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A Lyapunov Function-Based Condition for Nonoscillatory Behaviors in a Class of Memristor Circuits

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In recent years, many contributions have been reported in the literature showing that memristor circuits can exhibit very rich oscillatory and chaotic behaviors. In this paper, we look for conditions ensuring that memristor circuits are *nonoscillatory*, i.e. they do not display oscillations and more complex attractors and enable convergence toward some of the infinite nonisolated equilibrium points. Specifically, we consider the class of memristor circuits composed by the interconnection of a linear time-invariant two-terminal (one port) element and an ideal memristor, which can be either flux-controlled or charge-controlled and whose characteristic satisfies a slope-bounded condition. First, exploiting the well-known fact that any circuit with an ideal memristor admits *first integrals*, and hence its state space is decomposed in a continuum of invariant manifolds, a state space representation of the circuit dynamics on each invariant manifold is derived in an explicit way. Then, conditions to ensure the absence of oscillatory behaviors on each manifold are obtained by employing the 2-additive compound matrix of the Jacobian of these representations. It is shown that the memristor circuit is nonoscillatory if there exists a common Lyapunov function for two suitable 2-additive compound matrices, which is obtained by solving two linear matrix inequalities. Two memristor circuits are analyzed in some detail to illustrate the application of the results and their degree of conservatism.

Keywords: Memristor circuit; first integral; nonoscillatory behavior; compound matrix; Chua's circuit.

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1. Introduction

The memristor was introduced by [Chua, 1971] as the fourth basic passive circuit element. The interest in memristors only exploded many years later, thanks to the discovery of memristive effects in solid-state nanoscale electronic devices by Williams and his team [Strukov *et al.*, 2008]. Since then, the research activity on memristors has rapidly flourished in several directions, mainly motivated by their potential use for implementing power-efficient memories and their possible contribution to futuristic neuromorphic computers [Yang & Williams, 2013; Williams, 2017; Sebastian *et al.*, 2018; James *et al.*, 2018; Huang *et al.*, 2021; Sirakoulis *et al.*, 2022].

The ideal memristor is a circuit element defined by a nonlinear characteristic relating flux (the integral of voltage) and charge (the integral of current). In the voltage–current domain, the memristor satisfies a state-dependent Ohm's law where the state variable is either the flux, for a flux-controlled memristor, or the charge, for a charge-controlled memristor. Later on, wider classes of memristive systems have been proposed (see [Chua, 2015] and references therein), including extended memristors which are of practical interest for modeling real memristive devices [Mazumder *et al.*, 2012; Chen & Yu, 2015; Hajri *et al.*, 2019]. Beyond their importance in nonlinear circuit theory [Corinto & Forti, 2016; Corinto *et al.*, 2021; Chua *et al.*, 2019; Tetzlaff, 2014], ideal memristors can be employed to approximate the dynamics of physical components (see [Vista & Ranjan, 2019] and references therein). One relevant feature of memristors is that, due to their memory and nonlinearity, they are able to greatly enrich the range of circuit dynamic behaviors. For instance, this feature plays a key role for the use of analog circuits in future electronics [Itoh & Chua, 2008; Buscarino *et al.*, 2013; Kumar *et al.*, 2017; Liang *et al.*, 2020; Ascoli *et al.*, 2022a, 2022b].

In [Corinto & Forti, 2016], it has been pointed out that the flux–charge domain is more appropriate than the traditional voltage–current domain for the analysis of circuits with ideal memristors. By means of the flux–charge analysis method, it has been shown that any circuit with ideal memristors admits for structural reasons *first integrals*. Consequently, its state space can be foliated into infinitely many invariant manifolds where different dynamic behaviors can be displayed (see [Innocenti *et al.*, 2024] and references therein). This rigorously shows that these memristor circuits can possess infinitely many convergent, periodic and complex dynamics for a given set of circuits parameters, a property which is known in the literature as extreme multistability (see e.g. [Hens *et al.*, 2012; Bao *et al.*, 2016; Chang *et al.*, 2019]). In spite of literally many contributions dealing with memristor circuits displaying complex dynamics, less attention has been paid to the case where these circuits do not exhibit oscillations and their dynamics eventually converges toward the equilibrium points. A significant exception concerns some classes of memristor neural networks which have quite a different structure from the class here considered. For these classes, conditions ensuring complete stability, i.e. convergence toward equilibrium points, have been derived also in the case of extended memristors (see [Di Marco *et al.*, 2022a; Di Marco *et al.*, 2022b; Di Marco *et al.*, 2024] and references therein).

This paper aims to fill this gap by looking for memristor circuits which are *nonoscillatory*, i.e. they do not possess periodic solutions and more complex attractors, thus enabling convergence toward some of the infinite nonisolated equilibrium points. Indeed, identifying the parameter ranges for which the memristor circuit is nonoscillatory can also provide insights on how to choose these parameters for the synthesis of circuits capable to generate the complex dynamics usually sought. It is well known that the Poincaré–Bendixson criterion permits to ensure the absence of oscillations in second-order nonlinear systems, while sufficient conditions to rule out oscillatory and more complex behaviors in higher-order nonlinear systems were given by [Muldowney, 1990; Li & Muldowney, 1995]. More recently, these remarkable results have been reinterpreted within the context of 2-contraction theory [Wu *et al.*, 2022a], by showing that a system is nonoscillatory if it is 2-contractive, i.e. the variational equation associated to the 2-additive compound matrix of the system Jacobian is 1-contractive. Moreover, in [Angeli *et al.*, 2021], it has been shown that a nonlinear system is nonoscillatory if the variational equation associated to the 2-additive compound matrix of the Jacobian of a nonlinear system is exponentially stable, thus allowing for the use of classic Lyapunov stability tools. This approach has been extended in [Martini *et al.*, 2022] to provide sufficient conditions to rule out the existence of attractors with positive Lyapunov exponents and applied to some case studies in [Martini *et al.*, 2023]. For a nonlinear system of order n, these conditions require to solve Linear Matrix Inequalities (LMIs) involving matrices of order $n(n-1)/2$. To reduce the computational burden for large n, small gain like conditions have been proposed in [Angeli *et al.*, 2025] which, by suitably decomposing the nonlinear system into two subsystems of dimension n_1 and n_2, $n_1 + n_2 = n$, require the solution of LMIs of lower dimensions.

