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HARRODIAN INSTABILITY AND LEARNING

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It is a standard result in macro-dynamics that the actual law of motion of a model economy depends on the law of motion perceived by the agents populating the model. This yields a sort of indeterminacy of the prevailing dynamics. In particular, the nature of the shared beliefs, not less than the deep structural parameters, will often decide whether the actual dynamics may or may not exhibit systematic forecasting errors (Bullard [1994]), whether it is likely to converge to a perfect-foresight steady state and, possibly, which steady state is selected (see, for instance, the contributions in Kirman and Salmon [1995]).

Since agent's learning makes beliefs endogenous, introducing this feature of human activity into macroeconomic models may avoid indeterminacy, when it can show which types of beliefs are more likely to prevail. Moreover, if beliefs are attracted by the set of forecast rules leading to asymptotically-correct predictions, there is a sense in which belief formation can be regarded as self confirming.

The paper attempts at clarifying the difficulties encountered by this line of argument, when the forecasts of a variable are generated by expectation functions defined on a space of realized values of the variable, and learning is based on gradient-descent procedures. The analysis is carried out with reference to one of the first macroeconomic models in which the problem made itself apparent. More precisely, we reconsider the local stability of a steady state of a simple multiplier-accelerator model of Harrodian inspiration.

As is well known, under the assumption of extrapolative growth expectations, the macro-dynamic interaction between the instantaneous Keynesian multiplier and the simple accelerator causes the instability of the path on which entrepreneurs are satisfied of their decisions ('warranted path'). This corresponds to what came to be known as Harrod's *knife-hedge*, a notion which had a profound influence on growth theory and macrodynamics, more generally.

To justify the viewpoint adopted in this paper, it is worth recalling how Harrod himself was inclined to stress that the average propensity to save is not generally constant, but is more likely to be increasing in income. By implication, also the rate of growth on the warranted path (warranted rate of growth) is generally non constant. In particular, there may be points on a warranted path such that the warranted rate of growth coincides with the population growth rate[1]. Still, in so far as the warranted path is unstable, the actual path of the economy would be repelled from such steady states.

This paper considers the local dynamics around steady states on a warranted path with a view to address a number of related questions:

As a first step, we assume that agents (firms) do not know the structure of the economy and face the problem of predicting the future income level on the base of past realizations thereof. We identify a set of restrictions on economic structure and expectation functions such that a steady state is stable locally, when the income dynamics is driven by such 'stabilizing' expectations. Within the same framework, we can also reproduce the known result that extrapolative growth expectations make a steady state unstable. As emphasized by the literature on expectations and macroeconomics, these findings define the typical, but uncomfortable situation, where stability and instability come to depend upon the nature of exogenous beliefs.

The indeterminacy thus identified would be less disturbing if the set of 'stabilizing expectations' proves to be at least locally attractive in the appropriate belief space, when the income dynamics is coupled with that resulting from the endogenous revision of expectation functions[2]. In this paper belief revision takes place through a form of gradient-descent learning, a procedure extensively applied in the learning literature, including that on neural networks. The main findings are as follows.

Locally, if the rate of learning is positive and sufficiently low, or is controlled by appropriate euristics, income converges to the steady state and the parameters of the expectation functions converge to a set on which steady states are properly detected.

The same local analysis shows, however, that the set of 'stabilizing expectations' does not attract every belief trajectory starting in its neighbourhood, even when the definition of neighbourhood is particularly restrictive.

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[1] J. Marschack, commenting on Harrod [1939], hints at this possibility in his graphical illustration of the fluctuations of the warranted growth rate in the course of a trade cycle. See Young [1989].

[2] The difficulty here is that the attempt at explaining the otherwise exogenous parameters of a prediction rule, poses a new decision problem (e. g., the choice of a learning algorithm) and calls new parameters into being. Thus, we are faced with an apparently-infinite hierarchy of choice problems; still, every time we proceed further along this hierarchy, the arbitrariness of the primary choice tends to be lower and we may be able to give a more complete explanation thereof.

This lack of attractiveness of the set of stibilizing expectations is related to the way in which forecasting and learning are characterized within the model. Firms apply and then adaptively revise an expectation function mapping past output realizations into future output. However, the true function from past output relizations to future output depends on the forecast rule in use and changes with every change of this rule. Self referentiality of this sort implies that belief-revising firms are systematically chasing a shifting function, except the economy is in a perfect-foresight steady state and the expectation function in use correctly detects such a state.

In the situation just described, learning firms do not have at their disposal large sample relizations of a given true output function defined on the domain contemplated by the class of the forecast rules under use. This would *not* be necessarily the case if learning firms aim in the first place at knowing how the true law of motion of the economy depends on expectations, and then use their understanding of it to make forecasts. To clarify some implications of this point, in the final section we consider a situation where information on aggregate output forecasts is available, or can be inferred from data on the aggregate capital stock and from partial knowledge of the structural parameters. In this case, the learning environment is 'simple' in the sense that firms have access to the set of past sample realizations of the time-invariant true law of motion of the economy. Formal results from the field of neural networks can be used to show that the application of a (suitably defined) gradient-descent procedure to this 'simple' environment can avoid the lack of local convergence of beliefs encountered with the previous formulation of the problem.

Successful learning of the way in which expectations enter the output true law of motion does not solve firms' forecasting problem, but clarifies how this problem shares fundamental properties with a co-ordination game.

The plan of the paper is as follows. Part 1 introduces a simple multiplier-accelerator model and discusses its perfect-foresight dynamics in the neighbourhood of a per-capita-income steady state. Part 2 is concerned with the local stability of a steady state when the local dynamics is driven by given expectations functions. The expectation function $\Psi(\cdots)$ is defined here as a function from observations (lagged values of economic variables) to forecasts. It can be interpreted as the law of motion perceived by the boundedly-rational agent. Part 3 considers the coupled local dynamics of income and expectation functions when boundedly-rational firms try to learn from their past errors. The concluding section discusses the situation arising when gradient-descent learning procedures are applied by firms endowed with information on aggregate forecasts.

1. Fulfilled predictions

Let us consider a simple one-good economy with fixed prices. The existing good can be either consumed or used as a mean of production; in this case it depreciates at a constant proportional rate δ. Capital and labour are partial substitutes in production. For the sake of simplicity we hold to a descriptive approach to consumption which is partly reminiscent of Kaldor [1940]. Consumption per-capita C depends on income per-capita y and may also depend on wealth *per-capita* k, which is here identified with per-capita real capital.

Let us therefore introduce the per-capita aggregate consumption function $C = C(k, y)$ with partial derivatives $\partial C/\partial k \equiv C_1 \geq 0$, $\partial C/\partial y \equiv C_2 > 0$.

There is a given number of firms, each with a fixed market share. These firms make positive profits at the ongoing prices, so that their output is rationed by demand. Since prices and distribution are constant, and we abstract from adjustment costs, the desired long-run output-capital ratio γ is also constant. Still, capital adjustment takes time and firms operate, in the short-run, under the constraint of a given capital stock. For this reason, market demand is met in the short-run through profit-maximizing deviations from the long-run optimal factor proportion. In what follows, a position of the economic system is said to be fully adjusted if and only if the actual output-capital ratio is equal to its desired level γ.

Let K_t and Y_t be the aggregate capital stock and the aggregate output at time t. A warranted path, as defined in this paper[3], identifies a sequence of fully adjusted positions $\{K_t, Y_t\}$ such that Y_t is the good-market equilibrium output corresponding to gross investment $K_{t+1} - (1 - \delta)K_t$.

