu dependence*, *change of subjective perception* etc.

[13] Given an event $\Xi \subset \mathcal{P}(\Omega)$ and the pricing rule D, Ξ is relevant if $-N(-\Xi^*) > 0$, then $p(\Xi) > 0$, for all $p \in N$ and $k^{\Xi} : \{p(\Xi) \in P | k \in K\}$ is the set of conditional probabilities. The updated pricing rule N^{Ξ} is the unique pricing rule that satisfies the Full Bayes Rule (Araujo et al 2016).

is really worth to a man who buys it “for keeps”, but with what the market will value it at, under the influence of mass psychology, three months or a year hence” (Keynes 1936, p. 139).

5 Example: portfolio choice

In this section we apply the model of Section 4 (CHECK) to the case of stock returns. In particular, we consider a professional investor who needs to compute the expected returns of risky assets to decide her optimal portfolio strategy.

Uncertainty and investment opportunities. Consider a discrete-time model with 2 periods $(0, 1)$ and three states. There are 2 investment opportunities: a risk-less asset with initial price 1 and final value $1+r$ and a risky asset with initial price S_0. At time 1 the price of the risky asset can change by a factor u, m or d corresponding to the three state of the world. As a result the time-1 payoff of the risky asset is S_0u (probability p_u), S_0m (probability p_m) or S_0d (probability p_d) with $d < m < d$ and $d < 1+r < u$. The investor is endowed with initial wealth $w_0 = 1$ and selects the fraction of w_0 (say $0 \le \phi \le 1$) to be invested in the risky asset. We assume that the risky asset is represented by the $S\&P500$ and that $r = 0.02$.

Preferences. We consider two investors (experts): one equipped with linear utility function (risk neutral) and a second investor equipped mean-variance utility and risk aversion parameter γ. Standard results imply that the optimal portfolios are given by

$$\phi = \begin{cases} 0, & \text{if } \tilde{\mathbb{E}}[R] \le 1+r; \\ 1, & \text{if } \tilde{\mathbb{E}}[R] > 1+r. \end{cases} \tag{5.1}$$

for the risk-neutral investor and

$$\phi = \begin{cases} 0, & \text{if } \tilde{\mathbb{E}}[R] \le 1+r; \\ \frac{\tilde{\mathbb{E}}[R]-(1+r)}{w_0\gamma\sigma^2}, & \text{if } 0 < \frac{\tilde{\mathbb{E}}[R]-(1+r)}{w_0\gamma\sigma^2} < 1 \\ 1, & \text{if } \frac{\tilde{\mathbb{E}}[R]-(1+r)}{w_0\gamma\sigma^2} > 1. \end{cases}$$

where $\tilde{\mathbb{E}}[R]$ is the expectation of the risky asset return computed by the expert and σ the standard deviation. $\tilde{\mathbb{E}}[R]$ is computed using two sources of information: market-based information (e.g., observed prices) and other agents (e.g., forecasters) opinion about the probability distribution of the asset's payoff. The latter piece of information has to be aggregated to compute the consensus distribution of the asset's payoff.

Expected returns from market data. To compute the market-based expected payoff (return in this example) of investment the investors use observed returns using data[14] from $2013-01-18$ to $2023-01-17$. Let $\bar{R}$ be the average annual return then the probability of payoffs are selected in such a way that $p_h u + p_m m + p_d d = \bar{R}$ and $p_h + p_m + p_d = 1$ (see the Appendix for details).

Aggregation of agents' opinions and the consensus distribution. Assume that investors also obtain information from N agents, indexed with i, which provide an estimation on the upper (b_i) and lower (a_i) bounds of p_h, p_m and p_d. We consider two cases. Case 1: The investor has access to the opinion of three agents ($N = 3$). We assume that upper-lower bounds are given by

$$\begin{aligned} a_1 &= \left(\frac{15}{84}, \frac{23}{84}, \frac{24}{84}\right), \quad b_1 = \left(\frac{36}{84}, \frac{26}{84}, \frac{40}{84}\right) \\ a_2 &= \left(\frac{12}{84}, \frac{30}{84}, \frac{30}{84}\right), \quad b_2 = \left(\frac{27}{84}, \frac{30}{84}, \frac{39}{84}\right) \\ a_3 &= \left(\frac{23}{84}, \frac{18}{84}, \frac{32}{84}\right), \quad b_3 = \left(\frac{26}{84}, \frac{27}{84}, \frac{42}{84}\right). \end{aligned}$$

