Paolo Aglianò

Splittings in Subreducts of Hoops

Abstract. In this paper we extend to various classes of subreducts of hoops some results about splitting algebras. In particular we prove that every finite chain in the purely implicational fragment of basic hoops is splitting and that every finite chain in the $\{\wedge, \rightarrow\}$ fragment of hoops is splitting. We also produce explicitly the splitting equations in most cases.

1. Introduction

In the ASUBL Conference in Cagliari (June 2018) I gave a talk about splitting algebras in varieties of (divisible) residuated (semi)lattices; the material from which that talk originated has been published in [3–5]. However while in Cagliari I was asked how much of that theory could be naturally transferred to subreducts of those structures; it was a reasonable question, since the various possible subreducts share a great chunk of their theories with the parent algebras. Exactly for the same reason it was a deceptively hard question and my imprudent answer was that it should not be very hard to transfer most of the results.

It turns out that indeed we can transfer some of the results but it is not nearly as straightforward as we hoped; for instance the lack of expressivity of a language containing only the implication is a big obstacle. We chose to work in the framework of hoops, i.e. divisible and integral residuated semilattices since splitting hoops have been totally characterized in [3]. Here is a summary of the results contained in the current paper:

- every finite chain in the $\rightarrow$-fragment of basic hoops is splitting and for some it is possible to write explicitly the splitting equations;
- every finite chain in the $\{\rightarrow, \wedge\}$-fragment of hoops is splitting and it is possible to write explicitly the splitting equations.

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Finally we use a notation that is quite common in general algebra; for that, and for the elementary concepts in general algebra and lattice theory that we will be using, our textbook references are [13] and [23].

2. Residuated Semilattices and BCK-Algebras

A **commutative residuated semilattice** (short for *residuated semilattice ordered commutative monoid*) is an algebra $\mathbf{A} = \langle A, \wedge, \cdot, \rightarrow, 1\rangle$ where

- $\langle A, \wedge\rangle$ is a semilattice;
- $\langle A, \cdot, 1\rangle$ is a commutative monoid;
- $(\cdot, \rightarrow)$ form a residuated pair w.r.t. the semilattice ordering.

Residuated semilattices form a variety; for an axiomatization the reader can consult [1] where they have been studied under the (rather unfortunate) name of *BCI-monoids*. If 1 is the uppermost element in the ordering then we sat that the semilattice is **integral**; we will denote by CIRS the variety of integral and commutative residuated semilattices.

Commutative residuated semilattices share a good chunk of the theory with commutative residuated lattices; they are congruence permutable with Mal'cev term

$$m(x, y, z) = ((x \rightarrow y) \rightarrow z) \wedge ((z \rightarrow y) \rightarrow x).$$

Moreover, since they have a semilattice term, they are also congruence distributive; the theory of congruences (and filters) is identical to the one of residuated lattices. As a matter of fact it can be easily shown that the congruences of a residuated lattice are exactly the congruences of its semilattice reduct. Even more CIRS is exactly the class of $\vee$-less subreducts of the variety of commutative residuated lattices; this is a consequence of the results in Section 8 of [24], for instance Theorem 8.10.

In this paper we will deal with varieties of subreducts of subvarieties of CIRS; let's first examine the $\rightarrow$-subreducts. A BCK-algebra is an algebra $\mathbf{A} = \langle A, \rightarrow, 1\rangle$ satisfying the equations

$$(x \rightarrow y) \rightarrow ((z \rightarrow x) \rightarrow (z \rightarrow y)) \approx 1$$
$$x \rightarrow x \approx 1$$
$$x \rightarrow 1 \approx 1$$
$$1 \rightarrow x \approx x$$
$$x \rightarrow (y \rightarrow z) \approx y \rightarrow (x \rightarrow z)$$

and the quasi-identity

$$x \to y \approx y \to x \approx 1 \quad \Longrightarrow \quad x \approx y. \tag{M}$$

BCK-algebras were introduced in [19] and form a quasivariety that is not a variety [29]; the relation $x \le y$ if and only if $x \to y = 1$ is a partial ordering on any BCK-algebra. BCK-algebras coincide with the implicative subreducts of commutative and integral residuated semilattices; this has been known for a long time and it is again a consequence of the quoted result in [24].

We will mostly deal with totally ordered BCK-algebras and totally ordered CIRSs; a powerful tool for understanding them is the **ordinal sum**. This concept has been defined for CIRSs in [3] and the definition for BCK-algebras is simply its restriction. In details, let $\langle I, \le \rangle$ be a totally ordered set and let $(\mathbf{S}_i)_{i \in I}$ be a family of BCK-algebras; only in this definition we denote by 1_i the element of $\mathbf{S}_i$ that interprets the constant symbol 1 in the type. One can define the **ordinal sum** $\bigoplus_{i \in I} \mathbf{S}_i$ in the following way. The universe of $\bigoplus_{i \in I} \mathbf{S}_i$ is $\bigcup_{i \in I}(S_i \setminus \{1_i\}) \cup \{1\}$ and the operation is defined in the following way:

$$x \to y = \begin{cases} x \to^{\mathbf{S}_i} y & \text{if } x, y \in S_i \\ y & \text{if } x \in S_i \textit{ and } y \in S_j \textit{ with } i > j \\ 1 & \text{if } x \in S_i \setminus \{1\} \textit{ and } y \in S_j \textit{ with } i < j \end{cases}$$

It is easily seen that $\bigoplus_{i \in I} \mathbf{S}_i$ is always a BCK-algebra, where the ordering is obtained by stacking the BCK-algebras one over the other; the algebras $\mathbf{S}_i$, $i \in I$ are called the **components** of $\mathbf{S}$ and the set $\{\mathbf{S}_i : i \in I\}$ is a **decomposition** of $\mathbf{S}$. Let's introduce some notation; if $\boldsymbol{O}$ is a class operator such that for any class K, $\boldsymbol{O}(\mathsf{K}) \subseteq \boldsymbol{V}(\mathsf{K})$ and $\mathbf{A}_1, \dots, \mathbf{A}_n$ are BCK-algebras, then

$$\bigoplus_{i=1}^{n} \boldsymbol{O}(\mathbf{A}_i) = \{\bigoplus_{i=1}^{n} \mathbf{B}_i : \mathbf{B}_i \in \boldsymbol{O}(\mathbf{A}_i)\}.$$

Likewise if $\mathbf{A}_1, \dots, \mathbf{A}_n, \mathbf{B}$ are BCK-algebras we define

$$\mathbf{A}_1 \oplus \dots \oplus \mathbf{A}_n \oplus \boldsymbol{O}(\mathbf{B}) := \{\mathbf{A}_1 \oplus \dots \oplus \mathbf{A}_n \oplus \mathbf{C} : \mathbf{C} \in \boldsymbol{O}(\mathbf{B})\}.$$

The following lemma describes the behavior of ordinal sums w.r.t. to the usual class operators. It has been proved in [3] in a very general setting; however the proofs in [7] (Propositions 3.1, 3.2 and 3.3) go through with no change in our case.

Lemma 2.1. *Let $(\mathbf{S}_i)_{i \in I}$ be a family of* CIRS*s or* BCK*-algebras; then*

1. $\boldsymbol{S}(\bigoplus_{i\in I}\mathbf{S}_i) = \bigoplus_{i\in I}\boldsymbol{S}(\mathbf{S}_i)$;
2. $\boldsymbol{H}(\bigoplus_{i=1}^n \mathbf{S}_i) = \boldsymbol{H}(\mathbf{S}_1) \cup \mathbf{S}_1 \oplus \boldsymbol{H}(\mathbf{S}_2) \cup \cdots \cup \mathbf{S}_1 \oplus \cdots \oplus \mathbf{S}_{n-1} \oplus \boldsymbol{H}(\mathbf{S}_n)$;
3. *if J is any set and $\mathbf{S}_1^j, \ldots, \mathbf{S}_n^j$ are* CIRS*s (*BCK*-algebras) for $j \in J$, then*

$$\boldsymbol{P}_U(\{\bigoplus_{i=1}^n \mathbf{S}_i^j : j \in J\}) \subseteq \boldsymbol{P}_U(\{S_1^j : j \in J\}) \oplus \cdots \oplus \boldsymbol{P}_U(\{S_n^j : j \in J\}).$$

A BCK-algebra is **sum irreducible** if it cannot be expressed nontrivially as an ordinal sum; in other words $\mathbf{S}$ is sum irreducible if and only if it is nontrivial and whenever $\mathbf{S} \cong \bigoplus_{i\in I}\mathbf{S}_i$ then $\mathbf{S} \cong \mathbf{S}_i$ for some $i \in I$ (and all the other components are trivial). The proof of the following theorem can be easily extracted from the analogous statement for residuated semilattices (see [4]).

THEOREM 2.2. *Any* BCK*-algebra is the ordinal sum of sum irreducible* BCK*-algebras.*

The congruence structure of BCK-algebras is well-known [9] and it is similar to the one for residuated semilattices or lattices (see [1] or [12]); in particular congruences of a BCK-algebra $\mathbf{A}$ are in 1-1 correspondence with certain subsets of A, called the **(implicative) filters**. A subset F is a filter if $a, a \to b \in F$ implies $b \in F$; filters form an algebraic lattice, isomorphic with the lattice of **relative congruences** of $\mathbf{A}$, i.e. those $\theta \in \mathrm{Con}(\mathbf{A})$ such that $\mathbf{A}/\theta$ is a BCK-algebra. The filter lattice of a BCK-algebra is always distributive [26], therefore every variety of BCK-algebras is congruence distributive.

Nevertheless (see for instance [11]):

LEMMA 2.3. *Any relatively subdirectly irreducible (relatively simple)* BCK*-algebra is subdirectly irreducible (simple); hence a* BCK*-algebra is a subdirect product of subdirectly irreducible* BCK*-algebras.*

It follows that a BCK-algebra is simple if and only if it has no nontrivial proper filters and it is subdirectly irreducible if and only if it has a unique minimal filter. For $n \geq 1$ let us define some derived operations:

$$x \to^1 y := x \to y \qquad x \to^{n+1} y := x \to (x \to^n y).$$

The following lemma is well-known and easy to prove.

LEMMA 2.4. *If $\mathbf{A}$ is a* BCK*-algebra, $X \subseteq A$ and $F_{\mathbf{A}}(X)$ is the filter generated by X in $\mathbf{A}$, then*

$$F_{\mathbf{A}}(X) = \{b :\ \exists n \in \mathbb{N} and a_1, \ldots, a_n \in X such that a_1 \to (a_2 \to \ldots (a_n \to b) \ldots) = 1\}.$$

So $F_{\mathbf{A}}(a) = \{b : \text{there exists an } n \in \mathbb{N} \text{ with } a \to^n b = 1\}$.

Corollary 2.5. *Let* $\mathbf{A}$ *be any* BCK-*algebra; then* $\mathbf{A}$ *is simple if and only if for any* $a, b \in A$ *there exists an* $n \in \mathbb{N}$ *with* $a \to^n b = 1$. *Moreover if* $\mathbf{A}$ *is simple and* $a \to b = b$, *then either* $a = 1$ *or* $b = 1$.

If $\mathbf{A}$ is a BCK-algebra, a **cut** of $\mathbf{A}$ is a pair (F, S) of subsets of A with the following properties:

1. $F \cap S = \{1\}$;
2. F is the universe of a subalgebra of $\mathbf{A}$;
3. for all $a \in F \setminus \{1\}$, $b \in S$, $a \leq b$;
4. for all $a \in F$, $b \in S$, $b \to a = a$.

The following lemma is a consequence of the results in [3]; however a direct proof is easy enough.

Lemma 2.6. *Let* $\mathbf{A}$ *be a totally ordered* BCK-*algebra; then* $\mathbf{A}$ *is sum irreducible if and only if it has no nontrivial cut.*

Proof. Assume $\mathbf{A}$ is not sum irreducible and let $\bigoplus_{i \in I} \mathbf{A}_i$ with $|I| > 1$ a decomposition of $\mathbf{A}$ into its sum irreducible components. For $i \in I$ let $\mathbf{A}^i = \bigoplus_{j > i} \mathbf{A}_j$. We claim that A^i is a filter of $\mathbf{A}$ for all $i \in I$ and $(A \setminus A^i, A^i)$ is a nontrivial cut of $\mathbf{A}$. By definition of filter, it is enough to prove that for any $a \in A^i$ and $b \in A$, $a, a \to b \in A^i$ implies $b \in A^i$. Since $\mathbf{A}^i$ is a subalgebra (by Lemma 2.1), this must only be checked when $a \in A^i$ and $b \in A \setminus (A^i \cup \{1\})$; but in this case $a \to b = b$ by definition and the conclusion follows. That $(A \setminus A^i, A^i)$ is a nontrivial cut of $\mathbf{A}$ is obvious.

Conversely suppose we have a nontrivial cut (F, S) of $\mathbf{A}$; we claim that $\mathbf{A} = \mathbf{F} \oplus \mathbf{S}$. Since $F \setminus \{1\}$ is downward closed and $F \cap S = \{1\}$, S must be upward closed; hence if $a, b \in S$ $a \leq b \to a \in S$ and $b \to a \leq b \in S$ so S is the universe of a subalgebra. Everything else follows from the definition and so $\mathbf{A} = \mathbf{F} \oplus \mathbf{S}$ and it is not sum irreducible. ∎

Note that if $\mathbf{A}$ is totally ordered and finitely generated then, by Lemma 2.1(1), $\mathbf{A}$ can have only finitely many irreducible components, so it is a finite ordinal sum.

