Bollettino dell'Unione Matematica Italiana
https://doi.org/10.1007/s40574-025-00466-7

Interpolation in higher codimension

Luca Chiantini[1]

Received: 27 December 2024 / Accepted: 28 January 2025

Abstract
Following the suggestions contained in [5], I will discuss constructions that generalizes the interpolation problems for divisors to the case of varieties of higher codimension, with emphasis on the case of curves arising as 0-loci of vector bundles of rank 2 in $\mathbb{P}^3$.

Keywords algebraic interpolation · curves · vector bundles

1 Introduction

It is a pleasure for me to illustrate some aspects of Edoardo's researches, expecially those which can open new perspectives for future studies in Algebraic Geometry.

I will discuss in this note a construction that generalizes the interpolation problem for divisors to the case of curves in higher codimension (mainly in codimension 2, e.g. curves in $\mathbb{P}^3$).

An initial interpolation problem for linear systems can be described as follows. Consider a complete linear system $|\mathcal{L}|$ on a projective variety X, and call $\mathcal{V}_\delta(\mathcal{L})$ the subset of $|\mathcal{L}|$ which parametrizes divisors having δ nodes (i.e. δ ordinary double points) and no other singularities. $\mathcal{V}_\delta(\mathcal{L})$ has a natural structure of a quasi-projective variety. The problem consists of understanding the basic geometric properties of $\mathcal{V}_\delta(\mathcal{L})$, e.g. its dimension, its irreducibility, and its smoothness. Even for the (apparently elementary) case where $X = \mathbb{P}^n$ is a projective space, the problem is far from obvious. The dimension of the (smooth) component of $\mathcal{V}_\delta(\mathcal{L})$ whose generic elements have nodes at δ general points of $\mathbb{P}^n$ has been completely described only recently, in the celebrated paper [1] of Alexander and Hirschowitz.

The interpolation problem extends to non-necessarily complete linear systems $\mathcal{W} \subseteq |\mathcal{L}|$, and to singularities with higher multiplicities. In the extended form, no complete description of $\mathcal{V}_\delta(\mathcal{W})$ is available, even by looking at the components whose elements are singular at δ general points.

Any extension of the interpolation problem to subvarieties of higher codimension in X must face the initial obstruction that there is no standard analogue of linear systems for the

The author is member of INdAM-GNSAGA. No conflict of interest exists.
I am really honored to dedicate this short note to Edoardo, who was my Master during my first approaches to Algebraic Geometry, when we shared an office in the Mathematics Department of Brandeis University (1982), and with whom it has been always a delight to collaborate.

✉ Luca Chiantini
luca.chiantini@unisi.it

1 Universitá degli Studi di Siena, Siena, Italy

Published online: 19 February 2025

parametrization of subschemes in higher codimension. The Hilbert schemes, or the Chow schemes, provide a parametrization whose basis is not a linear space, so they are not an easy-to-handle substitute for ‘linear systems’ in the extensions of interpolation (see e.g. [17] for families of curves passing through a prescribed set).

Instead, one can use vector bundles (or even reflexive sheaves) to provide an analogue for linear systems for subvarieties of codimension greater than 1. The idea is that when E is a sufficiently positive rank 2 bundle over $X = \mathbb{P}^3$ then 0-loci of general sections of E determine curves in the projective space, so that a dense open set in the space $\mathbb{P}(H^0(E))$ determines families of such curves.

The aim of this note is to present a short account of the (few) results known in the study of singular curves arising as sections of a rank 2 bundle. The theory behind the study of such families has been developed, in its initial steps, in a joint paper with Edoardo, namely [5] in the reference list. Section 2 is devoted to a brief account of the results contained in the paper. The main result shows that when the bundle E is sufficiently positive then some fundamental property of the variety $\mathcal{V}_\delta(E) \subset \mathbb{P}(H^0(E))$, which parametrizes sections whose 0-loci Y have δ singular points, can be computed. In the statements, we do not care much about the position of the singularities of Y. By imposing, as in classical interpolation problems, that the singular points of Y are in general position, stronger results should be expected.

