Carleman Estimates and Controllability for a Degenerate Structured Population Model

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Abstract

In this paper we study the null controllability property for a single population model in which the population y depends on time t, space x, age a and size τ. Moreover, the diffusion coefficient k is degenerate at a point of the domain or both extremal points. Our technique is essentially based on Carleman estimates. The τ dependence requires us to modify the weight for the Carleman estimates, and accordingly the proof of the observability inequality. Thanks to this observability inequality we obtain a null controllability result for an intermediate problem and finally for the initial system through suitable cut off functions.

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1 Introduction

In this paper we consider the following degenerate structured population model

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial a} + \dfrac{\partial}{\partial \tau}(g(\tau)u) - k(x)u_{xx} - b(x)u_x + \mu(t,a,\tau,x)u = f(t,a,\tau,x) & \text{in } Q, \\ u(t,a,\tau,1) = u(t,a,\tau,0) = 0 & \text{on } Q_{T,A,\tau_2} \\ u(0,a,\tau,x) = u_0(a,\tau,x) & \text{in } Q_{A,\tau_2,1}, \\ u(t,a,\tau_1,x) = 0 & \text{in } Q_{T,A,1}, \\ u(t,0,x,\tau) = \int_0^A \beta(a,\tau,x)u(t,a,\tau,x)da & \text{in } Q_{T,\tau_2,1}, \end{cases} \tag{1.1}$$

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in the domain $Q := (0,T) \times (0,A) \times (\tau_1,\tau_2) \times (0,1)$ with fixed constant $T > 0$. Here, $Q_{T,A,\tau_2} := (0,T) \times (0,A) \times (\tau_1,\tau_2)$, $Q_{A,\tau_2,1} := (0,A) \times (\tau_1,\tau_2) \times (0,1)$ and $Q_{T,\tau_2,1} := (0,T) \times (\tau_1,\tau_2) \times (0,1)$; $u(t,a,\tau,x)$ is the distribution of certain individuals of size $\tau \in (\tau_1,\tau_2)$ and age $a \in (0,A)$ at location $x \in (0,1)$ and time $t \in (0,T)$. Here A is the maximal age of life, while β is the natural fertility. Thus, the formula $\int_0^A \beta u da$ denotes the distribution of newborn individuals at time t and location x. Moreover, μ is the natural death rate and satisfies $\mu \in C(\bar{Q})$ and $\mu \geq 0$ in Q. The function k is the dispersion coefficient and we assume that it depends on the space variable x and is degenerate at the boundary of the state space. Finally $\frac{\partial}{\partial \tau}(g(\tau)u)$ represents the growth effect and $g(\tau)$ is the growth modulus (i.e. $\int_{\tau_1}^{\tau_2} \frac{1}{g(\tau)} d\tau$ is a spending time to grow the individual from size τ_1 to size τ_2).

Clearly the asymptotic behavior of the solution for the Lotka-McKendrick system depends on the functions and the parameters which appear in the system (see, for example, [4] and [5]). Obviously, it is very worrying if the system represents the distribution of a damaging insect population or of a pest population and the solution grows exponentially. For this reason, recently great attention is given to null controllability issues. For example in [22], where (1.1) models an insect growth when y is independent of τ, the control corresponds to a removal of individuals by using pesticides.

In the case that the dispersion coefficient k *is positive*, (1.1) is considered in [26], where the authors prove some Carleman estimates and, as a consequence, a unique continuation. If k *is a constant or a positive function* and y *is independent of* τ, null controllability for (1.1) is studied, for example, in [4] (see [27] for the well-posedness). If k *is degenerate at the boundary or at an interior point of the domain* and y *is independent of* a *and* τ we refer, for example, to [3], [17], [18] and to [19], [20], [21] if μ is singular at the same point of k.

To our best knowledge, [2] is the first paper where the dispersion coefficient can degenerate at the boundary of the domain (for example $k(x) = x^\alpha$, being $x \in (0,1)$ and $\alpha > 0$), but y *is independent of* τ. Using Carleman estimates for the adjoint problem, the authors prove null controllability for (1.1) under the condition $T \geq A$. However, this assumption is not realistic when A is too large. In order to overcome this problem, [10] used Carleman estimates and a fixed point method via the Leray - Schauder Theorem. The case $T < A$ is considered in [6], [10], [15] and [16]. In [10] the problem is always in *divergence form* and the authors assume that k is degenerate only at a point of the boundary; moreover, they use the fixed point technique in which the birth rate β must be of class $C^2(Q)$ (necessary requirement in the proof of [10, Proposition 4.2]). A more general result is obtained in [15] where β is only a continuous function, but k can degenerate at both extremal points. In [6] the problem is always in *divergence form* and k is degenerate at an interior point x_0 and it belongs to $C[0,1] \cap C^1([0,1] \setminus \{x_0\})$; see also the recent paper [14] where the functions are less regular and both cases $T < A$ and $T > A$ are considered. The *non divergence form* is considered in [16], where we studied null controllability for (1.1) with a diffusion coefficient degenerating at a one point of the boundary domain or in an interior point (for a cascade system we refer to [7]). We underline again that in all the previous papers, except for [26], the *model is independent of* τ.