The goal of the paper is to derive conditions to assess if a memristor circuit is nonoscillatory. Specifically, we consider the class of circuits composed by the interconnection of a linear time-invariant two-terminal (one port) element and either a flux-controlled or a charge-controlled memristor. For such a class, it is possible to verify the nonoscillatory nature of the circuits via an approach that is more efficient than the one provided in [Martini *et al.*, 2023] for general nonlinear systems with first integral. Indeed, we pursue a strategy based on the two following facts: (i) the state space of memristor circuits is decomposed in a continuum of invariant manifolds which admit a closed form expression; (ii) a state space representation of the dynamics of the circuit on each invariant manifold can be derived in an explicit way. The first one permits to assess the nonoscillatory nature of a circuit by considering its dynamics on each invariant manifold, while the second ensures that this dynamics is governed by a state space model. Notably, models related to different manifolds only differ for a constant vector, thus implying that their Jacobian matrices (and consequently the 2-additive compound matrices) share the same structure. Moreover, if the memristor characteristic function is slope-bounded, then the 2-additive compound matrix belongs to the convex hull of two constant matrices, which do not depend on the considered invariant manifold. This makes it possible to ensure that the circuit is nonoscillatory if there exists a common quadratic (or homogeneous polynomial) Lyapunov function for the two matrices defining the convex hull, a problem which requires to solve two LMIs.

The rest of this paper is organized as follows. Section 2 introduces the considered class of circuits and formulates the addressed problem. Two examples of circuits, the first with a flux-controlled memristor and the second with a charge-controlled one, are presented in Sec. 2.1, while Sec. 2.2 concerns some known preliminary results on the approach pursued to derive LMI sufficient conditions for ensuring that nonlinear systems are nonoscillatory. Section 3 is devoted to the main result of the paper and its discussion. The two circuits presented in Sec. 2.1 are analyzed in Sec. 4 to illustrate the theory and to show its degree of conservatism and scope of applicability. Some final remarks and conclusions end the paper in Sec. 5.

2. Problem Formulation and Preliminaries

In this paper, we consider the class of memristor circuits composed by the interconnection of a two-terminal (one port) element $\mathbf{L}$, with voltage $v_{\mathbf{L}}$ and current $i_{\mathbf{L}}$, and a memristor, with voltage v_M and current i_M, as described in Fig. 1. The two-terminal element $\mathbf{L}$ can contain ideal operational amplifiers and linear time-invariant controlled sources, in addition to linear R, L, C components, while the memristor can be either flux-controlled or charge-controlled. The flux-controlled memristor is described by the flux–charge characteristic $N : \mathbb{R} \to \mathbb{R}$, i.e.

$$q_M = N(\varphi_M),$$

where φ_M and q_M are the memristor flux and charge, i.e.

$$\varphi_M(t) = \int_{-\infty}^{t} v_M(\tau)d\tau,$$

$$q_M(t) = \int_{-\infty}^{t} i_M(\tau)d\tau,$$

while the charge-controlled memristor is described by the charge–flux characteristic $N : \mathbb{R} \to \mathbb{R}$, i.e.

$$\varphi_M = N(q_M).$$

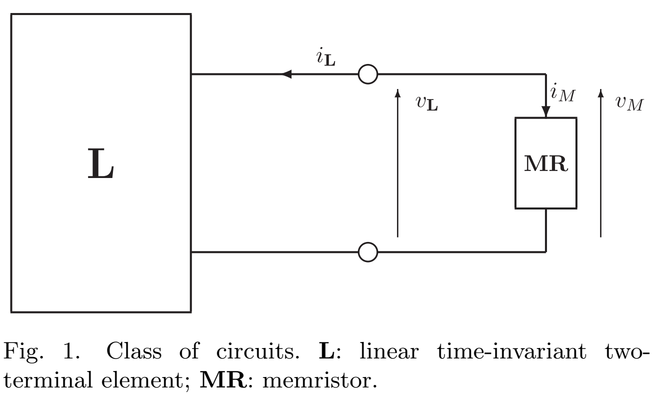

Fig. 1. Class of circuits. **L**: linear time-invariant two-terminal element; **MR**: memristor.

The flux-controlled memristor admits the following state space representation:

$$\begin{cases} \dot{\varphi}_M(t) = v_M(t), \\ i_M(t) = N'(\varphi_M(t))v_M(t), \end{cases} \tag{1}$$

where φ_M is the state variable, v_M is the scalar input, i_M is the scalar output, the characteristic function N is assumed of class $\mathcal{C}^1$ on the real axis and $N'(\varphi_M)$ is known as the memristor memconductance. Analogously, the dynamics of the charge-controlled memristor obeys

$$\begin{cases} \dot{q}_M(t) = i_M(t), \\ v_M(t) = N'(q_M(t))i_M(t), \end{cases} \tag{2}$$

where $N'(q_M)$ is known as the memristor memristance.

The two-terminal element $\mathbf{L}$ can be described by the following linear time-invariant finite-dimensional causal state space representation:

$$\begin{cases} \dot{z}_{\mathbf{L}}(t) = Az_{\mathbf{L}}(t) + Bu_{\mathbf{L}}(t), \\ y_{\mathbf{L}}(t) = Cz_{\mathbf{L}}(t), \end{cases} \tag{3}$$

where $z_{\mathbf{L}} \in \mathbb{R}^n$ is the state vector, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$, $C \in \mathbb{R}^{1\times n}$ are constant matrices, and $u_{\mathbf{L}}$ and $y_{\mathbf{L}}$ are the scalar input and output signals, respectively. Specifically, for consistency with (1) and (2), $u_{\mathbf{L}} = i_{\mathbf{L}}$ and $y_{\mathbf{L}} = v_{\mathbf{L}}$ in the case of flux-controlled memristor and $u_{\mathbf{L}} = v_{\mathbf{L}}$ and $y_{\mathbf{L}} = i_{\mathbf{L}}$ in the case of charge-controlled one. Consequently, the rational transfer function $G(s) = C(sI_n - A)^{-1}B$, where s denotes the complex variable and I_n is the identity matrix of order n, represents the impedance of $\mathbf{L}$ if the memristor is flux-controlled and the admittance if it is charge-controlled. The following assumptions are enforced on matrices A, B, and C.

Assumption 1

(a) The pair (A, B) is controllable and the pair (A, C) is observable.
(b) The matrix A is nonsingular.