[3] We abstract from more general definitions of a warranted path, where the initial position $\{K_0, Y_0\}$ may not be fully adjusted.

Since K_t can be expressed in terms of the aggregate-output prediction Y^e_t, taken at beginning of period t−1, aggregate gross investment is

$K_{t+1} - (1-\delta)K_t = Y^e_{t+1}/\gamma - (1-\delta)Y^e_t/\gamma$

If population grows at the constant proportional rate θ−1, the same relation can be written in per-capita terms as:

$\theta k_{t+1} - (1-\delta)k_t = \theta y^e_{t+1}/\gamma - (1-\delta)y^e_t/\gamma$

The equilibrium condition in the market for goods is

$y_t = C(y^e_t/\gamma, y_t) + \theta y^e_{t+1}/\gamma - (1-\delta)y^e_t/\gamma$ (1)

or, in implicit form

$T(y_t, y^e_t, y^e_{t+1}) = 0$ (2)

On the assumption that the partial derivative $\partial T/\partial y_t \neq 0$, the implicit function theorem makes it sure that y_t is a well defined function of y^e_t, y^e_{t+1}.

$y_t = F(y^e_{t+1}, y^e_t)$ (2.bis)

A warranted path is then obtained as the perfect-foresight dynamics resulting from the substitution of y_t, y_{t+1} for y^e_t, y^e_{t+1} into (1) or (2).

$y_{t+1} = G(y_t) \equiv [\gamma(y_t - C(y_t /\gamma, y_t)) + (1 - \delta)\, y_t]\theta^{-1}$ (3)

The corresponding formulation of the warranted rate of growth is:

$g(y_t) \equiv (Y_{t+1}/Y_t) - 1 = (\theta y_{t+1}/y_t) - 1 = (\theta G(y_t)/ y_t) - 1$ (4)

At a point y on a warranted path, the total derivative of $C(\cdot,\cdot)$ with respect to y is:

$dC/dy = C_1(y/\gamma, y)/\gamma + C_2(y/\gamma, y)$ (5)

which can be interpreted as the long-run marginal propensity to consume.

When predictions are fulfilled, the average propensity to consume is $c(y) \equiv C(y/\gamma, y)/y$. The case of a constant $c(y) = c$ corresponds to a particular situation where $dC/dy = c$. We may observe, in passing, that the empirical finding of an approximately constant ratio C/y does not mean that the *function* $c(y)$ is a constant, but may simply reflect observations in a neighbourhood of a steady state. Moreover, it turns out that even minor deviations of $c(y)$ from the non-generic constant case may have relevant implications for warranted growth. In what follows, it is assumed that in fully adjusted positions (that is, on a warranted path) a sufficiently-large (-low) per-capita income gives rise to a savings behaviour such the warranted rate of growth exceeds (is lower than) the population growth rate. This fits with the stylized fact that sufficiently higher income and wealth are normally associated to a higher propensity to save[4]. For the sake of later reference, this is stated as:

A.1. There is a finite y_M such that $g(y) > \theta - 1$ if $y > y_M$. There is $y_m > 0$ sufficiently small such that $g(y) < \theta - 1$ if $y < y_m$.

There is of course a continuum of 'warranted trajectories', that is, solutions to (3) parametrized by a non negative initial condition y_0.

y^* is a positive stationary equilibrium with perfect-foresight if and only if

$$g(y^*) \equiv \gamma[1 - c(y^*)] - \delta = \theta - 1 \qquad (6)$$

y^* is locally stable if $|G'(y^*)|<1$ and unstable if $|G'(y^*)|>1$.

If we had a constant $c(y)$, we would be in a situation where, either positive stationary equilibria do not exist, or there exists a continuum of them. The latter case must be regarded as an irrelevant fluke, but the condition $c(y) = c$ is itself non generic. A.1 implies that $c(y)$ is not constant; in particular, we have the following trivial proposition:

P.1. *Assume A.1. Then there exists a positive stationary equilibrium y^*; that is, $g(y^*) = \theta - 1$. The curve $(y_t, G(y_t))$ intersects the 45° line $y_{t+1} = y_t$ from below at y^*. Thus, $G'(y^*)>1$, and y^* is unstable under perfect foresight. A small upward shift of $c(y)$ gives rise to an increase in the level of y^*.*

Assumption A.1 is consistent with a locally-increasing average propensity to consume at a relatively low income y[5]. The possibility may arise for instance from the aggregation of interpersonal differences in consumer behaviour, and is not ruled out for the sake of generality. The interesting fact about a non-monotonic $c(y)$ is that it gives rise to the possibility of multiple steady-state equilibria. These may well originate from small-amplitude fluctuations of $c(y)$ around a constant value, hence even with an approximately-constant propensity to consume for the economy as a whole. To fix our ideas, let us consider the following case:

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[4] Harrod [1939], p. 25; Kaldor [1940].

[5] A locally-increasing specification of $c(y)$ gives rise to the possibility of endogenous fluctuations of the warranted trajectories only under the highly unrealistic assumption that per-capita output in one period is a locally *decreasing* function of per-capita output in the previous period. (In fact, the required condition is $G'(y^{**})<-1$).

A.2. There exists $y^{**} < y^*$ such that $g(y^{**}) = \theta - 1$ and the curve $(y_t, G(y_t))$ intersects the 45° line $y_{t+1} = y_t$ from above at y^{**}.

Under A.1, A.2 equation (4) has a multiplicity of stationary states. They can be partitioned into type-y^* and type-y^{**} stationary states. The highest stationary state is of type y^*, by A1. The dynamic and comparative -static properties of a type-y^{**} state are described by the following straightforward proposition:

P.2. *Assume A.1 and A.2. y^{**} is locally stable (in the perfect-foresight dynamics) if $y^{**} \gamma c'(y^{**})\theta^{-1} < 2$. This condition is necessarily fulfilled if $G'(y^{**}) > 0$, as assumed throughout in this paper. A small upward shift of $c(y)$ gives rise to an decrease in the level of y^{**}.*

We may observe, in passing, that the effect of a parametric change of the propensity to consume on steady-state output depends crucially on which steady state is selected. It is also worth stressing that assumption A.2 identifies a mere possibility, which can not be ruled out by theoretical considerations. Thus, if a type-y^{**} steady state may well not exist, a y^* steady state is much more in line with stylized facts and indeed is the implicit reference point of the macro-dynamic trade cycle literature of the forties and fifties.

2. Dynamics with fixed expectation functions

A crucial issue posed by the foregoing analysis is whether the local stability properties of the perfect-foresight dynamics characterizing a warranted trajectory, may reveal information of some kind on the stability properties of the dynamics which is induced when firms do not know the true structure of the economy and make mistakes in predictions. In particular, we may wonder about the stability property of a stationary state like y^* or y^{**} under plausible forms of expectation formation.

Let us consider a stationary state $\underline{y}$ of the perfect-foresight dynamics $y_{t+1} = G(y_t)$. Agents in the economy are boundedly rational. They locally form predictions on the base of past observations. Let y^e_{t+1} be the prediction on y_{t+1} made at the beginning of period t.

$y^e_{t+1} = \Psi(y_{t-1}, ..., y_{t-L})$ (7)

It is worth emphasizing how the first argument in (7) is y_{t-1}, rather than y_t. Since the capital stock can not be changed instantaneously[6], investment decisions in period t depend on y^e_{t+1} and determine current demand y_t. When firms form their prediction on y_{t+1} they can not observe y_t.