Case 2: the investor forms her opinion about a_i and b_i using information from the Federal Reserve Bank of Minneapolis which provides a time-varying estimation of the probability of 20% increase or a 20% decrease in the $S\&P500$ index over a 1 year index[15]. Thus, in this case $N = 1$, $u = 1.2$, $d = .8$ and we set $m = \bar{R}$. Upper an lower bounds of probabilities are estimated as maximum and minimum of the time series of these probabilities[16] (Figure 1). In this case we obtain

$$a_1 = (0.016, 0.352, 0.053), \quad b_1 = (0.361, 0.907, 0.351).$$

The opinion on upper/lower bounds obtained from different agents have to be aggregated to compute the consensus distribution ($\Pi = [\Pi_u, \Pi_m, \Pi_d]$) of stock returns for the two cases described above. The consensus distribution is computed using the Shapley's value (see Basili and Chateauneuf (2020) and the Appendix for details). The corresponding expected return (the conventional judgment) is then $\bar{R}_c = \Pi_u u + \Pi_m m + \Pi_d d$.

Investors can also update the consensus distribution using information on relevant events. Assume for instance that investors evaluate the possibility of tight/loose (T and L respectively) monetary policy. Let

$$P(T \mid R = u) = .6, \quad P(T \mid R = m) = 0.5, \quad P(T \mid R = d) = 0.4$$

[14]https://fred.stlouisfed.org/series/SP500Link

[15]https://www.minneapolisfed.org/banking/current-and-historical-market–based-probabilitiesLink

[16]Note that this is just an example to illustrate how an investors could obtain information about the probability of stock returns. Clearly other methods, possibly more complicated, can be used to estimate such probabilities. our point here is not the exact estimation of the probability of a given event. Instead, we want to illustrate how different sources of information can be aggregate into the consensus distribution os stock returns and then applied to a portfolio choice problem.

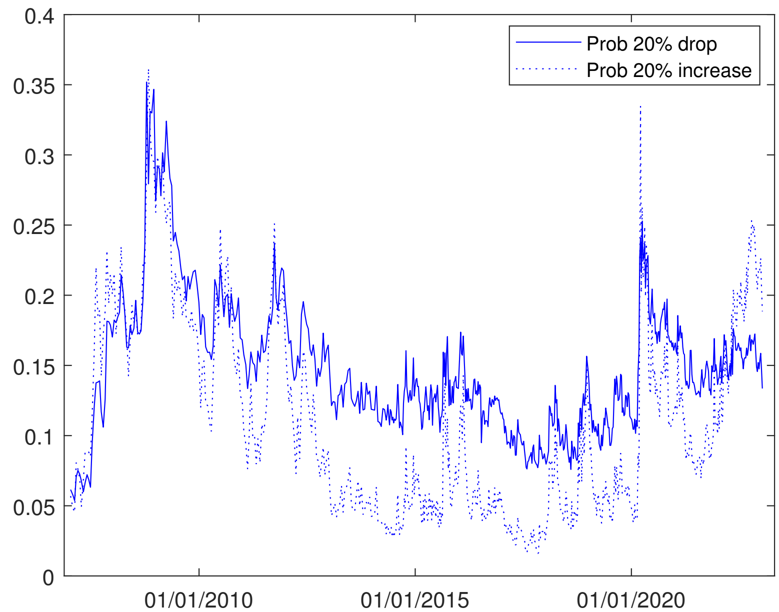

Figure 1: Market based probabilities of large changes in the $S\&P500$.

