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Exponential Stability for a Degenerate/Singular Beam-Type Equation in Non-Divergence Form

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Abstract: The paper deals with the stability of a degenerate/singular beam equation in non-divergence form. In particular, we assume that the degeneracy and the singularity are at the same boundary point and we impose clamped conditions where the degeneracy occurs and dissipative conditions at the other endpoint. Using the energy method, we provide some conditions to obtain the stability for the considered problem.

Keywords: degenerate wave equation; singular potentials; stabilization; exponential decay

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1. Introduction

This paper is devoted to studying the stability of a beam-type degenerate equation with a small singular perturbation through a linear boundary feedback. To be more precise, we consider the following problem:

$$\begin{cases} y_{tt}(t,x) + ay_{xxxx}(t,x) - \dfrac{\lambda}{d}y(t,x) = 0, & (t,x) \in (0,+\infty) \times (0,1), \\ y(t,0) = 0, \ y_x(t,0) = 0, & t > 0, \\ \beta y(t,1) - y_{xxx}(t,1) + y_t(t,1) = 0, & t > 0, \\ \gamma y_x(t,1) + y_{xx}(t,1) + y_{tx}(t,1) = 0, & t > 0, \\ y(0,x) = y_0(x), \ y_t(0,x) = y_1(x), & x \in (0,1), \end{cases} \tag{1}$$

where β and γ are non-negative constants and the function a is such that $a(0) = 0$ and $a(x) > 0$ for all $x \in (0,1]$. In particular, for the function a, we consider two types of degeneracy according to the following definitions:

Definition 1. *The function g: $[0,1] \to \mathbb{R}$ is weakly degenerate, (WD) for short, at 0 if $g \in C^0[0,1] \cap C^1(0,1]$ is such that $g(0) = 0, g > 0$ on $(0,1]$, and if*

$$\sup_{x\in(0,1]} \frac{x|g'(x)|}{g(x)} := K_g, \tag{2}$$

then $K_g \in (0,1)$.

Definition 2. *The function* g: $[0,1] \to \mathbb{R}$ *is strongly degenerate, (SD) for short, at 0 if* $g \in C^1[0,1]$ *is such that* $g(0) = 0, g > 0$ *on* $(0,1]$ *and in* (2) *we have* $K_g \in [1,2)$.

Roughly speaking, when $g(x) \sim x^K$, it is (WD) if $K \in (0,1)$ and (SD) if $K \in [1,2)$.

Problems similar to (1) are considered in several papers (see, for example, [1–7]). In particular, in [3,5], the following Euler–Bernoulli beam equation is considered:

$$my_{tt}(t,x) + EIy_{xxxx}(t,x) = 0, \quad x \in (0,1), \ t > 0, \tag{3}$$

with clamped conditions at the left end

$$y(t,0) = 0, \quad y_x(t,0) = 0, \tag{4}$$

and with dissipative conditions at the right end

$$\begin{cases} -EIy_{xxx}(t,1) + \mu_1 y_t(t,1) = 0, & \mu_1 \geq 0, \\ EIy_{xx}(t,1) + \mu_2 y_{tx}(t,1) = 0, & \mu_2 \geq 0. \end{cases} \tag{5}$$

Here, y is the vertical displacement, y_t is the velocity, y_x is the rotation, y_{tx} is the angular velocity, m is the mass density per unit length, EI is the flexural rigidity coefficient, $-EIy_{xx}$ is the bending moment, and $-EIy_{xxx}$ is the shear. In particular, the boundary conditions (5) mean that the shear is proportional to the velocity and the bending moment is negatively proportional to the angular moment. Observe that if we consider $\beta = \gamma = 0$ in (1), then we have boundary conditions analogous to those in (5). Thus, the dissipative conditions at 1 are not surprising. We remark that the conditions $\beta, \gamma \geq 0$ are necessary to study the well-posedness of the problem and to prove equivalence among all the norms introduced in this paper and that are crucial to obtain the stability result.

The qualitative behavior of (3)–(5) is studied in [4], where it is proved that if $\mu_1^2 > 0$ and $\mu_2^2 \geq 0$, the energy $E(t)$ of the vibration of the beam decays exponentially in a uniform way

$$E(t) \leq ke^{-\mu t}E(0) \tag{6}$$

for some $k, \mu > 0$.

Observe that in all the references above, the equation is non-degenerate; however, there are some papers where the equation is degenerate in the sense that a *degenerate damping* appears in the equation of (3) (see, for example, [8–10]). The first paper where the equation is degenerate in the sense that the fourth-order operator degenerates in a point as in (1) is [11]. However, to the best of our knowledge, [12] is the first paper where the *stability* for (1) with $\lambda = 0$ is considered. On the other hand, for a degenerate wave-equation, we refer to [13] (see also the arxiv version of 2015) for a problem in divergence form and to [14] for a problem in non-divergence form.

A position-dependent restoring force is introduced using the modified Euler–Bernoulli equation that includes a coefficient-dependent drift term. High stiffness, concentrated forces, or material discontinuities can be modeled by the term $\frac{\lambda}{d(x)}y$, which produces a single behavior at $x = 0$, suggesting a highly localized effect. Additionally, it denotes pre-stress or non-homogeneous stiffness, which applies to beams on uneven elastic foundations. Furthermore, if $d(x)$ is a distance function, the term affects tension in pre-stressed structures by acting like an inverse square law, similar to electrostatic or gravitational potentials. Additionally, the equation resembles singular potential quantum wave equations, which cause localized resonance effects in structural dynamics. All things considered, this formulation captures strong localized effects, pre-stress changes, and non-uniform limitations that are pertinent to practical physics and engineering (see [15]).

As far as we know, for beam-type equations simultaneously admitting degeneracy and singularity, only controllability problems have been faced (see the recent paper [16]), while nothing has been undertaken for stability. For this reason, in this paper, we focus on such a problem, proving that (1) permits boundary stabilization, provided that the singular term has a small coefficient (see Theorem 2 below). Hence, we may regard this result as a natural continuation of [12] and a perturbation of the related one in [16]. Clearly, the presence of the singular term $\frac{y}{d}$ introduces several difficulties with respect to [12], which let us treat only the case of a function d with weak degeneracy, according to the definition above. For a stability result for a degenerate/singular wave equation, we refer to [17].

Strategy method. In order to prove (6) for (1), we use a multiplier method. In particular, after defining the energy associated with the problem, we prove an estimate on it using a multiplier method (see Proposition 6). Obviously, the presence of a degenerate fourth-order operator brings more difficulties with respect to the ones for the second-order case. These difficulties are related to some new terms that we have to face; for example, we have to estimate from above $\int_s^T y^2(t,1)dt + \int_s^T y_x^2(t,1)dt$ for every $0 < s < T$ using the energy associated with the original problem. This is carried out in Proposition 7 thanks to a suitable fourth-order variational problem (see Proposition 3). Thanks to the estimates proved in Propositions 6 and 7 and using a result given in [18], we prove the main result of the paper, i.e. Theorem 2.

This paper is organized as follows: In Section 2, we give the functional setting and some preliminary results that we will use in the rest of the paper, together with the existence of solutions. In Section 3, we introduce the energy associated with a solution for the problem and we show that it decays exponentially as time diverges. In particular, we prove that if λ is small and a, d are not too degenerate (in the sense of Definitions 1 or 2), the energy satisfies (6), as in [4] for the non-degenerate and non-singular case. The last section is devoted to the conclusions and to some open problems.

2. Preliminary Results and Well-Posedness

In this section, we introduce the functional setting needed to treat (1). However, here, our assumptions are more general than those required to obtain the stability result in the next section.

We start by assuming a very modest requirement.

Hypothesis 1. *The functions $a, d \in C^0[0,1]$ are such that*

1. $a(0) = d(0) = 0$, $a, d > 0$ *on* $(0,1]$,
2. *there are* $K_a, K_d \in (0,2)$ *such that the functions*

$$x \longmapsto \frac{x^{K_a}}{a(x)} \tag{7}$$

and

$$x \longmapsto \frac{x^{K_d}}{d(x)} \tag{8}$$

are non-decreasing in a right neighborhood of $x = 0$.

It is clear that, if Hypothesis 1 holds, then

$$\lim_{x \to 0} \frac{x^\gamma}{a(x)} = 0, \tag{9}$$

for all $\gamma > K_a$, and

$$\lim_{x\to 0} \frac{x^\gamma}{d(x)} = 0, \tag{10}$$

for all $\gamma > K_d$.

Let us state that if a is (WD) or (SD), then (2) implies that (7) holds on the whole domain $(0,1]$ analogously for d.