Motivations to understand the interpolation problem for higher codimensions can be found in Sect. 3. I will mention several possible developments in the theory, following analogies with the theory of linear systems, which are still missing in the literature. Essentially all the interpolation problems for divisors have an analogue for curves 0-loci of sections of vector bundles. This includes a construction which mimicks, in Grassmannian, the definition of secant varieties (see Sect. 3.3). I hope that describing open problems in this circle of ideas can stimulate further researches in the listed topics.

For all the subjects touched in the list of developments described in Sect. 3 (rank 2 bundles and sheaves on projective spaces, interpolation with singularities, secant varieties and Terracin’s Lemma) Edoardo produced a huge amount of contributes, in many cases real cornerstones for the theory. I decided to limit the bibliography only to papers directly related with the extension of interpolation, because an inclusive list of Edoardo’s results in the mentioned topics could only be very partial.

2 Curves in higher codimension and vector bundles

Notation 2.1 In this section, let E be a rank 2 bundle on $\mathbb{P}^3$, with Chern classes c_1, c_2. If s is a section, element of $H^0(E)$, call $Y := (s)_0$ the 0-locus of s. Assume that for a general choice of s the 0-locus Y is an irreducible curve. Then we have the usual exact sequence (Serre’s extension):

$$0 \to \mathcal{O} \to E \to \mathcal{I}_Y(c_1) \to 0 \tag{1}$$

The curve Y has degree $d = c_2$. Its arithmetic and geometric genus are given respectively by:

$$p_a(Y) = (c_1 - 4)\frac{c_2}{2}$$
$$g = p_a(Y) - \delta.$$

The projective space $\mathbb{P}(H^0(E))$ can be considered as the analogue of the space that parametrizes linear systems of divisors. It determines a family of curves, parametrized by a linear space, which contains Y.

In the notation, one identifies $H^0(E)/(s)$ with the tangent space to $\mathbb{P}(H^0(E))$ at s.

We will focus on sections whose 0-locus Y is singular at δ points. We will assume that, in general, Y has δ nodes for singularities (i.e. Y is a nodal curve).

Definition 2.2 Call $\mathcal{V}_\delta(E)$ the subset of $\mathbb{P}(H^0(E))$ formed by those sections whose 0-locus is a nodal curve of geometric genus $p_a(Y) - \delta$.

It is a standard fact that $\mathcal{V}_\delta(E)$ is a quasi projective subvariety of $\mathbb{P}(H^0(E))$.

If Y is the 0-locus of s, then other section of E determines a surface of degree c_1 containing Y. Indeed we have:

Remark 2.3 A general section $s' \in H^0(E)$ defines, by sequence (1), a surface of degree c_1 containing Y. The surface defined by s' is exactly the 0-locus of the wedge product $s \wedge s'$ and it will be denoted by $F(s')$. If s, s' are defined by the pairs (f, g), (f', g') in the open set U, then in U the surface $F(s')$ has equation $fg' - f'g$.

Conversely, since the map $H^0(E) \to H^0 I_Y(c_1)$ surjects, every surface of degree c_1 containing Y is obtained in this way.

Next proposition determines the tangent space to $\mathcal{V}_\delta(E)$ as a subvariety of $\mathbb{P}(H^0(E))$ (see [5], Proposition 2.3).

Proposition 2.4 *The tangent space to $\mathcal{V}_\delta(E)$ at s is given by those sections $s' \in H^0(E)$ for which the surface $F(s') = (s \wedge s')_0$ is singular at the nodes of $Y = (s)_0$.*

Remark 2.5 Since the curve $Y \subset \mathbb{P}^3$ is a local complete intersection in an open neighbourhood of any point $P \in Y$, it follows that if P is a nodal point of Y, then it corresponds to a planar singularity of Y. Thus, among the surfaces containing Y, the condition of being singular in Y has codimension 1.