Assumption 1(a) ensures that $G(s)$ is a rational function of order n. Hence, it can be written as

$$G(s) = \frac{b_{n-1}s^{n-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}, \tag{4}$$

where $a_0, a_1, \ldots, a_{n-1}, b_0, b_1, \ldots, b_{n-1}$ are constant coefficients. Note that Assumption 1(b) is equivalent to $a_0 \neq 0$.

Taking into account that $v_{\mathbf{L}} = v_M$ and $i_{\mathbf{L}} = -i_M$, it can be verified that the dynamics of the circuit of Fig. 1 is completely described by the following system:

$$\begin{cases} \dot{\xi}(t) = \sigma Cz(t), \\ \dot{z}(t) = Az(t) - N'(\xi(t))BCz(t), \end{cases} \tag{5}$$

where $\xi \in \mathbb{R}$ and $z \in \mathbb{R}^n$ are the state variables, $N : \mathbb{R} \to \mathbb{R}$ is a nonlinear function of class $\mathcal{C}^1$, and σ is either equal to 1 or -1. Specifically, $z = z_{\mathbf{L}}$, $\xi = \varphi_M$, $\sigma = 1$ in the flux-controlled memristor case, $z = z_{\mathbf{L}}$, $\xi = q_M$, $\sigma = -1$ in the charge-controlled one.

It is known that system (5) has some peculiar features. We first observe that $(z, \xi) = (0, \bar{\xi})$ is an equilibrium point for all $\bar{\xi} \in \mathbb{R}$, i.e. system (5) possesses *infinitely many nonisolated equilibria*. Moreover, the structure of system (5) ensures the existence of first integrals [Innocenti *et al.*, 2024]. Indeed, the next result holds true.

Proposition 1. *The function* $H : \mathbb{R}^{n+1} \to \mathbb{R}$ *defined as*

$$H(\xi, z) = \xi - \sigma CA^{-1}z - CA^{-1}BN(\xi) \tag{6}$$

is constant along the solutions of (5), *i.e.*

$$H(\xi(t), z(t)) = H(\xi_0, z_0), \tag{7}$$

where $z_0 = z(t_0)$ *and* $\xi_0 = \xi(t_0)$ *are the initial conditions at* $t = t_0$.

Proof. From Assumption 1(b), the second equation of (5) can be rewritten as

$$\begin{aligned} z(t) &= A^{-1}(\dot{z}(t) + BN'(\xi(t))Cz(t)) \\ &= A^{-1}\left(\dot{z}(t) + \frac{1}{\sigma}B\dot{N}(\xi(t))\right), \end{aligned}$$

where the second equality follows from the following equivalences:

$$N'(\xi(t))BCz(t) = N'(\xi(t))B\frac{1}{\sigma}\dot{\xi}(t) = \frac{1}{\sigma}B\dot{N}(\xi(t)).$$

Consequently, the first equation of (5) boils down to

$$\dot{\xi}(t) - \sigma CA^{-1}\dot{z}(t) - CA^{-1}B\dot{N}(\xi(t)) = 0,$$

and hence $\dot{H}(\xi(t), z(t)) = 0$, i.e. $H(\xi(t), z(t))$ is constant over time. ■

We remark that the expression of the first integrals can be explicitly derived also if Assumption 1(b) does not hold (see e.g. [Di Marco *et al.*, 2021]).

In this paper, we are interested in investigating the dynamics of system (5) with the aim of determining conditions under which the system is *nonoscillatory*, i.e. it does not display oscillatory and more complex behaviors and enables convergence toward some of the infinite nonisolated equilibrium points. To derive these conditions, we refer to the approach developed in [Angeli *et al.*, 2021] for general time-invariant nonlinear systems and extended in [Martini *et al.*, 2022, 2023] to rule out the existence of attractors with positive Lyapunov exponents (see Sec. 2.2). However, instead of applying to (5) the procedure provided in [Martini *et al.*, 2023] for general nonlinear systems with first integral, we follow a different way which exploits the peculiar property of the memristor circuit state space which is foliated in invariant manifolds. Indeed, according to Proposition 1, the solution $z^0(t)$, $\varphi_M^0(t)$, $t \geq t_0$, of (5) with initial conditions $z(t_0) = z_0$, $\varphi_M^0(t_0) = \varphi_{M_0}$ is confined to lie onto the invariant manifold

$$\mathcal{M}_\mathcal{I} = \{\xi \in \mathbb{R}, z \in \mathbb{R}^n : H(\xi, z) = \mathcal{I}\}, \tag{8}$$

where the manifold index $\mathcal{I}$ is given by

$$\mathcal{I} = H(\xi_0, z_0), \tag{9}$$

i.e. it depends on the initial conditions of (5). Since for any fixed $\xi \in \mathbb{R}$ the equation $H(\xi, z) = \mathcal{I}$ is an hyperplane in $\mathbb{R}^n$, it follows that $\bigcup_{\mathcal{I}\in\mathbb{R}} \mathcal{M}_\mathcal{I} \equiv \mathbb{R}^{n+1}$, i.e. the state space of system (5) is decomposed into a continuum of invariant manifolds. Also, observe that the equilibrium points belonging to the invariant manifold $\mathcal{M}_\mathcal{I}$ are $(z, \xi) = (0, \bar{\xi}_\mathcal{I})$ with $\bar{\xi}_\mathcal{I}$ solving the scalar equation $\xi - CA^{-1}BN(\xi) = \mathcal{I}$.

This property of the state space suggests that we can assess the nonoscillatory nature of the memristor circuit by considering the dynamics on each manifold $\mathcal{M}_\mathcal{I}$. To proceed this way, we need to derive the equations governing the dynamics on $\mathcal{M}_\mathcal{I}$. This is accomplished in Sec. 3 together with the main result of this paper.