In order to treat (1), let us introduce the following Hilbert spaces with the related inner products and norms given by the following:

$$L^2_{\frac{1}{a}}(0,1) := \left\{ u \in L^2(0,1) \,\middle|\, \|u\|_{\frac{1}{a}} < \infty \right\},$$

$$\langle u,v\rangle_{\frac{1}{a}} := \int_0^1 uv\frac{1}{a}dx, \quad \|u\|^2_{\frac{1}{a}} = \int_0^1 \frac{u^2}{a}dx,$$

for all $u,v \in L^2_{\frac{1}{a}}(0,1)$;

$$H^i_{\frac{1}{a}}(0,1) := L^2_{\frac{1}{a}}(0,1) \cap H^i(0,1), i = 1,2,$$

$$\langle u,v\rangle_{i,\frac{1}{a}} := \langle u,v\rangle_{\frac{1}{a}} + \sum_{k=1}^{i} \int_0^1 u^{(k)}(x)v^{(k)}(x)dx$$

and

$$\|u\|^2_{H^i_{\frac{1}{a}}(0,1)} := \|u\|^2_{\frac{1}{a}} + \sum_{k=1}^{i} \left\| u^{(k)} \right\|^2_{L^2(0,1)},$$

$\forall\, u,v \in H^i_{\frac{1}{a}}(0,1)$, $i = 1,2$. In addition to the previous ones, we introduce the following important Hilbert spaces:

$$H^1_{\frac{1}{a},0}(0,1) := \left\{ u \in H^1_{\frac{1}{a}}(0,1) : u(0) = 0 \right\} \quad \text{and}$$

$$H^2_{\frac{1}{a},0}(0,1) := \left\{ u \in H^1_{\frac{1}{a},0}(0,1) \cap H^2(0,1) : u'(0) = 0 \right\},$$

with the previous inner products $\langle \cdot,\cdot\rangle_{i,\frac{1}{a}}$ and norms $\|\cdot\|_{H^i_{\frac{1}{a}}(0,1)}$, $i = 1,2$. Now, consider the scalar product

$$\langle u,v\rangle_{i,\circ} := \int_0^1 u^{(i)}(x)v^{(i)}(x)dx$$

for all $u,v \in H^i_{\frac{1}{a}}(0,1)$, which induces the norm

$$\|u\|_{i,\circ} := \|u^{(i)}\|_{L^2(0,1)}, \quad \forall\, u \in H^i_{\frac{1}{a}}(0,1),$$

$i = 1,2$. Observe that, if a is continuous, $a(0) = 0$ and (7) is satisfied, then the norms $\|\cdot\|_{H^i_{\frac{1}{a}}(0,1)}$, $\|\cdot\|_i$ and $\|\cdot\|_{i,\circ}$ are equivalent in $H^i_{\frac{1}{a},0}(0,1)$. Here,

$$\|u\|^2_i := \|u\|^2_{\frac{1}{a}} + \|u^{(i)}\|^2_{L^2(0,1)}, \quad \forall\, u \in H^i_{\frac{1}{a},0}(0,1),$$

$i = 1,2$ (see, e.g., [11]). Clearly, if $i = 1$, the previous equivalence is obviously satisfied. Indeed, $\|\cdot\|_{H^1_{\frac{1}{a}}(0,1)}$ and $\|\cdot\|_1$ coincide and, for ([19], Proposition 2.6), one proves that there is $C > 0$ such that

$$\int_0^1 \frac{u^2}{a} dx \leq C \int_0^1 (u')^2 dx, \tag{11}$$

for all $u \in H^1_{\frac{1}{a},0}(0,1)$. Let

$$C_{HP} \text{ be the best constant of (11).} \tag{12}$$

Now, assume $i = 2$ and fix $u \in H^2_{\frac{1}{a},0}(0,1)$. Proceeding as for $i = 1$ and applying the classical Hardy's inequality to $z := u'$ (observing that $z \in H^1_{\frac{1}{a},0}(0,1)$), we have

$$\int_0^1 (u')^2 dx \leq \int_0^1 \frac{z^2}{x^2} dx \leq 4 \int_0^1 (z')^2 dx = 4 \int_0^1 (u'')^2 dx = 4\|u\|^2_{2,\circ}.$$

Hence, $\| \cdot \|_{H^2_{\frac{1}{a}}(0,1)}$ and $\| \cdot \|_2$ are equivalent in $H^2_{\frac{1}{a},0}(0,1)$ (actually, they are equivalent in $H^2_{\frac{1}{a}}(0,1)$, see, e.g., [11]). Moreover, by the previous inequality,

$$\int_0^1 \frac{u^2}{a} dx \leq C_{HP} \int_0^1 (u')^2 dx \leq 4C_{HP}\|u\|^2_{2,\circ} \tag{13}$$

and the thesis follows. In particular, $\|u\|^2_1 \leq (C_{HP} + 1)\|u\|^2_{1,\circ}$ for all $u \in H^1_{\frac{1}{a},0}(0,1)$ and

$$\|u\|^2_2 \leq (4C_{HP} + 1)\|u\|^2_{2,\circ} \tag{14}$$

for all $u \in H^2_{\frac{1}{a},0}(0,1)$ (see ([12], Proposition 2.1)).

As in ([16], Proposition 2.3), one can prove the next result

Proposition 1. *Assume Hypothesis 1 and take K_a, K_d such that $K_a + K_d \leq 2$. If $u \in H^2_{\frac{1}{a},0}(0,1)$, then for $\frac{u}{\sqrt{ad}} \in L^2(0,1)$, there is a positive constant $C > 0$ such that*

$$\int_0^1 \frac{u^2(x)}{a(x)d(x)} dx \leq C \int_0^1 (u''(x))^2 dx. \tag{15}$$

Let

$$\tilde{C}_{HP} \text{ be the best constant of (15).} \tag{16}$$

As in ([20], Chapter V), we assume the next hypothesis:

Hypothesis 2. *The constant $\lambda \in \mathbb{R}$ is such that $\lambda \neq 0$ and*

$$\lambda < \frac{1}{\tilde{C}_{HP}}. \tag{17}$$

Observe that the case $\lambda = 0$ is already considered in [12]. Thus, it is not restrictive to assume $\lambda \neq 0$.

Moreover, if $\lambda \in \left(0, \frac{1}{\tilde{C}_{HP}}\right)$, we can take $\epsilon \in (0,1)$ such that

$$\lambda = \frac{1 - \epsilon}{\tilde{C}_{HP}} > 0. \tag{18}$$

Hence, as a consequence of Proposition 1, one has the next estimate (see ([16], Proposition 2.4)).

Proposition 2. *Assume Hypothesis 1 and* $\lambda \in \left(0, \frac{1}{C_{HP}}\right)$. *If* $u \in H^2_{\frac{1}{a},0}(0,1)$, *then*

$$\int_0^1 (u''(x))^2 dx - \lambda \int_0^1 \frac{u^2(x)}{a(x)d(x)} dx \geq \epsilon \int_0^1 (u''(x))^2 dx.$$

Under Hypotheses 1 and 2, one can consider in $H^2_{\frac{1}{a},0}(0,1)$ also the product

$$\langle u, v \rangle_{2,\sim} := \langle u, v \rangle_{2,\circ} - \lambda \int_0^1 \frac{uv}{ad} dx,$$

which induces the norm

$$\|u\|^2_{2,\sim} = \|u\|^2_{2,\circ} - \lambda \int_0^1 \frac{u^2}{ad} dx.$$

By Propositions 1 and 2, one can prove the following equivalence:

Corollary 1. *Assume Hypotheses 1 and 2 and* $K_a + K_d \leq 2$. *Then, the norms* $\| \cdot \|_{H^2_{\frac{1}{a}}(0,1)}$, $\| \cdot \|_2$, $\| \cdot \|_{2,\circ}$, *and* $\| \cdot \|_{2,\sim}$ *are equivalent in* $H^2_{\frac{1}{a}}(0,1)$.

In order to study the well-posedness of (1), we introduce the operator A: $D(A) \subset L^2_{\frac{1}{a}}(0,1) \to L^2_{\frac{1}{a}}(0,1)$ by $Au := au''''$, for all $u \in D(A) := \left\{ u \in H^2_{\frac{1}{a},0}(0,1) : au'''' \in L^2_{\frac{1}{a}}(0,1) \right\}$, where the next Gauss–Green formula holds

$$\int_0^1 u'''' v dx = u'''(1)v(1) - u''(1)v'(1) + \int_0^1 u''v'' dx \tag{19}$$

for all $(u, v) \in D(A) \times H^2_{\frac{1}{a},0}(0,1)$ (see [12]). Moreover, consider

$$A_\lambda u := Au - \frac{\lambda}{d} u, \quad \forall\, u \in D(A_\lambda),$$

where

$$D(A_\lambda) := \left\{ u \in H^2_{\frac{1}{a},0}(0,1) \mid A_\lambda u \in L^2_{\frac{1}{a}}(0,1) \right\}. \tag{20}$$

Observe that if $u \in H^2_{\frac{1}{a},0}(0,1)$ and $K_a + 2K_d \leq 2$, one proves that $\frac{u}{d} \in L^2_{\frac{1}{a}}(0,1)$; hence, $u \in D(A_\lambda)$ if and only if $u \in D(A)$, i.e., $D(A_\lambda) = D(A)$ (for more details, we refer to [16]). For this reason, in the following, we assume the next assumption:

Hypothesis 3. *Assume Hypothesis 1 and* $K_a + 2K_d \leq 2$.

Under this assumption, it is clear that d cannot be (SD). On the other hand, a can be (SD), but in this case, K_d has to be very small.

Finally, to prove the well-posedness of (1), we need to introduce the last Hilbert space $\mathcal{H}_0 := H^2_{\frac{1}{a},0}(0,1) \times L^2_{\frac{1}{a}}(0,1)$, with inner product and norm given by

$$\langle (u, v), (\tilde{u}, \tilde{v}) \rangle_{\mathcal{H}_0} := \langle u, \tilde{u} \rangle_{2,\circ} + \langle v, \tilde{v} \rangle_{\frac{1}{a}} + \beta u(1)\tilde{u}(1) + \gamma u'(1)\tilde{u}'(1)$$

and

$$\|(u, v)\|^2_{\mathcal{H}_0} := \|u''\|^2_{L^2(0,1)} + \|v\|^2_{\frac{1}{a}} + \beta u^2(1) + \gamma (u'(1))^2$$

for every $(u, v),\ (\tilde{u}, \tilde{v}) \in \mathcal{H}_0$, where $\beta, \gamma \geq 0$, and the matrix operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H}_0 \to \mathcal{H}_0$ given by

$$\mathcal{A} := \begin{pmatrix} 0 & Id \\ -A_\lambda & 0 \end{pmatrix}$$

with domain

$$D(\mathcal{A}) := \{(u,v) \in D(A) \times H^2_{\frac{1}{a},0}(0,1) : \beta u(1) - u'''(1) + v(1) = 0, \\ \gamma u'(1) + u''(1) + v'(1) = 0\}.$$

Thanks to (19), one can prove the next theorem that contains the main properties of the operator $(\mathcal{A}, D(\mathcal{A}))$. Since the proof is similar to the one of [21] or [22], we omit it.