The consensus distribution can then be updated using the Bayes rule. The updated distribution Π^b (assuming we observe a tight monetary policy) is given by

$$\Pi^b_u = \frac{P(T \mid R = u)\Pi_u}{P(T \mid R = m)\Pi_m + P(T \mid R = d)\Pi_d}$$
$$\Pi^b_m = \frac{P(T \mid R = m)\Pi_m}{P(T \mid R = m)\Pi_m + P(T \mid R = d)\Pi_d}$$
$$\Pi^b_d = \frac{P(T \mid R = d)\Pi_d}{P(T \mid R = m)\Pi_m + P(T \mid R = d)\Pi_d}$$

The corresponding expected value (i.e., Bayes-updated conventional judgment) is $\bar{R}_b = \Pi^b_u u + \Pi^b_m m + \Pi^b_d d$. Finally, the subjective expected return needed for portfolio choice is computed as $\tilde{E}[R] = \mu \bar{R} + (1-\mu)\bar{R}_i$ where i is either c or b. Results are summarized in Table 1. The consensus distribution of Case 1 gives more weight to the state with negative returns as compared with the consensus distribution obtained in Case 2. As a result the expected return computed using the former (latter) distribution is lower (bigger) than the risk-free rate (Figure 3 upper panel). Moreover, in both cases the expected return computed using

Table 1: Objective and consensus distribution of the $S\&P500$

	p_h	p_m	p_d	Expected Ret
Market-based prob.	0.378	0.500	0.121	10.31%
Case 1	0.297	0.297	0.404	0.92%
Case 2	0.181	0.622	0.195	6.14%
Case 1 (Bayesian update)	0.364	0.304	0.330	3.81%
Case 2 (Bayesian update)	0.218	0.624	0.156	7.67%

Table 2: The table describes probabilities of the 3 states computed i) using market returns (first row) 2) the example in Case 1 and iii) probabilities from the Federal Reserve Bank of Minneapolis described in Case 2 (second row) iii) Bayesian updates (third and fourth rows).

consensus probability is lower than the expected return computed using historical market data (first row of Table 1). In other words, in our examples (Case 1-2) the consensus distribution produces a more prudent estimation of expected returns as compared with historical market data. In both cases the Bayesian update increases the probability of high returns (because, by assumption, the tight monetary policy is more likely to occur in the high-return state) and, thus, the corresponding expected returns.

Portfolio choice. According to Eq 5 the risk-neutral investor selects $\phi = 1$ when $\tilde{\mathbb{E}}[R] > 1 + r$ and $\phi = 0$ otherwise. From Table 1 we notice that $\bar{R} > 1 + r$, $\bar{R}_b > 1 + r$ (Case 1 and Case 2), $\bar{R}_e < (>) 1 + r$ in Case 1 (2). As a result the risk-neutral investor invests 100% of wealth in the risky asset when subjective returns equal market-based expected returns ($\mu = 1$) or when computed as a weighted average between market-based returns and the Bayes-updated conventional judgment ($\bar{R}_b$). When subjective returns are given by the average between market-based returns and the conventional judgment ($\bar{R}_c$) the short selling constraint binds for low μ (i.e., $\mu \leq 0.1$) and the optimal portfolio is $\phi = 0$. For larger μ the optimal choice is $\phi = 1$. The unconstrained optimal portfolio of the mean-variance investor depends on the Sharpe ratio scaled by her risk aversion. As a result, the short selling constraint binds when $\tilde{\mathbb{E}}[R] < 1 + r$ (i.e., $\mu \leq 0.1$) and then increases monotonically in μ and coincides with the optimal portfolio of the investors which ignores the consensus probability when $\mu \to 1$. The optimal portfolio compute using the Bayes-updated conventional judgment ($\bar{R}_b$) lies in between these two extremes.