2.1. *Examples of memristor circuits*

In this section, we introduce a couple of memristor circuits. The first one is the memristor Chua's circuit depicted in Fig. 2, which is obtained from the celebrated Chua's circuit by replacing the classic nonlinear resistor (Chua's diode) with a flux-controlled memristor [Itoh & Chua, 2008], whose classic characteristic is the canonical piecewise-linear function

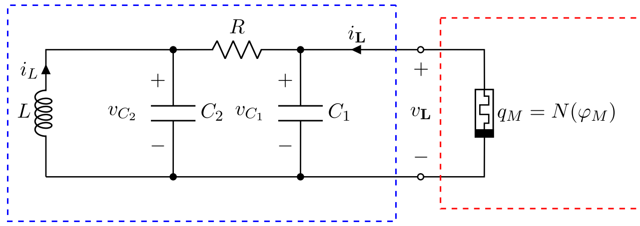

Fig. 2. Chua's memristor circuit: two-terminal element **L** (blue box); charge-controlled memristor (red box).

$$N_{pwl}(\varphi_M) = \frac{1}{2}(m_0 - m_1)(|\varphi_M + 1| - |\varphi_M - 1|) + m_1\varphi_M, \tag{10}$$

where the constants m_0 and m_1, $m_0 < m_1$, define the two different slopes. Since, in this paper, the memristor characteristic needs to be of class $\mathcal{C}^1$, we assume that $N(\varphi_M)$ is any $\mathcal{C}^1$ function on the real axis such that $N(0) = 0$ and $N'(\varphi_M) \in [m_0, m_1]$ for all $\varphi_M \in \mathbb{R}$. By exploiting the smooth-piecewise representation of (10) given in [Jimenez-Fernandez *et al.*, 2016], this assumption permits to consider also $N_{pwl}(\varphi_M)$ as a limiting case.

Applying the Kirchhoff laws to **L**, which contains two capacitors C_1 and C_2, an inductor L and a resistor R, we obtain the equations

$$\begin{cases} C_1\dot{v}_{C_1}(t) = \dfrac{1}{R}(v_{C_2} - v_{C_1}) + i_\mathbf{L}(t), \\ C_2\dot{v}_{C_2}(t) = \dfrac{1}{R}(v_{C_1} - v_{C_2}) + i_L(t), \\ L\dot{i}_L(t) = -v_{C_2}(t), \end{cases} \tag{11}$$

where v_{C_1} and v_{C_2} are the capacitor voltages and i_L is the inductor current. By setting $z_\mathbf{L} = (v_{C_1}, v_{C_2}, i_L)^\top$ and observing that $v_\mathbf{L} = v_{C_1}$, it can be readily verified that the two-terminal element **L** admits the structure (3) with

$$A = \begin{pmatrix} -\dfrac{1}{RC_1} & \dfrac{1}{RC_1} & 0 \\ \dfrac{1}{RC_2} & -\dfrac{1}{RC_2} & \dfrac{1}{C_2} \\ 0 & -\dfrac{1}{L} & 0 \end{pmatrix}, \quad B = \begin{pmatrix} \dfrac{1}{C_1} \\ 0 \\ 0 \end{pmatrix},$$

$$C = (1 \quad 0 \quad 0). \tag{12}$$

Hence, the dynamics of the memristor circuit obeys Eqs. (5) with $z = (v_{C_1}, v_{C_2}, i_L)^\top$, $\xi = \varphi_M$, $\sigma = 1$ and A, B, C as above. It can be readily checked that both (a) and (b) of Assumption 1 hold. The impedance of $\mathbf{L}$ is given by the following third-order rational function:

$$G(s) = \frac{\dfrac{1}{C_1}\left(s^2 + \dfrac{1}{RC_2}s + \dfrac{1}{LC_2}\right)}{s^3 + \dfrac{C_1 + C_2}{RC_1C_2}s^2 + \dfrac{1}{LC_2}s + \dfrac{1}{RLC_1C_2}}. \tag{13}$$

The expression of the invariant manifolds can be derived via Proposition 1, obtaining

$$\mathcal{M}_{\mathcal{I}} = \{(v_{C_1}, v_{C_2}, i_L)^\top \in \mathbb{R}^3, \varphi_M \in \mathbb{R} : \varphi_M + RC_1 v_{C_1} + L i_L + RN(\varphi_M) = \mathcal{I}\}. \tag{14}$$

The equilibrium points on the manifold $\mathcal{M}_{\mathcal{I}}$ are given by $v_{C_1} = v_{C_2} = i_L = 0$ and $\varphi_M = \bar{\varphi}_{M_{\mathcal{I}}}$ with $\bar{\varphi}_{M_{\mathcal{I}}}$ solving the scalar equation $\varphi_M + RN(\varphi_M) = \mathcal{I}$.

Consider the circuit of Fig. 3 which will be referred to as the RLC-memristor circuit. Applying the Kirchhoff laws to the two-terminal element $\mathbf{L}$, we obtain the equations

$$\begin{cases} C_0 \dot{v}_{C_0} = -\dfrac{1}{R} v_{C_0} + i_L, \\ L \dot{i}_L = -v_{C_0} - v_{\mathbf{L}}, \end{cases} \tag{15}$$

where v_{C_0} is the capacitor voltage and i_L is the inductor current. By setting $z_{\mathbf{L}} = (v_{C_0}, i_L)^\top$ and observing that $i_{\mathbf{L}} = -i_L$, it turns out that the two-terminal element admits the structure (3), with $i_{\mathbf{L}}$ and $v_{\mathbf{L}}$ exchanged, once

$$A = \begin{pmatrix} -\dfrac{1}{RC_0} & \dfrac{1}{C_0} \\ -\dfrac{1}{L} & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ -\dfrac{1}{L} \end{pmatrix}, \quad C = (0 \quad -1). \tag{16}$$

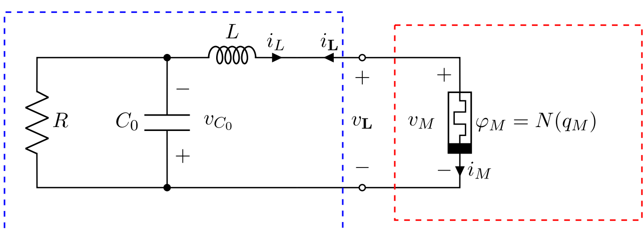

Fig. 3. RLC-memristor circuit: two-terminal element $\mathbf{L}$ (blue box); charge-controlled memristor (red box).