Theorem 1. *Assume a (WD) or (SD). Then, the operator $(\mathcal{A}, D(\mathcal{A}))$ is non-positive with a dense domain and generates a contraction semigroup $(S(t))_{t\geq 0}$.*

Thanks to the previous theorem, one has the next result, which can be proved as in ([16], Theorem 2.7).

Theorem 2. *Hypotheses 2 and 3 hold. If $(y_0, y_1) \in H^2_{\frac{1}{a},0}(0,1) \times L^2_{\frac{1}{a}}(0,1)$, then there is a unique mild solution*

$$y \in C^1\Big([0,+\infty); L^2_{\frac{1}{a}}(0,1)\Big) \cap C\Big([0,+\infty); H^2_{\frac{1}{a},0}(0,1)\Big)$$

of (1), which depends continuously on the initial data. In addition, if $(y_0, y_1) \in D(\mathcal{A}_1)$, then the solution y is classical in the sense that

$$y \in C^2\Big([0,+\infty); L^2_{\frac{1}{a}}(0,1)\Big) \cap C^1\Big([0,+\infty); H^2_{\frac{1}{a},0}(0,1)\Big) \cap C([0,+\infty); D(A))$$

and the equation of (1) holds for all $t \geq 0$.

Remark 1. *Due to the reversibility in time of the equation, solutions exist with the same regularity for $t < 0$. We will use this fact in the proof of the controllability result by considering a backward problem whose final time data will be transformed in initial data: this is the reason for the notation of the initial data in problem (1).*

The last important result of this section is given by the next proposition. Let us start with

Hypothesis 4. *Assume a (WD) or (SD), d (WD) with $K_a + 2K_d \leq 2$, $\lambda \neq 0$ with $\lambda < \frac{1}{C_{HP}}$ and $\beta, \gamma \geq 0$.*

Proposition 3. *Assume Hypothesis 4 and define*

$$|||z|||^2 := \int_0^1 (z'')^2 dx - \lambda \int_0^1 \frac{z^2}{ad} dx + \beta z^2(1) + \gamma (z'(1))^2$$

for all $z \in H^2_{\frac{1}{a},0}(0,1)$. Then, the norms $|||\cdot|||$ and $||\cdot||_{2,\circ}$ are equivalent in $H^2_{\frac{1}{a},0}(0,1)$. Moreover, for every $\rho, \mu \in \mathbb{R}$, the variational problem

$$\int_0^1 z'' \varphi'' dx - \lambda \int_0^1 \frac{z\varphi}{ad} + \beta z(1)\varphi(1) + \gamma z'(1)\varphi'(1) = \rho\varphi(1) + \mu\varphi'(1), \quad \forall\, \varphi \in H^2_{\frac{1}{a},0}(0,1),$$

admits a unique solution $z \in H^2_{\frac{1}{a},0}(0,1)$, which satisfies the estimates

$$\|z\|^2_{\frac{1}{a}} \leq \frac{4C_{HP}}{C_\epsilon}(|\rho| + |\mu|)^2, \quad \text{and} \quad |||z|||^2 \leq (|\rho| + |\mu|)^2, \tag{21}$$

where

$$C_\epsilon := \begin{cases} 1, & \lambda < 0, \\ \epsilon, & \lambda > 0. \end{cases} \tag{22}$$

In addition, $z \in D(A_\lambda)$ *and solves*

$$\begin{cases} A_\lambda z = 0, \\ \beta z(1) - z'''(1) = \rho, \\ \gamma z'(1) + z''(1) = \mu. \end{cases} \tag{23}$$

Proof. As a first step, observe that for all $z \in H^2_{\frac{1}{a},0}(0,1)$, one has

$$|z'(x)| = \left| \int_0^x z''(t)dt \right| \le \|z''\|_{L^2(0,1)} = \|z\|_{2,\circ}, \tag{24}$$

and

$$|z(x)| = \left| \int_0^x z'(t)dt \right| = \left| \int_0^x \int_0^t z''(\tau)d\tau\, dt \right| \le \|z''\|_{L^2(0,1)} = \|z\|_{2,\circ} \tag{25}$$

for all $x \in [0,1]$. Thus, $|||\cdot|||$ and $\|\cdot\|_{2,\circ}$ are equivalent. Indeed, for all $z \in H^2_{\frac{1}{a},0}(0,1)$, if $\lambda < 0$, one proves immediately that

$$\|z\|^2_{2,\circ} = \|z''\|^2_{L^2(0,1)} \le |||z|||^2. \tag{26}$$

If $\lambda \in \left(0, \dfrac{1}{\tilde{C}_{HP}}\right)$, by Proposition 2, one has

$$\int_0^1 (z''(x))^2 dx - \lambda \int_0^1 \frac{z^2(x)}{a(x)d(x)} dx \ge \epsilon \int_0^1 (z''(x))^2 dx. \tag{27}$$

for all $z \in H^2_{\frac{1}{a},0}(0,1)$, and so

$$\|z\|^2_{2,\circ} \le \frac{1}{\epsilon} |||z|||^2, \tag{28}$$

for all $z \in H^2_{\frac{1}{a},0}(0,1)$. In conclusion, we have

$$\|z\|^2_{2,\circ} \le \frac{1}{C_\epsilon} |||z|||^2. \tag{29}$$

Now, we prove that there is $C > 0$ such that

$$|||z|||^2 \le C\|z\|^2_{2,\circ},$$

for all $z \in H^2_{\frac{1}{a},0}(0,1)$. Clearly, (25) and (24) imply $\beta z^2(1) \le \beta\|z\|^2_{2,\circ}$ and $\gamma(z'(1))^2 \le \gamma\|z\|^2_{2,\circ}$, respectively; hence, if $\lambda > 0$, one proves immediately that $|||z|||^2 \le (1+\beta+\gamma)\|z\|^2_{2,\circ}$; if $\lambda < 0$, by (15), then

$$|||z|||^2 \le (1 - \lambda\tilde{C}_{HP} + \beta + \gamma)\|z\|^2_{2,\circ}.$$

In any case, the claim holds.

Now, consider the bilinear and symmetric form $\Lambda : H^2_{\frac{1}{a},0}(0,1) \times H^2_{\frac{1}{a},0}(0,1) \to \mathbb{R}$ such that

$$\Lambda(z,\varphi) := \int_0^1 z''\varphi'' dx - \lambda \int_0^1 \frac{z\varphi}{ad} dx + \beta z(1)\varphi(1) + \gamma z'(1)\varphi'(1).$$

As in [22] or in [17], one can easily prove that Λ is coercive and continuous. Now, consider the linear functional

$$\mathcal{L}(\varphi) := \rho\varphi(1) + \mu\varphi'(1),$$

with $\varphi \in H^2_{\frac{1}{a},0}(0,1)$ and $\rho, \mu \in \mathbb{R}$. Clearly, $\mathcal{L}$ is continuous and linear. Thus, by the Lax-Milgram Theorem, there is a unique solution $z \in H^2_{\frac{1}{a},0}(0,1)$ of

$$\Lambda(z,\varphi) = \mathcal{L}(\varphi) \tag{30}$$

for all $\varphi \in H^2_{\frac{1}{a},0}(0,1)$. In particular,

$$|||z|||^2 = \Lambda(z,z) = \int_0^1 (z'')^2 dx - \lambda \int_0^1 \frac{z^2}{ad} dx + \beta z^2(1) + \gamma(z'(1))^2 = \rho z(1) + \mu z'(1). \tag{31}$$

Concerning the other estimates, by (24)–(26) and (31), we have $|||z|||^2 = \rho z(1) + \mu z'(1) \leq (|\rho| + |\mu|)|||z|||$; thus,

$$|||z||| \leq |\rho| + |\mu|. \tag{32}$$

Moreover, by the equivalence of the norms in $H^2_{\frac{1}{a},0}(0,1)$, Proposition 2, and (13), one has

$$|||z|||^2 = \|z\|^2_{2,\circ} - \lambda \int_0^1 \frac{z^2}{ad} dx + \beta z^2(1) + \gamma(z'(1))^2 \geq C_\epsilon \|z\|^2_{2,\circ} \geq \frac{C_\epsilon}{4C_{HP}} \|z\|^2_{\frac{1}{a}},$$

where C_ϵ is as in (22). Thus, by (32), $\|z\|^2_{\frac{1}{a}} \leq \frac{4C_{HP}}{C_\epsilon} |||z|||^2 \leq \frac{4C_{HP}}{C_\epsilon}(|\rho| + |\mu|)^2$.

Now, we will prove that z belongs to $D(A)$ and solves (23). To this end, consider (30) again; clearly, it holds for every $\varphi \in C_c^\infty(0,1)$, so that $\int_0^1 z''\varphi'' dx - \lambda \int_0^1 \frac{z\varphi}{ad} dx = 0$ for all $\varphi \in C_c^\infty(0,1)$. Thus, $z'''' = \lambda \frac{z\varphi}{ad}$a.e. in $(0,1)$ (see, e.g., ([23], Lemma 1.2.1)) and so $az'''' - \lambda \frac{z\varphi}{d} = 0$ a.e. in $(0,1)$, in particular $A_\lambda z = 0 \in L^2_{\frac{1}{a}}(0,1)$; this implies that $z \in D(A)$.