Corollary 2.7.

1. *Any simple* BCK-*chain is sum irreducible.*
2. *A finitely generated* BCK-*chain is subdirectly irreducible if and only if it is totally ordered and its last component is subdirectly irreducible.*

At the moment we do not have a complete description of BCK-chains that are sum irreducible; note that by Lemma 2.4 the filter structure of a

CIRS is identical to the one of its BCK-reduct. Likewise if $\mathbf{A}$ is a CIRS whose irreducible components are $\mathbf{A}_i$, $i \in I$, then the BCK-reduct $\mathbf{B}$ of $\mathbf{A}$ is $\bigoplus_{i\in I} \mathbf{B}_i$ where each $\mathbf{B}_i$ is the (sum irreducible) reduct of $\mathbf{A}_i$. Hence any BCK-reduct of a sum irreducible CIRS is a sum irreducible BCK-algebra.

3. Some Varieties of BCK-Algebras

While BCK-algebras do not form a variety, the join (in the lattice of subquasivarieties) of two varieties of BCK-algebras is a variety of BCK-algebras. This is a very deep result, proved in [9], implying that varieties of BCK-algebras form a distributive lattice.

Let's consider first the equation

$$(x \to y) \to z \leq ((y \to x) \to z) \to z. \tag{B}$$

A CIRS satisfying (B) is said to be **basic**; it is well-known that any CIRS is basic if and only if it is **representable**, i.e. it is a subdirect product of totally ordered residuated semilattices (see Section 2 of [4], where this issue is discussed in greater generality). Since Pałasiǹski in [25] showed that BCK-algebras satisfying (B) are exactly those BCK-algebras that are subdirect products of BCK-chains, we conclude that (B) defines the quasivariety RBCK of representable BCK-algebras, which consists of implicative subreducts of representable CIRSs.

Next consider the equation

$$(((x \to y) \to y) \to x) \to x \approx (((y \to x) \to x) \to y) \to y \tag{J}$$

If V is a variety of CIRS*s* satisfying (J), then the class $\boldsymbol{S}^{\to}(\mathsf{V})$ of implicative subreducts of V is always a variety; this is again a consequence of the results in [9] (for a direct proof the reader may look at Proposition 1.2 in [6]). The variety of subreducts of the subvariety of CIRS defined by (J) is denoted by JBCK and we will deal with its subvarieties. A CIRS is **divisible** if its ordering is the *inverse divisibility* ordering i.e.

$$x \leq y \qquad \text{if and only if} \qquad \exists z\, x = yz.$$

Divisible CIRS are called **hoops** in the literature and their variety is denoted by HO [8]. It can be shown that hoops satisfy (J), hence implicational subreducts of hoops form a variety which will be denoted by HBCK. In [17] I. M. A. Ferreirim proved that the equation

$$(x \to y) \to (x \to z) \approx (y \to x) \to (y \to z). \tag{H}$$

implies (J) and axiomatizes the implicative subreducts of hoops modulo BCK. If a hoop satisfies also (B), then it is a **basic hoop** [6]; implicative subreducts of basic hoops are called **basic BCK-algebras** [6]. We denote their variety by BBCK and we observe that it is clearly axiomatized by (B) and (H) modulo BCK. Since BBCK is representable it follows at once that any subdirectly irreducible basic BCK-algebra is totally ordered.

Next we consider *Tanaka's equation*

$$(x \to y) \to y \approx (y \to x) \to x. \tag{T}$$

A hoop satisfying (T) is called a **Wajsberg hoop** [6] and a BCK-algebra satisfying (T) is said to be **commutative**; since it is easy to check that (T) implies (J) in BCK, commutative BCK-algebras form a variety denoted by TBCK. The variety of implicative subreducts of Wajsberg hoops is the variety LBCK of Łukasiewicz BCK-algebras; it is axiomatized relative to BCK by (T) and (B). This is a consequence of the nontrivial result that, in presence of (T), (H) and (B) are equivalent in BCK-algebras [14].

Finally a **Gödel hoop** is an idempotent Wajsberg hoop; the variety of subreducts of Gödel hoops is the variety GBCK of **Gödel** BCK-algebras; it is axiomatized relative to BCK by (B) and

$$x \to (x \to y) \approx x \to y. \tag{G}$$

As observed in [6], any Gödel BCK-algebra is a Hilbert algebra (i.e. an implicative subreduct of a Heyting algebra); therefore GBCK coincides with the variety of representable Hilbert algebras.

There is one more variety of BCK-algebras that is worth mentioning, i.e. the variety RJBCK of representable BCK-algebras satisfying (J); this is a variety containing BBCK but distinct from it (see Example 3.1) that has received no attention but that might be interesting to investigate.

Example 3.1. Let $\mathbf{A} = \langle \{a, b, c, 1\}, \to, 1\rangle$ where the operation table of $\to$ is

$\to$	a	b	c	1
a	1	1	1	1
b	c	1	1	1
c	c	c	1	1
1	a	b	c	1

Again one may check that **A** is a BCK-chain satisfying (J) but

$$(c \to b) \to (c \to a) = c \to c = 1 \neq c = 1 \to c = (b \to c) \to (b \to a),$$

so **A** does not satisfy (H). Observe also that, since the only two element BCK-algebra is the →-reduct of the two element Boolean algebra and that the two (necessarily totally ordered) three-element BCK-algebras are the →-reducts of the three-element Heyting algebra or the three-element Gödel algebra respectively, this example is of minimal size.

As a final observation we stress that the operator $\boldsymbol{S}^{\to}$ is not injective even on varieties of hoops; a **cancellative** hoop is a hoop in which the monoidal operation is cancellative in the usual sense. It is well-known that cancellative hoops form a variety denoted by C that is properly contained in the variety of Wajsberg hoops. However, as proven in [6],

Lemma 3.2. *Any Łukasiewicz* BCK*-algebra* **A** *is isomorphic with a subreduct of a cancellative hoop that can be taken to be totally ordered if* **A** *is totally ordered; hence* $\boldsymbol{S}^{\to}(\mathsf{C}) = \mathsf{LBCK}$.

This fact will become very useful in the sequel. It is clear that any nontrivial BCK-algebra contains the 2-element BCK-algebra, which therefore generates the only atom in the lattice of subvarieties of BCK-algebras. We will denote this algebra by $\mathbf{Ł}_1$ (and the reason will be clear in the next section); the algebras is $\boldsymbol{V}(\mathbf{Ł}_1$ are known as **Tarski algebras** and we denote the variety by TA. TA is the class of →-subreducts (in the obvious sense) of Boolean algebras and it is axiomatized relative to GBCK by

$$(x \to y) \to x \approx x.$$

Finally we mention the variety HA of **Hilbert algebras** [16], i.e. the class of →-subreducts of Heyting algebras; it is a non prelinear subvariety of HBCK. Figure 1 shows the inclusion relations between the varieties of BCK-algebras we have considered.

4. Sum Irreducible Chains in BBCK

In this section we develop the theory of sum irreducible BBCK-chains, in analogy with the same theory for basic hoops [6]; for any variety V we denote by $\Lambda(\mathsf{V})$ the lattice of subvarieties of V.

Lemma 4.1. *For a totally ordered algebra* **A** *in* BBCK *the following are equivalent:*

1. **A** *is sum irreducible;*

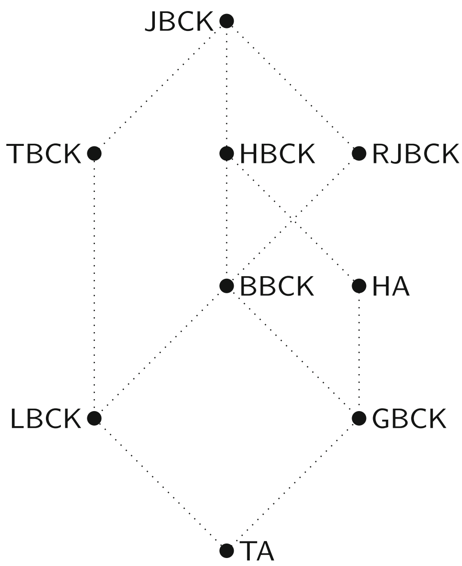

Figure 1. The inclusion relations between varieties of BCK-algebras

2. for all $a, b \in A$ $a \to b = b$ *implies* $a = 1$ *or* $b = 1$*;*

3. $\mathbf{A}$ *is an* LBCK*-algebra.*

Proof. The proof is quite similar to the ones of Lemmas 3.5 and 3.6 in [7]. Assume that (2) fails; then there is a $b \in A \setminus \{1\}$ such that the set $F = \{a : b \to a = a\} \neq \{1\}$ so if we set $S = (A \setminus F) \cup \{1\}$, then $F \cap S = \{1\}$. Note that for any $a \in F \setminus \{1\}$, $b \not\leq a$ and so, since $\mathbf{A}$ is a chain $a < b$.

If $a, c \in F$ then either $a = 1$ and $1 \to c = c \in F$, or else $a \neq 1$ and

$$\begin{aligned} b \to (a \to c) &= a \to (b \to c) = (b \to a) \to (b \to c) \\ &= (a \to b) \to (a \to c) = a \to c \end{aligned}$$

so F is the universe of a subalgebra of $\mathbf{A}$

Next take $c \leq a \in F \setminus \{1\}$; then $a = b \to a \geq b \to c$ so $(b \to c) \to a = 1$. Thus

$$\begin{aligned} (b \to c) \to c = 1 &\to ((b \to c) \to c) \\ &= ((b \to c) \to a) \to ((b \to c) \to c) \\ &= (a \to (b \to c)) \to (a \to c) \qquad \text{by (H)} \\ &= ((b \to a) \to (b \to c)) \to (a \to c) \\ &= ((a \to b) \to (a \to c)) \to (a \to c) \qquad \text{by (H)} \end{aligned}$$

$$= (1 \to (a \to c)) \to (a \to c) \qquad \text{since } a < b$$
$$= (a \to c) \to (a \to c) = 1.$$

So $b \to c = c$ and $c \in F \setminus \{1\}$ that is thus downward closed.

Finally if $d \in S$ and $a \in F$; if $d = 1$ then of course $d \to a = a$. Otherwise from $d \leq (d \to a) \to a$ (that clearly holds in any BCK-algebra) we get $d \notin S$, unless $(d \to a) \to a = 1$. Since the former does not hold, the latter must hold, i.e. $d \to a = a$. We have thus proved that (F, S) is a nontrivial cut and so $\mathbf{A}$ is not sum irreducible by Lemma 2.6.

The implication from (2) to (3) has been proved in [17]; it is a straightforward computation where equation (H) is used critically.

Finally assume the (1) fails i.e. that $\mathbf{A} = \mathbf{F} \oplus \mathbf{S}$ nontrivially. If $a \in F$ and $b \in S \setminus \{1\}$, then

$$(a \to b) \to b = 1 \to b = b \neq 1 = a \to a = (b \to a) \to a,$$

hence $\mathbf{A}$ does not satisfy (T) and it not a Łukasiewicz BCK-algebra. ∎

So any chain in BBCK is the ordinal sum of Łukasiewicz BCK-chains and this decomposition is also the maximal one.

Since Łukasiewicz BCK-chains are the building blocks of chains in BBCK it would be useful to know them; luckily Łukasiewicz BCK-algebras have been studied at length in [20] and a description of the relevant chains is available.

Lemma 4.2. *[20]*

1. *If a finite chain in* LBCK *has $n + 1$ elements, then it is isomorphic with $\mathbf{Ł}_n$; the universe of $\mathbf{Ł}_n$ is the set $\{0, 1/n, 2/n, \dots, (n-1)/n, 1\}$ and $x \to y = \min(1, 1 - x + y)$; clearly $\mathbf{Ł}_n$ is the $\to$-reduct of the $n + 1$ totally ordered Wajsberg hoop;*
2. *$\mathbf{Ł}_n$ is a subalgebra of $\mathbf{Ł}_m$ if and only if $n \leq m$;*
3. *the chain $\mathbf{Ł}_\omega$ is the reduct of the cancellative hoop $\mathbf{C}_\omega$ (see [7] for a definition);*
4. *every $\mathbf{Ł}_n$ is generated by two elements (e.g. the coatom and the bottom element);*
5. *$\mathbf{Ł}_\omega$ is not finitely generated and every finitely generated subalgebra of $\mathbf{Ł}_\omega$ is isomorphic to $\mathbf{Ł}_n$ for some n;*
6. *the only simple chains in* LBCK *are $\mathbf{Ł}_\omega$ and $\mathbf{Ł}_n$ for $n \in \mathbb{N}$;*
7. *if $\mathbf{L}$ is any infinite chain in* LBCK*, then $\mathbf{L}$ contains a subalgebra isomorphic with $\mathbf{Ł}_n$ for each n.*

Note that there are infinite chains that are finitely generated (take for instance the →-subreduct of Chang's algebra $\mathbf{Wa}_1^\infty$ [7]), so LBCK is not locally finite. In [20] a description of the lattice $\Lambda(\mathsf{LBCK})$ of subvarieties of LBCK is provided; we will produce the same description using a slightly different proof that will allow us to introduce some concepts that will be of use in the sequel. A class of algebras K has the **finite embeddability property** (FEP for short) if every finite partial subalgebra of an algebra in K can be embedded in a finite algebra in K; it is evident that every locally finite variety has the FEP and every variety with the FEP has the finite model property. We need a lemma that has been proved for basic hoops in [2]; needless to say the same proof goes true for basic BCK-algebras.