The novelty of this paper is to study, via Carleman estimates, the null controllability for (1.1) in the case that y *depends on* τ *and* k *is degenerate at a point of the domain or at both extremal points.* Clearly, the τ dependence leads to some consequences in the choice of the weight for the Carleman estimates and in the proof of the observability inequality for the associated adjoint problem (see Theorem 4.5). Thanks to this observability inequality we obtain a null controllability result for an intermediate problem and finally for (1.1). It is important to underline that with our technique we are not able to prove a *global null controllability* in the sense that

$$u(T,a,\tau,x) = 0 \quad \text{a.e. } (a,\tau,x) \in (0,A) \times (\tau_1,\tau_2) \times (0,1),$$

but only a *partial* global null controllability result, i.e.

$$u(T,a,\tau,x) = 0 \quad \text{a.e. } (a,\tau,x) \in (\Xi,A) \times (\tau_1,\tau_2) \times (0,1),$$

where $\Xi := \max\{A - \epsilon\bar{a}, \bar{a}\}$ for all $\epsilon \in (0,1)$.

Finally, observe that in this paper, as in [15] or [16], we do not consider the positivity of the solution, even if it is clearly an interesting question to face (see [23] for related results in non degenerate cases). This topic will be a subject of further investigations.

The paper is organized as follows. In Section 2 we study the well-posedness of the problem; in Section 3 we prove the Carleman estimates given in Theorem 3.1 (if the diffusion coefficient k is degenerate at

0) and Theorem 3.2 (if k is degenerate at 1). Section 4 is devoted to study, first of all, the observability inequality for the associated adjoint problem (see Proposition 4.1) via $\omega-$local Carleman estimates (see Theorems 4.1 and 4.3, if k vanishes at 0 and 1, respectively) and a Caccioppoli's inequality (for the reader's convenience, we give its proof in Appendix). Secondly, thanks to this observability inequality, we obtain a null controllability result for an intermediate system (see Theorem 4.6); hence, via cut off functions, we extend the null controllability result to the original problem if k is degenerate only at a point of the boundary (see Theorem 4.7) or at both extremal points (see Theorem 4.8).

A final comment on the notation: by C we shall denote *universal* positive constants, which are allowed to vary from line to line.

2 Well-posedness of the problem

To study the well-posedness, we assume that the dispersion coefficient k satisfies the following assumption:

Hypothesis 2.1. The diffusion function $k \in C[0,1]$ and there exist $M_1, M_2 \in (0,2)$ such that

$$k \in C^1(0,1), k > 0 \text{ in } (0,1), k(0) = k(1) = 0, xk'(x) \leq M_1 k(x) \text{ and } (x-1)k'(x) \leq M_2 k(x)$$

for all $x \in [0,1]$, or

$$k \in C^1(0,1], k > 0 \text{ in } (0,1], k(0) = 0 \text{ and } xk'(x) \leq M_1 k(x)$$

for all $x \in [0,1]$, or

$$k \in C^1[0,1), k > 0 \text{ in } [0,1) \text{ and } k(1) = 0, \text{ and } (x-1)k'(x) \leq M_2 k(x)$$

for all $x \in [0,1]$.

For example, as k one can consider $k(x) = x^\alpha$, $k(x) = (1-x)^\beta$ or $k(x) = x^\alpha(1-x)^\beta$, $\alpha, \beta > 0$.

On the rates b, g, μ and β we assume:

Hypothesis 2.2. The functions g, μ and β are such that

$$\begin{aligned} &\bullet\, b \in C[0,1] \text{ and } \frac{b}{k} \in L^1(0,1),\\ &\bullet\, g \in C^2[\tau_1, \tau_2] \text{ and } g > 0 \text{ on } [\tau_1, \tau_2],\\ &\bullet\, \beta \in C(\bar{Q}_{A,\tau_2,1}) \text{ and } \beta \geq 0 \text{ in } Q_{A,\tau_2,1},\\ &\bullet\, \mu \in C(\bar{Q}) \text{ and } \mu \geq 0 \text{ in } Q. \end{aligned} \tag{2.2}$$

Observe that b can vanish at 0 and/or at 1; indeed we just require that $\dfrac{b}{k} \in L^1(0,1)$ thus if $k(x) = x^\alpha(1-x)^\beta$, then we can consider as b the function given by $b(x) = x^{\alpha_1}(1-x)^{\beta_1}$ with $\alpha_1 > \alpha - 1$ and $\beta_1 > \beta - 1$.

In order to prove the well-posedness of (1.1), we introduce the well-known weight function

$$\eta(x) := exp\left\{\int_x^{\frac{3}{4}} \frac{b(y)}{k(y)} dy\right\}, \quad x \in [0,1],$$

which introduced by Feller [12] and used by several authors, see, e.g. [9], [11] and [24]. Define

$$\sigma(x) := k(x)\frac{1}{\eta(x)},$$

and observe that if u is sufficiently smooth, e.g. $u \in W^{2,1}_{\text{loc}}(0,1)$, then

$$\mathcal{A}_0 u := k u_{xx} + b u_x = \sigma(\eta u_x)_x,$$

for almost every $x \in (0,1)$. For this purpose, let us consider the following Hilbert spaces

$$\begin{aligned}
&L^2_{\frac{1}{\sigma}}(0,1) := \left\{ u \in L^2(0,1) \mid \|u\|_{\frac{1}{\sigma}} < \infty \right\}, && \|u\|^2_{\frac{1}{\sigma}} := \int_0^1 u^2 \tfrac{1}{\sigma} dx,\\
&H^1_{\frac{1}{\sigma}}(0,1) := L^2_{\frac{1}{\sigma}}(0,1) \cap H^1_0(0,1), && \|u\|^2_{1,\frac{1}{\sigma}} := \|u\|^2_{\frac{1}{\sigma}} + \int_0^1 u_x^2 dx,\\
&H^2_{\frac{1}{\sigma}}(0,1) := \left\{ u \in H^1_{\frac{1}{\sigma}}(0,1) \mid \mathcal{A}_0 u \in L^2_{\frac{1}{\sigma}}(0,1) \right\}, && \|u\|^2_{2,\frac{1}{\sigma}} := \|u\|^2_{1,\frac{1}{\sigma}} + \|\mathcal{A}_0 u\|^2_{\frac{1}{\sigma}}.
\end{aligned}$$