Hence, the dynamics of the RLC-memristor circuit is governed by Eqs. (5) with $z = (v_{C_0}, i_L)^\top$, $\xi = q_M$, $\sigma = -1$ and A, B, C as above. It can be checked that both (a) and (b) of Assumption 1 hold also in this case. The admittance of $\mathbf{L}$ reads

$$G(s) = \frac{\dfrac{1}{L}\left(s + \dfrac{1}{RC_0}\right)}{s^2 + \dfrac{1}{RC_0}s + \dfrac{1}{LC_0}}, \tag{17}$$

while (8) becomes

$$\mathcal{M}_{\mathcal{I}} = \left\{(v_{C_0}, i_L)^\top \in \mathbb{R}^2, q_M \in \mathbb{R} : q_M - C_0 v_{C_0} + \frac{L}{R} i_L + \frac{1}{R} N(q_M) = \mathcal{I}\right\}. \tag{18}$$

The equilibrium points belonging to $\mathcal{M}_{\mathcal{I}}$ are $(v_{C_0}, i_L, q_M) = (0, 0, \bar{q}_{M_{\mathcal{I}}})$, where $\bar{q}_{M_{\mathcal{I}}}$ is such that $R\bar{q}_{M_{\mathcal{I}}} + N(\bar{q}_{M_{\mathcal{I}}}) = R\mathcal{I}$.

2.2. *Preliminary results*

Muldowney [1990] provided a significant extension of the classic Poincaré–Bendixson criterion to nonlinear systems of order higher than two. Specifically, sufficient conditions to rule out the existence of nontrivial periodic solutions were given in terms of the logarithmic norm of the 2-additive compound matrix of the system Jacobian. Moreover, it was also shown that the conditions ensure that every solution with initial conditions lying within some forward invariant set of the system converges toward the set of equilibrium points contained in the invariant set [Li & Muldowney, 1995]. Recently, the Muldowney results have been reinterpreted and extended within the context of 2-contraction theory [Wu *et al.*, 2022b], by exploiting the powerful algebraic machinery of compound matrices [Bar-Shalom & Margaliot, 2021]. Specifically, it is shown that a system is nonoscillatory if it is 2-contractive which means that the variational equation associated to the 2-additive compound matrix of the system Jacobian is 1-contractive. Here, we recall the definition of 2-multiplicative and 2-additive compounds of a given matrix (see e.g. [Bar-Shalom & Margaliot, 2021]).

Definition 2.1. Consider a matrix $A \in \mathbb{R}^{n \times n}$ with $n \geq 2$. Then,

- the 2-multiplicative compound matrix of A, denoted as $A^{(2)} \in \mathbb{R}^{\binom{n}{2} \times \binom{n}{2}}$, is the matrix whose entries are all 2-dimensional minors of A listed in lexicographical order;

- the 2-additive compound matrix of A, denoted as $A^{[2]} \in \mathbb{R}^{\binom{n}{2}\times\binom{n}{2}}$, is given by

$$A^{[2]} = \frac{d}{d\varepsilon}(I_n + \varepsilon A)^{(2)}\Big|_{\varepsilon=0},$$

where $\varepsilon \in \mathbb{R}$ and I_n is the identity matrix of order n.

Some useful properties of these compound matrices are listed as follows (see e.g. [Wu *et al.*, 2022b]).

Property 1. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of matrix $A \in \mathbb{R}^{n\times n}$. Then, the eigenvalues of $A^{(2)}$ and $A^{[2]}$ are given by $\{\lambda_{i_1}\lambda_{i_2}, 1 \le i_1 < i_2 \le n\}$ and $\{\lambda_{i_1} + \lambda_{i_2}, 1 \le i_1 < i_2 \le n\}$, respectively.

Property 2. Let $T \in \mathbb{R}^{n\times n}$ be a nonsingular matrix. Then,

$$(T^{-1}AT)^{[2]} = T^{(2)^{-1}}A^{[2]}T^{(2)}. \tag{19}$$

Consider the time-invariant nonlinear system

$$\dot{x}(t) = f(x(t)), \tag{20}$$

where $x \in \mathbb{R}^n$ is the state vector, $f : \mathbb{R}^n \to \mathbb{R}^n$ is assumed to be of class $\mathcal{C}^1$, and $x(t)$ is the state time evolution with initial condition $x(0) = x_0$.

A natural way to assess the nature of the nontrivial attractors of (20) consists in analyzing the variational equation

$$\dot{\theta}(t) = \frac{\partial f}{\partial x}(x(t))\theta(t), \tag{21}$$

where $\frac{\partial f}{\partial x}(x(t)) \in \mathbb{R}^{n\times n}$ is the Jacobian of (20) evaluated at $x(t)$ and $\theta \in \mathbb{R}^n$ is a vector which propagates along solutions of (20) the effect of an infinitesimal perturbation of initial conditions in the direction $\theta(0)$. Indeed, several techniques are available to numerically compute from (21) the Lyapunov exponents of any attractor of (20) [Dieci & Elia, 2008; Dieci *et al.*, 2011].

In this paper, we refer to the approach developed in [Angeli *et al.*, 2021] to assess if (20) is a nonoscillatory system. The approach looks at the following, possibly higher-order, variational equation:

$$\dot{\psi}(t) = \frac{\partial f}{\partial x}^{[2]}(x(t))\psi(t), \tag{22}$$

where $\frac{\partial f}{\partial x}^{[2]} \in \mathbb{R}^{\binom{n}{2}\times\binom{n}{2}}$ is the 2-additive compound of the Jacobian and $\psi \in \mathbb{R}^{\binom{n}{2}}$ propagates along solutions of (20) with the effect of the infinitesimal area perturbation enclosed by the two vectors $\theta_1(0)$ and $\theta_2(0)$. In particular, if the initial condition $\psi(0)$ is set equal to the wedge product between $\theta_1(0)$ and $\theta_2(0)$, i.e. $\psi(0) = \theta_1(0) \wedge \theta_2(0)$, then the corresponding solution $\psi(t)$ is given by $\psi(t) = \theta_1(t) \wedge \theta_2(t)$, where $\theta_1(t)$ and $\theta_2(t)$ are the solutions of (21) with initial conditions $\theta_1(0)$ and $\theta_2(0)$, respectively. In [Angeli *et al.*, 2021], it is shown that if the linear time-varying system (22) is exponentially stable, then the solution $x(t)$ of (20) is not periodic. This permits to employ Lyapunov stability conditions to ensure that (20) is a nonoscillatory system. In particular, if the Jacobian of system (20) belongs to a convex hull of matrices, i.e.

$$\frac{\partial f}{\partial x}(x) \in \mathrm{conv}(J_1, \ldots, J_N), \quad \forall\, x \in \mathbb{R}^n \tag{23}$$

for fixed constant matrices $J_i \in \mathbb{R}^{n\times n}$, $i = 1, \ldots, N$, then it is possible to employ some Lyapunov functions $V(\psi; x)$ to obtain conditions, in terms of LMIs, ensuring exponential stability of (22) and hence the nonoscillatory nature of (20). The case of a quadratic Lyapunov function in ψ is reported next.