LEMMA 4.3. *Let* V *be a variety of basic* BCK*-algebras and let* $\mathsf{K} \subseteq \mathsf{V}$*.*

1. *If each totally ordered finitely generated member of* V *belongs to* $\boldsymbol{ISP}_U(\mathsf{K})$*, then* $\mathsf{V} = \boldsymbol{ISPP}_U(\mathsf{K})$*.*
2. *If in addition* K *has the FEP, then*

$$\mathsf{V} = \boldsymbol{ISPP}_U(\mathsf{K}_{fin}).$$

Since the class of totally ordered members of LBCK has the FEP ([6], Lemma 3.4) and LBCK is a representable variety of BCK-algebras, from Lemma 4.3 we get at once that LBCK is generated as a quasivariety by its finite algebras, i.e.

$$\mathsf{LBCK} = \boldsymbol{ISPP}_U(\text{Ł}_n : n \in \mathbb{N}).$$

By Lemma 4.2 this implies that $\mathsf{LBCK} = \boldsymbol{ISPP}_U(\mathbf{L})$ for any infinite chain in LBCK. We can also prove a useful corollary:

COROLLARY 4.4. *Let* $\mathbf{L}$ *be any infinite chain in* LBCK*; then* $\boldsymbol{ISP}_U(\mathbf{L}) = \boldsymbol{ISP}_U(\text{Ł}_\omega)$*.*

PROOF. Let $\mathbf{L}$ be any chain in LBCK; then by Lemma 3.2 it is a subreduct of a totally ordered cancellative hoop $\mathbf{C}$. By Theorem 6.4 in [7] $\mathbf{C} \in \boldsymbol{ISP}_U(\mathbf{C}_\omega)$ so $\mathbf{C} \leq \mathbf{C}_\omega^X/U$ for a suitable ultrafilter on X. Now it is immediate to verify that $\mathbf{L} \leq \text{Ł}_\omega^X/U$ so $\mathbf{L} \in \boldsymbol{ISP}_U(\text{Ł}_\omega)$.

Conversely, if $\mathbf{L}$ is infinite, $\text{Ł}_\omega \in \boldsymbol{ISPP}_U(\mathbf{L})$; since Ł_ω is simple, $\text{Ł}_\omega \in \boldsymbol{ISP}_U(\mathbf{L})$ as required. ■

In conclusion:

THEOREM 4.5. *The lattice* $\Lambda(\mathsf{LBCK})$ *is a chain of length* $\omega + 1$*; if* T *is the trivial subvariety, then*

$$\mathsf{T} \subsetneq \boldsymbol{V}(\text{Ł}_1) \subsetneq \cdots \subsetneq \boldsymbol{V}(\text{Ł}_n) \subsetneq \cdots \subsetneq \boldsymbol{V}(\mathbf{L}_\omega) = \mathsf{LBCK}.$$

Proof. Suppose that $\mathsf{V} \subseteq \mathsf{LBCK}$ is generated by a family T of chains; if T is infinite, then by Lemma 4.2 $\mathsf{V} = \boldsymbol{V}(T)$ contains all the finite chains, hence it is equal to LBCK. Hence a proper subvariety of LBCK must be generated by finitely many finite chains and hence, again by Lemma 4.2, by a single finite chain. Since any infinite chain generates LBCK the conclusion of the theorem holds. ∎

5. The Lattice of Subvarieties of BBCK

From now on when we write $\mathbf{A} = \bigoplus_{i \in I} \mathbf{A}_i$ for a chain in BBCK, we always assume that the $\mathbf{A}_i$'s are the sum irreducible components of $\mathbf{A}$.

Lemma 5.1. *Let $\mathbf{A}_1, \dots, \mathbf{A}_n$ be chains in* LBCK*; then*

$$\boldsymbol{ISP}_U(\bigoplus_{i=1}^n \mathbf{A}_i) = \boldsymbol{ISP}_U(\mathbf{A}_1) \oplus \cdots \oplus \boldsymbol{ISP}_U(\mathbf{A}_n).$$

Proof. The left-to-right inclusion follows from Lemma 2.1. For the other we induct on n and we observe that for $n = 1$ there is nothing to prove. So let $n > 1$ and let $\mathbf{B}_i \in \boldsymbol{ISP}_U(\mathbf{A}_i)$ for $i = 1, \dots, n$. By induction hypothesis

$$\mathbf{B}_1 \oplus \cdots \oplus \mathbf{B}_{n-1} \in \boldsymbol{ISP}_U(\mathbf{A}_1 \oplus \cdots \oplus \mathbf{A}_{n-1}).$$

Let $\mathbf{D} = \mathbf{B}_1 \oplus \cdots \oplus \mathbf{B}_{n-1}$, $\mathbf{C} = \mathbf{A}_1 \oplus \cdots \oplus \mathbf{A}_{n-1}$; then $\mathbf{D}$ embeds in some ultrapower $\mathbf{C}^X/U$, for a suitable ultrafilter U of X. It is readily seen that $\mathbf{D} \oplus \mathbf{B}_n$ embeds into $(\mathbf{D} \oplus \mathbf{B}_n)^X/U$. On the other hand $\mathbf{B}_n$ embeds in some ultrapower $\mathbf{A}_n^Y/V$ and again $\mathbf{C} \oplus \mathbf{B}_n$ embeds into $(\mathbf{C} \oplus \mathbf{A}_n)^Y/V$. Thus

$$\mathbf{B}_1 \oplus \cdots \oplus \mathbf{B}_n \in \boldsymbol{ISP}_U\boldsymbol{SP}_U(\mathbf{A}_1 \oplus \cdots \oplus \mathbf{A}_n) \subseteq \boldsymbol{ISP}_U(\bigoplus_{i=1}^n \mathbf{A}_i)$$

which proves the conclusion. ∎

An application of Lemma 2.1 gives at once:

Corollary 5.2. *Let $\mathbf{A}_1, \dots, \mathbf{A}_n$ be chains in* LBCK*; then*

$$\boldsymbol{HSP}_U(\bigoplus_{i=1}^n \mathbf{A}_i) = \boldsymbol{HSP}_U(\mathbf{A}_1) \cup \bigcup_{k=2}^n (\bigoplus_{i=1}^{k-1} \boldsymbol{ISP}_U(\mathbf{A}_i) \oplus \boldsymbol{HSP}_U(\mathbf{A}_k)).$$

Lemma 5.1 and Corollary 5.2 tell us that if $\mathbf{A} = \bigoplus_{i=1}^n \mathbf{A}_i$ then every chain in $\boldsymbol{HSP}_U(\mathbf{A})$ has at most n sum irreducible components. In particular if $\bigoplus_{i=1}^n \mathbf{A}_i = \mathbf{A} = \bigoplus_{j=1}^k \mathbf{B}_j$, then $n = k$ and if $\mathbf{A}$ is finite then $\mathbf{A}_i = \mathbf{B}_i$ (since in this case all components must be finite chains in LBCK). This allows us to define the **index** of a totally ordered chain $\bigoplus_{i\in I}\mathbf{A}_i$ in BBCK, to be n if $|I| = n$ and infinite otherwise. Moreover by applying Lemma 2.1 we get

Corollary 5.3. *Let* K *be a class of* LBCK*-chains whose index is at most* n*; then any* LBCK*-chain in* $\boldsymbol{V}(\mathsf{K})$ *has index at most* n*.*

We have already observed that $\boldsymbol{V}(\mathbf{Ł}_1)$ is the only atom in the lattice $\Lambda(\mathsf{BBCK})$ of subvarieties of BBCK. In [21] T. Kowalski showed that there are only two almost minimal varieties (i.e. covers of the atom) of BCK-algebras and they are generated by the LBCK-algebra $\mathbf{Ł}_2$ and the GBCK-algebra $\mathbf{Ł}_1 \oplus \mathbf{Ł}_1$; therefore they are also the only almost minimal varieties in $\Lambda(\mathsf{BBCK})$. Using the description of finite chains is easy enough to climb higher in $\Lambda(\mathsf{BBCK})$. For instance it is easily seen that the covers of the almost minimal varieties are exactly $\boldsymbol{V}(\mathbf{Ł}_3)$, $\boldsymbol{V}(\mathbf{Ł}_2, \mathbf{Ł}_1 \oplus \mathbf{Ł}_1)$ and $\boldsymbol{V}(\mathbf{Ł}_1 \oplus \mathbf{Ł}_1 \oplus \mathbf{Ł}_1)$.

Finite chains can also be used to describe the entire $\Lambda(\mathsf{BBCK})$; for any subvariety V of BBCK we define

$$\mathcal{F}_{\mathsf{V}} = \{\mathbf{C} \in \mathsf{V} : \mathbf{C} \text{ is a chain of finite index}\}$$

and

$$\Phi_{\mathsf{V}} = \{\boldsymbol{V}(\mathbf{A}) : \mathbf{A} \in \mathcal{F}_{\mathsf{V}}\};$$

Φ_{V} is clearly a poset under inclusion. A **join dense completion** of a poset $\mathbf{P}$ is a complete lattice $\mathbf{L}$ with an order embedding $\alpha : \mathbf{P} \longrightarrow \mathbf{L}$ such that $\alpha(\mathbf{P})$ is **join dense** in $\mathbf{L}$, i.e. for every $x \in L$, $x = \bigvee\{\alpha(p) : p \in P, \alpha(p) \leq x\}$.

Theorem 5.4. *For any variety* V *of* BBCK*-algebras,* $\Lambda(\mathsf{V})$ *is the join dense completion of* Φ_{V}*.*

Proof. Let $\mathbf{A}$ be a finitely generated totally ordered algebra in V; then, $\mathbf{A}$ has index $\leq n$ and so $\mathbf{A} \in \mathcal{F}_{\mathsf{V}}$. Since every subvariety W of V is generated by its totally ordered members and every algebra is in the variety generated by its finitely generated subalgebras, every subvariety W of V is the supremum of all varieties $\boldsymbol{V}(\mathbf{A})$, where $\mathbf{A} \in \mathcal{F}_{\mathsf{W}}$. We have thus shown that for every $\mathsf{W} \in \Lambda(\mathsf{V})$

$$\mathsf{W} = \bigvee\{\boldsymbol{V}(\mathbf{A}) : \mathbf{A} \in \mathcal{F}_{\mathsf{W}}\}.$$

This implies that $\Lambda(\mathsf{V})$ is the join dense completion of Φ_{V}. ■

So varieties generated by chains of finite index form the *backbone* of $\Lambda(\mathsf{BBCK})$. The next step in exploring $\Lambda(\mathsf{BBCK})$ is to find the strictly join irreducible varieties, i.e. those varieties that are strictly join irreducible elements of the lattice $\Lambda(\mathsf{BBCK})$. First, by Birkhoff's Theorem on subdirect decompositions, any such variety must be generated by a single BBCK-chain; we have the following lemma.

LEMMA 5.5. *If* $\mathbf{A}$ *is a* BBCK*-chain of infinite index, then* $\boldsymbol{V}(\mathbf{A})$ *is not strictly join irreducible.*

PROOF. Let $\mathsf{V} = \boldsymbol{V}(\mathbf{A})$ and consider $\mathcal{F}_{\mathsf{V}}$; for any $\mathbf{C} \in \mathcal{F}_{\mathsf{V}}$, by Corollary 5.2, the index of any chain in $\boldsymbol{V}(\mathbf{C})$ is at most the finite index of $\mathbf{C}$, so $\boldsymbol{V}(\mathbf{A}) \nsubseteq \boldsymbol{V}(\mathbf{C})$. On the other hand any equation failing in $\mathbf{A}$ must fail in a finitely generated subalgebra $\mathbf{C}$ of $\mathbf{A}$ and the index of $\mathbf{C}$ cannot exceed the cardinality of the set of generators. It follows that $\mathbf{C} \in \mathcal{F}_{\mathsf{V}}$ and therefore $\bigvee_{\mathbf{C}\in\mathcal{F}_{\mathsf{V}}} \boldsymbol{V}(\mathbf{C}) = \boldsymbol{V}(\mathbf{A})$; hence $\boldsymbol{V}(\mathbf{A})$ is not strictly join irreducible. ■

THEOREM 5.6. *A variety of* BBCK*-algebras is strictly join irreducible if and only if it is generated by a finite chain.*

PROOF. Let $\mathbf{A}$ be BBCK-chain of finite index that is not finite; then we may assume that at least one of the components is an infinite chain. Really by Lemma 2.1 and Corollary 4.4 we my assume w.l.o.g. that one of the components is equal to $\mathbf{L}_\omega$. So suppose that $\mathbf{A} = \bigoplus_{i=1}^n \mathbf{A}_i$ and let $\mathbf{A}_k = \mathbf{L}_\omega$; for any $m \in \mathbb{N}$ we define the chain $\mathbf{B}^m = \bigoplus_{i=1}^n \mathbf{B}_i^m$ where

$$\mathbf{B}_i^m = \begin{cases} \mathbf{A}_i, & \text{if} i \neq k; \\ \text{Ł}_m, & \text{if} i = k. \end{cases}$$

If an equation fails in $\mathbf{A}$, then it must fail in some finitely generated subalgebra of $\mathbf{A}$; and by Lemma 2.1 and Lemma 4.2(5) such subalgebra is a subalgebra of $\mathbf{B}^m$ for some m. It follows that $\bigvee_{m\in\mathbb{N}} \boldsymbol{V}(\mathbf{B}^m) = \boldsymbol{V}(\mathbf{A})$ and the latter is not strictly join irreducible.