As a consequence of the remark above, by imposing δ nodes to Y one obtains the following:

Corollary 2.6 *Let E be a rank 2 bundle and let $s \in H^0(E)$ be a global section such that the 0-locus $Y = (s)_0$ is a nodal curve, with nodes at $P_1, \ldots, P_\delta$. Then the dimension of the tangent space to $\mathcal{V}_\delta(E)$ at s is greater or equal than $h^0(E) - \delta$. Equality holds if and only the conditions of singularity imposed by the nodes $P_1, \ldots, P_\delta$ to surfaces of degree c_1 through Y are independent.*

Notation 2.7 The number $h^0(E) - \delta$ is the *expected dimension* of the components of $\mathcal{V}_\delta(E)$ (which, in principle, could be reducible).

A component of $\mathcal{V}_\delta(E)$ is *regular* if its dimension equals the expected one.

By Corollary 2.6, the dimension of a component of $\mathcal{V}_\delta(E)$ is greater or equal to the expected dimension, if it is non-empty.

As in the case of linear systems of divisors, there are natural regularity results if the vector bundle E turns out to be strongly positive.

Theorem 2.8 *Assume that $E(-1)$ is generated by global sections. Then for all $m \gg 0$ and for $\delta \leq m + 1$, the Severi variety $\mathcal{V}_\delta(E(m))$ is either empty or smooth, of the (expected) dimension $h^0(E(m)) - 1 - \delta$.*

One can prove the non-emptyness of $\mathcal{V}_\delta(E(m))$ in the numerical range of the previous theorem. Indeed, non-emptyness always holds when $h^0(E) \geq 4\delta$ (see [5], Remark 2.7).

Example 2.9 With the assumptions of Theorem 2.8 then $\mathcal{V}_\delta E(m)$ is non empty.

The construction of elements in $\mathcal{V}_\delta E(m)$ can be obtained by using a reduction to affine open subsets of $\mathbb{P}^3$, together with a Bertini argument. We refer to [5] Proposition 3.3 for details.

Notice that if we relax the numerical assumptions, then Theorem 2.8 fails to hold.

Example 2.10 (See [5] Example 3.2). Even for the decomposable rank 2 bundle $E = \mathcal{O}(1) \oplus \mathcal{O}(4)$ and for $\delta = m+2$ the variety $\mathcal{V}_\delta E(m)$ has some singular point. The complete intersection curve Y corresponding to a singular point of $\mathcal{V}_\delta E(m)$ has $m + 2$ collinear nodes. Notice that $E(-1)$ is generated by global sections.

3 Extensions

This section contains suggestions for further extensions of the construction oulined in Sect. 2. Most of the proposed topics are essentially unexplored.

3.1 Reflexive sheaves

Curves C associated to sections of rank 2 vector bundles in $\mathbb{P}^3$ turn out to be quite special. They are *subcanonical* curves, in the sense that the canonical class ω_C of C is a multiple of the hyperplane class $\mathcal{O}_C(1)$ (see e.g. [16]). For general curves, Serre's construction determines a description as the vanishing loci of sections of *reflexive sheaves*, sheaves F with a finite number of singular points, The singular points of F represent the differnce between ω_C and a multiple of the hyperplane class of C (see [15]). The analysis of the Sect. 2 can be extended, in principle, to the case where the rank 2 bundle E is replaced by a reflexive sheaf F. Of course the analysis must take care of the singularities of F, with special arguments around these points. As far as I know, nothing like that has been investigated in the literature.

3.2 Interpolation at generic points

It is relevant, for interpolation theory over linear systems, what happens if we impose singularities in δ *general* points of a projective space. The Alexander-Hirschowitz analysis [1] takes care exactly of the dimension of the space of divisors singular at general points. In general, it is almost immediate to realize that there exists a unique component V of $\mathcal{V}_\delta E$ which is dominant in the map which sends sections to the point of the symmetric product $(\mathbb{P}^3)^{(\delta)}$ which represents the singular locus. The dimension of V has an expected value, but it is far from obvious even to guess a classification of bundles E for which the dimension of V fails to be as expected. The problem is linked to the study of the pull back of bundles over the blow up of $\mathbb{P}^3$ at δ general points. The corresponding analysis is totally open.