Proposition 2. *Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ in (20) is of class $\mathcal{C}^1$ and its Jacobian satisfies (23) for some given matrices $J_i \in \mathbb{R}^{n\times n}$, $i = 1, \ldots, N$. Then, system (20) is nonoscillatory if there exists a quadratic Lyapunov function $V(\psi) = \psi^\top P\psi$ with $P \in \mathbb{R}^{\binom{n}{2}\times\binom{n}{2}}$ being a symmetric and positive definite matrix such that*

$$J_i^{[2]^\top}P + PJ_i^{[2]} < 0, \quad i = 1, \ldots, N, \tag{24}$$

where $J_i^{[2]}$ is the 2-additive compound matrix of J_i.

Proof. It readily follows from [Angeli *et al.*, 2021] once it is observed that conditions (23) and (24) imply that

$$\dot{V}(\psi) = \psi^\top\left(\frac{\partial f}{\partial x}^{[2]^\top}(x)P + P\frac{\partial f}{\partial x}^{[2]}(x)\right)\psi < 0,$$

$$\forall\, x \in \mathbb{R}^n, \quad \forall\, \psi \in \mathbb{R}^{\binom{n}{2}},\ \psi \neq 0,$$

which ensures that (22) is exponentially stable. ■

Remark 2.1. Proposition 2 ensures that the system is a nonoscillatory in some subset $\mathcal{D} \subseteq \mathbb{R}^n$ if condition (23) holds for all $x \in \mathcal{D}$. Moreover, if $\mathcal{D}$ is a forward invariant set of (20), i.e. $x(t) \in \mathcal{D}$, $\forall t \ge 0$ if $x(0) = x_0 \in \mathcal{D}$, then every solution with initial conditions in $\mathcal{D}$ converges to the set $\mathcal{E} \cap \mathcal{D}$, where $\mathcal{E}$ denotes the set of equilibrium points of (20). Finally, we observe that in [Martini *et al.*, 2022] it is shown that if condition (24) holds as a nonstrict inequality within some forward invariant set $\mathcal{D}$, then this set cannot contain attractors with positive Lyapunov exponents. Note that in this case, the presence of stable periodic solutions is not excluded.

Proposition 2 simply requires the existence of a common quadratic Lyapunov function for the matrices $J_i^{[2]}$, $i = 1, \ldots, N$ and condition (24) amounts to solve a finite number of LMIs, a problem for which efficient software is available [Lofberg, 2004]. Other LMI conditions can be derived by employing Lyapunov functions more general than the quadratic one. Indeed, less conservative sufficient conditions can be looked for by using the class of Homogeneous Polynomial Lyapunov Functions (HPLF) of order $2m$, $m \geq 1$ [Chesi *et al.*, 2003]. Specifically, by introducing for $i = 1, \ldots, N$ the *extended matrices* $J_{i,\{m\}}^{[2]} \in \mathbb{R}^{d \times d}$ of $J_i^{[2]}$, with $d = \binom{\binom{n}{2}+m-1}{m}$, condition (24) can be replaced by the following one:

$$J_{i,\{m\}}^{[2]\top} P_m + P_m J_{i,\{m\}}^{[2]} + L(\gamma) < 0, \quad i = 1, \ldots, N, \tag{25}$$

where $P_m \in \mathbb{R}^{d \times d}$ is symmetric positive definite and $L(\gamma)$, $\gamma \in \mathbb{R}^{d_L}$ denotes the linear parameterization of all the matrices $L \in \mathbb{R}^{d \times d}$ such that $\psi^{\{m\}^\top} L \psi^{\{m\}} = 0$, $\forall\, \psi^{\{m\}} \in \mathbb{R}^d$, with $\psi^{\{m\}}$ being the vector of all monomials of degree m of the vector ψ. It is known that the dimension of the parameter vector γ is given by $d_L = \frac{1}{2}d(d+1) - \binom{\binom{n}{2}+2m-1}{2m}$. For instance, in the case of an HPLF of order 4 ($m = 2$), we have $d = 6$ and $d_L = 6$. In this case, we have that system (20) is nonoscillatory if there exists a symmetric and positive definite matrix $P_2 \in \mathbb{R}^{6\times 6}$ and a vector $\gamma \in \mathbb{R}^6$ such that (25) holds. The use of HPLFs will be treated in some detail in Sec. 4 to show how the conservatism of condition (24) could be reduced. Finally, we mention that condition (25) is one instance of how several analysis and control problems can be reduced to polynomial optimization problems (see e.g. [Chesi, 2010] and references therein).

3. Main Results

To apply the results of Sec. 2.2, we first need to determine the equations governing the dynamics of system (5) on the invariant manifold $\mathcal{M}_\mathcal{I}$. For this purpose, we proceed as in [Innocenti *et al.*, 2020] by first observing that for any linear transformation of coordinates $\bar{z} = T_L^{-1} z$, with $T_L \in \mathbb{R}^{n\times n}$ being nonsingular, we obtain a system dynamically equivalent to (5) with matrices $\overline{A} = T_L^{-1} A T_L$, $\overline{B} = T_L^{-1} B$, and $\overline{C} = C T_L$ and the same impedance or admittance $G(s)$ in (4). Among all the possible choices, we find it convenient to assume that the matrices $\overline{A}$, $\overline{B}$, and $\overline{C}$ enjoy the following structure:

$$\overline{A} = \begin{pmatrix} -a_{n-1} & 1 & 0 & \cdots & 0 \\ -a_{n-2} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_1 & 0 & 0 & \cdots & 1 \\ -a_0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad \overline{B} = \begin{pmatrix} b_{n-1} \\ b_{n-2} \\ \vdots \\ b_1 \\ b_0 \end{pmatrix},$$

$$\overline{C} = (1 \quad 0 \quad \cdots \quad 0 \quad 0). \tag{26}$$

Note that the $\overline{A}$ and $\overline{B}$ directly depend on the coefficients of the polynomial denominator and numerator of $G(s)$, respectively. Also, it can be verified that $T_L = O^{-1}\overline{O}$, where $O \in \mathbb{R}^{n\times n}$ and $\overline{O} \in \mathbb{R}^{n\times n}$ are the observability matrices of the two equivalent systems.