When the consensus distribution is computed as in Case 2, the the conven-

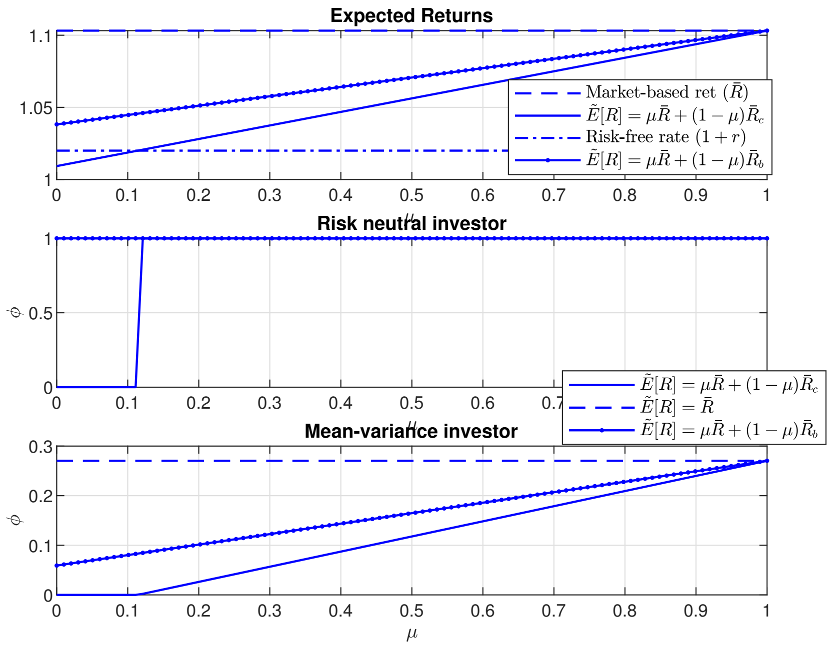

Figure 2: Portfolio choice when the consensus distribution is obtained from Basili and Chateauneuf (2020). $w_0 = 1$, $\gamma = 10$, $\sigma = 17.6\%$.

tional judgments ($\bar{R}_c$ and $\bar{R}_b$) exceed the risk-free rate. As a result the subjective return is above $1 + r$ for any μ and the constraint of the risk-neutral investor ($\phi \leq 1$) is always binding, that is $\phi = 1$. Differently, the optimal portfolio of the mean-variance investor in this case lies between zero and 1 and increases with μ for any assumption about subjective returns.

6 Concluding remarks

This paper proposes a different interpretation of Keynes's theory of long-term expectation and agents' ambiguity based on the $\varepsilon-$ *contamination* approach of probability distributions. The $\varepsilon-$ *contamination* interpretation of Keynes's long-term expectation theory makes direct and explicit the relationship between his long-term expectation notion and contemporary decision theory originated by the Ellsberg Paradox. The paper introduces a new representation of conventional judgement based on the Steiner point of the set of common opinions among agents. This work can give a formal description of the process by which professional investors try to anticipate the change in conventional judgment. The new representation of long-term expectation is also coherent with the behavior of competitors in the Keynes's beauty contest. Remarkably, this new

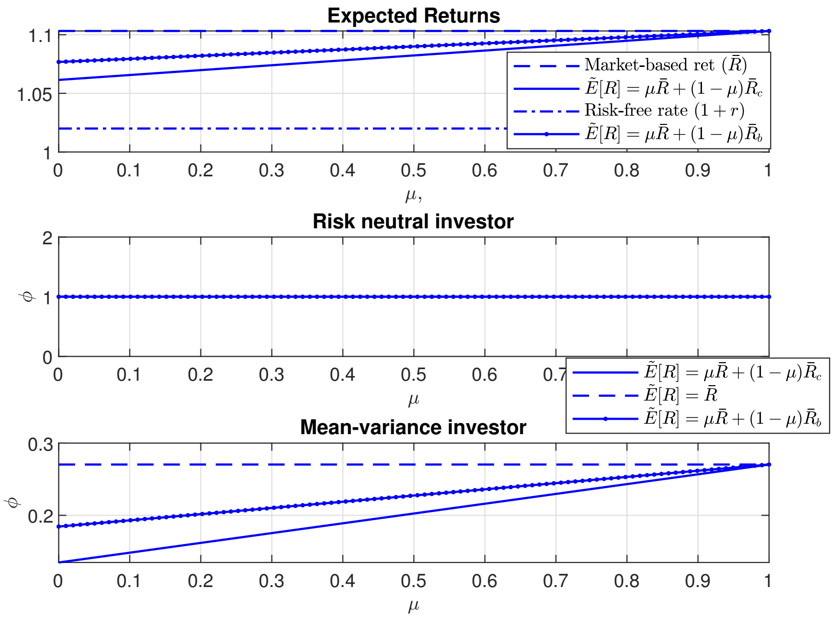

Figure 3: Portfolio choice when the consensus distribution is obtained from the Federal Reserve Bank of Minneapolis. $w_0 = 1$, $\gamma = 10$, $\sigma = 17.6\%$.