Observe that since $\dfrac{b}{k} \in L^1(0,1)$, $\eta \in C^0[0,1] \bigcap C^1(0,1)$ is a strictly positive function. Thus, in the sense of Banach spaces, one has that

$$\left(L^2_{\frac{1}{\sigma}}(0,1),\ H^1_{\frac{1}{\sigma}}(0,1),\ H^2_{\frac{1}{\sigma}}(0,1) \right) \simeq \left(L^2_{\frac{1}{k}}(0,1),\ H^1_{\frac{1}{k}}(0,1),\ H^2_{\frac{1}{k}}(0,1) \right),$$

where the last triplet is the triplet related to well-posedness as, for example, in [8] or [16]. Observe that, if k is nondegenerate, the spaces $L^2_{\frac{1}{k}}(0,1)$, $H^1_{\frac{1}{k}}(0,1)$ and $H^2_{\frac{1}{k}}(0,1)$ coincide, respectively, with $L^2(0,1)$, $H^1_0(0,1)$ and $H^2(0,1) \cap H^1_0(0,1)$.

As in [9, Lemma 1], one can prove

Lemma 2.1. *For all* $(u,v) \in H^2_{\frac{1}{\sigma}}(0,1) \times H^1_{\frac{1}{\sigma}}(0,1)$ *one has*

$$\langle \mathcal{A}_0 u, v \rangle_{\frac{1}{\sigma}} = -\int_0^1 \eta u_x v_x dx. \tag{2.3}$$

Thanks to the previous Green's formula one can prove, as in [8], [9] or [18], that the operator $(\mathcal{A}_0, D(\mathcal{A}_0))$, where $D(\mathcal{A}_0) := H^2_{\frac{1}{\sigma}}(0,1)$, is self–adjoint, nonpositive and generates a contraction semigroup on the space $L^2_{\frac{1}{\sigma}}(0,1)$.

Now, setting $\mathcal{A}_a u := \dfrac{\partial u}{\partial a}$ and $\mathcal{A}_\tau u := \dfrac{\partial (gu)}{\partial \tau}$, we have that

$$\mathcal{A} u := \mathcal{A}_a u + \mathcal{A}_\tau u - \mathcal{A}_0 u,$$

for

$$u \in D(\mathcal{A}) = \left\{ u \in L^2(Q_{A,\tau_2}; D(\mathcal{A}_0)) : \frac{\partial u}{\partial a}, \frac{\partial u}{\partial \tau} \in L^2(Q_{A,\tau_2}; H^1_{\frac{1}{\sigma}}(0,1)), \right.$$
$$\left. u(0,\tau,x) = \int_0^A \beta(a,\tau,x) u(a,\tau,x) da \right\},$$

generates a strongly continuous semigroup on $L^2_{\frac{1}{\sigma}}(Q_{A,\tau_2,1}) := L^2(Q_{A,\tau_2}; L^2_{\frac{1}{\sigma}}(0,1))$ (see also [27, Chapter 1.4]). Here $Q_{A,\tau_2} := (0,A) \times (\tau_1, \tau_2)$. Moreover, the operator $B(t)$ defined as

$$B(t)u := \mu(t,a,\tau,x)u,$$

for $u \in D(\mathcal{A})$, can be seen as a bounded perturbation of $\mathcal{A}$ (see, for example, [3]); thus also $(\mathcal{A} + B(t), D(\mathcal{A}))$ generates an evolution family.

Setting $L^2_{\frac{1}{\sigma}}(Q) := L^2(Q_{T,A,\tau_2}; L^2_{\frac{1}{\sigma}}(0,1))$, the following well-posedness result holds (see [25, Theorem 2,1]):

Theorem 2.1. *Assume that Hypotheses* 2.1 *and* 2.2 *are satisfied. For all* $f \in L^2_{\frac{1}{\sigma}}(Q)$ *and* $u_0 \in L^2_{\frac{1}{\sigma}}(Q_{A,\tau_2,1})$, *the system* (1.1) *admits a unique solution*

$$u \in \mathcal{U} := C\big([0,T]; L^2_{\frac{1}{\sigma}}(Q_{A,\tau_2,1})\big) \cap L^2\big(0,T; H^1(0,A; H^1((\tau_1,\tau_2); H^1_{\frac{1}{\sigma}}(0,1)))\big).$$

In addition, if $f \equiv 0$, $u \in C^1\big([0,T]; L^2_{\frac{1}{\sigma}}(Q_{A,\tau_2,1})\big)$.