Conversely assume that $\mathbf{A} = \bigoplus_{i=1}^n \text{Ł}_{k_i}$ is a finite BBCK-chain. Let $\{\mathbf{B}_s : s \leq l\}$ be the set of proper subalgebras of $\mathbf{A}$; we claim that $\bigvee_{s\leq l} \boldsymbol{V}(\mathbf{B}_s)$ is the unique lower cover of $\boldsymbol{V}(\mathbf{A})$. Now since all the $\mathbf{B}_s$ are finite and there are finitely many

$$\boldsymbol{HSP}_U(\{\mathbf{B}_s : s \leq l\}) = \boldsymbol{HS}(\{\mathbf{B}_s : s \leq l\})$$

and, since each one of them is proper, $\mathbf{A} \notin \boldsymbol{HS}(\{\mathbf{B}_s : s \leq l\})$. On the other hand $\boldsymbol{V}(\{\mathbf{B}_s : s \leq l\})$ is clearly maximal in the set of all proper subvarieties of $\boldsymbol{V}(\mathbf{A})$, so it is its lower cover. ■

6. Splitting Algebras in Subvarieties of BBCK

A powerful tool in investigating lattices of subvarieties is the description of splitting algebras. An algebra $\mathbf{A}$ is **splitting in** a variety V if there is a subvariety $\mathsf{W}_{\mathbf{A}} \subseteq \mathsf{V}$ (called the **conjugate variety** of $\mathbf{A}$) such that for every subvariety $\mathsf{U} \subseteq \mathsf{V}$, either $\mathbf{A} \in \mathsf{U}$ or $\mathsf{U} \subseteq \mathsf{W}_{\mathbf{A}}$. The main properties of splitting algebras can be found in [22] and [28] (but see also Section 2.5 of [3] for a detailed discussion). Summarizing:

- every splitting algebra is finitely generated and subdirectly irreducible;
- if $\mathbf{A}$ is splitting with conjugate variety $\mathsf{W}_{\mathbf{A}}$, then $\mathsf{W}_{\mathbf{A}}$ is axiomatized by a single equation, called the **splitting equation** of $\mathbf{A}$;
- if we are inside a congruence distributive variety and $\mathbf{A}$ is finite, then $\mathsf{W}_{\mathbf{A}}$ is totally determined by $\mathbf{A}$ up to isomorphism; in other words $\mathsf{W}_{\mathbf{A}}$ cannot be the conjugate variety of any algebra not isomorphic with $\mathbf{A}$.

Clearly if V has the finite model property, then each splitting algebra must be finite; *a fortiori* the same holds if V has the FEP.

We have already seen that $\Lambda(\mathsf{LBCK})$ is a chain and therefore any finite LBCK chain $\mathbf{Ł}_n$ is splitting with conjugate variety $\boldsymbol{V}(\mathbf{Ł}_{n-1})$; a splitting equation for $\mathbf{Ł}_n$ is therefore any equation not holding in $\mathbf{Ł}_n$ but holding in $\mathbf{Ł}_{n-1}$ (and hence in any $\mathbf{L}_k$ with $k < n$). By examining Theorem 3.13 in [20], it is easy to see that the splitting equation for $\mathbf{Ł}_n$ is $x \to^{n-1} y \approx 1$.

From now on if α is any ordinal and $\mathbf{L}$ is any LBCK-chain, by $\alpha\mathbf{L}$ we mean the ordinal sum of α copies of $\mathbf{L}$; it an easy consequence of our theory that the only finite chains in GBCK are $n\mathbf{Ł}_1$ and that every infinite chain generates GBCK (the proof is an easy exercise). Moreover $\boldsymbol{V}((n-1)\mathbf{Ł}_1) \subsetneq \boldsymbol{V}(n\mathbf{Ł}_1)$ (since $n\mathbf{Ł}_1 \notin \boldsymbol{HSP}_U((n-1)\mathbf{Ł}_1)$) and so each $n\mathbf{Ł}_1$ is splitting in GBCK with conjugate variety $\boldsymbol{V}((n-1)\mathbf{Ł}_1)$.

However in this case finding a splitting equation is harder and we will attack the problem from a more global perspective. The class of totally ordered algebras in HBCK has the FEP ([6], Proposition 3.5); therefore BBCK is generated as a quasivariety by its finite algebras, has the finite model property and therefore any splitting algebra in BBCK must be finite. A finite chain in BBCK is a finite ordinal sum of finite sum irreducible chains in BBCK; by Lemma 4.1 if $\mathbf{A}$ is such a chain, then there are $n_1, \ldots, n_k \in \mathbb{N}$ such that

$$\mathbf{A} = \mathbf{Ł}_{n_1} \oplus \cdots \oplus \mathbf{Ł}_{n_k}.$$

Let's define now a class of chains $\mathsf{K}_{\mathbf{A}}$ for any finite BBCK-chain $\mathbf{A} = \mathbf{Ł}_{n_1} \oplus \cdots \oplus \mathbf{Ł}_{n_k}$. A BBCK-chain $\bigoplus_{i \in I} \mathbf{B}_i$ is in $\mathsf{K}_{\mathbf{A}}$ if we cannot find any sequence $i_1 < i_2, < \cdots < i_k \in I$ such that $\mathbf{Ł}_{n_j} \in \boldsymbol{S}(\mathbf{B}_{i_j})$, for $j = 1, \ldots, k$.

Theorem 6.1. *Let* $\mathbf{A} = \bigoplus_{j=1}^{k} \mathbf{Ł}_{n_j}$ *be a finite* BBCK*-chain; then* $\mathbf{A}$ *is splitting in* BBCK *with conjugate variety* $\boldsymbol{V}(\mathsf{K}_{\mathbf{A}})$ *if and only if* $\mathbf{A} \notin \boldsymbol{V}(\mathsf{K}_{\mathbf{A}})$.

Proof. The "only if"part is obvious. Let U be a subvariety of BBCK and suppose that $\mathbf{A} \in \mathsf{U}$; then, clearly $\mathsf{U} \not\subseteq \boldsymbol{V}(\mathsf{K}_{\mathbf{A}})$ since $\mathbf{A} \notin \boldsymbol{V}(\mathsf{K}_{\mathbf{A}})$. Conversely suppose that $\mathbf{A} \notin \mathsf{U}$ and let $\mathbf{B} = \bigoplus_{i\in I} \mathbf{B}_i$ be any chain in U; clearly it is not possible to find $i_1 < \cdots < i_k \in I$ such that $\mathbf{Ł}_{n_j} \in \boldsymbol{S}(\mathbf{B}_{i_j})$ otherwise $\mathbf{A} \in \mathsf{U}$, a contradiction. Therefore any chain in U belongs to $\mathsf{K}_{\mathbf{A}}$ and so $\mathsf{U} \subseteq \boldsymbol{V}(\mathsf{K}_{\mathbf{A}})$. This proves the thesis. ∎

Let $\mathsf{BBCK}^{(n)}$ be the variety generated by all the BBCK-chains with index at most n; by Corollary 5.2 all the chains in $\mathsf{BBCK}^{(n)}$ have index at most n so

$$\mathsf{BBCK}^{(n)} = \boldsymbol{V}(\{\mathbf{A}_1 \oplus \cdots \oplus \mathbf{A}_n : \mathbf{A}_1, \ldots, \mathbf{A}_n \text{ are chains in} \mathsf{LBCK}\}).$$

Let now $\mathbf{A} = n\mathbf{Ł}_1$; since $\mathbf{Ł}_1$ is embeddable in any BBCK-chain it is clear that no chain of index at least n can be in $\mathsf{K}_{\mathbf{A}}$. On the other hand all chains of index at most $n-1$ are in $\mathsf{K}_{\mathbf{A}}$ and so $\boldsymbol{V}(\mathsf{K}_{\mathbf{A}}) = \mathsf{BBCK}^{(n-1)}$. Let $\mathsf{BBCK}^{(0)}$ denote the trivial variety; since $n\mathbf{Ł}_1 \notin \mathsf{BBCK}^{(n-1)}$ an application of Theorem 6.1 yields:

Theorem 6.2. *For any $n \geq 1$, $n\mathbf{Ł}_1$ is splitting in* BBCK *with conjugate variety* $\mathsf{BBCK}^{(n-1)}$.

It follows from our argument above that if $\mathbf{A}$ is splitting, then a splitting equation for $\mathbf{A}$ is simply any equation holding in $\mathsf{K}_{\mathbf{A}}$ but failing in $\mathbf{A}$. In fact if σ is such an equation, then $\boldsymbol{V}(\mathsf{K}_{\mathbf{A}}) \vDash \sigma$ and if $\mathbf{B} \vDash \sigma$, then $\mathbf{A} \notin \boldsymbol{V}(\mathbf{B})$ (since $\mathbf{A} \nvDash \sigma$); so $\boldsymbol{V}(\mathbf{B}) \subseteq \boldsymbol{V}(\mathsf{K}_{\mathbf{A}})$ and thus $\mathbf{B} \in \boldsymbol{V}(\mathsf{K}_{\mathbf{A}})$.

Now we introduce some notation; let $\mathbf{A} = \bigoplus_{i\in I} \mathbf{A}_i$ be a chain in BBCK. For $a, b \in A$ we write $a \ll b$ if either $a \neq 1$ and $b = 1$ or else a lies in a component strictly below the component in which b lies. We write $a \sim b$ if $a \not\ll b$ and $b \not\ll a$; this means that either $a = b = 1$ or else a, b lie in the same component.

Constructing the splitting equation for each $n\mathbf{Ł}_1$ is useful in that it illustrates a particular instance of a formula that we will need in the sequel; the definition is rather baroque, mainly due to the lack of expressivity of a language containing only $\to$. First we define

$$m(x, y) = (x \to y) \to y;$$

then we observe that

- if $a \ll b$, then $m(a, b) = b$ and $m(b, a) = 1$;
- if $a \sim b$ then $m(a, b) = a \vee b = m(b, a)$ and $a \leq m(a, b)$;
- for any a, $m(a, 1) = m(1, a) = 1$.

Next for all k we define a $(k+1)$-ary term inductively by

$$q_1(x_1, x_2) := m(x_2, x_1) \to m(x_1, x_2)$$
$$q_2(x_1, x_2, x_3) := m(x_3, x_2) \to m(q_1(x_1, x_2), m(x_2, x_3))$$
$$q_k(x_1, \dots, x_{k+1}) := m(x_{k+1}, x_k) \to m(q_{k-1}(x_1, \dots, x_k), m(x_k, x_{k+1})).$$

LEMMA 6.3. *Let* $\mathbf{A}$ *be a chain in* BBCK *an let* $a_1, \dots, a_{k+1} \in A$*; then*

$$q(a_1, \dots, a_{k+1}) = \begin{cases} a_{k+1}, & if a_1 \ll a_2 \ll \cdots \ll a_{k+1} \ll 1; \\ 1, & otherwise. \end{cases}$$

PROOF. The proof is by induction; let $k = 1$ and let $a_1, a_2 \in A$. If $a_1 \ll a_2$ then

$$q_1(a_1, a_2) = m(a_2, a_1) \to m(a_1, a_2) = 1 \to a_2 = a_2.$$

Conversely if $a_1 \not\ll a_2$, either $a_1 \sim a_2$ or $a_2 \ll a_1$ and in both cases it is easily checked that $q_1(a_1, a_2) = 1$.