The expectation function $\Psi(\ )$ is said to detect period k (Grandmont and Laroque [1986], [1989], [1990]) if for any sequence $(y_{t-1},..., y_{t-L})$ of prime period k, $\Psi(y_{t-1},..., y_{t-L}) = y_{t-k+1}$. In part 2. of this paper it is assumed that $\Psi(\ )$ detects *at least* period 1; in other words, agents are prepared to extrapolate constant sequences, *i. e.* $\Psi(\underline{y},..., \underline{y}) = \underline{y}$. This restriction appears to be quite weak and is equally consistent with cautious or daring behaviour in expectation formation. For instance, while prepared to extrapolate a constant sequence an agent may or may not be prepared to extrapolate a growing or periodic sequence. The assumption will be more thoroughly motivated in part 3 of this paper.

Remark 1: On the assumption that firms extrapolate the last-observable growth performance $y^e_{t+1} = y_{t-1} (1 + (y_{t-1} - y_{t-2}) / y_{t-2})^2$. Thus, $\Psi(y_{t-1}, ..., y_{t-L})$ takes the form: $\Psi(y_{t-1}, ..., y_{t-L}) = (y_{t-1})^3 / (y_{t-2})^2$.

$\Psi(\ )$ is assumed to be continuous and differentiable; its partial derivative with respect to y_{t-j} evaluated at $(\underline{y}, \underline{y}, ..., \underline{y})$ is Ψ_j, for j = 1, ..., L. Since $\Psi(\ )$ detects period 1, $\sum_1^L \Psi_j = 1$. One can also write:

$(y^e_{t+1}, y_{t-1},..., y_{t-L+1})' = Q((y_{t-1},..., y_{t-L})')$.

The Jacobian matrix **DQ** of Q( ) evaluated at $(\underline{y}, \underline{y}, ..., \underline{y})$ and here shown for the case L=4 is:

$$\mathbf{DQ} \equiv \begin{bmatrix} \Psi_1 & \Psi_2 & \Psi_{L-1} & \Psi_L \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad (8)$$

with characteristic polynomial:

[6]The fact that consumption at time t depends on income at time t and meanwhile contributes to determine the market-clearing output at the same date, reflects the simplifying convention followed in this paper, which describes consumer decisions, the adjustment of labor input and production as instantaneous. This assumption makes the model more exposed to dynamic instability and is therefore admissible in the context of the present attempt at showing if and to what extent Harrodian instability is inhibited by learning.

$P_Q(z) = z^L - \sum_1^L \Psi_j z^{L-j}$ (9)

In a neighbourhood of $\underline{y}$, the actual dynamics driven by the expectation function (7) is obtained from the equilibrium condition (1) or (2) substituting for y^e_{t+1} and y^e_t with the expressions $\Psi(y_{t-1},...,y_{t-L})$ and $\Psi(y_{t-2},..., y_{t-L-1})$, respectively.

$y_t = C(\Psi(y_{t-2},..., y_{t-L-1})/\gamma, y_t)+\theta\Psi(y_{t-1},..., y_{t-L})/\gamma -(1-\delta)\Psi(y_{t-2},..., y_{t-L-1})/\gamma$ (10)

The actual dynamics can be equivalently written

$(y_t, y_{t-1},..., y_{t-L})' = W(y_{t-1}, y_{t-2},..., y_{t-L-1})'$ (11)

Since $\Psi()$ detects period 1, $(\underline{y}, \underline{y}, ..., \underline{y})$ is a stationary point of (11). The characteristic polynomial of the Jacobian matrix of (11) evaluated at $(\underline{y}, \underline{y}, ..., \underline{y})$ is:

$P_W(z) = z^{L+1} - \sum_1^L \Psi_j z^{L-j}[\theta z-(1-\delta-C_1)]/[\gamma(1- C_2)]$ (12)

where $C_1 \equiv C_1(\underline{y}/\gamma, \underline{y})$ and $C_2 \equiv C_2(\underline{y}/\gamma, \underline{y})$.

The instability of the perfect-foresight dynamics (3) around a steady state like y^{**} is established here following upon a suggestion developed for a different type of environment by Grandmont and Laroque [1986], [1989], [1990] and Grandmont [1994].

P.3. *Let $\underline{y}$ be a locally stable stationary equilibrium of the perfect-foresight dynamics (3). Assume that the expectation function (7) detects period 1. The stationary equilibrium $(\underline{y}, \underline{y}, ..., \underline{y})$ of the actual dynamics with learning induced by this expectation function is unstable.*

Proof: Since $\underline{y}$ is locally stable under perfect-foresight, and realism requires $\theta > 0$, $0 < C_2 < 1$, $1 - \delta - C_1 > 0$, it must be the case that:
$0 < G'(\underline{y}) = [\gamma(1 - C_2) + 1 - \delta - C_1]\theta^{-1} < 1$ and therefore $[\theta-(1-\delta- C_1)]/[\gamma(1- C_2)] > 1$.
The expectation function $\Psi()$ detects period 1, hence 1 is a root of $P_Q(z)$. Re-arrangement of (9) yields $-\sum_1^L \Psi_j z^{L-j} = P_Q(z) - z^L$.
Thus, $P_W(z) = z^{L+1} + (P_Q(z) - z^L)\,[\theta z-(1-\delta- C_1)]/[\gamma(1- C_2)]$. At $z = 1$ this expression boils down to: $P_W(1) = 1 - [\theta-(1-\delta- C_1)]/[\gamma(1- C_2)] < 0$
Let z vary on the real line. If the modulus of z is sufficiently large, the term z^{L+1} dominates the sign of the polynomial $P_W(z)$, hence there exists a real number $z° > 1$ such that $P_W(z°) > 0$. Since $P_W(z)$ is a continuous function of z, it must cross $P_W = 0$ at a point $z > 1$ of the real line. This proves the proposition.

Remark 2: The proof of P.3 relies upon the implicit assumption that the number of arguments of the expectation function (7) is finite. However, this assumption can be dropped without invalidating the instability of $(\underline{y}, \underline{y}, ..., \underline{y})$ stated by P.3. For instance, let $\overline{y}^e_t$ be the prediction on y_t made at the beginning of period t, and assume:

$\overline{y^e_t} = \lambda\, y_{t-1} + (1-\lambda)\overline{y^e_{t-1}} = \sum_{i=1}^{\infty} \lambda\, y_{t-i}\,(1-\lambda)^{i-1}$; $y^e_{t+1} = \overline{y^e_t}$

This assumption on expectations is clearly consistent with (7) if L = ∞. It can be shown that, if $\underline{y}$ is locally stable under perfect foresight (like in P.3), then it is unstable in the actual dynamics induced by the adaptive expectations specified above.

In synthesis, and ruling out totally unplausible circumstances, a steady state of type *y*** is unstable for *any* expectation function of the form (7). We now turn our attention to a steady state like *y**.

Let $q \equiv \theta / \gamma(1-C_2) > 0$; $r \equiv -(1-\delta-C_1)/\gamma(1-C_2) < 0$.

Pw(*z*) can be written as:

$$Pw(z) = z^{L+1} - q\Psi_1 z^L - \sum_1^{L-1} (q\Psi_{i+1} + r\Psi_i)\, z^{L-i} - r\Psi_L$$

P.4 *Let $\underline{y}$ be an unstable stationary state of the perfect foresight dynamics (4). Assume that the expectation function Ψ( ) detects period 1, that each partial derivative Ψ_j, (with j = 1, ..., L) evaluated at $\underline{y}$ is sufficiently close to 1/L, and L is sufficiently large. The stationary equilibrium ($\underline{y}$, ..., $\underline{y}$) of the actual dynamics induced by Ψ( ) is locally stable.*

Remark 3: Since y* is unstable under perfect foresight and realism requires G'(y) > 0, then G'(y) > 1, that is $0 < q + r < 1$.