3 Carleman estimates

From the general theory, it is known that null controllability for a linear parabolic system is, roughly speaking, equivalent to the observability for the associated homogeneous adjoint problem (see, for example, [13]). In particular, in this case the adjoint problem of (1.1) is the following system:

$$\begin{cases} \dfrac{\partial v}{\partial t} + \dfrac{\partial v}{\partial a} + g(\tau)\dfrac{\partial v}{\partial \tau} + k(x)v_{xx} + b(x)v_x - \mu(t,a,\tau,x)v + \beta(a,\tau,x)v(t,0,\tau,x) = 0, & (t,a,\tau,x) \in Q, \\ v(t,a,\tau,0) = v(t,a,\tau,1) = 0, & (t,a,\tau) \in Q_{T,A,\tau_2}, \\ v(T,a,\tau,x) = v_T(a,\tau.x), & (a,\tau,x) \in Q_{A,\tau_2,1}, \\ v(t,A,\tau,x) = 0, & (t,\tau,x) \in Q_{T,\tau_2,1}, \\ v(t,a,\tau_2,x) = 0, & (t,a,x) \in Q_{T,A,1}, \end{cases} \tag{3.4}$$

and we prove, in the next section, that the observability inequality implies a null controllability result for (1.1). Thus, the key point is to prove such an inequality. A usual strategy in showing the observability inequality is to prove that certain global Carleman estimates hold true for the adjoint operator. For this reason, we consider the following system:

$$\begin{cases} \dfrac{\partial z}{\partial t} + \dfrac{\partial z}{\partial a} + g(\tau)\dfrac{\partial z}{\partial \tau} + k(x)z_{xx} + b(x)z_x - \mu(t,a,,\tau,x)z = f, & (t,a,\tau,x) \in Q, \\ z(t,a,\tau,0) = z(t,a,\tau,1) = 0, & (t,a,\tau) \in Q_{T,A,\tau_2}, \\ z(t,A,\tau,x) = 0, & (t,\tau,x) \in Q_{T,\tau_2,1}, \\ z(t,a,\tau_2,x) = 0, & (t,a,x) \in Q_{T,A,1}. \end{cases} \tag{3.5}$$

In this subsection we will consider separately the case when $k(0) = 0$ or $k(1) = 0$. In both cases we assume that b, g, β and μ satisfy Hypothesis 2.2. On the other hand, on k we make different assumptions:

Hypothesis 3.1. Hypothesis 2.2 is satisfied, the function $k \in C^0[0,1] \bigcap C^2(0,1]$ is such that $k(0) = 0$, $k > 0$ on $(0,1]$ and there exists $M_1 \in (0,2)$ so that $\dfrac{xk_x(x)}{k(x)} \leq M_1$ a.e. in $[0,1]$ and $\dfrac{x(b-k_x(x))}{k(x)} \in L^\infty(0,1)$. Finally, there exist $\varepsilon \in (0,1]$ and a function $C_1 = C_1(\varepsilon) > 0$, defined in $(0,\varepsilon)$, such that

$$C_1(\varepsilon) \to 0 \quad as \quad \varepsilon \to 0^+,$$

and

$$\left| \left(\frac{x(b(x) - k_x(x))}{k(x)} \right)_{xx} - \frac{b(x)}{k(x)} \left(\frac{x(b(x) - k_x(x))}{k(x)} \right)_x \right| \leq C_1(\varepsilon)\frac{1}{x^2}, \tag{3.6}$$

$\forall\, x \in (0,\varepsilon)$.

Hypothesis 3.2. Hypothesis 2.2 is satisfied, the function $k \in C^0[0,1] \bigcap C^2[0,1)$ is such that $k(1) = 0$, $k > 0$ on $(0,1)$ and there exists $M_2 \in (0,2)$ so that $\dfrac{(x-1)k_x(x)}{k(x)} \leq M_2$ a.e. in $[0,1]$ and $\dfrac{(x-1)(b-k_x(x))}{k(x)} \in L^\infty(0,1)$. Finally, there exist $\varepsilon \in (0,1]$ and a function $C_2 = C_2(\varepsilon) > 0$, defined in $(0,\varepsilon)$, such that

$$C_2(\varepsilon) \to 0 \quad as \quad \varepsilon \to 0^+,$$

and

$$\left| \left(\frac{(x-1)(b(x) - k_x(x))}{k(x)} \right)_{xx} - \frac{b(x)}{k(x)} \left(\frac{(x-1)(b(x) - k_x(x))}{k(x)} \right)_x \right| \leq C_2(\varepsilon)\frac{1}{(x-1)^2}, \tag{3.7}$$

$\forall\, x \in (1-\varepsilon,1)$.

We observe that (3.6) and (3.7) are similar to the assumpsions made in [9].

Moreover, in order to estimate the distributed terms that appear in Lemma 3.1 (see below), we suppose the following:

Hypothesis 3.3. Setting

$$\rho(x) := \frac{x(b(x) - k_x(x))}{k(x)},$$

the functions b and k are such that $\rho_{xx}(x)$ and $\rho_x(x)$ exist for $0 < x < 1$.

Now, let us introduce the weight functions

$$\varphi(t,a,\tau,x) := \Theta(t,a,\tau)(p(x) - 2\|p\|_{L^\infty(0,1)}), \tag{3.8}$$

and

$$\bar\varphi(t,a,\tau,x) := \Theta(t,a,\tau)(\bar p(x) - 2\|\bar p\|_{L^\infty(0,1)}), \tag{3.9}$$

if k satisfies Hypothesis 3.1 or Hypothesis 3.2, respectively, where

$$\Theta(t,a,\tau) := \frac{1}{t^4(T-t)^4 a^4(\tau-\tau_1)^4}, \tag{3.10}$$

$$p(x) := \int_0^x \frac{y}{k(y)}\, e^{Ry^2} dy \text{ and } \bar p(x) := \int_0^x \frac{y-1}{k(y)}\, e^{R(y-1)^2} dy,$$

with $R > 0$. Observe that $\varphi(t,a,\tau,x), \bar\varphi(t,a,\tau,x) \to -\infty$ as $t \to 0^+, T^-$, $a \to 0^+$ or $\tau \to \tau_1^+$. The following estimates hold:

Theorem 3.1. *Assume Hypotheses* 3.1 *and* 3.3*. Then, there exist two positive constants C and s_0 such that every solution v of* (3.5) *in*

$$\mathcal{V} := L^2\big(Q_{T,A,\tau_2}; H^2_{\frac{1}{\sigma}}(0,1)\big) \cap H^1\big(0,T; H^1(0,A; H^1(\tau_1,\tau_2; H^1_{\frac{1}{\sigma}}(0,1)))\big)$$

satisfies, for all $s \geq s_0$,

$$\int_Q \Big(s\eta\Theta v_x^2 + s^3\eta\Theta^3\Big(\frac{x}{k}\Big)^2 v^2\Big) e^{2s\varphi} dx d\tau da dt \leq C\int_Q f^2 \frac{e^{2s\varphi}}{\sigma} dx d\tau da dt$$
$$+ s\,C\int_{Q_{T,A,\tau_2}} \Theta(t,a,\tau)\Big[v_x^2 e^{2s\varphi}\Big](t,a,\tau,1) d\tau da dt.$$

Theorem 3.2. *Assume Hypotheses* 3.2 *and* 3.3*. Then, there exist two strictly positive constants C and s_0 such that every solution v of* (3.5) *in $\mathcal{V}$ satisfies, for all $s \geq s_0$,*

$$\int_Q \Big(s\eta\Theta v_x^2 + s^3\eta\Theta^3\Big(\frac{x-1}{k}\Big)^2 v^2\Big) e^{2s\bar\varphi} dx d\tau da dt \leq C\int_Q f^2 \frac{e^{2s\bar\varphi}}{\sigma} dx d\tau da dt$$
$$+ sC\int_{Q_{T,A,\tau_2}} \Theta(t,a,\tau)\Big[v_x^2 e^{2s\bar\varphi}\Big](t,a,\tau,0) d\tau da dt.$$

Clearly the previous Carleman estimates hold for every function v that satisfies (3.5) in $(0,T) \times (0,A) \times (\tau_1,\tau_2) \times (0,B)$ or $(0,T) \times (0,A) \times (\tau_1,\tau_2) \times (B,1)$ as long as $(0,1)$ is substituted by $(0,B)$ or $(B,1)$ and k satisfies Hypothesis 3.1 in $(0,B)$ or Hypothesis 3.2 in $(B,1)$, respectively.

In the following, we will prove only Theorem 3.1 since the proof of Theorem 3.2 is analogous.

Proof of Theorem 3.1 As a first step, as in [15] or [16], assume that $\mu \equiv 0$. In order to prove Theorem 3.1, we define, for $s > 0$, the function

$$w(t,a,\tau,x) := e^{s\varphi(t,a,\tau,x)} v(t,a,\tau,x)$$

where v solves (3.5) in $\mathcal{V}$. Clearly, this implies that $w \in \mathcal{V}$ and satisfies

$$\begin{cases} (e^{-s\varphi}w)_t + (e^{-s\varphi}w)_a + g(\tau)(e^{-s\varphi}w)_\tau + k(x)(e^{-s\varphi}w)_{xx} - b(e^{-s\varphi}w)_x = f(t,a,\tau,x), & (t,a,\tau,x) \in Q,\\ w(0,a,\tau,x) = w(T,a,\tau,x) = 0, & (a,\tau,x) \in Q_{A,\tau_2,1},\\ w(t,A,\tau,x) = w(t,0,\tau,x) = 0, & (t,\tau,x) \in Q_{T,\tau_2,1},\\ w(t,a,\tau,0) = w(t,a,\tau,1) = 0, & (t,a,\tau) \in Q_{T,A,\tau_2},\\ w(t,a,\tau_1,x) = w(t,a,\tau_2,x) = 0, & (t,a,x) \in Q_{T,A,1}. \end{cases} \tag{3.11}$$

Defining $Lw := w_t + w_a + gw_\tau + kw_{xx} - bw_x$ and $L_s w := e^{s\varphi}L(e^{-s\varphi}w)$, the equation of (3.11) can be recast as follows

$$L_s w = e^{s\varphi} f.$$

In particular, the following equality holds:

Proposition 3.1. *The operator $L_s w$ can be rewritten as*

$$L_s w = L_s^+ w + L_s^- w,$$

where L_s^+ and L_s^- denote the (formal) selfadjoint and skewadjoint parts of L_s. In this case

$$\begin{cases} L_s^+ w := \mathcal{A}_0 w - s(\varphi_t + \varphi_a + \varphi_\tau g(\tau))w + s^2 k\varphi_x^2 w - \frac{1}{2} g_\tau(\tau) w, \\ L_s^- w := w_t + w_a + g w_\tau + \frac{1}{2} g_\tau(\tau) w - 2sk\varphi_x w_x - s\mathcal{A}_0 \varphi w. \end{cases}$$

Proof. Computing $L(e^{-s\varphi} w)$ one has

$$\begin{aligned} L(e^{-s\varphi} w) = {} & e^{-s\varphi} \left(-s\varphi_t w - s\varphi_a w - s\varphi_\tau g(\tau) w + sb(x)\varphi_x w\right) \\ & + e^{-s\varphi} \left(w_t + w_a + g(\tau) w_\tau - b(x) w_x\right) \\ & + e^{-s\varphi} k \left(-s\varphi_{xx} w + s^2 \varphi_x^2 w - 2s\varphi_x w_x + w_{xx}\right). \end{aligned}$$