Now suppose that the statement holds for $k-1$ and let $a_1, \dots, a_{k+1} \in A$ so that

$$q_k(a_1, \dots, a_{k+1}) = m(a_{k+1}, a_k) \to m(q_{k-1}(a_1, \dots, a_k), m(a_k, a_{k+1})).$$

Suppose first that $a_1 \ll \cdots \ll a_k \ll a_{k+1} \ll 1$; then by induction $q_{k-1}(a_1, \dots, a_k) = a_k$ and so

$$\begin{aligned} q_k(a_1, \dots, a_{k+1}) &= m(a_{k+1}, a_k) \to m(a_k, m(a_k, a_{k+1})) \\ &= 1 \to m(a_k, a_{k+1}) = a_{k+1}. \end{aligned}$$

Next observe that if $a_i = 1$ for some $i \leq k+1$, then $q_k(a_1, \dots, a_{k+1}) = 1$. So suppose that $\alpha_i \not\ll \alpha_{i+1}$ for some $i \leq k-1$; then by induction $q_{k-1}(a_1, \dots, a_k) = 1$ so $q_k(a_1, \dots, a_k) = 1$ as well. Hence we are left with the case $a_1 \ll \cdots \ll a_{k-1} \ll a_k \ll 1$ and $a_k \not\ll a_{k+1}$; if $a_{k+1} \ll a_k$ then $m(a_k, a_{k+1}) = 1$ and again $q_k(a_1, \dots, a_{k+1}) = 1$. If $a_k \sim a_{k+1}$, observe that by induction $q_{k-1}(a_1, \dots, a_k) = a_k$. Therefore

$$q_k(a_1, \dots, a_{k+1}) = m(a_{k+1}, a_k) \to m(a_k, m(a_k, a_{k+1})) = 1$$

and the proof is finished. ∎

Let's now define the equation λ_n as

$$q_n(x_1, \dots, x_{n+1}) \approx 1. \qquad (\lambda_n)$$

By Lemma 6.3 any chain in $\mathsf{BBCK}^{(n)}$ satisfies λ_n; by the same token $(n+1)\mathbf{Ł}_1$ does not satisfy λ_n, since it possible to find there $a_1 \ll \cdots \ll a_n \ll a_{n+1} \ll 1$. Therefore:

Corollary 6.4. *For any n, λ_n is the splitting equation of $(n+1)\mathbf{Ł}_1$ in* BBCK *(and hence also in* GBCK*).*

Before looking for other splitting equations we better be sure that there is something to look for; from now on, to save space, we will write $\mathsf{V}_{\mathbf{A}}$ for $\boldsymbol{V}(\mathsf{K}_{\mathbf{A}})$ for a finite totally ordered algebra $\mathbf{A} \in \mathsf{BBCK}$.

Let $\mathbf{A} = \mathbf{Ł}_{n_1} \oplus \cdots \oplus \mathbf{Ł}_{n_k}$; since we totally worked out the case in which $n_1, \ldots, n_k = 1$ we may assume that $n_j > 1$ for at least one $j \leq k$. In this case the set $R_{\mathbf{A}} = \{n_i : n_i > 1\}$ is nonempty and if $|R_{\mathbf{A}}| = r$ we call r the **reduced index** of $\mathbf{A}$. We can also enumerate the elements of $R_{\mathbf{A}} = \{m_1, \ldots, m_r\}$. For each $m_s \in R_{\mathbf{A}}$ we define a chain $\mathbf{A}^s = \bigoplus_{i=1}^k \mathbf{A}_i^s$ in the following way

$$\mathbf{A}_i^s = \begin{cases} \mathbf{Ł}_{n_i-1}, & \text{if} n_i = m_s; \\ \mathbf{Ł}_{\omega}, & \text{otherwise.} \end{cases}$$

Lemma 6.5. *If $\mathbf{A} = \mathbf{Ł}_{n_1} \oplus \cdots \oplus \mathbf{Ł}_{n_k}$ is a chain of reduced index $r > 0$, then every chain in $\mathsf{V}_{\mathbf{A}}$ of index at most k belongs to $\boldsymbol{HSP}_U(\mathbf{A}^s)$ for some $m_s \in R_{\mathbf{A}}$.*

Proof. Since in any $\mathbf{A}^s$ there is exactly one component that is different from $\mathbf{Ł}_\omega$ it follows from Lemma 4.2, Corollary 4.4 and Corollary 5.2 that any chain of index strictly less then k belongs to $\boldsymbol{HSP}_U(\mathbf{A}^s)$ for all s. So suppose that $\mathbf{B} = \bigoplus_{j=1}^k \mathbf{B}_i \in \mathsf{K}_{\mathbf{A}}$; since $\mathbf{Ł}_1$ is a subalgebra of any nontrivial chain, there must be an $m_s = n_j \in R_{\mathbf{A}}$ such that $\mathbf{Ł}_{n_j} \notin \boldsymbol{S}(\mathbf{B}_j)$. It follows that $\mathbf{B}_j = \mathbf{Ł}_l$ for some $l \leq n_j - 1$, so that $\mathbf{B}_j \in \boldsymbol{S}(\mathbf{Ł}_{n_j-1}) = \boldsymbol{S}(\mathbf{A}_j^s)$. The conclusion now follows again from the lemma and corollaries mentioned above. ■

Theorem 6.6. *Every finite totally ordered algebra $\mathbf{A} \in \mathsf{BBCK}$ is splitting, with conjugate variety $\mathsf{V}_{\mathbf{A}}$.*

Proof. Let $\mathbf{A}$ be a finite BBCK-chain; then there are $n_1, \ldots, n_k \in \mathbb{N}$ with $\mathbf{A} \cong \mathbf{Ł}_{n_1} \oplus \cdots \oplus \mathbf{Ł}_{n_k}$. Clearly we may assume that its reduced index is greater or equal to 1, so that $|R_{\mathbf{A}}| = |\{m_1, \ldots, m_r\}| = r \geq 1$.

Now let $m_s \in R_{\mathbf{A}}$ and pick i such that $n_i = m_s$; then $\mathbf{A}_i^s = \mathbf{Ł}_{n_{i-1}}$. It follows immediately that $\mathbf{Ł}_{n_i} \notin \boldsymbol{SP}_U(\mathbf{A}_i^s)$ if $i < k$ and $\mathbf{Ł}_{n_k} \notin \boldsymbol{HSP}_U(\mathbf{A}_k^s)$; therefore $\mathbf{A} \notin \boldsymbol{HSP}_U(\mathbf{A}^s)$. Since m_s was generic, Lemma 6.5 implies that $\mathbf{A} \notin \mathsf{V}_{\mathbf{A}}$; by Theorem 6.1, $\mathbf{A}$ is splitting with conjugate variety $\mathsf{V}_{\mathbf{A}}$. ■

Finding the splitting equations is a different story: for $n \geq 2$ we define

$$j_n(x, y) := (x \to^n y) \to (x \to^{n-1} y) \quad \text{for} n > 1.$$

It is well known that for $n > 1$, $\mathbf{Ł}_k \vDash j_n(x, y) \approx 1$ if and only if $k \leq n-1$.

First observe that if $\mathbf{A} = \mathbf{Ł}_n$, $n \geq 2$, then $j_n(x, y) \approx 1$ is a splitting equation for $\mathbf{A}$; in fact $\mathbf{A} \nvDash j_n(x, y)$, but if $\mathbf{B} \in \mathsf{V}_{\mathbf{A}}$) then no component of $\mathbf{B}$ can have $\mathbf{Ł}_n$ as a subalgebra. It follows that all the components of $\mathbf{B}$ are of the form $\mathbf{L}_k$ with $k \leq n-1$ and hence they all satisfy $j_n(x,y) \approx 1$. Since $j_n(x,y) \approx 1$ holds trivially also if x, y lie in different components, we conclude that $\mathbf{B} \vDash j_n(x,y) \approx 1$ and therefore $\mathsf{V}_{\mathbf{A}} \vDash j_n(x,y) \approx 1$.

The next step is to consider the finite chains with reduced index equal to 1; i.e. those finite chains $\mathbf{Ł}_{n_1} \oplus \cdots \oplus \mathbf{Ł}_{n_k}$ such that there exists exactly one r, called the **lone dissenter**, with $n_r \neq 1$.

Given such $\mathbf{A}$ we define binary terms $\beta_1, \ldots, \beta_k$ in the following way:

$$\text{if} i = 1 and i \neq r, then \beta_1(x,y) = m(y,x) \to m(x,y)$$
$$\text{if} i \neq 1 and i \neq r, then \beta_i(x,y) = m(x,y)$$
$$\text{if} i = r, then \beta_r(x,y) = j_{n_r}(x,y)$$

Then we define

$$\alpha_1(x_1,x_2) := \beta_1(x_1,x_2)$$
$$\alpha_2(x_1,x_2,x_3) := \beta_2(x_3,x_2) \to m(\alpha_1(x_1,x_2), \beta_2(x_2,x_3))$$
$$\alpha_k(x_1,\ldots,x_{k+1}) := \beta_k(x_{k+1},x_k) \to m(\alpha_{k-1}(x_1,\ldots,x_k), \beta_k(x_k,x_{k+1}))$$

Finally we set $s^{\mathbf{A}}(x_1,\ldots,x_{k+1}) = \alpha_k(x_1,\ldots,x_{k+1})$. We have the following obvious lemma.

Lemma 6.7.

1. *If* $\mathbf{A} = k\mathbf{Ł}_1$ *then* $s^{\mathbf{A}} = q_k$.
2. *If* $\mathbf{A} = \mathbf{Ł}_{n_1} \oplus \cdots \oplus \mathbf{Ł}_{n_k}$ *and* $\mathbf{A}' = \mathbf{Ł}_{n_1} \oplus \cdots \oplus \mathbf{Ł}_{n_{k-1}}$ *then*

$$s^{\mathbf{A}}(x_1,\ldots,x_{k+1}) = \beta_k(x_{k+1},x_k) \to m(s^{\mathbf{A}'}(x_1,\ldots,x_k), \beta_k(x_k,x_{k+1}))$$

The key step in finding splitting equations for finite chain of reduced index equal to 1 is the following lemma.

Lemma 6.8. *Let* $\mathbf{A} = \mathbf{Ł}_{n_1} \oplus \cdots \oplus \mathbf{Ł}_{n_k}$ *be a chain of reduced index equal to* 1*; then if* $\mathbf{B} \in \mathsf{V}_{\mathbf{A}}$ *and* $a_1, \ldots, a_{k+1} \in B$ *then*

$$s^{\mathbf{A}}(a_1,\ldots,a_{k+1}) = 1.$$

Proof. The proof is by induction on $k \geq 2$. If $k = 2$, then either $\mathbf{A} = \mathbf{Ł}_n \oplus \mathbf{Ł}_1$ or $\mathbf{A} = \mathbf{Ł}_1 \oplus \mathbf{Ł}_n$ for some $n > 1$.

Assume the first; in this case

$$s^{\mathbf{A}}(x_1,x_2,x_3) := m(x_3,x_2) \to m(j_n(x_1,x_2), m(x_2,x_3)).$$

Let $\mathbf{B} = \bigoplus_{i\in I} \mathbf{B}_i \in \mathsf{K}_{\mathbf{A}}$ and let $a_1, a_2, a_3 \in B$. First if $a_1 \ll a_2 \ll a_3$ or if $a_i = 1$ for some $i \leq 3$, then $s(a_1, a_2, a_3) = 1$.

So we may assume that $a_1 \not\ll a_2$ or $a_2 \not\ll a_3 \ll 1$; on the other hand if $a_3 \ll a_2$ or $a_2 \ll a_1$ then again $s(a_1, a_2, a_3) = 1$. So we are left with only two possibilities: either $a_1 \sim a_2 \ll a_3$ or else a_1, a_2, a_3 belong to the same component.

First suppose that the latter happens; since $m(x, y) = x \vee y$ on any component, we get

$$s(a_1, a_2, a_3) = (a_3 \vee a_2) \rightarrow (j_n(a_1, a_2) \vee (a_2 \vee a_3)) = 1.$$

Otherwise there are i_1, i_2 such that $a_1, a_2, a_3 \in B_{i_1} \cup B_{i_2}$; since $\mathbf{B} \in \mathsf{K}_{\mathbf{A}}$ and Ł_1 is embeddable in any BBCK-chain we must have $\text{Ł}_n \notin \boldsymbol{S}(\mathbf{B}_{i_1})$, i.e. $\mathbf{B}_{i_1} = \text{Ł}_l$ with $l \leq n-1$. Since $a_1, a_2 \in B_{i_1}$ then $j_n(a_1, a_2) = 1$, since $\text{Ł}_l \vDash j_n(x, y) \approx 1$; then again $s(a_1, a_2, a_3) = 1$ and the base step holds for $\text{Ł}_n \oplus \text{Ł}_1$.

The proof that the same happens if $\mathbf{A} = \text{Ł}_1 \oplus \text{Ł}_n$ is similar so the base step of the induction holds.

Assume now that $\mathbf{A} = \text{Ł}_{n_1} \oplus \cdots \oplus \text{Ł}_{n_{k-1}} \oplus \text{Ł}_{n_k}$ and let n_r be the lone dissenter; let $\mathbf{A}' = \text{Ł}_{n_1} \oplus \cdots \oplus \text{Ł}_{n_{k-1}}$, so that

$$s^{\mathbf{A}}(x_1, \dots, x_{k+1}) = \beta_k(x_{k+1}, x_k) \rightarrow m(s^{\mathbf{A}'}(x_1, \dots, x_k), \beta_k(x_k, x_{k+1})).$$

If $a_i = 1$ for some i, then either $s^{\mathbf{A}'}(a_1, \dots, a_k) = 1$ (by induction) or else $a_{k+1} = 1$; in both cases $s^{\mathbf{A}}(a_1, \dots, a_{k+1}) = 1$, so we may assume that $a_i \neq 1$ for all $i \leq k+1$.