Now, coming back to (30) and using (19) and the fact that $A_\lambda z = 0$, we have

$$\begin{aligned} &\int_0^1 z''\varphi'' dx - \lambda \int_0^1 \frac{z\varphi}{ad} dx + \beta z(1)\varphi(1) + \gamma z'(1)\varphi'(1) = \rho\varphi(1) + \mu\varphi'(1) \\ &\iff -z'''(1)\varphi(1) + (z''\varphi')(1) + \beta z(1)\varphi(1) + \gamma(z'\varphi')(1) = \rho\varphi(1) + \mu\varphi'(1) \end{aligned}$$

for all $\varphi \in H^2_{\frac{1}{a},0}(0,1)$. Thus, $-z'''(1) + \beta z(1) = \rho$ and $\gamma z'(1) + z''(1) = \mu$, that is, z solves (23). □

3. Energy Estimates and Exponential Stability

In this section, we prove the main result of this paper. In particular, proving some estimates of the energy associated with (1), we obtain the exponential stability.

To begin with, we give the next definition:

Definition 3. *For a mild solution y of (1), we define its energy as the continuous function*

$$E_y(t) := \frac{1}{2}\int_0^1 \left(\frac{y_t^2(t,x)}{a(x)} + y_{xx}^2(t,x) - \frac{\lambda}{ad} y^2(t,x) \right) dx + \frac{\beta}{2} y^2(t,1) + \frac{\gamma}{2} y_x^2(t,1), \qquad \forall\, t \geq 0. \tag{33}$$

Recalling that $\beta, \gamma \geq 0$, one proves that if y is a mild solution and if $\beta, \gamma \neq 0$, then

$$y^2(t,1) \leq \frac{2}{\beta} E_y(t) \quad \text{and} \quad y_x^2(t,1) \leq \frac{2}{\gamma} E_y(t);$$

on the other hand, thanks to Equations (24)–(26), for all $\beta \geq 0$ and $\gamma \geq 0$,

$$y^2(t,1) \leq \frac{2}{C_\epsilon} E_y(t) \quad \text{and} \quad y_x^2(t,1) \leq \frac{2}{C_\epsilon} E_y(t),$$

where C_ϵ is as in (22). Thus, we have

$$y^2(t,1) \leq C_\beta E_y(t) \quad \text{and} \quad y_x^2(t,1) \leq C_\gamma E_y(t), \tag{34}$$

where $C_\beta := \begin{cases} 2\min\left\{\frac{1}{C_\epsilon}, \frac{1}{\beta}\right\}, & \beta \neq 0, \\ \frac{2}{C_\epsilon}, & \beta = 0 \end{cases}$ and $C_\gamma := \begin{cases} 2\min\left\{\frac{1}{C_\epsilon}, \frac{1}{\gamma}\right\}, & \gamma \neq 0, \\ \frac{2}{C_\epsilon}, & \gamma = 0. \end{cases}$

Observe that if $\beta > 1$, being $\frac{1}{\beta} < 1$ and $\frac{1}{C_\epsilon} \geq 1$ (recall that $\epsilon \in (0,1)$), $\min\left\{\frac{1}{C_\epsilon}, \frac{1}{\beta}\right\} = \frac{1}{\beta}$. Analogously for γ.

As in ([21], Thoerem 3.1), it is possible to prove that the energy is a non-increasing function.

Theorem 1. *Assume Hypothesis 4 and let y be a classical solution of (1). Then, the energy is non-increasing. In particular,*

$$\frac{dE_y(t)}{dt} = -y_t^2(t,1) - y_{tx}^2(t,1), \qquad \forall\, t \geq 0.$$

Actually, one can prove that the previous monotonicity result also holds under weaker assumptions on the functions a and d.

Proposition 4. *Assume Hypothesis 4. For the fixed $T > 0$, if y is a classical solution of (1), then*

$$\begin{aligned} 0 = {} & 2\int_0^1 \left[y_t \frac{x}{a} y_x\right]_{t=s}^{t=T} dx - \frac{1}{a(1)} \int_s^T y_t^2(t,1)dt + \int_{Q_s} \frac{y_t^2}{a}\left(1 - \frac{xa'}{a}\right) dx\,dt \\ & - \frac{\lambda}{a(1)d(1)} \int_0^T y^2(t,1)dt + 3\int_{Q_s} y_{xx}^2 dx\,dt + 2\beta \int_s^T y_x(t,1)y(t,1)dt \\ & + 2\int_s^T y_x(t,1)y_t(t,1)dt + 2\gamma \int_s^T y_x^2(t,1)dt + 2\int_s^T y_x(t,1)y_{tx}(t,1)dt \\ & - \int_s^T y_{xx}^2(t,1)dt + \lambda \int_{Q_s} \left(1 - \frac{xa'}{a} - \frac{xd'}{d}\right)\frac{y^2}{ad} dxdt, \end{aligned} \tag{35}$$

for every $0 < s < T$. Here, $Q_s := (s,T) \times (0,1)$.

Proof. Since some computations are similar to the ones of ([21], Proposition 4.7), we sketch them. Fix $s \in (0,T)$. Multiplying the equation in (1) by $\frac{xy_x}{a}$ and integrating over Q_s, we have

$$0 = \int_{Q_s} y_{tt}\frac{xy_x}{a} dx\,dt + \int_{Q_s} a y_{xxxx}\frac{xy_x}{a} dx\,dt - \lambda \int_{Q_s} \frac{xyy_x}{ad} dxdt. \tag{36}$$

As in [21], one proves that

$$\begin{aligned} & \int_{Q_s} y_{tt}\frac{xy_x}{a} dx\,dt + \int_{Q_s} a y_{xxxx}\frac{xy_x}{a} dx\,dt \\ & = \int_0^1 \left[y_t \frac{xy_x}{a}\right]_{t=s}^{t=T} dx - \frac{1}{2a(1)} \int_s^T y_t^2(t,1)dt + \frac{1}{2}\int_{Q_s} \frac{y_t^2}{a}\left(1 - \frac{xa'}{a}\right) dx\,dt \\ & + \frac{3}{2}\int_{Q_s} y_{xx}^2 dxdt + \beta \int_s^T y_x(t,1)y(t,1)dt + \int_s^T y_x(t,1)y_t(t,1)dt \\ & + \gamma \int_s^T y_x^2(t,1)dt + \int_s^T y_x(t,1)y_{tx}(t,1)dt - \frac{1}{2}\int_s^T y_{xx}^2(t,1)dt. \end{aligned}$$

Hence, it remains to compute $-\lambda\int_{Q_s}\frac{xyy_x}{ad}dxdt$. As in [17], one proves

$$-\lambda\int_{Q_s}\frac{xyy_x}{ad}dxdt = -\frac{\lambda}{2}\int_s^T\left[\frac{xy^2}{ad}\right]_{x=0}^{x=1}dt + \frac{\lambda}{2}\int_{Q_s}\left(\frac{ad-x(a'd+ad')}{(ad)^2}\right)y^2dxdt$$
$$= -\frac{\lambda}{2}\int_s^T\left[\frac{xy^2}{ad}\right]_{x=0}^{x=1}dt + \frac{\lambda}{2}\int_{Q_s}\left(1-\frac{xa'}{a}-\frac{xd'}{d}\right)\frac{y^2}{ad}dxdt.$$

By ([17], Lemma 1)

$$\lambda\int_0^T\left[y^2\frac{x}{ad}\right]_{x=0}^{x=1} = \frac{\lambda}{a(1)d(1)}\int_0^T y^2(t,1)dt$$

and the thesis follows. □

As a consequence of the previous equality, we have the next relation:

Proposition 5. *Assume Hypothesis 4 and fix $T > 0$. If y is a classical solution of (1), then for every $0 < s < T$, we have*

$$\int_{Q_s}\frac{y_t^2}{a}\left(\frac{K_a}{2}+1-\frac{xa'}{a}\right)dx\,dt + \int_{Q_s}y_{xx}^2\left(3-\frac{K_a}{2}\right)dx\,dt$$
$$+\lambda\int_{Q_s}\left(1-\frac{xa'}{a}-\frac{xd'}{d}+\frac{K_a}{2}\right)\frac{y^2}{ad}dxdt = (B.T.), \tag{37}$$

where

$$(B.T.) = \frac{K_a}{2}\int_0^1\left[\frac{yy_t}{a}\right]_{t=s}^{t=T}dx - 2\int_0^1\left[y_t\frac{x}{a}y_x\right]_{t=s}^{t=T}dx + \frac{K_a\beta}{2}\int_s^T y^2(t,1)dt$$
$$+\frac{K_a}{2}\int_s^T y(t,1)y_t(t,1)dt + \gamma\left(\frac{K_a}{2}-2\right)\int_s^T y_x^2(t,1)dt$$
$$+\left(\frac{K_a}{2}-2\right)\int_s^T y_x(t,1)y_{tx}(t,1)dt + \int_s^T\frac{y_t^2(t,1)}{a(1)}dt - 2\beta\int_s^T y_x(t,1)y(t,1)dt$$
$$-2\int_s^T y_x(t,1)y_t(t,1)dt + \int_s^T y_{xx}^2(t,1)dt + \frac{\lambda}{a(1)d(1)}\int_s^T y^2(t,1)dt.$$

Proof. Let y be a classical solution of (1) and fix $s \in (0,T)$. Multiplying the equation in (1) by $\frac{y}{a}$, integrating by parts over Q_s, and using (19), we obtain

$$0 = \int_0^1\left[y_t\frac{y}{a}\right]_{t=s}^{t=T}dx - \int_{Q_s}\frac{y_t^2}{a}dx\,dt + \int_s^T(yy_{xxx})(t,1)dt$$
$$-\int_s^T(y_xy_{xx})(t,1)dt + \int_{Q_s}y_{xx}^2dx\,dt - \lambda\int_{Q_s}\frac{y^2}{ad}dxdt. \tag{38}$$

Obviously, all the previous integrals make sense, and by multiplying (38) by $\frac{K_a}{2}$, one has

$$0 = \frac{K_a}{2}\int_{Q_s}\left(-\frac{y_t^2}{a}+y_{xx}^2-\lambda\frac{y^2}{ad}\right)dx\,dt + \frac{K_a}{2}\int_0^1\left[y_t\frac{y}{a}\right]_{t=s}^{t=T}dx$$
$$+\frac{K_a}{2}\int_s^T y(t,1)y_{xxx}(t,1)dt - \frac{K_a}{2}\int_s^T y_x(t,1)y_{xx}(t,1)dt. \tag{39}$$

By summing (35) and (39) and using the boundary conditions at 1, we obtain the thesis. □

An immediate consequence of (37) is the next result. However, to prove it, we assume an additional hypothesis on functions a and d.