Clearly, this transformation leads to the following equivalent system:

$$\begin{cases} \dot{\xi}(t) = \sigma \overline{C} \bar{z}(t), \\ \dot{\bar{z}}(t) = \overline{A}\bar{z}(t) - N'(\xi(t))\overline{B}\,\overline{C}\bar{z}(t), \end{cases} \tag{27}$$

while the expression of the invariant manifold (8) becomes

$$\mathcal{M}_\mathcal{I} = \left\{ \xi \in \mathbb{R}, \bar{z} \in \mathbb{R}^n : \right.$$
$$\left. H(\xi, \bar{z}) = \xi + \frac{\sigma}{a_0}\bar{z}_n + \frac{b_0}{a_0} N(\xi) = \mathcal{I} \right\}. \tag{28}$$

Consider now the nonlinear state transformation $T_N : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ from $(\xi, \bar{z}^\top)^\top \in \mathbb{R}^{n+1}$ to $(\xi, \zeta^\top)^\top \in \mathbb{R}^{n+1}$, $\eta \in \mathbb{R}$ and $\zeta \in \mathbb{R}^n$, given by

$$\begin{cases} \eta = \xi, \\ \zeta = \sigma\bar{z} - \overline{A} e_1 \xi + \overline{B} N(\xi), \end{cases} \tag{29}$$

where $e_1 = (1, 0, \ldots, 0)^\top \in \mathbb{R}^n$. It turns out that T_N can be readily reversed obtaining

$$\begin{cases} \xi = \eta, \\ \bar{z} = \sigma\zeta + \sigma\overline{A} e_1 \eta - \sigma\overline{B} N(\eta). \end{cases} \tag{30}$$

This implies that, under the hypothesis that $N(\xi)$ is of class $\mathcal{C}^1$ on the real axis, T_N is a diffeomorphism on $\mathbb{R}^{n+1}$. Hence, by applying the transformation T_N, we get a new system which is dynamically equivalent to (27). Tedious though straightforward computations show that this equivalent system obeys

$$\begin{cases} \begin{pmatrix} \dot{\eta}(t) \\ \dot{\zeta}_1(t) \\ \dot{\zeta}_2(t) \\ \vdots \\ \dot{\zeta}_{n-2}(t) \\ \dot{\zeta}_{n-1}(t) \end{pmatrix} = \overline{A} \begin{pmatrix} \eta(t) \\ \zeta_1(t) \\ \zeta_2(t) \\ \vdots \\ \zeta_{n-2}(t) \\ \zeta_{n-1}(t) \end{pmatrix} - \overline{B}N(\eta(t)) + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ \zeta_n(t) \end{pmatrix} \\ \dot{\zeta}_n(t) = 0. \end{cases} \tag{31}$$

The last equation in (31) states that ζ_n is constant over time, since it can be shown that the following relation

$$\zeta_n = a_0\mathcal{I} \tag{32}$$

holds. Indeed, noticing that $\eta = \xi$, the last equality in (29) can be written as

$$\zeta_n = \sigma\bar{z}_n + a_0\xi + b_0N(\xi).$$

Therefore, (32) readily follows from the equality in (28) and it provides an equivalent description of the invariant manifold $\mathcal{M}_{\mathcal{I}}$. As a consequence, the remaining n equations of (31) are the sought equations governing the dynamics on the invariant manifolds, as stated next.

Proposition 3. *Let Assumption 1 hold and assume that $N : \mathbb{R}^n \to \mathbb{R}^n$ is of class $\mathcal{C}^1$ on the real axis. Then, the dynamics on the invariant manifold $\mathcal{M}_{\mathcal{I}}$ obeys*

$$\dot{x}(t) = \overline{A}x(t) - \overline{B}N(\overline{C}x(t)) + e_n a_0\mathcal{I}, \tag{33}$$

where $x \in \mathbb{R}^n$ and $e_n = (0, \ldots, 0, 1)^\top \in \mathbb{R}^n$.

Proof. Since the assumptions ensure that system (31) is dynamically equivalent to (5) and the last equation of (31) exactly defines the manifold $\mathcal{M}_{\mathcal{I}}$, the proof follows by observing that for $x = (\eta, \zeta_1, \ldots, \zeta_{n-1})^\top$ the equations in (33) boil down to the first n equation of (31) . ■

Remark 3.1. Once a solution $x(t) = (\eta(t), \zeta_1(t), \ldots, \zeta_{n-1}(t))^\top$ of (33) is available, the corresponding solution $z(t)$ of (5) can be obtained by first computing $\bar{z}$ via the second equation in (30) with $\zeta_n = a_0\mathcal{I}$ and then using the relation $z = T_L\bar{z}$. Also, we observe that the manifold index $\mathcal{I}$ acts as a constant external input on the dynamics of (33) and, in particular, its variation influences the equilibrium points of the system.

We can now derive the main result of the paper by applying the results of Sec. 2.2 to system (33). First, we observe that the Jacobian $\overline{J}(x)$ of (33) reads

$$\overline{J}(x) = \overline{A} - N'(\overline{C}x)\overline{B}\overline{C}. \tag{34}$$

Note that it does not depend explicitly on the manifold index $\mathcal{I}$. Moreover, if the derivative of N is bounded on the real axis, i.e.

$$N'(\xi) \in [k_1, k_2], \quad \forall\, \xi \in \mathbb{R}, \tag{35}$$

then on any invariant manifold $\mathcal{M}_{\mathcal{I}}$, the Jacobian $\overline{J}(x)$ satisfies the following condition:

$$\overline{J}(x) \in \mathrm{conv}(\overline{A} - k_1\overline{B}\overline{C}, \overline{A} - k_2\overline{B}\overline{C}), \quad \forall\, x \in \mathbb{R}^n. \tag{36}$$

The next result holds true.

Theorem 1. *Consider system (5) which describes the dynamics of the memristor circuit of Fig. 1. Let $\mathcal{E}$ denote the equilibrium points of (5) and assume that the memristor characteristic $N : \mathbb{R}^n \to \mathbb{R}^n$ is of class $\mathcal{C}^1$ on the real axis and it satisfies the slope-bounded condition (35) for some given constants $k_i \in \mathbb{R}$, $i = 1, 2$. If there exists a symmetric positive definite matrix $P \in \mathbb{R}^{\binom{n}{2}\times\binom{n}{2}}$ such that*

$$\begin{cases} (A - k_1BC)^{[2]^\top}P + P(A - k_1BC)^{[2]} < 0, \\ (A - k_2BC)^{[2]^\top}P + P(A - k_2BC)^{[2]} < 0, \end{cases} \tag{37}$$

then the memristor circuit of Fig. 1 is nonoscillatory. Moreover, every solution with initial conditions in a forward invariant set $\mathcal{D}$ of (5) converges toward the set $\mathcal{E} \cap \mathcal{D}$.