First suppose that $r = k$; then $\mathbf{A}' = (k-1)\text{Ł}_1$ and $s^{\mathbf{A}'} = q_{k-1}$. Suppose first that $a_i \not\ll a_{i+1}$ for some $i \leq k-1$; then by Lemma 6.3 $s^{\mathbf{A}'}(a_1, \dots, a_k) = q_{k-1}(a_1, \dots, a_k) = 1$, so that $s^{\mathbf{A}}(a_1, \dots, a_{k+1}) = 1$.

Hence we may assume that $a_1 \ll \cdots \ll a_{k-1}$ and either $a_k \sim a_{k+1}$ or $a_{k+1} \ll a_k$. In the latter case $\beta_k(a_k, a_{k+1}) = 1$, so $s^{\mathbf{A}}(a_1, \dots, a_{k+1}) = 1$. In the former, let $B_{i_1}, \dots, B_{i_k}$ such that

$$a_1, \dots, a_{k+1} \in B_{i_1} \oplus \cdots \oplus B_{i_k}.$$

Then since Ł_1 is embeddable in any chain and $r = k$ it must be that $\mathbf{B}_{i_k} = \text{Ł}_m$ with $m \leq n_k - 1$ and $\beta_k(x, y) = j_{n_k}(x, y)$. Since clearly $a_k, a_{k+1} \in B_{i_k}$ we have $\beta_k(a_k, a_{k+1}) = 1$ and again the conclusion follows.

Suppose now that $r < k$; by induction we may assume that $a_i \not\ll a_{i+1}$ for some $i \leq k+1$. Let $B_{i_1}, \dots, B_{i_{k-1}}$ such that $a_1, \dots, a_k \in B_{i_1} \oplus \cdots \oplus B_{i_{k-1}}$; then $\mathbf{B}' \in \mathsf{V}_{\mathbf{A}'}$ (since $\mathbf{B} \in \mathsf{V}_{\mathbf{A}}$). So by induction $s^{\mathbf{A}'}(a_1, \dots, a_k) = 1$ implying as usual $s^{\mathbf{A}}(a_1, \dots, a_{k+1}) = 1$.

Hence as before we may assume that $a_1 \ll a_2 \ll \cdots \ll a_k$ and either $a_k \sim a_{k+1}$ or $a_{k+1} \ll a_k$. In the latter case $\beta_k(a_k, a_{k+1}) = m(a_k, a_{k+1}) = 1$. In the other

$$s^{\mathbf{A}}(a_1, \ldots, a_{k+1}) = m(a_{k+1}, a_k) \to m(s^{\mathbf{A}'}(a_1, \ldots, a_k), m(a_k, a_{k+1}))$$

and $a_k \sim a_{k+1}$. Now either $s^{\mathbf{A}'}(a_1, \ldots, a_k) \ll a_k$, and in this case

$$\begin{aligned} s^{\mathbf{A}}(a_1, \ldots, a_{k+1}) &= m(a_{k+1}, a_k) \to m(s^{\mathbf{A}'}(a_1, \ldots, a_k), m(a_k, a_{k+1})) \\ &= m(a_{k+1}, a_k) \to m(a_k, a_{k+1}) = 1 \end{aligned}$$

or $s^{\mathbf{A}'}(a_1, \ldots, a_k) \sim a_k \sim a_{k+1}$ and in this case

$$\begin{aligned} s^{\mathbf{A}}(a_1, \ldots, a_{k+1}) &= m(a_{k+1}, a_k) \to m(s^{\mathbf{A}'}(a_1, \ldots, a_k), m(a_k, a_{k+1})) \\ &= (a_{k+1} \vee a_k) \to (s^{\mathbf{A}'}(a_1, \ldots, a_k) \vee a_k \vee a_{k+1}) = 1 \end{aligned}$$

Therefore the proof is finished. ■

Theorem 6.9. *If* $\mathbf{A}$ *is a finite chain of index* k *and reduced index* 1*, then*

$$s^{\mathbf{A}}(x_1, \ldots, x_{k+1}) \approx 1$$

is the splitting equation for $\mathbf{A}$*.*

Proof. By Lemma 6.8 every generator of $\mathsf{V}_{\mathbf{A}}$ satisfies the equation, hence so does $\mathsf{V}_{\mathbf{A}}$.

However $\mathbf{A}$ does not satisfy the equation; if $\mathbf{A} = \mathbf{Ł}_{n_1} \oplus \cdots \oplus \mathbf{Ł}_{n_k}$ and n_r is the lone dissenter, we pick $a_1, \ldots, a_{k+1} \in A$ in the following way:

$$\begin{aligned} &\text{if} i < r pick a_i \in Ł_{n_i} with a_i \neq 1 \\ &\text{if} i = r, pick a_r, a_{r+1} \in Ł_{n_r} with j_{n_r}(a_r, a_{r+1}) \neq 1 \\ &\text{if} r + 2 \leq i \leq k + 1, pick a_i \in Ł_{n_{i-1}} with a_i \neq 1. \end{aligned}$$

Clearly, since $j_{n_r}(a_r, a_{r+1}) \neq 1$, $a_r, a_{r+1} \neq 1$ and $a_r \not\leq a_{r+1}$; since we are in a chain $a_{r+1} < a_r < 1$.

First suppose that $r \leq k - 1$; then $j_{n_r}(a_r, a_{r+1}) \ll a_{r+2}$. In this case we leave it to the reader to prove that $s^{\mathbf{A}}(a_1, \ldots, a_{k+1}) = a_{k+1} < 1$ (this can be done by direct inspection or by induction, using the properties of $m(x, y)$).

Finally suppose $r = k$. Then $\mathbf{A} = (k-1)\mathbf{Ł}_1 \oplus \mathbf{Ł}_{n_k}$; if $\mathbf{A}' = (k-1)\mathbf{Ł}_1$, then

$$s^{\mathbf{A}'}(a_1, \ldots, a_k) = a_k$$

and thus

$$\begin{aligned} s^{\mathbf{A}}(a_1, \ldots, a_{k+1}) &= j_{n_k}(a_{k+1}, a_k) \to m(s^{\mathbf{A}'}(a_1, \ldots, a_k), j_{n_k}(a_k, a_{k+1})) \\ &= 1 \to m(a_k, j_{n_k}(a_k, a_{k+1})) = a_k \vee j_{n_k}(a_k, a_{k+1}). \end{aligned}$$

But $a_k, j_{n_k}(a_k, a_{k+1}) \neq 1$ and since we are in a chain 1 is join irreducible; therefore $s^{\mathbf{A}}(a_1, \ldots, a_{k+1}) \neq 1$.

This proves that $\mathbf{A} \notin \mathsf{V}_{\mathbf{A}}$ and so, by Theorem 6.1, $s^{\mathbf{A}}(x_1, \ldots, x_{k+1}) \approx 1$ is a splitting equation for $\mathbf{A}$. ■

We were not able to find explicit splitting equations for chains of reduced index greater than 1, not even in the simplest cases (more on the reason why we believe it difficult is in Section 8). On the other hand there might be equations that are simpler than the ones we have found and can be more readily generalized to the case in which the reduced index is greater than 1. We can neither confirm nor deny that possibility so we leave it as open question.

What about HBCK? Since any totally ordered HBCK-algebra is a BBCK-algebra, then HBCK has the FEP as well, therefore every splitting HBCK-algebra must be finite. Moreover:

Theorem 6.10. *[17] An algebra in* HBCK *is subdirectly irreducible if and only if it is isomorphic to* $\mathbf{F} \oplus \mathbf{S}$ *for some subdirectly irreducible (hence totally ordered)* $\mathbf{S}$ *in* LBCK.

So again $2\mathbf{Ł}_1$ is embeddable in any subdirectly irreducible $\mathbf{F} \oplus \mathbf{S} \in \mathsf{HBCK}$, unless $\mathbf{F}$ is trivial ($\mathbf{S}$ cannot be trivial, being subdirectly irreducible). So it is clear that $2\mathbf{Ł}_1$ is splitting for HBCK, where the splitting equation is clearly Tanaka's equation (T). The question if there are other splitting algebras is open.

7. A Meet is a Wonderful Thing

The obvious next step would be to consider $\{\to, \cdot, 1\}$-subreducts of CIRSs; these structures are usually called **pocrims** (acrostic for partially ordered commutative residuated integral monoids) [10] and they form a proper quasivariety. As for BCK-algebras it is readily shown that any quasivariety of pocrims satisfying (J) is a variety; moreover if a quasivariety of pocrims satisfies (H), then the ordering is a meet semilattice ordering where the meet is

$$x \wedge y = (x \to y)x.$$

In other words pocrims satisfying (H) are termwise equivalent to hoops and splitting hoops have been completely classified in [3].

The situation is different if we consider non-monoidal fragments of the language. A BCK-semilattice is an algebra $\langle A, \to, \wedge, 1\rangle$, where $\langle A, \to, 1\rangle$ is a

BCK-algebra, $\wedge$ is a semilattice operation and the ordering defined by $\wedge$ and $\rightarrow$ is the same. These algebras (or better a supervariety of these algebras) have been investigated at length in [1].

THEOREM 7.1. *(see Proposition 2.4 in [1]) An algebra $\langle A, \rightarrow, \wedge, 1\rangle$ is a BCK-semilattice if and only if*

1. *$\langle A, \rightarrow, 1\rangle$ is a BCK-algebra;*
2. *$\langle A, \wedge\rangle$ is a semilattice;*
3. *for all $a, b \in A$, $a \wedge ((a \rightarrow b) \rightarrow b) = a$;*
4. *for all $a, b, c \in A$ $(a \rightarrow b) \wedge (a \rightarrow c) = a \rightarrow (b \wedge c)$.*

BCK-semilattices form a variety $\mathsf{BCK}^{\wedge}$ since the ordering induced by $\rightarrow$ is a semilattice ordering and hence the quasiidentity (M) in the definition of BCK-algebras is a consequence of the other axioms; they are also congruence permutable with Mal'cev term

$$m(x, y, z) = ((x \rightarrow y) \rightarrow z) \wedge ((z \rightarrow y) \rightarrow x).$$

and congruence distributive since they are congruence modular and have a semilattice term. The congruence structure is equally transparent; a **filter** F of a BCK-semilattice is a BCK-filter that is also a semilattice filter and the filters form an algebraic lattice isomorphic with $\mathrm{Con}(\mathbf{A})$; the proofs are quite standard but the reader may consult [1].

The structure of BCK-semilattices is even closer to that of BCK-algebras if we consider the totally ordered members of $\mathsf{BCK}^{\wedge}$; in fact in a chain any subset is closed under $\wedge$, so filters and subalgebras of a $\mathsf{BCK}^{\wedge}$-chain depend only on the underlying BCK-structure. Not surprisingly we will see that equation (B) isolates the variety of BCK-semilattices that are subdirect product of totally ordered semilattices. However we have to be careful: we cannot invoke Palasiǹski's results directly, since in principle the filters of a BCK-semilattice are different from the filters of its BCK-reduct. However the principal filters turn out to be the same, and this is enough to prove the result.

LEMMA 7.2. *If $\mathbf{A}$ is a BCK-semilattice and $F_{\mathbf{A}}(a)$ is the filter generated by $a \in A$; then*

$$F_{\mathbf{A}}(a) = \{b : \textit{there exists ann} \in \mathbb{N} \textit{witha} \rightarrow^n b = 1\}.$$

PROOF. Note that the set on the right-hand side contains a and it must be contained in any filter containing a. So we have only to show that $F_{\mathbf{A}}(a)$ is a filter; clearly F is a BCK-filter so it is enough to show that it is closed under meets. Let $b, c \in F_{\mathbf{A}}$; then there are $n, m \in \mathbb{N}$ then $a \to^n b = a \to^m c = 1$. We may assume clearly that $n = m$ and a straightforward induction shows that for $a, b, c \in A$

$$a \to^n (b \wedge c) = (a \to^n b) \wedge (a \to^n c),$$

from which we get $a \to^n (b \wedge c) = 1$. So $F_{\mathbf{A}}(a)$ is closed under meets and it is a filter of $\mathbf{A}$. ■

Corollary 7.3.

1. *A* BCK*-semilattice* $\mathbf{A}$ *is simple if and only if for all* $a \in A \setminus \{1\}$ *and for all* $b \in A$ *there exists an* n *with* $a \to^n b = 1$.
2. *A* BCK*-semilattice is subdirectly irreducible if and only if there exists an* $a_\star \in A$ *(the* **monolithical element***) such that for all* $b \in A \setminus \{1\}$ *there exists an* $n \in \mathbb{N}$ *with* $b \to^n a_\star = 1$. *In this case* $F_{\mathbf{A}}(a_\star)$ *is the filter corresponding to the monolith of* $\mathbf{A}$.

The following lemma is the key to prove the next theorem.

Lemma 7.4. *Let* $\mathbf{A}$ *be a* BCK*-semilattice and let* $a, b \in A$*; if* $\sup(a, b) = 1$, *then* $F_{\mathbf{A}}(a) \cap F_{\mathbf{A}}(b) = \{1\}$.