Hypothesis 5. *Assume a (WD) or (SD), d (WD) with* $K_a + 2K_d < 2$, $\lambda \neq 0$ *with* $\lambda < \frac{1}{\tilde{C}_{HP}}$ *and* $\beta, \gamma \geq 0$.

Observe that this hypothesis is more restrictive than Hypothesis 4; indeed, in Hypothesis 5, we exclude the case $K_a + 2K_d = 2$. In fact, as we can see already from the next result, the condition $K_a + 2K_d < 2$ is important for the technique used in the following proposition:

Proposition 6. *Assume Hypothesis 5, fix* $T > 0$ *and let y be a classical solution of (1). Then, for every* $0 < s < T$ *and for all* $\varepsilon_0 \in (0, 2 - K_a - 2K_d)$*, one proves*

$$\frac{\varepsilon_0}{2}\int_{Q_s}\left(\frac{y_t^2}{a} + y_{xx}^2 - \lambda\frac{y^2}{ad}\right)dx\,dt \leq \left(4\vartheta + \varrho + \frac{C_\gamma}{2}\left(2 - \frac{K_a}{2}\right)\right)E_y(s)$$
$$+ \left(\frac{K_a}{4} + \beta + \frac{K_a\beta}{2} + \frac{\lambda}{a(1)d(1)}\right)\int_s^T y^2(t,1)dt + (\beta + 1 + 2\gamma^2)\int_s^T y_x^2(t,1)dt,$$

if $\lambda > 0$*, and*

$$\frac{\varepsilon_0}{2}\int_{Q_s}\left(\frac{y_t^2}{a} + y_{xx}^2 - \lambda\frac{y^2}{ad}\right)dx\,dt \leq \left(4\vartheta + \varrho + \frac{C_\gamma}{2}\left(2 - \frac{K_a}{2}\right)\right)E_y(s)$$
$$+ \left(\frac{K_a}{4} + \beta + \frac{K_a\beta}{2}\right)\int_s^T y^2(t,1)dt + (\beta + 1 + 2\gamma^2)\int_s^T y_x^2(t,1)dt$$
$$- 4\lambda\tilde{C}_{HP}\left(1 + \frac{3}{2}K_a + K_d\right)\int_s^T E_y(t)dt,$$

if $\lambda < 0$.

Here, $\vartheta := \max\left\{\frac{1}{\epsilon}\left(\frac{4}{a(1)} + K_a C_{HP}\right), 1 + \frac{K_a}{4}\right\}$ *and* $\varrho := \max\left\{2, \frac{K_a}{4} + 1 + \frac{1}{a(1)}\right\}$.

Proof. By assumption, we can take $\varepsilon_0 \in (0, 2 - K_a - 2K_d)$; thus,

$$\begin{aligned} &1 - \frac{xa'}{a} + \frac{K_a}{2} > \frac{\varepsilon_0}{2}, \\ &3 - \frac{K_a}{2} > \frac{\varepsilon_0}{2} \\ &1 - \frac{xa'}{a} - \frac{xd'}{d} + \frac{K_a}{2} \geq \frac{\varepsilon_0}{2}. \end{aligned} \tag{40}$$

Now, we distinguish between the cases $\lambda > 0$ and $\lambda < 0$.
Case $\lambda > 0$.

In this case, the distributed terms in (37) can be estimated from below in the following way:

$$\begin{aligned} &\int_{Q_s}\frac{y_t^2}{a}\left(\frac{K_a}{2} + 1 - \frac{xa'}{a}\right)dx\,dt + \int_{Q_s} y_{xx}^2\left(3 - \frac{K_a}{2}\right)dx\,dt + \lambda\int_{Q_s}\left(1 - \frac{xa'}{a} - \frac{xd'}{d} + \frac{K_a}{2}\right)\frac{y^2}{ad}dxdt \\ &\geq \frac{\varepsilon_0}{2}\int_{Q_s}\left(\frac{y_t^2}{a} + y_{xx}^2 + \lambda\frac{y^2}{ad}\right)dx\,dt \geq \frac{\varepsilon_0}{2}\int_{Q_s}\left(\frac{y_t^2}{a} + y_{xx}^2 - \lambda\frac{y^2}{ad}\right)dx\,dt. \end{aligned} \tag{41}$$

Now, we estimate the boundary terms in (37) from above. First of all, consider the integral $\int_0^1\left(-2y_t\frac{x}{a}y_x + \frac{K_a}{2}\frac{yy_t}{a}\right)(\tau, x)dx$ for all $\tau \in [s, T]$. Using the fact that $\frac{x^2}{a(x)} \leq \frac{1}{a(1)}$, together with the classical Hardy inequality (12) and proceeding as in ([12], Proposition 3.3), one proves

$$\int_0^1 \left(-2y_t \frac{x}{a} y_x + \frac{K_a}{2} \frac{y y_t}{a}\right)(\tau, x) dx \leq$$
$$\leq \frac{4}{a(1)} \int_0^1 y_{xx}^2(\tau, x) dx + \left(1 + \frac{K_a}{4}\right) \int_0^1 \frac{y_t^2}{a}(\tau, x) dx + K_a C_{HP} \int_0^1 y_{xx}^2(\tau, x) dx.$$

Hence, by Proposition 2,

$$\begin{aligned}
&\int_0^1 \left(-2y_t \frac{x}{a} y_x + \frac{K_a}{2} \frac{y y_t}{a}\right)(\tau, x) dx \\
&\leq \left(\frac{4}{a(1)} + K_a C_{HP}\right) \int_0^1 y_{xx}^2(\tau, x) dx + \left(1 + \frac{K_a}{4}\right) \int_0^1 \frac{y_t^2}{a}(\tau, x) dx \\
&\leq \frac{1}{\epsilon}\left(\frac{4}{a(1)} + K_a C_{HP}\right)\left(\int_0^1 y_{xx}^2(\tau, x) dx - \lambda \int_0^1 \frac{y^2}{ad}(\tau, x) dx\right) + \left(1 + \frac{K_a}{4}\right) \int_0^1 \frac{y_t^2}{a}(\tau, x) dx \\
&\leq 2 \max\left\{\frac{1}{\epsilon}\left(\frac{4}{a(1)} + K_a C_{HP}\right), 1 + \frac{K_a}{4}\right\} E_y(\tau),
\end{aligned}$$

for all $\tau \in [s, T]$. Hence, since the energy is non-increasing,

$$\int_0^1 \left[-2y_t \frac{x}{a} y_x + \frac{K_a}{2} \frac{y y_t}{a}\right]_{t=s}^{t=T} dx \leq 4 \max\left\{\frac{1}{\epsilon}\left(\frac{4}{a(1)} + K_a C_{HP}\right), 1 + \frac{K_a}{4}\right\} E_y(s). \tag{42}$$

Now, based on (34) and the fact that $K_a < 2$, we have

$$\begin{aligned}
&\gamma\left(\frac{K_a}{2} - 2\right) \int_s^T y_x^2(t, 1) dt + \left(\frac{K_a}{2} - 2\right) \int_s^T y_x(t, 1) y_{tx}(t, 1) dt \\
&\leq \left(\frac{K_a}{2} - 2\right) \frac{1}{2} \int_2^T (y_x(t, 1)^2)_t dt = \left(\frac{K_a}{2} - 2\right) \frac{1}{2} (y_x^2(T, 1) - y_x^2(s, 1)) \\
&\leq \left(2 - \frac{K_a}{2}\right) \frac{1}{2} y_x^2(s, 1) \leq \left(2 - \frac{K_a}{2}\right) \frac{C_\gamma}{2} E_y(s).
\end{aligned} \tag{43}$$

Obviously,

$$\frac{K_a}{2} \int_s^T y(t, 1) y_t(t, 1) dt \leq \frac{K_a}{4} \int_s^T y^2(t, 1) dt + \frac{K_a}{4} \int_s^T y_t^2(t, 1) dt, \tag{44}$$

$$-\beta \int_s^T 2y_x(t, 1) y(t, 1) dt \leq \beta \int_s^T 2|y_x(t, 1) y(t, 1)| dt \leq \beta \int_s^T y_x^2(t, 1) dt + \beta \int_s^T y^2(t, 1) dt \tag{45}$$

and

$$-\int_s^T 2y_x(t, 1) y_t(t, 1) dt \leq \int_s^T 2|y_x(t, 1) y_t(t, 1)| dt \leq \int_s^T y_x^2(t, 1) dt + \int_s^T y_t^2(t, 1) dt. \tag{46}$$

Furthermore, recalling that $\gamma y_x(t, 1) + y_{xx}(t, 1) + y_{tx}(t, 1) = 0$,

$$\int_s^T y_{xx}^2(t, 1) dt \leq 2\gamma^2 \int_s^T y_x^2(t, 1) dt + 2 \int_s^T y_{tx}^2(t, 1) dt. \tag{47}$$