Thus,

$$\begin{aligned} L_s w = {} & -s\varphi_t w - s\varphi_a w - s\varphi_\tau g(\tau) w - s\mathcal{A}_0 \varphi w \\ & + w_t + w_a + g(\tau) w_\tau + \mathcal{A}_0 w + k\left(s^2 \varphi_x^2 w - 2s\varphi_x w_x\right). \end{aligned}$$

By (2.3), one has that the adjoint operator L_s^* of L_s is

$$\begin{aligned} L_s^* w = {} & -s\varphi_t w - s\varphi_a w - s\varphi_\tau g(\tau) w - s\mathcal{A}_0 \varphi w + s^2 k\varphi_x^2 w \\ & - w_t - w_a - (g(\tau) w)_\tau + \mathcal{A}_0 w + 2sk\varphi_x w_x + 2sk\varphi_{xx} w - 2sb\varphi_x w \\ = {} & -s\varphi_t w - s\varphi_a w - s\varphi_\tau g(\tau) w - s\mathcal{A}_0 \varphi w + s^2 k\varphi_x^2 w \\ & - w_t - w_a - (g(\tau) w)_\tau + \mathcal{A}_0 w + 2sk\varphi_x w_x + 2s\mathcal{A}_0 \varphi w. \end{aligned}$$

Thus

$$L_s^+ w = \frac{L_s w + L_s^* w}{2} = -s\varphi_t w - s\varphi_a w - s\varphi_\tau g(\tau) w + s^2 k\varphi_x^2 w - \frac{1}{2} g_\tau(\tau) w + \mathcal{A}_0 w,$$

and

$$L_s^- w = \frac{L_s w - L_s^* w}{2} = w_t + w_a + g w_\tau + \frac{1}{2} g_\tau(\tau) w - 2sk\varphi_x w_x - 2s\mathcal{A}_0 \varphi w.$$

□

Moreover, setting $< u, v >_{\frac{1}{\sigma}} := \int_Q uv \frac{1}{\sigma} dx d\tau da dt,$ one has

$$\|L_s^+ w\|_{\frac{1}{\sigma}}^2 + \|L_s^- w\|_{\frac{1}{\sigma}}^2 + 2 < L_s^+ w, L_s^- w >_{\frac{1}{\sigma}} = \|f e^{s\varphi}\|_{\frac{1}{\sigma}}^2. \tag{3.12}$$

Now, we compute the inner product $< L_s^+ w, L_s^- w >_{\frac{1}{\sigma}}$ whose first expression is given in the following lemma

