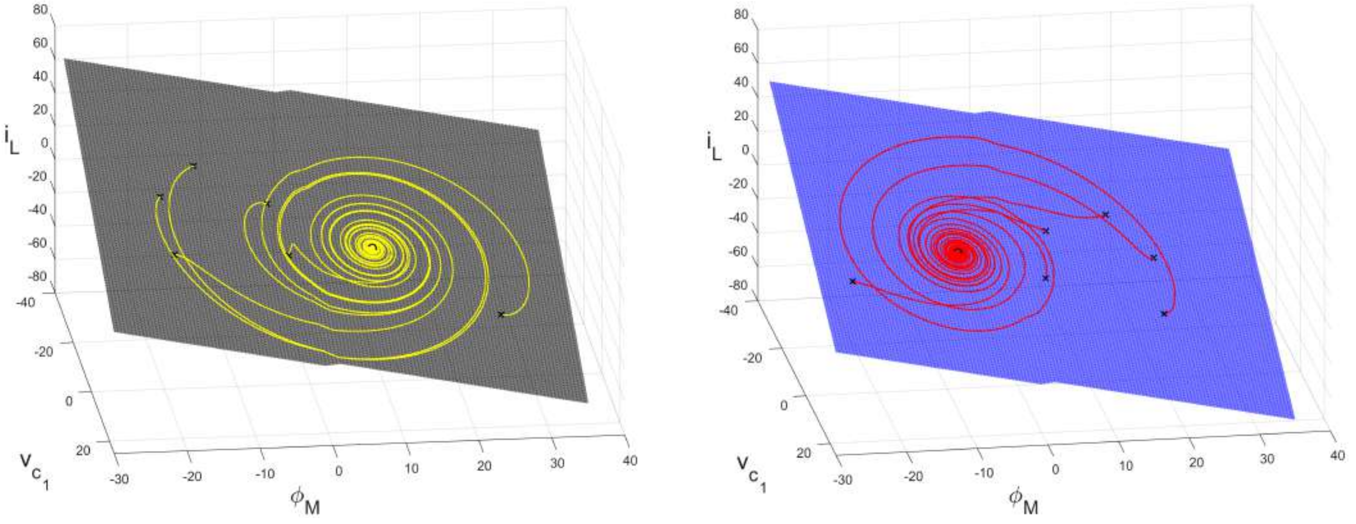

(a) (b)

Fig. 8. State space trajectories of the Chua's memristor circuit with $R = 1/0.7$, $C_1 = 1$, $C_2 = 2$, $L = 1.36$ [corresponding to the point marked with $\star$ in Fig. 5(b)] and $N(\varphi_M)$ being a smooth-piecewise approximation of $N_{pwl}(\varphi_M)$. The initial conditions (marked with $\times$) are chosen to ensure that the trajectories lie on the invariant manifolds $\mathcal{M}_{\mathcal{I}}$: (a) Manifold with $\mathcal{I} = 2$ (the unique equilibrium point is marked with $\circ$); (b) Manifold with $\mathcal{I} = -2$ (the unique equilibrium is marked with $\circ$).

toward one stable equilibrium point. Instead, if we set $R = 1/0.7$, $C_1 = 1$, $C_2 = 2$, $L = 1.83$, corresponding to the point $(\alpha, \beta) = (2, 2.3)$ [marked with $\square$ in Fig. 5(b)], then the system displays oscillatory behaviors on some invariant manifold. For instance, Fig. 9 shows that on the invariant manifold with $\mathcal{I} = 0$, there are two stable periodic solutions. Simulations show that similar oscillatory behaviors are present also on manifolds with small values of $|\mathcal{I}|$, while for larger values of $|\mathcal{I}|$, the dynamics on the manifolds displays convergence toward the equilibrium points.