Hence, by (37), (41)–(47) and Theorem 1, we have

$$\frac{\varepsilon_0}{2}\int_{Q_s}\left(\frac{y_t^2}{a}+y_{xx}^2-\lambda\frac{y^2}{ad}\right)dx\,dt \le 4\max\left\{\frac{1}{\epsilon}\left(\frac{4}{a(1)}+K_aC_{HP}\right),1+\frac{K_a}{4}\right\}E_y(s)$$
$$+\left(2-\frac{K_a}{2}\right)\frac{C_\gamma}{2}E_y(s)+\left(\frac{K_a}{4}+1+\frac{1}{a(1)}\right)\int_s^T y_t^2(t,1)dt+2\int_s^T y_{tx}^2(t,1)dt$$
$$+\left(\frac{K_a}{4}+\beta+\frac{K_a\beta}{2}+\frac{\lambda}{a(1)d(1)}\right)\int_s^T y^2(t,1)dt+(\beta+1+2\gamma^2)\int_s^T y_x^2(t,1)dt$$
$$\le 4\max\left\{\frac{1}{\epsilon}\left(\frac{4}{a(1)}+K_aC_{HP}\right),1+\frac{K_a}{4}\right\}E_y(s)+\left(2-\frac{K_a}{2}\right)\frac{C_\gamma}{2}E_y(s)$$
$$+\max\left\{2,\frac{K_a}{4}+1+\frac{1}{a(1)}\right\}\int_s^T -\frac{d}{dt}E_y(t)dt$$
$$+\left(\frac{K_a}{4}+\beta+\frac{K_a\beta}{2}+\frac{\lambda}{a(1)d(1)}\right)\int_s^T y^2(t,1)dt+(\beta+1+2\gamma^2)\int_s^T y_x^2(t,1)dt$$
$$\le 4\max\left\{\frac{1}{\epsilon}\left(\frac{4}{a(1)}+K_aC_{HP}\right),1+\frac{K_a}{4}\right\}E_y(s)+\left(2-\frac{K_a}{2}\right)\frac{C_\gamma}{2}E_y(s)$$
$$+\max\left\{2,\frac{K_a}{4}+1+\frac{1}{a(1)}\right\}E_y(s)$$
$$+\left(\frac{K_a}{4}+\beta+\frac{K_a\beta}{2}+\frac{\lambda}{a(1)d(1)}\right)\int_s^T y^2(t,1)dt+(\beta+1+2\gamma^2)\int_s^T y_x^2(t,1)dt.$$

Hence,

$$\frac{\varepsilon_0}{2}\int_{Q_s}\left(\frac{y_t^2}{a}+y_{xx}^2-\lambda\frac{y^2}{ad}\right)dx\,dt \le \left(4\vartheta+\varrho+\left(2-\frac{K_a}{2}\right)\frac{C_\gamma}{2}\right)E_y(s)$$
$$+\left(\frac{K_a}{4}+\beta+\frac{K_a\beta}{2}+\frac{\lambda}{a(1)d(1)}\right)\int_s^T y^2(t,1)dt+(\beta+1+2\gamma^2)\int_s^T y_x^2(t,1)dt$$

and the thesis follows.

Case $\lambda < 0$. In this case, based on the definition of energy and (15), one proves

$$\int_{Q_s}\frac{y^2}{ad}dxdt \le \tilde{C}_{HP}\int_{Q_s}y_{xx}^2(t,x)dxdt \le 2\tilde{C}_{HP}\int_s^T E_y(t)dt;$$

hence,

$$-\lambda\int_{Q_s}\frac{y^2}{ad}dxdt \le -2\lambda\tilde{C}_{HP}\int_s^T E_y(t)dt. \tag{48}$$

Moreover, by (37) and (40), one has

$$\begin{aligned}\frac{\varepsilon_0}{2}\int_{Q_s}\left(\frac{y_t^2}{a}+y_{xx}^2-\lambda\frac{y^2}{ad}\right)dx\,dt &\le \int_{Q_s}\frac{y_t^2}{a}\left(\frac{K_a}{2}+1-\frac{xa'}{a}\right)dx\,dt+\int_{Q_s}y_{xx}^2\left(3-\frac{K_a}{2}\right)dx\,dt\\ &-\lambda\int_{Q_s}\left(1-\frac{xa'}{a}-\frac{xd'}{d}+\frac{K_a}{2}\right)\frac{y^2}{ad}dxdt\\ &=(B.T.)-2\lambda\int_{Q_s}\left(1-\frac{xa'}{a}-\frac{xd'}{d}+\frac{K_a}{2}\right)\frac{y^2}{ad}dxdt,\end{aligned} \tag{49}$$

where (B.T.) is the boundary terms in (37). Now, by (48),

$$-2\lambda\int_{Q_s}\left(1-\frac{xa'}{a}-\frac{xd'}{d}+\frac{K_a}{2}\right)\frac{y^2}{ad}dxdt \le -4\lambda\tilde{C}_{HP}\left(1+\frac{3}{2}K_a+K_d+M\right)\int_s^T E_y(t)dt.$$

Proceeding as for the case $\lambda > 0$ and using the fact that $\frac{\lambda}{a(1)d(1)} < 0$, one can estimate the boundary terms in the following way:

$$
\begin{aligned}
(B.T.) \leq\ & 4\max\left\{\frac{1}{\epsilon}\left(\frac{4}{a(1)} + K_a C_{HP}\right), 1 + \frac{K_a}{4}\right\} E_y(s) \\
& + \left\{\max\left\{2, \frac{K_a}{4} + 1 + \frac{1}{a(1)}\right\} + \left(2 - \frac{K_a}{2}\right)\frac{C_\gamma}{2}\right\} E_y(s) \\
& + \left(\frac{K_a}{4} + \beta + \frac{K_a\beta}{2}\right)\int_s^T y^2(t,1)dt + (\beta + 1 + 2\gamma^2)\int_s^T y_x^2(t,1)dt.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
& \frac{\varepsilon_0}{2}\int_{Q_s}\left(\frac{y_t^2}{a} + y_{xx}^2 - \lambda\frac{y^2}{ad}\right)dx\,dt \leq \left(4\vartheta + \varrho\left(2 - \frac{K_a}{2}\right)\frac{C_\gamma}{2}\right)E_y(s) \\
& + \left(\frac{K_a}{4} + \beta + \frac{K_a\beta}{2}\right)\int_s^T y^2(t,1)dt + (\beta + 1 + 2\gamma^2)\int_s^T y_x^2(t,1)dt \\
& - 4\lambda\tilde{C}_{HP}\left(1 + \frac{3}{2}K_a + K_d\right)\int_s^T E_y(t)dt
\end{aligned}
$$

and the thesis follows. □

In the next proposition, we will find an estimate from above for $\int_s^T y^2(t,1)dt + \int_s^T y_x^2(t,1)dt$. To this end, set

$$
\tilde{C}_\beta := \begin{cases} \frac{1}{\beta}, & \beta \neq 0, \text{ for all considered } \lambda, \\ \frac{1}{C_\epsilon}, & \beta = 0, \text{ for all considered } \lambda, \end{cases}
$$

$$
\tilde{C}_\gamma := \begin{cases} \frac{1}{\gamma}, & \gamma \neq 0, \text{ for all considered } \lambda, \\ \frac{1}{C_\epsilon}, & \gamma = 0, \text{ for all considered } \lambda, \end{cases}
$$

and

$$
\nu := \frac{1}{\tilde{C}_\beta + \tilde{C}_\gamma}. \tag{50}
$$

Proposition 7. *Assume Hypothesis 5 and fix $T > 0$. If y is a classical solution of (1), then for every $0 < s < T$ and for every $\delta \in (0, \nu)$, we have*

$$
\begin{aligned}
& \int_s^T y^2(t,1)dt + \int_s^T y_x^2(t,1)dt \leq \frac{\delta}{C_\delta}\int_s^T E_y(t)dt \\
& + \frac{1}{C_\delta}\left[\frac{2}{C_\epsilon}\left(1 + 2C_{HP}(C_\beta + C_\gamma)\right) + \frac{1}{\delta}\left(\frac{4C_{HP}}{C_\epsilon} + \frac{1}{2}\right)\right]E_y(s),
\end{aligned}
$$

where

$$
C_\delta := 1 - \delta(\tilde{C}_\beta + \tilde{C}_\gamma).
$$

Proof. Set $\rho = y(t,1)$, $\mu = y_x(t,1)$, where $t \in [s,T]$, and let $z = z(t,\cdot) \in H^2_{\frac{1}{a},0}(0,1)$ be the unique solution of

$$
\int_0^1 z_{xx}\varphi'' dx - \lambda\int_0^1 \frac{z\varphi}{ad}dx + \beta z(t,1)\varphi(1) + \gamma z'(t,1)\varphi'(1) = \rho\varphi(1) + \mu\varphi'(1),
$$

for all $\varphi \in H^2_{\frac{1}{a},0}(0,1)$. By Proposition 3, $z(t,\cdot) \in D(A_\lambda)$ for all t and solves

$$\begin{cases} A_\lambda z = 0, \\ \beta z(t,1) - z_{xxx}(t,1) = \rho, \\ \gamma z_x(t,1) + z_{xx}(t,1) = \mu. \end{cases} \tag{51}$$

By (21), we also have

$$\|z(t)\|^2_{L^2_{\frac{1}{a}}(0,1)} \leq \frac{8C_{HP}}{C_\epsilon}(y^2(t,1) + y_x^2(t,1)) \quad \text{and} \quad |||z(t)|||^2 \leq 2(y^2(t,1) + y_x^2(t,1)), \tag{52}$$

where C_ϵ is defined in (22). Moreover, if $\beta \neq 0$ and $\gamma \neq 0$, then by Proposition 2 if $\lambda > 0$, one proves

$$z^2(t,1) \leq \frac{1}{\beta}|||z|||^2 \leq \frac{2}{\beta}(y^2(t,1) + y_x^2(t,1))$$

and

$$z_x^2(t,1) \leq \frac{1}{\gamma}|||z|||^2 \leq \frac{2}{\gamma}(y^2(t,1) + y_x^2(t,1)).$$