representation of long-term expectation sheds light on Keynes's view of stock exchanges like casino, where speculators make the market by anticipating the change of conventional judgment.

7 Appendix

7.1 Computation of probabilities using historical market returns

Assume that the average annualized return of the market is $\bar{R}$. The investor need to find p_h, p_m and p_d such that

$$p_h u + p_m m + p_d d = \bar{R}$$
$$p_h + p_m + p_d = 1$$

with the additional constraint $0 \leq p_i \leq 1$, $i = h, m, d$. The system above must be solved for given u, m, d. To stay consistent with the data from the Federal reserve bank of Minneapolis we set $u = 1.2$, $d = 0.8$ and we set $m = \bar{R}$. The solution of the previous system is

$$p_h = \frac{R - d - (m-d)p_m}{u-d}$$
$$p_d = \frac{u - R - (u-m)p_m}{u-d}$$

where $0 \leq p_m \leq 1$ must be selected in such a way that $0 \leq p_h \leq 1$ and $0 \leq p_d \leq 1$which implies $0 \leq p_m \leq \frac{u-R}{u-m}$. Therefore the set of probabilities is defined by

$$p_m = \alpha \frac{U-R}{u-m}$$
$$p_h = \frac{R-d}{u-d} - \alpha \frac{(u-R)(m-d)}{(u-m)(m-d)}$$
$$p_d = (1-\alpha)\frac{u-R}{u-d}$$

for $\alpha \in (0,1)$. For any α probabilities above are such that $p_h u + p_m m + p_d d = \bar{R}$ and, thus, the choice of α is irrelevant for expected returns and portfolio choice (probabilities of Table 1 are computed for $\alpha = 0.5$.).

7.2 Computation of "probability interval" capacity and Shapley value

We show here the computations needed to compute the Shapley value for Case 2. Computations for Case 1 can be found in Basili and Chateauneuf (2020). Let the 3 states be s_1, s_2 and s_3 (good, intermediate and bad state respectively). The Shapley value is defined by

Table 3: "Probability interval" capacity

A	$\{s_1\}$	$\{s_2\}$	$\{s_3\}$	$\{s_1, s_2\}$	$\{s_1, s_3\}$	$\{s_2, s_3\}$	S
$\upsilon(A)$	0.016	0.352	0.053	0.648	0.092	0.638	1

$$\forall i \in [1, N] \quad \Pi_i = \sum_i \frac{(\mid A \mid -1)!(N - \mid A \mid)!}{N!} [\upsilon(A) - \upsilon(A \backslash \{i\})]$$

and we obtain $\Pi = [0.181, 0.622, 0.195]$.

8 References

Araujo A., Chateauneuf A., Faro J.H., 2012. Pricing rules and Arrow-Debreu ambiguous valuation. Economic Theory 49. 1-35.

Araujo A., Chateauneuf A., Faro J.H., 2018. Financial market structures revealed by pricing rules: Efficient complete markets are prevalent. Journal of Economic Theory 173, 257-288.

Araujo A., Chateauneuf A., Faro J.H., Holanda B., 2016. Updating pricing rules. INSPER working paper 369/2016, 1-25.

Asano T., 2008. Portfolio inertia and ε-contamination. Theory and Decision 68, 341-365.

Basili M., Chateauneuf A., 2016. Aggregation of Coherent Experts' Opinion: A Tractable Extreme-Outcomes Consistent Rule, in Fusion Methodologies in Crisis Management. Higher Level Fusion and Decision Making, G. Rogova and P. Scott (Eds.). New York, Springer-Verlag.