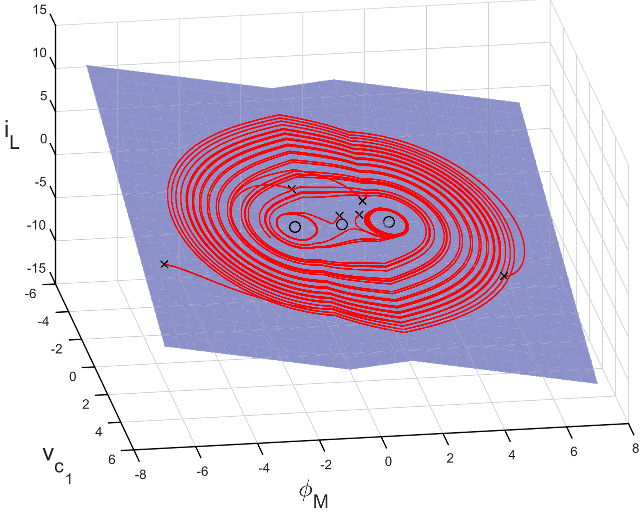

Fig. 9. State space trajectories of the Chua's memristor circuit with $R = 1/0.7$, $C_1 = 1$, $C_2 = 2$, $L = 1.83$ [corresponding to the point marked with $\square$ in Fig. 5(b)] and $N(\varphi_M)$ being a smooth-piecewise approximation of $N_{pwl}(\varphi_M)$. The initial conditions (marked with $\times$) are chosen to ensure that the trajectories lie on the invariant manifolds $\mathcal{M}_{\mathcal{I}}$ with $\mathcal{I} = 0$. The three equilibrium points (marked with $\circ$) are all unstable.

4.2. *The RLC-memristor circuit*

Let us assume that the memristor characteristic $N(q_m)$ satisfies the slope-bounded condition (35) for some k_1, k_2, $k_2 > k_1$ and consider the matrices A, B, C in (16). Since $(A - k_iBC) \in \mathbb{R}^2$, $i = 1, 2$, its 2-additive compound matrix reads

$$(A - k_iBC)^{[2]} = \mathrm{Tr}(A - k_iBC)$$

$$= -\frac{1}{RC_0} - \frac{k_i}{L},$$

where $\mathrm{Tr}(\cdot)$ denotes the matrix trace. Hence, from Theorem 1, we get that the memristor circuit is nonoscillatory if the following condition

$$k_1 > -\frac{L}{RC_0} \tag{47}$$

holds. Indeed, it can be shown that the above condition is exactly the condition derived by applying the Poincaré–Bendixson criterion to the state space model (33) which describes the circuit dynamics on the manifold $\mathcal{M}_{\mathcal{I}}$. From (17), we get that the matrices in (26) boil down to

$$\overline{A} = \begin{pmatrix} -\dfrac{1}{RC_0} & 1 \\ -\dfrac{1}{LC_0} & 0 \end{pmatrix}, \quad \overline{B} = \begin{pmatrix} -\dfrac{1}{LRC_0} \\ -\dfrac{1}{L} \end{pmatrix},$$

$$\overline{C} = (1 \quad 0).$$

Therefore, the dynamics on $\mathcal{M}_{\mathcal{I}}$ obeys

$$\begin{cases} \dot{x}_1 = -\dfrac{1}{RC_0}x_1 + x_2 - \dfrac{1}{L}N(x_1), \\ \dot{x}_2 = -\dfrac{1}{LC_0}x_1 - \dfrac{1}{RLC_0}N(x_1) + \dfrac{1}{LC_0}\mathcal{I}, \end{cases} \tag{48}$$

where, according to Remark 3.1 and (29), $x_1 = q_M$ and $x_2 = i_L + q_M/(RC_0) + (1/L)N(q_M)$. From these two relations and the expression of $M_{\mathcal{I}}$ in (18), it turns out that the original state variables of the circuit are related to x_1 and x_2 as follows:

$$\begin{aligned} q_M &= x_1, \\ v_{C_0} &= \frac{1}{C_0}\left(1 - \frac{L}{R^2C_0}\right)x_1 + \frac{L}{RC_0}x_2 - \frac{1}{C_0}\mathcal{I}, \\ i_L &= -\frac{1}{RC_0}x_1 + x_2 - \frac{1}{L}N(x_1). \end{aligned} \tag{49}$$

By applying the Poincaré–Bendixson criterion to (48), we get that the dynamics is nonoscillatory on each manifold $\mathcal{M}_{\mathcal{I}}$ if

$$-\frac{1}{RC_0} - \frac{1}{L}N'(x_1) < 0,$$

which turns out to be equivalent to condition (47) since $N'(x_1) \in [k_1, k_2]$.