Proof. In this lemma we will make extensive use without mention of equations that are well-known to hold in BCK-algebras; the most used will be

$$x \to (y \to z) \approx y \to (x \to z).$$

Let $c \in F_{\mathbf{A}}(a) \cap F_{\mathbf{A}}(b)$ with $c < 1$; then by 7.2 there exists an $n \in \mathbb{N}$ with $a \to^n c = 1$ and $a \to^{n-1} c < 1$. Since $c \leq a \to^{n-1} c$, $a \to^{n-1} c \in F_{\mathbf{A}}(b)$ so there exists an $m \in \mathbb{N}$ such that $b \to^m (a \to^{n-1} c) = 1$ and $b \to^{m-1} (a \to^{n-1} c) < 1$.

Now from $b \to^m (a \to^{n-1} c) = 1$ we get $b \to (b \to^{m-1} (a \to^{n-1} c)) = 1$, so $b \leq b \to^{m-1} (a \to^{n-1} c)$; on the other hand $a \to^n c = 1$ implies

$$1 = b \to^{m-1} (a \to^n c) = a \to^n (b \to^{m-1} c),$$

so as before $a \leq a \to^{n-1} (b \to^{m-1} c) = b \to^{m-1} (a \to^{n-1} c)$. Since $\sup(a, b) = 1$ it must be

$$b \to^{m-1} (a \to^{n-1} c) = 1$$

a clear contradiction. Hence $c = 1$ and $F_{\mathbf{A}}(a) \cap F_{\mathbf{A}}(b) = \{1\}$, as wished. ■

Theorem 7.5. *A subdirectly irreducible* BCK*-semilattice is totally ordered if and only if it satisfies (B).*

PROOF. Any totally ordered BCK-semilattice satisfies (B) trivially. So assume that $\mathbf{A}$ is a BCK-semilattice satisfying (B); then for any $a, b \in A$

$$(a \to b) \to (b \to a) \leq ((b \to a) \to (b \to a)) \to (b \to a) = b \to a$$

so $(a \to b) \to (b \to a) = b \to a$ and similarly $(b \to a) \to (a \to b) = a \to b$. Suppose now that $a \to b, b \to a \leq c$; then

$$\begin{aligned} 1 = (a \to b) \to c &= ((b \to a) \to (a \to b)) \to c \\ &\leq (((a \to b) \to (b \to a)) \to c) \to c \quad \text{by (B)} \\ &= ((b \to a) \to c) \to c = 1 \to c = c. \end{aligned}$$

Hence $\sup(a \to b, b \to a) = 1$ and by Lemma 7.4

$$F_{\mathbf{A}}(a \to b) \cap F_{\mathbf{A}}(b \to a)) = \{1\}.$$

If $\mathbf{A}$ is also subdirectly irreducible, then $\{1\}$ must be (completely) meet-irreducible, so either $F_{\mathbf{A}}(a \to b) = \{1\}$ or $F_{\mathbf{A}}(b \to a) = \{1\}$. But then either $a \to b = 1$ or $b \to a = 1$ and so $\mathbf{A}$ is totally ordered. ■

It follows that the class of $\mathsf{RBCK}^{\wedge}$ of BCK-semilattices satisfying (B) is a variety in which every subdirectly irreducible is totally ordered; the variety $\mathsf{RBCK}^{\wedge}$ has another interesting characteristic:

LEMMA 7.6. *Let* $\mathbf{A}$ *be a totally ordered* BCK*-semilattice; then for all* $a, b \in A$

$$\sup(a, b) = ((a \to b) \to b) \wedge ((b \to a) \to a).$$

PROOF. Let $j(a, b)$ the expression on the right hand side; then $j(a, b) = j(b, a)$, $j(a, a) = 1$, $a, b \leq j(a, b)$ and if $a \leq b$, then $j(a, b) = b$. But in a totally ordered structure this is exactly the least upper bound of a and b. ■

So in $\mathsf{RBCK}^{\wedge}$ there is a term definable join operation $\vee$ satisfying the equation

$$(x \vee y) \to z \approx (x \to y) \wedge (x \to z).$$

Similarly:

- $\mathsf{HBCK}^{\wedge}$ is the variety of BCK-semilattice satisfying (H) and coincides with the class of $\{\to, \wedge, 1\}$-subreducts of hoops;
- $\mathsf{BBCK}^{\wedge}$ is the variety of **basic** BCK-semilattices, i.e. those satisfying (H) and (B), and coincides with the class of $\{\to, \wedge, 1\}$-subreducts of basic hoops;

- $\mathsf{LBCK}^\wedge$ is the variety of **Łukasiewicz** BCK-semilattices, i.e. those satisfying (B) and (T) (and therefore (H)) and coincides with the class of $\{\rightarrow, \wedge, 1\}$-subreducts of Wajsberg hoops.

The **ordinal sum** of a family of BCK-semilattices is their ordinal sum as BCK-algebras; it is obvious that it is always a BCK-semilattice and that the ordering is the one induced by the ordinal sum.

There is one more property of BCK-semilattices that we need; an algebra $\mathbf{A}$ has the **congruence extension property** (CEP for short) if for any subalgebra $\mathbf{B}$ of $\mathbf{A}$ every congruence of $\mathbf{B}$ is the restriction to $\mathbf{B}$ of a congruence of $\mathbf{A}$. The **principal** CEP is the same property restricted to principal congruences. These two properties are in general distinct for a single algebra but they coincide for varieties: every algebra in V has the CEP if and only if every algebra in V has the principal CEP [15]. For BCK-semilattices this property can be expressed in terms of filters and it is clear from Lemma 7.2 that BCK-semilattices have the principal CEP, so they have the CEP. In particular this means that any subalgebra of a subdirectly irreducible BCK-semilattice must be subdirectly irreducible as well, and this is exactly what we need.

Due to the obvious similarities, many results about BCK-algebras extend to BCK-semilattices without any problem. In particular:

- every totally ordered BCK-semilattice is the ordinal sum of sum irreducible BCK-semilattices;
- the classes of $\mathsf{LBCK}^\wedge$-chains and of HBCK-chains have the FEP, so $\mathsf{HBCK}^\wedge$, $\mathsf{BBCK}^\wedge$ and $\mathsf{LBCK}^\wedge$ all have the FEP;
- the sum irreducible $\mathsf{HBCK}^\wedge$-chains are the $\mathsf{LBCK}^\wedge$-chains;
- an algebra in $\mathsf{HBCK}^\wedge$ is subdirectly irreducible if and only if it is isomorphic to $\mathbf{F} \oplus \mathbf{S}$ for some subdirectly irreducible (hence totally ordered) $\mathbf{S}$ in $\mathsf{LBCK}^\wedge$;
- a finite algebra in $\mathsf{BBCK}^\wedge$ is subdirectly irreducible if and only if

$$\mathbf{A} \cong \mathbf{Ł}^\wedge_{k_1} \oplus \cdots \oplus \mathbf{Ł}^\wedge_{k_n}$$

  where each $\mathbf{Ł}^\wedge_{k_j}$ is the obvious expansion of $\mathbf{L}_{k_j}$ to the type of $\mathsf{BBCK}^\wedge$.

The main difference between BCK-algebras and BCK-semilattices is that finite BCK-semilattices can *speak about themselves*; in other words if $\mathbf{A}$ is a finite BCK-semilattice we can write equations that encode information about $\mathbf{A}$. If $A = \{a_1, \ldots, a_n, 1\}$ and $\mathbf{x} = (x_1, \ldots, x_n)$ is a n-tuple of distinct variables, the **diagram** of $\mathbf{A}$ is the set $\mathrm{Diag}_{\mathbf{A}}(\mathbf{x})$ of equations defined in the following way ($x \leftrightarrow y := (x \to y) \wedge (y \to x)$):

- $(x_i \to x_j) \approx x_k \in \mathrm{Diag}_{\mathbf{A}}(\mathbf{x})$ if and only if $a_i \to a_j = a_k$;
- $(x_i \wedge x_j) \approx x_k \in \mathrm{Diag}_{\mathbf{A}}(\mathbf{x})$ if and only if $a_i \wedge a_j = a_k$;

Let now

$$T_{\mathbf{A}}(\mathbf{x}) = \bigwedge\{p(\mathbf{x}) \leftrightarrow q(\mathbf{x}) : p(\mathbf{x}) \approx q(\mathbf{x}) \in \mathrm{Diag}_{\mathbf{A}}(\mathbf{x})\};$$

it is evident that $T_{\mathbf{A}}(\mathbf{x}) \approx 1$ encodes the operation tables of $\mathbf{A}$.

Suppose now that $\mathbf{A}$ is a finite subdirectly irreducible HBCK-semilattice; then $\mathbf{A} = \mathbf{F} \oplus \mathbf{S}$, where $\mathbf{S}$ is a totally ordered LBCK-semilattice. Let $A = \{a_1, \ldots, a_n, 1\}$ and $S = \{b_1 < b_2 < \cdots < b_m < 1\}$ and let $\mathbf{z} = (z_1, \ldots, z_{n+m})$ be distinct variables where we assume that $z_i = x_i$ if $i \leq n$ and $z_{n+j} = y_j$ otherwise. We observe that $(\mathbf{x}, \mathbf{y}) = \mathbf{z}$ and we define:

$$\begin{aligned}
D^1_{\mathbf{A}}(\mathbf{x}, \mathbf{y}) &:= \bigwedge_{1 \leq i < j \leq n+m} ((z_i \leftrightarrow z_j) \to z_{m+n}) \\
D^2_{\mathbf{A}}(\mathbf{x}, \mathbf{y}) &:= \bigwedge_{i=1}^{m+n} (z_i \to z_{m+n}) \\
D^3_{\mathbf{A}}(\mathbf{x}, \mathbf{y}) &:= \bigwedge_{j=1}^{m} (((y_m \to y_i) \to y_i) \to y_m) \\
D_{\mathbf{A}}(\mathbf{x}, \mathbf{y}) &:= D^1_{\mathbf{A}}(\mathbf{x}, \mathbf{y}) \wedge D^2_{\mathbf{A}}(\mathbf{x}, \mathbf{y}) \wedge D^3_{\mathbf{A}}(\mathbf{x}, \mathbf{y}) \\
J_{\mathbf{A}}(\mathbf{x}, \mathbf{y}) &:= D_{\mathbf{A}}(\mathbf{x}, \mathbf{y}) \to (T_{\mathbf{A}}(\mathbf{x}, \mathbf{y}) \to y_m).
\end{aligned}$$

In the next lemma we will use the fact that, if $\mathbf{A}$ is an LBCK-semilattice and $a, b \in A$, then

$$(a \to b) \to b = \sup(a, b),$$

because of Lemma 7.6 and the fact that $(a \to b) \to b = (b \to a) \to a$ in any LBCK-algebra.

Lemma 7.7. *Let* $\mathbf{A} = \mathbf{F} \oplus \mathbf{S}$ *be a finite subdirectly irreducible* HBCK*-semilattice where* $F = \{a_1, \ldots, a_n, 1\}$ *and* $S = \{b_1 < b_2 < \cdots < b_m < 1\}$*. Let* $\mathbf{B}$ *be any subdirectly irreducible* HBCK*-semilattice such that there are* $c_1, \ldots, c_n, d_1, \ldots, d_m \in B$ *with*

$$J_{\mathbf{A}}(\mathbf{c}, \mathbf{d}) \neq 1;$$

then $\mathbf{A} \in \boldsymbol{S}(\mathbf{B})$.

Proof. Let $\mathbf{B}_1$ be the subalgebra generated in $\mathbf{B}$ by $\{c_1, \dots, c_n, d_1, \dots, d_m\}$; since $\mathbf{B}$ is subdirectly irreducible, so is $\mathbf{B}_1$. Therefore $\mathbf{B}_1 \cong \mathbf{F}_1 \oplus \mathbf{S}_1$, where $\mathbf{S}_1$ is a totally ordered LBCK-semilattice.

First we show that d_m belongs to S_1; since $\mathbf{S}_1$ cannot be trivial either $c_j \in S_1$ of $d_k \in S_1$ for some j, k. If $d_m \notin S_1$ and $d_k \in S_1$, then $d_k \to d_m = d_m$. So

$$D_{\mathbf{A}}(\mathbf{c}, \mathbf{d}) \leq (d_k \to d_m) = d_m \leq T_{\mathbf{A}}(\mathbf{c}, \mathbf{d}) \to d_m$$

contradicting $J_{\mathbf{A}}(\mathbf{c}, \mathbf{d}) \neq 1$. An analogous argument can be made if $c_j \in S_1$, so $d_m \in S_1$.

Next we show that $d_i \in S_1$ for all i; if $d_i \notin S_1$, then $d_m \to d_i = d_i$ and

$$D_{\mathbf{A}}(\mathbf{c}, \mathbf{d}) \leq ((d_m \to d_i) \to d_i) \to d_m = d_m \leq T_{\mathbf{A}}(\mathbf{c}, \mathbf{d}) \to d_m$$

again contradicting $J_{\mathbf{A}}(\mathbf{c}, \mathbf{d}) \neq 1$.