Lemma 3.1. *Assume Hypothesis* 3.1. *The following identity holds*

$$
\begin{aligned}
< L_s^+ w, L_s^- w >_{\frac{1}{\sigma}} &= \left.\begin{aligned}
& s\int_Q \eta(k\varphi_{xx}+(k\varphi_x)_x)w_x^2 dxd\tau dadt - \frac{1}{2}s\int_Q (\eta(\mathcal{A}_0\varphi)_x)_x w^2 dxd\tau dadt \\
&+ s^2\int_Q \big(\eta\frac{\varphi_t}{k}\mathcal{A}_0\varphi - \eta\varphi_x\varphi_{xt} - (\eta\varphi_x\varphi_t)_x\big)w^2 dxd\tau dadt \\
&+ s^3\int_Q \eta\varphi_x^2\big((k\varphi_x)_x + k\varphi_{xx}\big)w^2 dxd\tau dadt \\
&+ \frac{1}{2}\int_Q \eta g_\tau w_x^2 dxd\tau dadt - s^2\int_Q \eta\varphi_x\varphi_{xa} w^2 dxd\tau dadt \\
&+ \frac{s}{2}\int_Q \frac{\eta}{k}w^2(\varphi_{t\tau}+\varphi_{tt}+2\varphi_{ta}+\varphi_{aa}+\varphi_{t\tau}g+\varphi_{\tau a}g)dxd\tau dadt \\
&- s^2\int_Q \eta\varphi_x\varphi_{x\tau}w^2 dxd\tau dadt \\
&+ \frac{1}{2}\int_Q \frac{\eta}{k}g_\tau w^2(-s\varphi_t + s^2k\varphi_x^2)dxdad\tau dt \\
&+ s^2\int_Q (\varphi_a+\varphi_\tau g)(\eta\varphi_x)_x w^2 dxd\tau dadt \\
&+ \frac{s}{2}\int_Q \frac{\eta}{k}(\varphi_a+\varphi_\tau g)_\tau g w^2 dxd\tau dadt - s^2\int_Q [\eta(\varphi_a+\varphi_\tau g)\varphi_x]_x w^2 dxd\tau dadt \\
&+ \frac{1}{2}\int_Q \frac{\eta}{k}(g^2)_{\tau\tau}w^2 dxd\tau dadt - \frac{1}{4}\int_Q \frac{\eta}{k}(g_\tau)^2 w^2 dxd\tau dadt
\end{aligned}\right\}\{D.T.\} \\
\{B.T.\} &\left\{\begin{aligned}
&- \frac{1}{2}\int_{Q_{A,\tau_2,1}} \eta\big[w_x^2\big]_0^T dxd\tau da + \int_{Q_{T,A,\tau_2}} \big[\eta w_x w_t\big]_0^1 d\tau dadt \\
&+ \frac{1}{2}s\int_{Q_{T,A,\tau_2}} \big[\eta(\mathcal{A}_0\varphi)_x w^2\big]_0^1 d\tau dadt - s\int_{Q_{T,A,\tau_2}} \big[\eta\mathcal{A}_0\varphi w w_x\big]_0^1 d\tau dadt \\
&- s\int_{Q_{T,A,\tau_2}} \big[\eta k\varphi_x w_x^2\big]_0^1 d\tau dadt + \frac{1}{2}\int_{Q_{A,\tau_2,1}} \eta\big[\big(s^2\varphi_x^2 - s\frac{\varphi_t}{k}\big)w^2\big]_0^T dxd\tau da \\
&- \int_{Q_{T,A,\tau_2}} \big[\eta\big(s^3k\varphi_x^3 - s^2\varphi_x\varphi_t\big)w^2\big]_0^1 d\tau dadt - \frac{1}{2}\int_{Q_{T,\tau_2,1}} \big[\eta w_x^2\big]_0^A dxd\tau dt \\
&+ \int_{Q_{T,A,\tau_2}} \big[\eta w_x w_a\big]_0^1 d\tau dadt + \int_{Q_{T,A,\tau_2}} \big[g\eta w w_\tau\big]_0^1 d\tau dadt \\
&- \int_{Q_{T,A,1}} \big[g\eta w_x^2\big]_{\tau_1}^{\tau_2} dxdadt + \frac{1}{2}\int_{Q_{T,A,\tau_2}} \big[\eta g_\tau w w_x\big]_0^1 d\tau dadt \\
&- \frac{s}{2}\int_{Q_{T,\tau_2,1}} \big[\frac{\eta}{k}w^2\varphi_t\big]_0^A dxd\tau dt + \frac{s^2}{2}\int_{Q_{T,\tau_2,1}} \big[\eta\varphi_x^2 w^2\big]_0^A dxd\tau dt \\
&- \frac{s}{2}\int_{Q_{T,A,1}} \big[\frac{\eta}{k}w^2\varphi_t\big]_{\tau_1}^{\tau_2} dxdadt + \frac{s^2}{2}\int_{Q_{T,A,1}} \big[\eta\varphi_x^2 w^2\big]_{\tau_1}^{\tau_2} dxdadt \\
&- s\int_{Q_{A,\tau_2,1}} \frac{\eta}{k}\big[(\varphi_a+\varphi_\tau g)\frac{w^2}{2}\big]_0^T dxd\tau da - s\int_{Q_{T,\tau_2,1}} \frac{\eta}{k}\big[(\varphi_a+\varphi_\tau g)\frac{w^2}{2}\big]_0^A dxd\tau dt \\
&- s\int_{Q_{T,A,1}} \frac{\eta}{k}\big[(\varphi_a+\varphi_\tau g)g\frac{w^2}{2}\big]_{\tau_1}^{\tau_2} dxdadt + s^2\int_{Q_{T,A,\tau_2}} \frac{\eta}{k}\big[(\varphi_a+\varphi_\tau g)k\varphi_x w^2\big]_0^1 d\tau dadt \\
&- \frac{1}{4}\int_{Q_{A,\tau_2,1}} \left[\frac{1}{\sigma}g_\tau w^2\right]_0^T dxd\tau da - \frac{1}{4}\int_{Q_{T,\tau_2,1}} \left[\frac{1}{\sigma}g_\tau w^2\right]_0^A dxd\tau dt \\
&- \frac{1}{2}\int_{Q_{T,A,1}} \left[\frac{1}{\sigma}(g^2)_\tau w^2\right]_{\tau_1}^{\tau_2} dxdadt + \frac{s}{2}\int_{Q_{T,A,\tau_2}} \big[\eta g_\tau\varphi_x w^2\big]_0^1 d\tau dadt.
\end{aligned}\right.
\end{aligned}
\tag{3.13}
$$

Proof. It results, integrating by parts, that

$$< L_s^+ w, L_s^- w >_{\frac{1}{\sigma}} = \sum_{i=1}^{i=6} I_i,$$

where

$$I_1 = \int_Q \frac{1}{\sigma}\mathcal{A}_0 w(w_t - 2sk\varphi_x w_x - s\mathcal{A}_0\varphi w)dxd\tau dadt,$$

$$I_2 = \int_Q \frac{1}{\sigma}\big(- s\varphi_t w + s^2 k\varphi_x^2 w\big)(w_t - 2sk\varphi_x w_x - s\mathcal{A}_0\varphi w)dxd\tau dadt,$$

$$I_3 = \int_Q \frac{1}{\sigma}\mathcal{A}_0 w(w_a + gw_\tau + \frac{1}{2}g_\tau w)dxd\tau dadt,$$

$$I_4 = \int_Q \frac{1}{\sigma}(-s\varphi_t w + s^2 k\varphi_x^2 w)(w_a + gw_\tau + \frac{1}{2}g_\tau w)dxd\tau dadt,$$

$$I_5 = -s\int_Q \frac{1}{\sigma}(\varphi_a + \varphi_\tau g)w(w_t + w_a + gw_\tau + \frac{1}{2}g_\tau w - 2sk\varphi_x w_x - s\mathcal{A}_0\varphi w)dxd\tau dadt$$

and

$$I_6 = -\frac{1}{2}\int_Q \frac{1}{\sigma}g_\tau w(w_t + w_a + gw_\tau + \frac{1}{2}g_\tau w - 2sk\varphi_x w_x - s\mathcal{A}_0\varphi w)dxd\tau dadt.$$