Basili M., Chateauneuf A., 2020 : Aggregation of Coherent Experts' Opinions and Conditional Consensus by the Steiner Point. Int. J. Approx. Reason. 123, 17-25.

Berger J., 1984. The Robust Bayesian Viewpoint. In J. Kadane, ed. Robust Bayesian Analysis. Amsterdam, North-Holland.

Berger J., Berliner M., 1986. Robust Bayes and Empirical Bayes Analysis with Epsilon-Contaminated Priors. The Annals of Statistics 14, 461–486.

Castaing C., Valadier M., 1977. Convex Analysis and Measurable Multifunctions. Springer-Verlag, Berlin-Heidelberg.

Carlier G., Dana R., Shahdi N., 2003. Efficient insurance contracts under epsilon contaminated utilities. The Geneva Papers on Risk and Insurance Theory 28, 59-71.

Castagnoli E., Maccheroni F., Marinacci M., 2002. Insurance premia consistent with the market. Insurance. Mathematics and Economics 31, 267-284.

Cerreia-Vioglio S., Maccheroni F. , Marinacci M., Montrucchio L., 2013. Ambiguity and robust statistics. Journal of Economic Theory 148, 974-1049.

Chateauneuf A., Gajdos T., Jaffay J-Y, 2011. Regular updating. Theory and Decision 71, 111-128.

Chateauneuf A., Jaffray J-Y., 1995. Local Möbius transforms of monotone capacities. In: Symbolic and Quantitative Approaches to Reasoning and Uncertainty, 115-124. Springer Berlin Heidelberg.

Chateauneuf A., Kast R., Lapied A., 1996. Choquet pricing for financial markets with frictions. Mathematical Finance 6, 323-330.

Choquet, G., 1954. Theory of Capacities. Annales de l'institut Fourier (Grenoble), 5, 131-292.

de Finetti B., 1954. Media di decisioni e media di opinioni. Bollettino dell'lstituto Internazionale di Statistica 34, 144-157.

DeGroot M., Montera J., 1991. Optimal Linear Opinion Pools. Management Science 37, 546-558.

Eichberger J., Kelsey D., 1999. E-capacities and the Ellsberg Paradox. Theory and Decision 46, 107-140.

Epstein L., Wang T., 1994. Intertemporal asset pricing and Knightian uncertainty. Econometrica 62, 283–382.

Gajdos T., Hayashi T., Tallon J.-M., Vergnaud J.-C., 2008. Attitude towards imprecise information. Journal of Economic Theory 140, 27-65.

Huber P., 1981 Robust Statistics. John Wiley and Sons New York.

Jaffray J-Y, 1992. Bayesian updating and belief functions. IEEE Transactions on Systems, Man, and Cybernetics 22, 1144-1152.

Jouini E.: Price functionals with bid-ask spreads: An axiomatic approach. Journal of Mathematical Economics, 34. 547-558, (2000).

Jouini E., Kallal H., 2001. Efficient trading strategies in the presence of market frictions. The Review of Financial Studies 14, 343-369.

Keynes J.M., 1973. The General Theory of Employment, Interest and Money. The Collected Writings of John Maynard Keynes, Vol. VIII. [1936]. London, Macmillan.

Kopylov, I., 2009. Choice deferral and ambiguity aversion. Theoretical Economics 4, 199-225.

Moreno E., Cano J., 1991. Robust bayesian analysis with epsilon contaminations partially known. Journal of the Royal Statistical Society, Part B, 53, 143-155.

Nishimura K. and H. Ozaki: Search and Knightian uncertainty. Journal of Economic Theory. 119:299-333, (2004).

Nishimura K., Ozaki H., 2006. An Axiomatic Approach to Epsilon-Contamination. Economic Theory 27, 333-340.

Owen G., 1968. Game theory. Saunders Company,Philadelphia.

Vitale R., 1985. The Steiner point in infinite dimensions. Israel Journal of Mathematics 52, 245-250.

Wolintzky A., 2016. Mechanism design with maxmin agents: theory and an application to bilateral trade.Theoretical Economics 11, 971-1004.

https://fred.stlouisfed.org/series/SP500