Proof of P.4: Under the stated assumptions, for any *z* on the complex plane, the following inequality holds true.

$|z^{L+1} - Pw(z)| = |q\Psi_1 z^L + \sum_1^{L-1} (q\Psi_{i+1} + r\Psi_i)\, z^{L-i} + r\Psi_L| \leq$

$q\Psi_1 |z^L| + \sum_1^{L-1} (q\Psi_{i+1} + r\Psi_i) |z^{L-i}| + |r\Psi_L| \equiv Z(z)$. Since $0 < q + r < 1$, Z(1) < 1, thus implying that $|z^{L+1} - Pw(z)| < 1$ for any *z* on the unit circle, or equivalently that Pw(z) has no roots on the unit circle. If Pw(z) had a real or complex root $\underline{z}$ on $|z| > 1$, then, as a consequence, $|z^{L+1} - Pw(z)| > 1$ at $\underline{z}$. That this can not be the case follows from the fact that, under the stated assumptions, $|z^{L+1}|$ grows faster than Z(*z*) as $|z|$ increases on $|z| \geq 1$. This completes the proof.

The expectation functions of the form (7) with partial derivatives evaluated at *y** which meet the assumptions stated under P.4 are called in this paper 'stabilizing expectations'[7]. Such expectations share the property of being little responsive to

[7]The local stability of *y** under 'stabilizing expectations' would justify comparative-statics considerations. In particular, a small upward shift of the propensity to consume would effectively increase the steady-state output per capita. We would have here a long-run equivalent of the standard

single deviations from average observation. The following remark clarifies how extrapolative expectations are not ‘stabilizing’.

Remark 4: Assume that $\Psi(y_{t-1}, ..., y_{t-L})$ corresponds to the case of extrapolative expectations as defined by remark 1. Then L = 2, $\Psi_1 = 3$, $\Psi_2 = -2$. The Jacobian matrix of (11) evaluated at y^* is

$$\begin{bmatrix} 3q & 3r-2q & -2r \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

and has at least one eigenvalue outside the unit circle for any plausible value of q and r meeting the restriction at remark 3.

Whether the prevalence of stabilizing expectations can or can not be regarded as the relevant case is a completely open issue. To gain some information on this point we drop the assumption that expectation functions are fixed and detect period 1 with a view to considering the coupled (local) dynamics of income and expectation functions, when the latter are revised in the light of the past forecasting errors. It is to this task that we now turn.

3. Adaptively changing expectation functions

3. 1.

Let us choose a local parametrization of the expectation functions (7) in terms of their partial derivatives Ψ_j at (y^*, ..., y^*). An expectation function held at t-1 is locally identified by a vector of expectation parameters, or weights ($\Psi_{1,\,t-1}, ..., \Psi_{L,\,t-1}$):

$$y^e_{t+1} = \Psi_{1,t-1}\, y_{t-1} + ... + \Psi_{L,t-1}\, y_{t-L} \qquad (13)$$

For the scope of the present analysis we can also linearize the per-capita consumption function around y*. Re-defining the partial derivatives C_1 and C_2 at y*, we have:

$$C = \sigma + C_1 k + C_2 y \qquad (14)$$

Recalling (1), the aggregate output y_t as function of the forecasts y^e_t, y^e_{t+1} is then:

$$y_t = (\sigma + (C_1 + \delta - 1)\, y^e_t /\gamma + \theta\, y^e_{t+1}/\gamma)/(1 - C_2) \qquad (15)$$

Keynesian multiplier in a model with endogenous investment. A small parametric increase of the propensity to consume may not permanently drive the economy out of its balanced-growth path (as or, equivalently,

$$y_t = p + r\sum_{j=1}^{L} \Psi_{j,t-2}\, y_{t-1-j} + q\sum_{j=1}^{L} \Psi_{j,t-1}\, y_{t-j} \qquad (16)$$

where $p \equiv \sigma /(1-C_2)$, $r \equiv (C_1 + \delta - 1) / (\gamma (1-C_2))$, $q \equiv \theta / (\gamma(1-C_2))$; moreover, the restatement of *remark 3* yields: $0 < q + r < 1$.

Learning, if any, takes place through the observation of the past squared prediction error E_t [8].

$$E_t = 1/2\, (y^e_t - y_t)^2 \qquad (17)$$

This information induces a revision of the vector $\mathbf{\Psi}$ to be used in the next prediction y^e_{t+2}.

$$\Psi_{j,t} = \Psi_{j,t-1} - h\, \partial E_t / \partial \Psi_{j,t-1} \qquad j = 1, ..., L \qquad (18)$$

This corresponds to a gradient-descent rule widely used in the learning literature, where the adjustment parameter h is often referred to as the rate of learning (Hassoun [1995], Mehra and Wah [1992]). In words, the information that an increase of $\Psi_{j,t-1}$ would have increased the squared error E observed at t, is taken as evidence that the appropriate level of $\Psi_{j,t}$ should be lower than $\Psi_{j,t-1}$.

The learning rule interprets y_t as the 'target' of the prediction y^e_t. To understand the nature of this 'target', it is worth stressing how the true model of the world is unknown and subjective forecasts affect the observed realizations of *y*. More formally, the arguments in the output true law of motion (2) and the definition of expectation function (7) show that the actual function mapping the output realizations (y_{t-1}, ..., y_{t-L}) into current output y_t changes with every change in the expectation function used to produce the forecast y^e_{t+1} and/or y^e_t. Self-referentiality of this sort implies that y_t is not the 'true' target of the prediction y^e_t. In other words, learning firms do not have at their disposal sample relizations of a given true output function defined on the domain contemplated by the class of forecast rules under use. Such a 'true' function $y_t = f(y_{t-1}, ..., y_{t-L})$ simply does not exist. For ease of reference, learning environments with this property are here referred to as 'complex' and are distinguished from 'simple' environments, where agents can observe the true targets

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implied by Harrod), but drive it to a growth path with identical growth rate and a *higher* income per capita (as opposed to what is the case in Solow's growth model).

[8] It may be worth observing that under any plausible circumstance, a perfect-foresight trajectory (3) can not have oscillations of any kind, periodic or aperiodic (see above, footnote 5). Thus, if $\{y_t\}$ is a bounded trajectory and E_t converges to zero, then $\lim_{t\to\infty} \Psi_t = y_{t+1} = y_t = y^*$. In other words, if $\{y_t\}$ does not explode, then E_t converges to zero *only if* $\lim_{t\to\infty} y_t = y^*$.

of their acts. We shall see below (section 4) how information on past aggregate expectations and a properly defined learning task can change the environment from complex to simple.