On the other hand, if $\beta = 0$ and $\gamma = 0$, then by (24), (25) and (29), it results in $z^2(t,1), z_x^2(t,1) \leq \|z\|^2_{2,\circ} \leq \frac{1}{C_\epsilon}|||z|||^2 \leq \frac{1}{C_\epsilon}(|\rho| + |\mu|)^2 \leq \frac{2}{C_\epsilon}(y^2(t,1) + y_x^2(t,1))$. In every case, for all considered λ, we have

$$z^2(t,1) \leq 2\tilde{C}_\beta(y^2(t,1) + y_x^2(t,1)), \tag{53}$$

and

$$z_x^2(t,1) \leq 2\tilde{C}_\gamma(y^2(t,1) + y_x^2(t,1)). \tag{54}$$

Finally, observe that for all considered λ and all $t > 0$, we have

$$\frac{1}{2}\int_0^1 \left(\frac{1}{a}y_t^2(t,x) + y_{xx}^2(t,x)\right)dx \leq \frac{1}{C_\epsilon}E_y(t). \tag{55}$$

Indeed, consider, first of all, $\lambda < 0$, then

$$\frac{1}{2}\int_0^1 \left(\frac{1}{a}y_t^2(t,x) + y_{xx}^2(t,x)\right)dx \leq E_y(t).$$

If $\lambda \in \left(0, \dfrac{1}{\tilde{C}_{HP}}\right)$ and $\epsilon \in (0,1)$ is as in (18), we obtain (27), which implies that

$$\begin{aligned} 2E_y(t) &\geq \epsilon \int_0^1 y_{xx}^2(t,x)dx + \int_0^1 \frac{y_t^2(t,x)}{a(x)}dx + \beta y^2(t,1) + \gamma y_x^2(t,1) \\ &\geq \epsilon\left(\int_0^1 y_{xx}^2(t,x)dx + \int_0^1 \frac{y_t^2(t,x)}{a(x)}dx\right) + \beta y^2(t,1) + \gamma y_x^2(t,1); \end{aligned}$$

in particular,

$$\frac{1}{2}\int_0^1 \left(\frac{1}{a}y_t^2(t,x) + y_{xx}^2(t,x)\right)dx \leq \frac{1}{\epsilon}E_y(t),$$

for all $t \geq 0$.

Now, multiplying the equation in (1) by $\dfrac{z}{a}$ and integrating over Q_s, we have

$$\begin{aligned} 0 &= \int_{Q_s} y_{tt}\frac{z}{a}dx\,dt + \int_{Q_s} z y_{xxxx} dx\,dt - \lambda\int_{Q_s} \frac{y}{ad} z dx dt \\ &= \int_0^1 \left[y_t \frac{z}{a}\right]_{t=s}^{t=T} dx - \int_{Q_s} \frac{y_t z_t}{a} dx\,dt + \int_s^T z(t,1) y_{xxx}(t,1)dt - \int_s^T z_x(t,1) y_{xx}(t,1)dt \\ &+ \int_{Q_s} z_{xx} y_{xx} dx\,dt - \lambda \int_{Q_s} \frac{y}{ad} z dx dt. \end{aligned} \tag{56}$$

Hence, (56) reads

$$\int_0^1 \left[y_t \frac{z}{a}\right]_{t=s}^{t=T} dx - \int_{Q_s} \frac{y_t z_t}{a} dx\,dt - \lambda \int_{Q_s} \frac{y}{ad} z dx dt \\ = -\int_s^T z(t,1) y_{xxx}(t,1) dt + \int_s^T z_x(t,1) y_{xx}(t,1) dt - \int_{Q_s} z_{xx} y_{xx} dx\,dt. \tag{57}$$

On the other hand, multiplying the equation in (51) by $\frac{y}{a}$ and integrating over Q_s, we have $\int_{Q_s} z_{xxxx} y\,dx\,dt = 0$. By (19), we obtain $\int_{Q_s} z_{xx} y_{xx} dx\,dt = -\int_s^T z_{xxx}(t,1) y(t,1) dt + \int_s^T y_x(t,1) z_{xx}(t,1) dt$. Substituting in (57), using the fact that $z_{xxx}(t,1) = \beta z(t,1) - \rho$, $z_{xx}(t,1) = -\gamma z_x(t,1) + \mu$, $\rho = y(t,1)$, $\mu = y_x(t,1)$ and proceeding as in [12], we have

$$\begin{aligned}
&\int_0^1 \left[y_t \frac{z}{a}\right]_{t=s}^{t=T} dx - \int_{Q_s} \frac{y_t z_t}{a} dx\,dt - \lambda \int_{Q_s} \frac{y}{ad} z dx dt \\
&= -\int_s^T z(t,1) y_{xxx}(t,1) dt + \int_s^T z_x(t,1) y_{xx}(t,1) dt + \int_s^T y(t,1)[\beta z(t,1) - \rho] dt \\
&- \int_s^T y_x(t,1)[-\gamma z_x(t,1) + \mu] dt \\
&= \int_s^T z(t,1)[\beta y(t,1) - y_{xxx}(t,1)] dt \\
&+ \int_s^T z_x(t,1)[y_{xx}(t,1) + \gamma y_x(t,1)] dt - \int_s^T y^2(t,1) dt - \int_s^T y_x^2(t,1) dt.
\end{aligned}$$

Then,

$$\int_s^T y^2(t,1) dt + \int_s^T y_x^2(t,1) dt = -\int_s^T (y_t z)(t,1) dt - \int_s^T (z_x y_{tx})(t,1) dt \\ - \int_0^1 \left[y_t \frac{z}{a}\right]_{t=s}^{t=T} dx + \int_{Q_s} \frac{y_t z_t}{a} dx\,dt + \lambda \int_{Q_s} \frac{y}{ad} z dx dt. \tag{58}$$

Thus, in order to estimate $\int_s^T y^2(t,1) dt + \int_s^T y_x^2(t,1) dt$, we have to consider the four terms in the previous equality.

So, by (21), (34), and Theorem 1, we have, for all $\tau \in [s,T]$,

$$\begin{aligned}
\int_0^1 \left|\frac{y_t z}{a}(\tau,x)\right| dx &\le \frac{1}{2}\int_0^1 \frac{y_t^2(\tau,x)}{a(x)} dx + \frac{1}{2}\int_0^1 \frac{z^2(\tau,x)}{a(x)} dx \\
&\le \frac{1}{2}\int_0^1 \frac{y_t^2(\tau,x)}{a(x)} dx + \frac{2C_{HP}}{C_\epsilon}\left(y^2(\tau,1) + y_x^2(\tau,1)\right) \\
&\le \frac{1}{C_\epsilon} E_y(\tau) + \frac{2C_{HP}}{C_\epsilon}(C_\beta + C_\gamma) E_y(\tau) \\
&\le \left(1 + 2C_{HP}(C_\beta + C_\gamma)\right)\frac{1}{C_\epsilon} E_y(s).
\end{aligned}$$

By Theorem 1, we have

$$\left|\int_0^1 \left[\frac{y_t z}{a}\right]_{t=s}^{t=T} dx\right| \le 2\left(1 + 2C_{HP}(C_\beta + C_\gamma)\right)\frac{1}{C_\epsilon} E_y(s). \tag{59}$$

Moreover, for any $\delta > 0$, we have

$$\int_s^T |(y_t z)(t,1)| dt \le \frac{1}{2\delta}\int_s^T y_t^2(t,1) dt + \frac{\delta}{2}\int_s^T z^2(t,1) dt \\ \le \frac{1}{2\delta}\int_s^T y_t^2(t,1) dt + \delta \tilde{C}_\beta \int_s^T (y^2 + y_x^2)(t,1) dt, \tag{60}$$

by (53). In a similar way, using (54), it is possible to find the next estimate

$$\int_s^T |(z_x y_{tx})(t,1)|dt \le \frac{1}{2\delta}\int_s^T y_{tx}^2(t,1)dt + \delta\tilde{C}_\gamma \int_s^T (y^2+y_x^2)(t,1)dt. \tag{61}$$

Therefore, by summing (60) and (61) and applying Theorem 1, we obtain

$$\begin{aligned} &\int_s^T |(y_t z)(t,1)|dt + \int_s^T |(z_x y_{tx})(t,1)|dt \\ &\le \frac{1}{2\delta}\int_s^T -\frac{d}{dt}E_y(t)dt + \delta\left(\tilde{C}_\beta + \tilde{C}_\gamma\right)\int_s^T (y^2+y_x^2)(t,1)dt \\ &\le \frac{E_y(s)}{2\delta} + \delta\left(\tilde{C}_\beta + \tilde{C}_\gamma\right)\int_s^T (y^2+y_x^2)(t,1)dt. \end{aligned} \tag{62}$$

Finally, we estimate the integral $\int_{Q_s}\left|\frac{y_t z_t}{a}\right| dx\,dt$. To this end, consider again the problem (23) and differentiate with respect to t. Thus,

$$\begin{cases} a(z_t)_{xxxx} - \lambda\frac{z_t}{d} = 0, \\ \beta z_t(t,1) - (z_t)_{xxx}(t,1) = y_t(t,1), \\ \gamma(z_t)_x(t,1) + (z_t)_{xx}(t,1) = (y_x)_t(t,1). \end{cases}$$