To investigate the circuit dynamics, we assume that the memristor characteristic function is given by

$$N(q_M) = -aq_M + b\frac{q_M^3}{1 + cq_M^2}, \tag{50}$$

where a, b, and c are positive constants. It can be verified that

$$-a \le N'(q_M) \le -a + \frac{b}{c}, \quad \forall\, q_M \in \mathbb{R},$$

i.e. $N(q_M)$ satisfies the slope-bounded condition (35) with $k_1 = -a$ and $k_2 = -a + b/c$. Hence, the nonoscillatory condition (47) becomes

$$a < \frac{L}{RC_0}. \tag{51}$$

The equilibrium points $(x_1, x_2) = (\bar{x}_1, \bar{x}_1)$ of (48) depend on the value of the manifold index $\mathcal{I}$. In particular, $\bar{x}_1$ is the solution of the following equation:

$$\left(1 - \frac{a}{R}\right)x_1 + \frac{b}{R}\frac{x_1^3}{1 + cx_1^2} = \mathcal{I},$$

while $\bar{x}_2$ is obtained by the first equation of (48) and, according to (49), the corresponding equilibria in the original state variables are $(v_{C_0}, i_L, q_M) = (0, 0, \bar{x}_1)$. It can be verified that if $a \le R$, then there is a unique equilibrium point on each invariant manifold $\mathcal{M}_{\mathcal{I}}$ which is locally asymptotically stable. If $a > R$ and $c < b/(a - R)$, then $\mathcal{M}_{\mathcal{I}}$ can have either one or three equilibrium points, the latter case requiring sufficiently small values of $|\mathcal{I}|$. For instance, if $\mathcal{I} = 0$, there are three equilibria with $\bar{x}_1 = 0$ and $\bar{x}_1 = \pm\sqrt{(a/R - 1)/(b/R - c(a/R - 1))}$. Moreover, it turns out that the equilibrium point with $\bar{x}_1 = 0$ is unstable, while the other two are locally asymptotically stable. For increasing values of $|\mathcal{I}|$, we get a unique equilibrium point which is locally asymptotically stable. This implies that at $\mathcal{I} = \pm\mathcal{I}_F$, $\mathcal{I}_F > 0$, one of the two stable equilibrium points and the unstable one collapse and disappear, thus implying the occurrence of a fold bifurcation in the circuit state space, without varying the parameters. For instance, in the case $a/R = 2$, $b = 1/3$, $c = 0.2$, this bifurcation occurs at $\mathcal{I}_F = 0.574$. Hence, if the

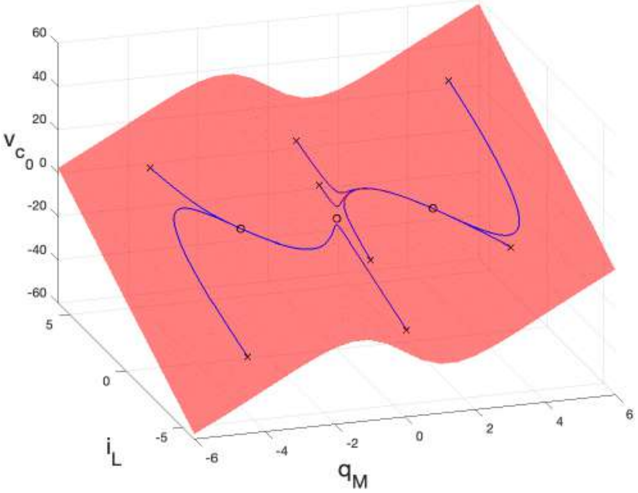

Fig. 10. State space trajectories of the RLC-memristor circuit with $R = 1$, $L = 0.5$, $C_0 = 0.1$ and $a = 2$, $b = 1/3$, $c = 0.2$ in memristor characteristic (50). The initial conditions (marked with ×) are chosen to ensure that the trajectories lie on the invariant manifolds $\mathcal{M}_{\mathcal{I}}$ with $\mathcal{I} = 0$. The three equilibrium points are marked with ∘.