Now we show that no c_i can belong to S_1. In fact suppose $c_i \in S_1$; then $\sup(c_i, d_m) = (d_m \to c_i) \to c_i$. So

$$\begin{aligned} c_i \to d_m &= (c_i \to d_m) \wedge (d_m \to d_m) \\ &= \sup(c_i, d_m) \to d_m = ((d_m \to c_i) \to c_i) \to d_m. \end{aligned}$$

Observe also that $y_m \to x_i \approx x_i \in \mathrm{Diag}_{\mathbf{A}}(\mathbf{x}, \mathbf{y})$ so

$$\begin{aligned} D_{\mathbf{A}}(\mathbf{c}, \mathbf{d}) &\leq c_i \to d_m = ((d_m \to c_i) \to c_i) \to d_m \\ &\leq ((d_m \to c_i) \leftrightarrow c_i) \to d_m \leq T_{\mathbf{A}}(\mathbf{c}, \mathbf{d}) \to d_m \end{aligned}$$

again contradicting the hypothesis.

Next we show that all the c_i and d_i are distinct. Since $\{c_1, \dots, c_n\} \subseteq F_1$ and $\{d_1, \dots, d_m\} \subseteq S_1$ it is enough to show that $c_i \neq c_j$ if $i \neq j$ and $d_k \neq d_l$ if $k \neq l$. So suppose that $c_i = c_j$; then

$$D_{\mathbf{A}}(\mathbf{c}, \mathbf{d}) \leq (c_i \leftrightarrow c_j) \to d_m = d_m \leq T_{\mathbf{A}}(\mathbf{c}, \mathbf{d}) \to d_m$$

again a contradiction. An analogous argument is used in case $d_l = d_k$.

Finally we show that $h : a_i \longmapsto c_i$ is an embedding of $\mathbf{F}$ in $\mathbf{F}_1$. Since we have already seen above that the map is injective we have to show that it is a homomorphism. Suppose that in $\mathbf{F}$ for $* \in \{\to, \wedge\}$ and a_i, a_j we have $a_i * a_j = a_k$ but $(c_i * c_j) \leftrightarrow b_k \neq 1$; then, since $x_i * x_j \approx x_k \in \mathrm{Diag}_{\mathbf{A}}(\mathbf{x}, \mathbf{y})$, we get

$$1 = (c_i * c_j \leftrightarrow c_k) \to d_m \leq T(\mathbf{c}, \mathbf{d}) \to d_m$$

which again contradicts out hypothesis.

So $\mathbf{F}$ is embeddable in $\mathbf{F}_1$; on the other hand we have proved that $\mathbf{S}_1$ has at least $m+1$ elements, therefore $\mathbf{S}$ is embeddable in $\mathbf{S}_1$. In conclusion $\mathbf{F}\oplus\mathbf{S}$ embeds in $\mathbf{B}_1$ and hence in $\mathbf{B}$. ∎

Now we are ready to prove the main theorem of this section.

THEOREM 7.8. *Let* V *be a variety of* HBCK*-semilattices and let* $\mathbf{A} = \mathbf{F}\oplus\mathbf{S}$ *be a finite subdirectly irreducible algebra in* V*. Then* $\mathbf{A}$ *is splitting in* V *with splitting equation* $J_{\mathbf{A}}(\mathbf{x},\mathbf{y}) \approx 1$*. Moreover the conjugate variety is* $\mathsf{V}_{\mathbf{A}} = \{\mathbf{B} \in \mathsf{V} : \mathbf{A} \notin \boldsymbol{HS}(\mathbf{B})\}$*.*

PROOF. Let $F = \{a_1,\dots,a_n\}$ and $S = \{b_1,\dots,b_n\}$; observe that $D_{\mathbf{A}}(\mathbf{a},\mathbf{b}) = T_{\mathbf{A}}(\mathbf{a},\mathbf{b}) = 1$ so $\mathbf{A} \not\models J_{\mathbf{A}}(\mathbf{x},\mathbf{y}) \approx 1$. Therefore if $\mathbf{B} \models J_{\mathbf{A}}(\mathbf{x},\mathbf{y}) \approx 1$, then $\mathbf{A} \notin \boldsymbol{HS}(\mathbf{B})$.

So suppose that $\mathbf{B} \not\models J_{\mathbf{A}}(\mathbf{x},\mathbf{y}) \approx 1$; then there are $c_1,\dots,c_n,d_1,\dots,d_m \in B$ with $J_{\mathbf{A}}(\mathbf{c},\mathbf{d}) \neq 1$. Let $\mathbf{B}_1$ be the subalgebra of $\mathbf{B}$ generated by $\{c_1,\dots,c_n, d_1,\dots,d_m\}$, and let θ be maximal in the set $\{\alpha \in \mathrm{Con}(\mathbf{B}_1) : (d_m,1) \notin \alpha\}$. Then $\mathbf{B}_1/\alpha$ is subdirectly irreducible and

$$J_{\mathbf{A}}(c_1/\theta,\dots,c_n/\theta,d_1/\theta,\dots,d_m/\theta) \neq 1.$$

By Lemma 7.7,

$$\mathbf{A} \in \boldsymbol{S}(\mathbf{B}_1) = \boldsymbol{SHS}(\mathbf{B}) \subseteq \boldsymbol{HS}(\mathbf{B}).$$

This proves the thesis. ∎

8. But If We Have a Join ...

Clearly Theorem 6.1 tells also that every finite subdirectly irreducible algebra in $\mathsf{BBCK}^\wedge$ is splitting in $\mathsf{BBCK}^\wedge$ but this is not a great novelty since it is easy to verify that the results on BBCK-algebras extends easily to BBCK-semilattices. Surely the possibility of writing a splitting equation for any finite chain in $\mathsf{BBCK}^\wedge$ is an advantage, but it is obvious that the equations so obtained are awkward, difficult to use in practice and do not say much about the algebra itself. For $\mathsf{HBCK}^\wedge$ they were relevant, since they allowed us to prove that every finite subdirectly irreducible algebra in $\mathsf{HBCK}^\wedge$ is splitting, a thing that was far from obvious.

However in $\mathsf{BBCK}^\wedge$ there is also a term definable join operation and this allows some progress. Let λ'_n be the following equation:

$$\bigwedge_{i=1}^{n}((x_{i+1} \to x_i) \to x_i) \leq \bigvee_{i=1}^{n+1} x_i \qquad (\lambda'_n)$$

Lemma 8.1. *Let* $\mathbf{A}$ *be a* BBCK*-chain and let* $\bigoplus_{i\in I}\mathbf{A}_i$ *be a decomposition of* $\mathbf{A}$ *into sum irreducible components; then* $|I|\le n$ *if and only if* λ'_n *holds in* $\mathbf{A}$.

Proof. Suppose that $|I|\le n$; then if $a_1,\ldots,a_{n+1}\in A$ either $a_i=1$ for some $i\le n+1$ or else there exists a $k<n$ such that a_k,a_{k+1} lie in the same Łukasiewicz component. In the first case the right hand side of λ'_n is 1, so the equation holds. In the second case $(a_{k+1}\to a_k)\to a_k=a_k\vee a_{k+1}$ and thus

$$\bigwedge_{i=1}^{n}((a_{i+1}\to a_i)\to a_i)\le a_k\vee a_{k+1}\le\bigvee_{i=1}^{n+1}a_i$$

so λ'_n holds as well. Conversely if $|I|>n$ we can pick $a_1\ll\cdots\ll a_n\ll a_{n+1}\ll 1$; so $(a_{i+1}\to a_i)\to a_i=a_i\to a_i=1$ and thus

$$\bigwedge_{i=1}^{n}((a_{i+1}\to a_i)\to a_i)=1\ne a_{n+1}=\bigvee_{i=1}^{n+1}a_i.$$

■

Hence λ'_n can replace λ_n as the splitting equation of $(n+1)\mathbf{Ł}_n^{\wedge}$ in $\mathsf{BBCK}^{\wedge}$. However having a join has a more relevant consequence; given a finite $\mathsf{BBCK}^{\wedge}$-chain $\mathbf{A}=\mathbf{Ł}_{n_1}\oplus\cdots\oplus\mathbf{Ł}_{n_k}$ and reduced index $r\ge 1$, we define for each $s\le r$ an algebra $\mathbf{A}(s)$ (of index k) in the following way:

$$\mathbf{A}(s)_i=\begin{cases}\mathbf{Ł}_1, & \text{if} n_i\ne m_s;\\ \mathbf{Ł}_{n_s}, & \text{if} n_i=m_s.\end{cases}$$

Clearly each $\mathbf{A}(s)$ is a chain of reduced index equal to 1 and so by Theorem 6.9 has a splitting equation $s^{\mathbf{A}(s)}(x_1,\ldots,x_{k+1})\approx 1$; we define $s^{\mathbf{A}}(x_1,\ldots,x_{k+1}):=\bigvee_{s=1}^{r}s^{\mathbf{A}(s)}(x_1,\ldots,x_{k+1})$.

Theorem 8.2. *Let* $\mathbf{A}$ *be a finite* $\mathsf{BBCK}^{\wedge}$*-chain of index* k *and reduced index* $r\ge 1$*; then a splitting equation for* $\mathbf{A}$ *is*

$$s^{\mathbf{A}}(x_1,\ldots,x_{k+1})\approx 1.$$

Proof. Let $\mathbf{B}\in\mathsf{V}_{\mathbf{A}}$ and let $a_1,\ldots,a_{k+1}\in B$; the same argument as in Theorem 6.9 shows that if $a_1\ll a_2\ll\cdots\ll a_{k+1}$ or if $a_i=1$ for some $i\le k+1$ or if $a_{i+1}\ll a_i$ for some i, then $s^{\mathbf{A}(s)}(a_1,\ldots,a_{k+1})=1$ for every $s\le r$ and so $s^{\mathbf{A}}(a_1,\ldots,a_{k+1})=1$.

So we may assume that there is a j with $a_j\sim a_{j+1}$ hence there are $B_{i_i},\ldots,B_{i_k}$ such that

$$a_1,\ldots,a_{k+1}\in B_{i_1}\oplus\cdots\oplus B_{i_k}.$$

Since $\mathbf{B} \in \mathsf{V}_{\mathbf{A}}$ there must be an $s \leq r$ such that $\mathbf{Ł}_{n_s}$ is not embeddable in $\mathbf{B}_{n_s}$. Then $\mathbf{B}_{i_1} \oplus \cdots \oplus \mathbf{B}_{i_k} \in \mathsf{V}_{\mathbf{A}(s)}$ and by Lemma 6.8 $s^{\mathbf{A}(s)}(a_1, \ldots, a_{k+1}) = 1$. This implies that $\mathbf{B} \vDash s^{\mathbf{A}(s)}(x_1, \ldots, x_{k+1})) \approx 1$ and so that

$$\mathsf{V}_{\mathbf{A}} \vDash s^{\mathbf{A}(s)}(x_1, \ldots, x_{k+1}) \approx 1.$$

To prove that $\mathbf{A} \nvDash s^{\mathbf{A}(s)}(x_1, \ldots, x_{k+1}) \approx 1$ we will show that for any $s \leq r$ we can produce $a_1, \ldots, a_{k+1} \in A$ such that $s^{\mathbf{A}(s)}(a_1, \ldots, a_{k+1}) \neq 1$. Let $s \leq r$ with $m_s = n_j$; we pick $a_1, \ldots, a_{k+1} \in A$ in the following way:

$$\text{if} i < j, then a_i \in Ł_{n_i}$$
$$\text{if} i = j, then a_j, a_{j+1} \in Ł_{n_j} with j_{n_j}(a_j, a_{j+1}) \neq 1$$
$$\text{if} j < i \leq k+1, then a_i \in Ł_{n_i - 1}$$

Since we picked only one element from each $\mathbf{Ł}_{n_i}$ except for $i = j$ we may suppose that $a_1, \ldots, a_{k+1}$ belong to (an algebra isomorphic with) $\mathbf{A}(s)$. Then by Theorem 6.9 we conclude that $s^{\mathbf{A}(s)}(a_1, \ldots, a_{k+1}) \neq 1$. Since s was generic this proves the thesis. ∎

Finally we observe that BCK-algebras with a join operation have been considered in the literature [18], [27]; such algebras form a variety $\mathsf{BCK}^{\vee}$ axiomatized by the axioms of BCK-algebras, those that say that $\vee$ is a semilattice operation and

$$x \to (x \vee y) = 1.$$

Now, since $a \to (a \vee b) = 1$, the congruence lattices of any algebra therein coincide with the congruences of the underlying BCK-structure. Therefore congruences are determined by BCK-filters and, since the filter lattices of BCK-algebras are congruence distributive, $\mathsf{BCK}^{\vee}$ is a congruence distributive variety.

For the same reason the variety of $\mathsf{BCK}^{\vee}$-algebras that are representable is axiomatized (relative to $\mathsf{BCK}^{\vee}$) by the equation (B) and $\mathsf{BBCK}^{\vee}$-algebras are simply $\mathsf{BCK}^{\vee}$-algebras satisfying (B) and (H). It turns out (and it is easily proved) that $\mathsf{BBCK}^{\vee}$-algebras are exactly the $\{\vee, \to, 1\}$ subreducts of basic hoops and of course any finite finite $\mathsf{BBCK}^{\vee}$-chain is splitting in $\mathsf{BBCK}^{\vee}$.

The situation for $\mathsf{HBCK}^{\vee}$-algebras is different, since they are not subreducts of hoops, but rather of divisible and commutative residuated *lattices*; this class deserves further investigation.

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P. Aglianò
University of Siena
Siena
Italy
agliano@live.com