On the other hand, recently the theory of secant varieties suggested to study the locus of subsets $\Delta \subset \mathbb{P}^n$ which fail to impose the expected number of conditions to elements of a linear system with singularities along the subset. These varieties are a first instance of *Terracini loci*, a subject introduced in [3, 4, 8], which promises to be helpful in the analysis of secant varieties to Veronese varieties. For general projective curves, results on the dimensions of Terracini loci are contained in [9].

The analogue of Terracini loci for sections of rank 2 bundles E of $\mathbb{P}^3$ is the locus of sets $\Delta \in (\mathbb{P}^3)^{(\delta)}$ such that the space of sections of E singular at Δ fails to have the expected dimension. Almost nothing is known on the structure of such sets.

3.3 Schubert secant varieties

Interpolation problems for linear systems on projective spaces have a strong connection with the study of secant varieties of Veronese varieties. Thanks to the Terracini's Lemma, the dimension of the space of divisors in $|\mathcal{O}_{\mathbb{P}^n}(m)|$ which are singular at δ general points determines the dimension of the δ-secant variety of the Veronese embedding of degree m of $\mathbb{P}^n$.

Which analogue can be introduced for varieties of sections of a rank 2 bundle E on $\mathbb{P}^3$ whose 0-locus Y is singular?

The bundle E determines a natural map from $\mathbb{P}^3$ to the Grassmannian G of codimension 2 spaces in $\mathbb{P}(H^0(E))$, which sends a point P to the subspace of $\mathbb{P}(H^0(E))$ defined by sections vanishing at P (see, e.g., [2]). When E is sufficiently positive, the map is an embedding and defines a subvariety X of dimension 3 in G. For a general choice of a set Δ of δ points in X the 'Schubert analogue' $< \Delta >$ of the linear space spanned by the δ points is the set of codimension 2 subspaces of $\mathbb{P}(H^0(E))$, i.e. the set of points in G, which contain the space W_Δ of sections vanishing at all the points of Δ. In fact, $< \Delta >$ corresponds to a Schubert cycle in G. As Δ moves, these cycles define a 'Schubert' analogue of a secant variety Σ_δ, whose properties are widely unknown. For instance, unless E is very positive it is possible that Σ_δ has several components, arising when the points specialize. The component $\Sigma \subset \Sigma_\delta$ which dominates the symmetric product $(\mathbb{P}^3)^{(\delta)}$ has a dimension whose expected value is the sum of 3δ (the choice of δ points in X) plus $2(2\delta - 3)$ (the dimension of the Grassmannian of 2-codimensional spaces in $\mathbb{P}(H^0(E))$ which contain W_Δ), i.e. the expected dimension is $7\delta - 3$.

It is widely unknown if, at least for very positive bundles E, the dimension of Σ attains the expected value. In order to compute the dimension, it would be relevant to know who is the general tangent space to Σ. The same proof of the Terracini's Lemma indicates that the tangent space of Σ at a general point of $< \Delta >$ should be the set of sections which are singular at the points of Δ. Under these lines one can find an extension of the theory of secant varieties, yielding applications for the study of sections of rank 2 bundles which are singular at a given set. Also the notion of Terracini loci can be rephrased in the setting of Grassmannians and secant Schubert cycles.

3.4 Interpolation with higher multiplicities

One can ask questions similar to those introduce in Sect. 2 when imposing to the 0-loci of sections singularities of multiplicity higher than 2. The study of linear systems with singularities of higher order is a deeply studied topic in Algebraic Geometry, and presents still amazing open problems, like the Segre–Harbourne–Hirschowitz–Gimigliano conjecture on the dimension of linear systems in $\mathbb{P}^2$ with singularities at general points. The conjecture essentially says that the unique exceptions to the attainement of the expected dimension for linear systems with singularities is the existence of (-1)-curves, i.e. singular curves whose strict transform in the blow up of the plane at δ general points have self-intersection -1

(see e.g. [14]). For linear systems on $\mathbb{P}^n$, $n \geq 3$, a conjecture about the attainement of the expected dimension is not yet known, at least in general. Of course, even in the case of rank 2 bundles E on $\mathbb{P}^3$, the existence of (-1)-curves in the blow-up at δ general points can force the dimension of the variety of sections with singular 0-locus to be bigger than the expected one. On the contrary, a big number of points of multiplicity 2 has the effect of making more regular the behavior of varieties of singular elements of linear systems, see e.g. [6, 7], and we wonder if a similar principle holds for curves arising from sections of vector bundles. A detailed analysis of the problems is beyond the target of this note.