Complexity as defined above implies that the learning rule (18) has a further property deserving special attention. This is that firms are not in the position of computing the 'true' derivative $\partial E/\partial w$ in (18), but only an approximation thereof. A plausible assumption is that firms disregard the unknown relation between the signal y_t and the vector $\mathbf{\Psi}$. Thus, y_t is treated *as if* it were an exogenous signal (this approximation has a robust motivation in 'large' economies) and the learning rule (18) is approximated by:

$$\Psi_{j,t} = \Psi_{j,t-1} - h(y^e_t - y_t)\, y_{t-j-1} \qquad j = 1, ..., L \qquad (19)$$

The coupled dynamics of income and expectation functions is then locally described by the following system of 3L + 1 equations:

$$y_t = p + r\textstyle\sum_{j=1}^{L} \Psi_{j,t-2}\, y_{t-1-j} + q\sum_{j=1}^{L} \Psi_{j,t-1}\, y_{t-j} \qquad (20.1)$$

$$y_{t-j} = y_{t-j} \qquad j = 1, ..., L \qquad (20.2)$$

$$\Psi_{j,t} = \Psi_{j,t-1} + h\,(y_t - \textstyle\sum_{j=1}^{L} \Psi_{j,t-1}\, y_{t-j-1})\, y_{t-j-1} \qquad j = 1, ..., L \qquad (20.3)$$

$$\Psi_{t-j} = \Psi_{t-j} \qquad j = 1, ..., L \qquad (20.4)$$

This is more compactly written as

$$\mathbf{x}_t = V_h(\mathbf{x}_{t-1}) \qquad (21)$$

When $h = 0$ expectation functions are fixed and we are back to the case already considered by proposition 4 of section 2. We call $\{\mathbf{\Psi^*}\} \subset \Re^L$ the set of vectors of the form $(\Psi_1, ..., \Psi_L)$ such that y^* is locally stable in the actual dynamics (20.1), (20.2) under the restriction $\mathbf{\Psi}_t = \mathbf{\Psi}_{t-1}$. $\{\mathbf{\Psi^*}\}$ corresponds to the set of stabilizing expectations, introduced in our comment to proposition 4 of section 2. Clearly, every vector in $\{\mathbf{\Psi^*}\}$ is such that $\sum \Psi_j = 1$; hence $\{\mathbf{\Psi^*}\}$ is a subset of the unit simplex S^L of $\Re^L$. The question posed at the beginning of this paper amounts to asking whether, with endogenous expectation formation, hence at $h > 0$, the local dynamics (20) drives expectation parameters to the set of stabilizing expectations, and, simultaneously, income to the steady state y^*. More formally, the question is whether the set $\boldsymbol{Q} \equiv \boldsymbol{y}^* \otimes \{\boldsymbol{\Psi}^*, \boldsymbol{\Psi}^*\}$, where $\boldsymbol{y}^* \equiv (y^*, ..., y^*) \in \Re^{L+1}$ and $\boldsymbol{\Psi}^* \in \{\boldsymbol{\Psi}^*\}$, is a local attractor of the dynamics (20) under $h > 0$.

A moment reflection reveals that this can not be the case. If initial beliefs are described by *any* expectation vector $\boldsymbol{\Psi}_0$ in S^L, then $(y^*, ..., y^*, \boldsymbol{\Psi}_0, \boldsymbol{\Psi}_0)$ is a rest point of (20). The reason is, of course, that, a stationary state would be detected without error by any such expectation function $\boldsymbol{\Psi}_0$; thus, firms would not have any incentive to revise their beliefs.

The above property is mirrored by a salient feature of the Jacobian matrix $\mathbf{J}(h)$ of (21) evaluated at a point in $\boldsymbol{Q}$.

$$\mathbf{J}(h) = \begin{bmatrix} A & B \\ F(h) & G(h) \end{bmatrix}$$

where $\mathbf{A} \in \Re^{(L+1)} \otimes \Re^{(L+1)}$, $\mathbf{B} \in \Re^{L+1} \otimes \Re^{2L}$, $\mathbf{F}(h) \in \Re^{2L} \otimes \Re^{L+1}$, $\mathbf{G}(h) \in \Re^{2L} \otimes \Re^{2L}$. These sub-matrixes are shown below for the case L = 2.

$$\mathbf{A} = \begin{bmatrix} q\Psi_1 & q\Psi_2 + r\Psi_1 & r\Psi_2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} qy & qy & ry & ry \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{F}(h) = \begin{bmatrix} hyq\Psi_1 & hy(q\Psi_2 + (r-1)\Psi_1 & hy(r-1)\Psi_2 \\ hyq\Psi_1 & hy(q\Psi_2 + (r-1)\Psi_1 & hy(r-1)\Psi_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{G}(h) = \begin{bmatrix} 1 + h(q-1)y^2 & h(q-1)y^2 & hr\,y^2 & hr\,y^2 \\ h(q-1)y^2 & 1 + h(q-1)y^2 & hr\,y^2 & hr\,y^2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

P.5 *At any* $h \geq 0$, $z = 1$ *is eigenvalue of* $\mathbf{J}(h)$ *with multiplicity at least* $L - 1$.

Proof of P.5: Let $\boldsymbol{e}_i$ be the ith unit-co-ordinate (column) vector in $\Re^{3L+1}$:

$\boldsymbol{e}_i^T \equiv (0 ... 0 \quad 1 \quad 0 \quad ... \quad 0)$

It can be easily checked that for i = 2, ..., L $(\boldsymbol{e}_{L+i} - \boldsymbol{e}_{L+1+i})^T$ is a row eigenvector of $\mathbf{J}(h)$ associated to the eigenvalue $z = 1$.

For trajectories $\{\mathbf{x}_t\}$ generated by (21) with initial conditions in a 'spherical' ε-neighbourhood $\mathbf{U}_\varepsilon$ around $\boldsymbol{Q}$ we prove the following result[9]:

P.6 *Assume that each coefficient* Ψ_j j = 1, ..., L *in the matrix* $\mathbf{J}(h)$ *is sufficiently close to* 1/L, *and* L *is sufficiently large. There exist* $\varepsilon > 0$ *and* $\underline{h} > 0$ *sufficiently small such that the law of motion* (21) *restricted by* $0 < h < \underline{h}$ *induces the following properties on every trajectory* $\{\mathbf{x}_t\}$ *with generic initial condition in* $\mathbf{U}_\varepsilon$:

(a) *the first* L + 1 *elements of* $\mathbf{x}_t$ *specifying the co-ordinates of the recent income history* $(y_{t-1}, ..., y_{t-L-1})$ *converge to* $(y^*, ..., y^*)$;

(b) *the last* 2L *elements of* $\mathbf{x}_t$ *specifying the co-ordinates of the expectation functions* $(\mathbf{\Psi}_{t-1}, \mathbf{\Psi}_{t-2})$ *converge to a vector* $(\mathbf{\Psi}, \mathbf{\Psi})$ *in* $S^L \otimes S^L$.

In words, for appropriate levels of the learning parameter h the coupled local dynamics of income and expectation functions drives income to the stationary state y^* and forces expectations functions to detect period 1, asymptotically.

Properties (a) and (b) above are revealed by the following result.

P.7 *Assume that each coefficient* Ψ_j j = 1, ..., L *in the matrix* $\mathbf{J}(h)$ *is sufficiently close to* 1/L, *and* L *is sufficiently large. There exists* $\underline{h} > 0$ *such that for* $0 < h < \underline{h}$ *the 2L+2 eigenvalues of the Jacobian matrix* $\mathbf{J}(h)$*, that are left undetermined by P.5, have modulus less than* 1.