Clearly, z_t satisfies (52), in particular

$$\|z_t(t)\|^2_{L^2_{\frac{1}{a}}(0,1)} \le \frac{8C_{HP}}{C_\epsilon}(y_t^2(t,1) + y_{tx}^2(t,1))$$

and

$$|||z_t(t)|||^2 \le 2(y_t(^2t,1)| + y_{tx}^2(t,1)).$$

Thus, by (55) and the previous estiamte, for $\delta > 0$, we find

$$\begin{aligned} \int_{Q_s}\left|\frac{y_t z_t}{a}\right| dx\,dt &\le \frac{\delta}{2}\int_{Q_s}\frac{y_t^2}{a}dx\,dt + \frac{1}{2\delta}\int_{Q_s}\frac{z_t^2}{a}dx\,dt \\ &\le \delta\int_s^T E_y(t)dt + \frac{4C_{HP}}{\delta C_\epsilon}\int_s^T (y_t^2(t,1) + y_{tx}^2(t,1))dt \\ &= \delta\int_s^T E_y(t)dt + \frac{4C_{HP}}{\delta C_\epsilon}\int_s^T -\frac{d}{dt}E_y(t)dt \\ &\le \delta\int_s^T E_y(t)dt + \frac{4C_{HP}}{\delta C_\epsilon}E_y(s). \end{aligned} \tag{63}$$

Coming back to (58) and using (59), (62), and (63), we find

$$\begin{aligned} \int_s^T (y^2+y_x^2)(t,1)dt \ &\le 2\left(\frac{1}{C_\epsilon} + \frac{2C_{HP}}{C_\epsilon}(C_\beta + C_\gamma)\right)E_y(s) + \frac{E_y(s)}{2\delta} \\ &+ \delta\left(\tilde{C}_\beta + \tilde{C}_\gamma\right)\int_s^T (y^2+y_x^2)(t,1)dt \\ &+ \delta\int_s^T E_y(t)dt + \frac{4C_{HP}}{\delta C_\epsilon}E_y(s). \end{aligned}$$

Hence, for every $\delta \in (0,\nu)$,

$$\begin{aligned} C_\delta\int_s^T (y^2(t,1) + y_x^2(t,1))dt \le 2\left(\frac{1}{C_\epsilon} + \frac{2\tilde{C}_{HP}}{C_\epsilon}(C_\beta + C_\gamma)\right)E_y(s) \\ + \delta\int_s^T E_y(t)dt + \frac{1}{\delta}\left(\frac{4C_{HP}}{C_\epsilon} + \frac{1}{2}\right)E_y(s), \end{aligned}$$

and the thesis follows. □

As a consequence of Propositions 6 and 7, we can formulate the main result of the paper, whose proof is based on ([18], Theorem 8.1).

Set

$$C_1 := \frac{1}{C_\delta}\left[\frac{2}{C_\epsilon}(1+2C_{HP}(C_\beta+C_\gamma))+\frac{1}{\delta}\left(\frac{4C_{HP}}{C_\epsilon}+\frac{1}{2}\right)\right],$$

$$C_2 := \left(4\vartheta+\varrho+\frac{C_\gamma}{2}\left(2-\frac{K_a}{2}\right)\right),$$

$$C_3 := \begin{cases}\left(\frac{K_a\beta}{2}+\frac{K_a}{4}+\beta+\varepsilon_0\frac{\beta}{2}+\frac{\lambda}{a(1)d(1)}\right), & \lambda>0,\\ \left(\frac{K_a}{4}+\beta+\frac{K_a\beta}{2}+\varepsilon_0\frac{\beta}{2}\right), & \lambda<0,\end{cases}$$

and

$$C_4 := \left(\beta+1+2\gamma^2+\varepsilon_0\frac{\gamma}{2}\right).$$

Theorem 2. *Assume Hypothesis 5, fix $T>0$ and if $\lambda<0$, then $\lambda\in\left(\frac{-\varepsilon_0}{4\tilde{C}_{HP}(1+\frac{3}{2}K_a+K_d)},0\right)$. Let y be a mild solution for (1). Then, for all $t>0$ and for all $\delta\in(0,\min\{\nu,\mu\})$*

$$E_y(t)\le E_y(0)e^{1-\frac{t}{M}}, \tag{64}$$

where

$$\mu := \begin{cases}\frac{\varepsilon_0 C_\delta}{\max\{C_3,C_4\}}, & \lambda>0,\\ \frac{\varepsilon_0+4\lambda\tilde{C}_{HP}(1+\frac{3}{2}K_a+K_d)}{\max\{C_3,C_4\}+(\varepsilon_0+4\lambda\tilde{C}_{HP}(1+\frac{3}{2}K_a+K_d))(\tilde{C}_\beta+\tilde{C}_\gamma)}, & \lambda<0\end{cases}$$

and

$$M := \begin{cases}\frac{C_\delta(C_2+\max\{C_3,C_4\}C_1)}{\varepsilon_0 C_\delta-\delta\max\{C_3,C_4\}}, & \lambda>0,\\ \frac{C_\delta(C_2+\max\{C_3,C_4\}C_1)}{C_\delta(\varepsilon_0+4\lambda C_{HP}(1+\frac{3}{2}K_a+K_d))-\delta\max\{C_3,C_4\}}, & \lambda<0.\end{cases}$$

Here, ν is defined as in (50).

Proof. As a first step, consider y a classical solution of (1) and $\lambda>0$. Take $\delta\in\left(0,\min\left\{\nu,\frac{\varepsilon_0 C_\delta}{\max\{C_3,C_4\}}\right\}\right)$. Then, based on the definition of E_y and Propositions 6 and 7, we have

$$\begin{aligned}\varepsilon_0\int_s^T E_y(t)dt &= \frac{\varepsilon_0}{2}\int_{Q_s}\left(\frac{y_t^2(t,x)}{a(x)}+y_{xx}^2(t,x)-\frac{\lambda}{ad}y^2(t,x)\right)dxdt\\ &+\varepsilon_0\frac{\beta}{2}\int_s^T y^2(t,1)dt+\varepsilon_0\frac{\gamma}{2}\int_s^T y_x^2(t,1)dt\\ &\le C_2E_y(s)+C_3\int_s^T y^2(t,1)dt+C_4\int_s^T y_x^2(t,1)dt\\ &\le C_2E_y(s)+\max\{C_3,C_4\}\left(\int_s^T y^2(t,1)+y_x^2(t,1)\right)dt\\ &\le C_2E_y(s)+\max\{C_3,C_4\}\frac{\delta}{C_\delta}\int_s^T E_y(t)dt+\max\{C_3,C_4\}C_1E_y(s).\end{aligned}$$

This implies $\left[\varepsilon_0 - \max\{C_3, C_4\}\frac{\delta}{C_\delta}\right] \int_s^T E_y(t)dt \le (C_2 + \max\{C_3, C_4\}C_1)E_y(s)$. Hence, we can apply ([18], Theorem 8.1) with $M := \frac{C_\delta(C_2+\max\{C_3,C_4\}C_1)}{\varepsilon_0 C_\delta - \delta\max\{C_3,C_4\}}$ and (64) holds.

Now, consider $\lambda < 0$. By Propositions 6 and 7, we have

$$\begin{aligned}
\varepsilon_0 \int_s^T E_y(t)dt &= \frac{\varepsilon_0}{2}\int_{Q_s}\left(\frac{y_t^2(t,x)}{a(x)} + y_{xx}^2(t,x) - \frac{\lambda}{ad}y^2(t,x)\right)dxdt \\
&+ \varepsilon_0\frac{\beta}{2}\int_s^T y^2(t,1)dt + \varepsilon_0\frac{\gamma}{2}\int_s^T y_x^2(t,1)dt \\
&\le C_2E_y(s) + \max\{C_3, C_4\}\left(\int_s^T y^2(t,1)dt + \int_s^T y_x^2(t,1)dt\right) \\
&- 4\lambda\tilde{C}_{HP}\left(1 + \frac{3}{2}K_a + K_d\right)\int_s^T E_y(t)dt \\
&\le (C_2 + \max\{C_3, C_4\}C_1)E_y(s) + \max\{C_3, C_4\}\frac{\delta}{C_\delta}\int_s^T E_y(t)dt \\
&- 4\lambda\tilde{C}_{HP}\left(1 + \frac{3}{2}K_a + K_d\right)\int_s^T E_y(t)dt.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\left(\varepsilon_0 + 4\lambda\tilde{C}_{HP}\left(1 + \frac{3}{2}K_a + K_d\right) - \max\{C_3, C_4\}\frac{\delta}{C_\delta}\right)\int_s^T E_y(t)dt \\
&\le (C_2 + \max\{C_3, C_4\}C_1)E_y(s).
\end{aligned}$$

Recalling that $C_\delta := 1 - \delta(\tilde{C}_\beta + \tilde{C}_\gamma)$, where $\delta < \mu$, one proves that

$$\varepsilon_0 + 4\lambda\tilde{C}_{HP}\left(1 + \frac{3}{2}K_a + K_d\right) - \max\{C_3, C_4\}\frac{\delta}{C_\delta} > 0.$$

Hence, again by ([18], Theorem 8.1) with $M := \frac{C_\delta(C_2+\max\{C_3,C_4\}C_1)}{C_\delta(\varepsilon_0+4\lambda C_{HP}(1+\frac{3}{2}K_a+K_d)) - \delta\max\{C_3,C_4\}}$, (64) holds.

If y is the mild solution for the problem, we can proceed as in [22], obtaining the thesis. □

4. Conclusions and a Open Problems

In this paper, we study the exponential stability of the energy related to (1). In particular, in Theorem 2, we show that

If λ is small and a, d are not too degenerate, then
the energy of the solution to (1) converges exponentially to 0 as time diverges.

This result leads to some open problems. The first one is to prove the stability if $K_a + 2K_d \ge 2$. As we have seen, the condition $K_a + 2K_d < 2$ is crucial in Proposition 6 and the condition $K_a + 2K_d \le 2$ is important for finding that the two domains $D(A)$ and $D(A_\lambda)$ coincide. On the other hand, $K_a + K_d \le 2$ is crucial to obtain the estimate given in Proposition 1.

Another important open problem is to prove the stability of (1), when $\lambda \le \frac{-\varepsilon_0}{4\tilde{C}_{HP}(1+\frac{3}{2}K_a+K_d)}$.

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