3.5 Further developments

Further developments like interpolation problems for curves in threefolds or higher codimensional schemes in n-folds different from projective spaces can be introduced. The analysis of the imposition of singularities to sections of rank 2 bundles over threefolds X can be considerably different from the case where X is a projective space, e.g. when the Picard group of X has many generators (see, for instance, [10–12]). When X is a manifold (even a projective space) of dimension bigger than 3 the problem becomes considerably harder (and totally missing in the literature). It presents stimulating perspectives, together with non trivial obstacles, for a geometric investigation.

Funding Open access funding provided by Università degli Studi di Siena within the CRUI-CARE Agreement.

Data Availability Not applicable.

References

1. Alexander, J., Hirschowitz, A.: Polynomial interpolation in several variables. J. Algebraic Geom. **4**, 201–222 (1995)
2. Arrondo, E., Paoletti, R.: Characterization of Veronese varieties via projection in Grassmannians. In: Projective Varieties with Unexpected properties, De Gruyter, pp. 1–12 (2005)
3. Ballico, E.: Terracini loci et espaces homogenes. Adv. Pure Appl. Math. **13**, 62–74 (2022)
4. Ballico, E., Bernardi, A., Santarsiero, P.: Terracini locus for three points on a Segre variety. Asian J. Math. **27**, 375–404 (2023)
5. Ballico, E., Chiantini, L.: A look into the Severi varieties of curves in higher codimension. Collect. Math. **49**, 191–201 (1998)
6. Ballico, E., Chiantini, L.: Imposing singular points and many nodes to curves on surfaces. Monatshefte Math. **135**, 89–96 (2002)
7. Ballico, E., Chiantini, L.: Good failures for fat points in $\mathbb{P}^2$. Collect. Math. **49**, 191–201 (1998)
8. Ballico, E., Chiantini, L.: On the Terracini locus of projective varieties. Milan J. Math. **89**, 1–17 (2021)
9. Ballico, E., Chiantini, L.: Terracini loci of curves. Rev. Mat. Complutense **37**, 447–465 (2024)
10. Ballico, E., Huh, S., Malaspina, F.: Globally generated vector bundles of rank 2 on a smooth quadric threefold. J. Pure Appl. Algebra **218**, 197–207 (2014)
11. Ballico, E., Huh, S., Malaspina, F.: Globally generated vector bundles on the Segre threefold with Picard number two. Math. Nach. **289**, 523–536 (2016)

12. Ballico, E., Benzo, L., Fontanari, C.: Families of nodal curves in $\mathbb{P}^r$ with the expected number of moduli. Boll. UMI **7**, 183–192 (2014)
13. Ballico, E., Migliore, J.: Smooth curves whose hyperplane section is a given set of points. Comm. Algebra **18**, 3015–3040 (1990)
14. Ciliberto, C., Miranda, R.: The Segre and Harbourne-Hirschowitz Conjectures. In: Applications of algebraic geometry to coding theory, physics and computation (Eilat 2001), NATO Sci. Ser. II Math. Phys. Chem. Kluwer Acad. Publ. 36, pp. 37– 51 (2001)
15. Hartshorne, R.: Stable reflexive sheaves. Math. Ann. **254**, 121–176 (1980)
16. Hartshorne, R.: Stable vector bundles of rank 2 in $\mathbb{P}^3$. Math. Ann. **238**, 229–280 (1978)
17. Perrin, D.: Courbes passant par m points généraux de $\mathbb{P}^3$. Mem. Soc. Mat. Fr. pp. 28–29 (1987)