Proof. The assumptions on N ensure that system (5) is dynamically equivalent to system (31) and that the dynamics on each manifold $\mathcal{M}_{\mathcal{I}}$ is completely described by (33). To apply the results of Sec. 2.2, we observe that, according to the expression of the Jacobian $\overline{J}(x)$ in (34), the variational equations (21) and (22) boil down to

$$\dot{\theta}(t) = (\overline{A} - N'(\overline{C}x(t))\overline{B}\overline{C})\theta(t)$$

and

$$\dot{\psi}(t) = (\overline{A} - N'(\overline{C}x(t))\overline{B}\overline{C})^{[2]}\psi(t),$$

respectively. Since the memristor characteristic N satisfies (35) and thus (36) holds on each manifold $\mathcal{M}_{\mathcal{I}}$, Proposition 2 and (36) ensure that the memristor circuit is nonoscillatory if there exists a symmetric positive definite matrix $\overline{P}$ such that

$$(\overline{A} - k_i\overline{B}\overline{C})^{[2]^\top}\overline{P} + \overline{P}(\overline{A} - k_i\overline{B}\overline{C})^{[2]} < 0, \quad i = 1, 2. \tag{38}$$

By taking into account that $\overline{A} = T_L^{-1}AT_L$, $\overline{B} = T_L^{-1}B$, $\overline{C} = CT_L$ and exploiting Property 2, (38) can be rewritten as

$$(T_L^{(2)^{-1}}(A - k_iBC)^{[2]}T_L^{(2)})^\top\overline{P} + \overline{P}T_L^{(2)^{-1}}(A - k_iBC)^{[2]}T_L^{(2)} < 0, \quad i = 1, 2. \tag{39}$$

To complete the proof, we show that conditions (38) are solved for $\overline{P} = T_L^{(2)^\top}PT_L^{(2)}$ with P such that (37) holds. Indeed, with this choice, (39) boils down to

$$(T_L^{(2)})^\top((A - k_iBC)^{[2]^\top}P + P(A - k_iBC)^{[2]})T_L^{(2)} < 0, \quad i = 1, 2,$$

which is clearly satisfied if and only if (37) holds.

In the case that $\mathcal{D}$ is any forward invariant contained in $\mathcal{M}_{\mathcal{I}}$, we observe that $\overline{J}(x)$ in (34) satisfies condition (36) for all $x \in \mathcal{D}$. Hence, from Proposition 2, it follows that (37) guarantees that every solution with initial conditions in $\mathcal{D}$ converges toward the set of equilibrium points there contained. ■

Remark 3.2. Condition (37) does not depend on $\mathcal{I}$, thus implying that oscillatory and more complex behaviors are ruled out on all the invariant manifolds. Moreover, according to Remark 2.1, if the strict inequalities in condition (37) are replaced by the nonstrict ones, then we can conclude that no attractors with positive Lyapunov exponents are displayed on the invariant manifolds.

Remark 3.3. We observe that (37) holds only if $(A - k_1BC)^{[2]}$ and $(A - k_2BC)^{[2]}$ are Hurwitz matrices, i.e. all their eigenvalues have a negative real part. Indeed, it can be readily verified that this must hold for any matrix $(A - kBC)^{[2]}$ with $k \in [k_1, k_2]$. Hence, Property 1 implies that all the matrices $(A - kBC)$, $k \in [k_1, k_2]$, should have no more than one eigenvalue with a positive real part. Moreover, this positive real part should be smaller than the magnitude of the real part of all the remaining $n-1$ eigenvalues.

Remark 3.4. Less conservative conditions (37) can be obtained by looking at an HPLF of degree $2m$, $m > 1$, instead of a quadratic one as in Theorem 1. Indeed, according to (25) and (37), we get the following condition:

$$(A - k_iBC)_{\{m\}}^{[2]^\top}P_m + P_m(A - k_iBC)_{\{m\}}^{[2]} + L(\gamma) < 0, \quad i = 1, 2, \tag{40}$$

where $(A - k_iBC)_{\{m\}} \in \mathbb{R}^{d\times d}$ is the extended matrix of $(A - k_iBC)$, $P_m \in \mathbb{R}^{d\times d}$ is symmetric positive definite and $\gamma \in \mathbb{R}^{d_L}$, with $d = \binom{\binom{n}{2}+m-1}{m}$ and $d_L = \frac{1}{2}d(d+1) - \binom{\binom{n}{2}+2m-1}{2m}$.

Remark 3.5. It can be verified that $\overline{J}(x)$ in (34) is equivalent to the Jacobian of the Lur'e system of Fig. 4, where the linear time-invariant subsystem in the forward path is described by the rational transfer function $G(s) = C(sI_n - A)^{-1}B$, i.e. the impedance (or the admittance) of the two-terminal element **L**, y and w are the voltage and current (or the current and voltage) of the nonlinear resistor in the feedback path whose voltage–current (or current–voltage) characteristic is equal to the flux–charge (or charge–flux) characteristic $N(\cdot)$ of the memristor. This equivalence suggests that to investigate the nonoscillatory nature of the memristor circuit of Fig. 1, we can replace the memristor with

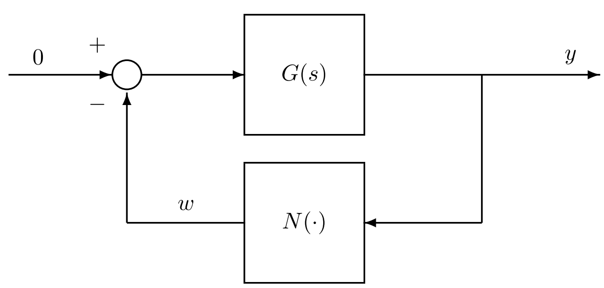

Fig. 4. Lur'e system: $G(s) = C(sI_n - A)^{-1}B$ is the impedance (or the admittance) of the two-terminal element **L**; y and w are the voltage and current (or the current and voltage) of the nonlinear resistor whose characteristic is equal to the memristor characteristic $N(\cdot)$.