Proof of P.7: The eigenvalues of $\mathbf{J}(h)$ are continuous functions of the coefficients of the characteristic polynomial of $\mathbf{J}(h)$; in turn, these coefficients are continuous functions of h. The eigenvalues of $\mathbf{J}(h)$ can then be identified by the vector complex-valued function $\boldsymbol{z}(h)$. Consider the Jacobian matrix $\mathbf{J}(0)$. This has eigenvalues z = 1 and z = 0, both with multiplicity L. The remaining L + 1 eigenvalues of $\mathbf{J}(0)$ are the roots of the characteristic polynomial of the sub-matrix $\mathbf{A}$. The argument used in the proof of P.4 shows that under the stated assumptions the modulus of these L + 1 eigenvalues is bounded away from 1 from above. Let us write the L eigenvalues $z = 1$ of $\mathbf{J}(0)$ as $z_i(0) = 1$, i = 1, ..., L. P.5 implies that there exist $z_i(h)$ i = 2, ..., L such that $z_i(h) = 1$ at $h \geq 0$. Appendix A proves that at $h \geq 0$ and sufficiently small: *(i)* $z_1(h)$ is a real eigenvalue of $\mathbf{J}(h)$; *(ii)* $z_1(h)$ decreases locally as h increases from $h = 0$. Proposition P.7 is then implied by continuity of $\boldsymbol{z}(h)$.

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[9] The 'spherical' ε-neighbourhood $\mathbf{U}_\varepsilon$ is the set of points in $\mathfrak{R}^{3L+1}$ with distance less than ε from $\boldsymbol{Q}$.

The choice of initial expectation functions detecting period 1, but outside the set of stabilizing expectations, shows that there are trajectories of (21) starting from points arbitrarily close to ***Q*** and which are not attracted by this set. We can at best expect convergence to ***Q*** only if the above possibility is ruled out. To this end it is worth introducing the notion of a cylindric ε-neighbourhood.

Definition: The cylindric ε-neighbourhood $\mathbf{N}_\varepsilon$ of ***Q*** is the set of vectors in $\mathbf{U}_\varepsilon$ that are orthogonal to ***Q***.

The analysis of the coupled dynamics of income and expectation functions described by (21) is completed by the following negative result, extending to the present framework an argument originally developed by Fuchs [1979] for the case where (unlike here) expectation functions are restricted to detect period 1. The argument exploits the fact that the local trajectories of (21) do not approach ***Q*** orthogonally.

P.8 *Under the conditions of P.6 and P.7 there exist trajectories* $\{\mathbf{x}_t\}$ *of* (21) *with initial conditions in* $\mathbf{N}_\varepsilon$ *that do not converge to* ***Q***.

Proof of P.8: As it can be easily verified, the proposition is false *only if* the subspace spanned by the eigenvectors of **J**(*h*), associated to the eigenvalues with modulus less than 1, is orthogonal to ***Q***. The orthogonality requirement is illustrated in *fig*. 1. It is trivially verified at $h = 0$, but not at $h > 0$.

3.2.

As implied by the conditions stated in *P.6* and *P.7*, an excessively high rate of learning *h* is a potential source of the lack of (local) convergence of y_t to y^*.

Formal criteria for determining the time-varying learning rate which is optimal for fast convergence rely upon the knowledge of the true derivative $\partial E / \partial \Psi_j$. Partly because these criteria are computationally expensive[10], or because the knowledge in

[10] Cf. Hassoun [1995], p. 211-13 and the literature quoted therein.

question is not even available in 'complex' environments, various euristics have been suggested for the automatic control of the learning rate by means of rules of thumb.

A suggested euristics is that a sign reversal of the approximated derivative $\partial E/\partial \Psi_j$, hence of the prediction error $(y^e_t - y_t)$, is taken as evidence of a too high learning rate (Hassoun[1995]). It is worth observing in this respect how the sign of the error $(y^e_t - y_t)$ determines the *direction* in which the expectation coefficients Ψ_j are revised. Thus, a sign reversal of the prediction error signals an incoherence of the adjustment process and suggests a more cautious implementation thereof. Under this interpretation, incoherence should be detected not only by the number of sign reversals, but also by the amplitude of the oscillations of $(y^e_t - y_t)$.

Effective control on the learning rate is achieved in our simulations by an automatic determination of the learning rate reflecting the above euristics[11] (*fig*. 2).

4.

A distinctive feature of the approach followed in section 2 and 3 is that the boundedly-rational firms do not try to model *explicitly* how the aggregate-income dynamics depends on aggregate forecasts. Agents use instead expectation functions mapping directly the recent history of y into the expected future level of the same variable. We have seen how this approach is bound to imply a lack of attractiveness of the set of stabilizing expectations when firms try to learn from their past errors.

[11] We simulate a model economy where h_t is substituted for h in (18), (19), (20), and h_t is defined as:

$$h_t = \frac{\left|\sum_0^t (y^e_s - y_s)\right|}{\sum_0^t \left|y^e_s - y_s\right|} h_m + h_0 \qquad (22)$$

$h_m + h_0$ and h_0 identify the upper and lower bound on h_t, respectively. A recursive formulation of h_t is easily obtained by means of the auxiliary variables u and v.

$$h_t = (|u_t| / v_t)\, h_m + h_0 \qquad (23.1)$$

$$u_t = u_{t-1} + (y^e_t - y_t) \qquad (23.2)$$

$$v_t = v_{t-1} + |(y^e_t - y_t)| \qquad (23.3)$$

Although a formal proof would be cumbersome, it should be easy to see that the same result carries over to learning environments which are more general than (21) in that initial expectations functions are allowed to differ across firms.

A strictly-related implication of this approach to forecasting is that learning firms do not have at their disposal *large* sample relizations of a given true income function defined on the domain contemplated by the class of the forecast rules under use. This is because the actual map from (y_{t-1}, ..., y_{t-L-1}) to y_t depends on which (aggregate) expectation function is in use, hence the map changes, as long as firms are still learning.

In this section we contrast the above situation with that arising when firms try to model explicitly how the aggregate-income dynamics depends on expectations. In this case the forecasting problem can be separated into a learning and a coordination task. Unlike the situation described in section 3, the learning task admits a formulation where firms are applying the gradient-descent procedure in a 'simple' environment, that is, one in which they can compare the output of their learning algorithm with its true target. It is as if firms were in a situation of supervised learning.

Let us therefore assume that agents in the model economy have access at time t to the time series of the aggregate output forecasts $\{y^e_t, y^e_{t-1}, ..., y^e_{t-L}, ...\}$ or that they can infer this information from the time series of *k* and from their knowledge that the *desired* output-capital ratio γ is uniform across firms. For the scope of the present discussion we may well assume also that firms correctly perceive that true law of motion of aggregate income has the general form $y_t = F(y^e_{t+1}, y^e_t)$, but they do not know about the functional form of F(·,·). A weaker, but equally admissible assumption, is that firms wrongly perceive an income function of the general form $y_t = \Phi(y^e_{t+1}, y^e_t, s_t)$ with *s* representing some (possibly vector valued) characteristics of the model economy. In this case firms would have to learn, among other things, that *y* does not depend on *s*.

That in the stated conditions firms can in principle approximate locally the function F(·,·) to an arbitrary degree of accuracy, is stated by the following result (Hornik et al. [1989]).

Theorem (see Haykin [1995], p.182): *Let φ(·) be a non constant, bounded and monotone-increasing function. Let* I_p *denote the p-dimensional unit hypercube*$[0,1]^p$. *The space of continuous functions on* I_p *is denoted* $C(I_p)$. *Then, give any function* $F \in$

$C(I_p)$ and $\varepsilon > 0$*, there exists an integer M and sets of real constants* α_i , θ_i, w_{ij}*, where i* = 1, ..., *M and j = 1, ..., p such that we may define*

$$f(x_1, ..., x_p) = \sum_{i=1}^{M} \alpha_i \, \varphi\Big(\sum_{j=1}^{p} w_{ij}\, x_j - \theta_i\Big) \qquad (24)$$

as an approximate realization of the function F, that is,

$| F(x_1, ..., x_p) - f(x_1, ..., x_p) | < \varepsilon$ *for all* $(x_1, ..., x_p) \in I_p$.

Mathematical architectures of the form (24), where $\varphi(\cdot)$ is the sigmoid function, M is 'sufficiently large' and the coefficients α_i and w_{ij} are sequentially apdated by means of an appropriate gradient descent procedure (see below), have been proved to yield successful implementations of the approximation (24), provided that the number of point realizations $(x_1, ..., x_p, F(x_1, ..., x_p))$ used to 'train' the coefficients α_i and w_{ij} is sufficiently large with respect to *M*. The fact that the derivatives of the function *f*() built with this procedure have been proved to approximate the derivatives of the 'target' function *F*() (Hornik et al. [1990]) gives an explanation of the good extrapolation properties of *f*() in the neighbourhood of the training points $(x_1, ..., x_p, F(x_1, ..., x_p))$.

The gradient-descent procedure for the training of each coefficient w_{ij} has the familiar form (the same procedure applies to α_i):

$$w(t) = w(t-1) - \partial E(t-1) / \partial w(t-1) \qquad (25)$$

where E(t – 1) is the square error

$$E(t-1) = (½)[\, f(x_1(t-1), ..., x_p(t-1)) - F(x_1(t-1), ..., x_p(t-1))]^2$$

and the index t refers here to the number of training examples. We may observe, in passing, that given the 'simple' nature of the learning task, firms are in the position of computing the true derivative $\partial E / \partial w$.

Constraints on the capacity of (25) to generate coefficients with the desired approximation properties are posed in particular by the sample size (number of examples) of the training set[12].

To sharpen the contrast between the situation in the focus of this section and that considered in sections 2. and 3., it is worth expanding upon some implications of the envisaged possibility that firms come to learn the true law of motion (2.bis), namely, $y_t = F(y^e_{t+1}, y^e_t)$.

[12] On this and other problems see, for instance Haykin [1995], Hassoun [1995].

In the first place, firms knowing (2.bis) can easily compute the perfect-foresight dynamics (3) and know about the stability properties of it.

In the second place, and more importantly, the knowledge of the true law of motion (2.bis) makes firms' forecasting and decision problem similar, in fundamental respects, to a tacit co-ordination game. Firms, although playing non co-operatively, would be aware of their common, rather than conflicting, interests (recall that market shares are fixed). This can be identified with *the selection of behaviours leading to a persistently high aggregate income*. The analysis of this game-like situation is well beyond the scope of this paper. Still, it may be worth observing how the perfect-foresight high-income steady state y^* would presumably stand out as a *salient equilibrium* of the game: 'one that stands out from the rest by its uniqueness in some conspicuous respect' and which is unique 'in some way the subjects will notice, expect each other to notice, and so on'[13].

Should we replace our toy representation of a model economy with a less extreme characterization thereof, agents' attempts at *(a)* understanding of how expectations feed in to the true law of motion and *(b)* solving co-ordination problems would presumably have much more controversial outcomes. The fact remains, however, that *(a)* and *(b)* are conceptually separate activities, a property which is concealed by the modelling approach to forecasting and learning considered in section 3.

Appendix A

Let us consider the matrix $\mathbf{J}(h)$ of section 3. Tedious, but trivial, calculations yield the following:

Proposition 9: *For any given real scalar* z_1*, let the real scalars* $\alpha(z_1)$ *and* $h(z_1)$ *be determined by*:

$$\alpha(z_1) = [z_1^{L+1} - z_1^{L} q\Psi_1 - z_1^{L-1}(q\Psi_2 + r\Psi_1) - z_1(q\Psi_L + r\Psi_{L-1}) - r\Psi_L] \,/\, Ly^*(z_1 q + r)$$

$$h(z_1) = \alpha(z_1)(z_1^2 - z_1)(1/y^*) \cdot$$
$$\cdot [z_1^{L} q\Psi_1 + z_1^{L-1}(q\Psi_2 + (r-1)\Psi_1) + \cdots + z_1(q\Psi_L + (r-1)\Psi_{L-1}) + (r-1)\Psi_L + \alpha(z_1) Ly^*(z_1(q-1) + r)]^{-1}$$

Then, the vector in $\Re^{3L+1}$

$(z_1^{L}, z_1^{L-1}, \ldots, z_1, 1, \alpha(z_1) z_1, \ldots, \alpha(z_1) z_1, \alpha(z_1), \ldots, \alpha(z_1))^{T}$

(where T *denotes transposition) is the column eigenvector of* $\mathbf{J}(h(z_1))$ *associated to the eigenvalue* z_1.

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[13] Sugden [1996], p. 249.

We can now state the main result of this appendix.

Proposition 10:

(i) $h(1) = 0$; $\alpha^* \equiv \alpha(1) = [(q + r)^{-1} - 1] / y^* > 0$.

(ii) Assume that each coefficient Ψ_j j = 1, ..., L *in the matrix* $\mathbf{J}(h)$ *is sufficiently close to* 1/L, *and* L *is sufficiently large. There exists a real scalar* $\pi > 0$ *sufficiently small, such that* $h(z_1)$ *and* $\alpha(z_1)$ *are invertible functions on the domain* $[1 - \pi, 1]$, *with* $h(z_1) \geq 0$, $\alpha(z_1) > 0$.*and* $h(z_1) > 0$ if $z_1 \neq 1$.

Proof: Statement *(i)* is directly obtained from the definitions of $h(z_1)$ and $\alpha(z_1)$ using the fact that from the definitions of y^* and $\mathbf{J}(h)$ we have: $0 < q + r < 1$; $\sum\Psi_j = 1$.
To prove statement *(ii)* we may first notice that, since q > 1, $0 < q + r < 1$, then for Ψ_j sufficiently close to 1/L (j = 1, ..., L) there exists $\pi > 0$ such that $\alpha(z_1)$ and $h(z_1)$ are C^1 functions (are continuous and have a continuous derivative) on the domain $\{z_1 > 1 - \pi\}$.
The sign of $\partial\alpha(z_1) / \partial z_1$, evaluated at $z_1 = 1$, is determined by the sign of:

$$[L+1 - Lq\Psi_1 - (L-1)(q\Psi_2 + r\Psi_1) - (L-2)(q\Psi_3 + r\Psi_2) - \cdots - (q\Psi_L + r\Psi_{L-1})](q + r) - q[1 - (q + r)]$$

The sign of this term is positive for Ψ_j sufficiently close to 1/L, j = 1, ..., L and L sufficiently large. This is shown by the fact that, when each Ψ_j tends to 1/L, the above expression converges to:

$$[L+1 - q - (q + r)(L-1)/L - (q + r)(L-2)/L - \cdots - (q + r)/L](q + r) - q[1 - (q + r)]$$

The sign of $\partial h(z_1) / \partial z_1$, evaluated at $z_1 = 1$, is determined by the sign of the expression:

$$\alpha^* (q + r - 1)(1 + y^*\alpha^*L)/y^* < 0$$

These findings reveal that $h(z_1)$ is monotonic decreasing, $\alpha(z_1)$ monotonic increasing on $[1 - \pi, 1]$, proving *(ii)*.

Corollary: *There exists* $\underline{\underline{h}}$, $0 < \underline{\underline{h}} \leq h(\pi)$, *such that at every* h *in the interval* $(0, \underline{\underline{h}})$ z_1 *is a real eigenvalue of* $\mathbf{J}(h)$, *with modulus less than* 1.

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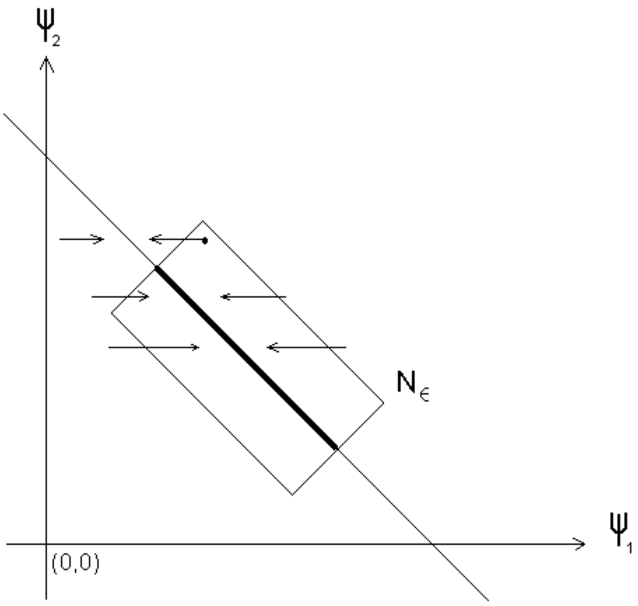
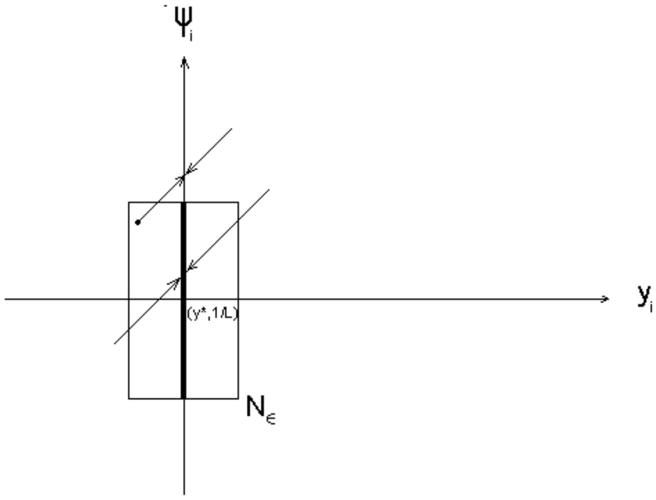


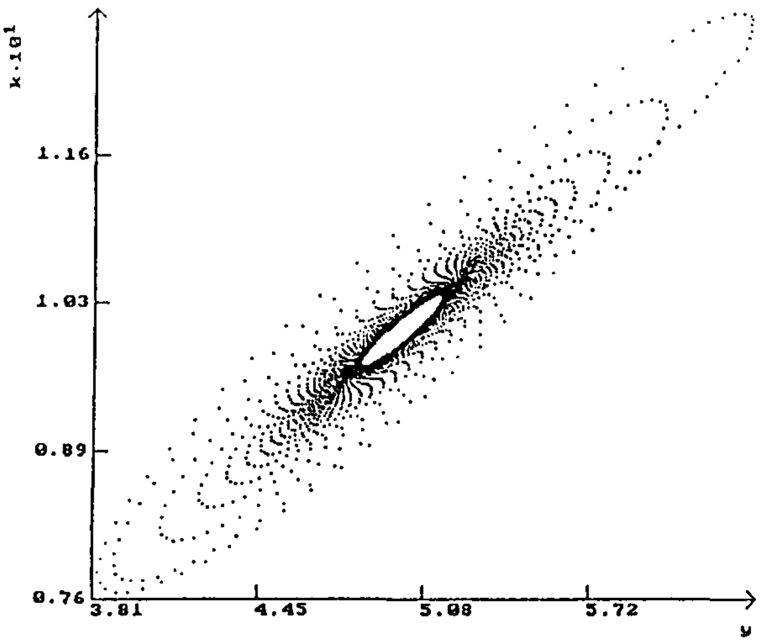
1.*a*

Fig. 1.*a* and 1.*b* illustrate the implications of the non orthogonality between the set ***Q*** (bold segment) and the trajectories of (21) approaching this set. *Fig.* 1.*a* shows the co-ordinates $\boldsymbol{\Psi}_1$ and $\boldsymbol{\Psi}_2$ of the points in the intersection between $\mathbf{N}_\varepsilon$ and the plane through ***Q*** which has every *y* co-ordinate fixed at y*, and every $\boldsymbol{\Psi}$ co-ordinate, except $\boldsymbol{\Psi}_1$, $\boldsymbol{\Psi}_2$, fixed at 1/L.

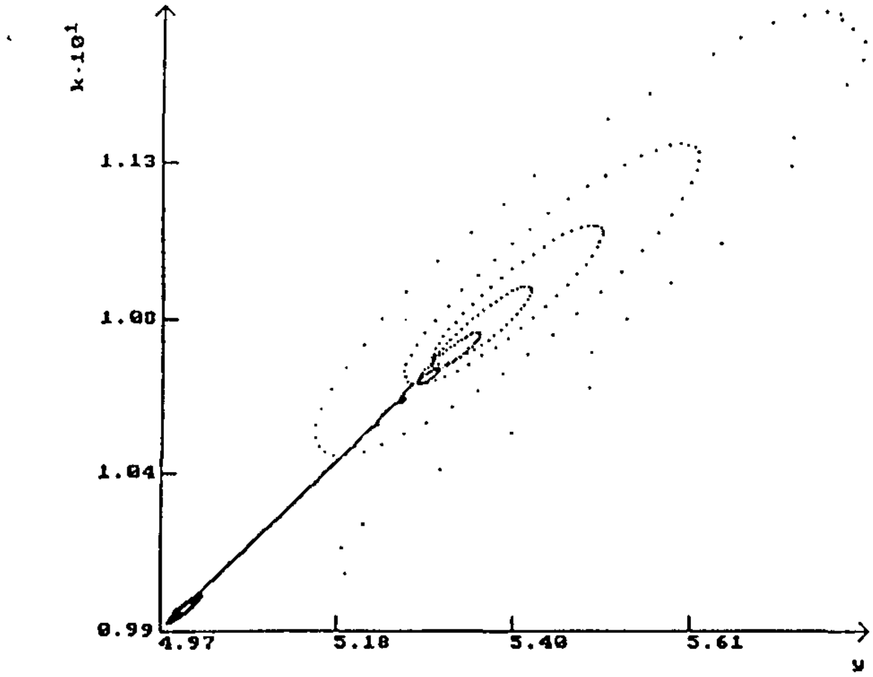


1.*b*

Fig. 1.*b* shows the co-ordinates y_i and $\mathbf{\Psi}_i$ of the points in the intersection between $\mathbf{N}_\varepsilon$ and the plane through ***Q*** (bold segment) which has every *y* co-ordinate equal to y*, except y_i , $\mathbf{\Psi}_i$ + $\mathbf{\Psi}_j$ = 2/L and every other $\mathbf{\Psi}$ co-ordinate equal to 1/L.



2.*a*



2.b

Figures 2.a, 2.b illustrate how the ability to learn is conditional upon a positive, but sufficiently low learning rate *h*. In *fig.* 2.a the trajectory $\{y_t, k_t\}$ first approaches and then visits a small neighbourhood of a closed orbit around $(y^*, y^*/\gamma)$. The trajectory is eventually repelled from the orbit. In *fig*. 2.b the learning rate at t = 0 is ten times larger than in *fig*. 2.a, but is controlled through the mechanisms explained in the text and formalized in footnote 11. *Fig*. 2.b shows a $\{y_t, k_t\}$ trajectory eventually converging to $(y^*, y^*/\gamma)$.