By several integrations by parts in space and in time (see [9, Lemma 3]), we get

$$\begin{aligned} I_1 + I_2 = &-\frac{1}{2}\int_{Q_{A,\tau_2,1}} \eta\big[w_x^2\big]_0^T dxd\tau da + \int_{Q_{T,A,\tau_2}} \big[\eta w_x w_t\big]_0^1 d\tau dadt \\ &+ s\int_Q \eta\mathcal{A}_0\varphi w_x^2 dxd\tau dadt - \frac{1}{2}s\int_Q (\eta(\mathcal{A}_0\varphi)_x)_x w^2 dxd\tau dadt \\ &+ \frac{1}{2}s\int_{Q_{T,A,\tau_2}} \big[\eta(\mathcal{A}_0\varphi)_x w^2\big]_0^1 d\tau dadt - s\int_{Q_{T,A,\tau_2}} \big[\eta\mathcal{A}_0\varphi w w_x\big]_0^1 d\tau dadt \\ &+ s\int_Q \eta((k\varphi_x)_x + b\varphi_x)w_x^2 dxd\tau dadt - s\int_{Q_{T,A,\tau_2}} \big[\eta k\varphi_x w_x^2\big]_0^1 d\tau dadt \\ &+ \frac{s}{2}\int_Q \eta\frac{\varphi_{tt}}{k}w^2 dxdt + s^2\int_Q \big(\eta\frac{\varphi_t}{k}\mathcal{A}_0\varphi - \eta\varphi_x\varphi_{xt} - (\eta\varphi_x\varphi_t)_x\big)w^2 dxdt \\ &+ s^3\int_Q \big((\eta k\varphi_x^3)_x - \eta\varphi_x^2\mathcal{A}_0\varphi\big)w^2 dxdt + \frac{1}{2}\int_{Q_{A,\tau_2,1}} \eta\big[\big(s^2\varphi_x^2 - s\frac{\varphi_t}{k}\big)w^2\big]_0^T dxd\tau da \\ &- \int_{Q_{T,A,\tau_2}} \big[\eta\big(s^3 k\varphi_x^3 - s^2\varphi_x\varphi_t\big)w^2\big]_0^1 d\tau dadt. \end{aligned} \tag{3.14}$$

Next, we compute I_i with $i = 3, 4, 5, 6$:

$$\begin{aligned} I_3 = &-\frac{1}{2}\int_{Q_{T,\tau_2,1}} \big[\eta w_x^2\big]_0^A dxd\tau dt + \int_{Q_{T,A,\tau_2}} \big[\eta w_x w_a\big]_0^1 d\tau dadt + \int_{Q_{T,A,\tau_2}} \big[g\eta w w_\tau\big]_0^1 d\tau dadt \\ &- \int_{Q_{T,A,1}} \big[g\eta w_x^2\big]_{\tau_1}^{\tau_2} dxdadt + \frac{1}{2}\int_{Q_{T,A,\tau_2}} \big[\eta g_\tau w w_x\big]_0^1 d\tau dadt \\ &+ \int_Q \eta g_\tau w_x^2 dxd\tau dadt - \frac{1}{2}\int_Q \eta g_\tau w_x^2 dxd\tau dadt. \end{aligned} \tag{3.15}$$

On the other hand

$$
\begin{aligned}
I_4 = &-\frac{s}{2}\int_{Q_{T,\tau_2,1}}\big[\frac{\eta}{k}w^2\varphi_t\big]_0^A dxd\tau dt+\frac{s^2}{2}\int_{Q_{T,\tau_2,1}}\big[\eta\varphi_x^2w^2\big]_0^A dxd\tau dt\\
&-\frac{s}{2}\int_{Q_{T,A,1}}\big[\frac{\eta}{k}w^2\varphi_t\big]_{\tau_1}^{\tau_2} dxd\tau dt+\frac{s^2}{2}\int_{Q_{T,A,1}}\big[\eta\varphi_x^2w^2\big]_{\tau_1}^{\tau_2} dxd\tau dt\\
&+\frac{s}{2}\int_Q\frac{\eta}{k}w^2\varphi_{ta}dxd\tau dadt-s^2\int_Q\eta\varphi_x\varphi_{xa}w^2dxd\tau dadt\\
&+\frac{s}{2}\int_Q\frac{\eta}{k}w^2\varphi_{t\tau}dxd\tau dadt-s^2\int_Q\eta\varphi_x\varphi_{x\tau}w^2dxd\tau dadt\\
&+\frac{1}{2}\int_Q\frac{\eta}{k}g_\tau w^2(-s\varphi_t+s^2k\varphi_x^2)dxdad\tau dt.
\end{aligned}
\tag{3.16}
$$

Moreover

$$
\begin{aligned}
I_5 = &-\frac{s}{2}\int_Q\frac{\eta}{k}(\varphi_a+\varphi_\tau g)g_\tau w^2dxd\tau dadt+s^2\int_Q(\varphi_a+\varphi_\tau g)(\eta\varphi_x)_xw^2dxd\tau dadt\\
&-s\int_Q\frac{\eta}{k}(\varphi_a+\varphi_\tau g)\left(\frac{w_t^2}{2}+\frac{w_a^2}{2}+g\frac{w_\tau^2}{2}-sk\varphi_xw_x^2\right)dxd\tau dadt\\
= &-\frac{s}{2}\int_Q\frac{\eta}{k}(\varphi_a+\varphi_\tau g)g_\tau w^2dxd\tau dadt+s^2\int_Q(\varphi_a+\varphi_\tau g)(\eta\varphi_x)_xw^2dxd\tau dadt\\
&-s\int_{Q_{A,\tau_2,1}}\frac{\eta}{k}\big[(\varphi_a+\varphi_\tau g)\frac{w^2}{2}\big]_0^T dxd\tau da-s\int_{Q_{T,\tau_2,1}}\frac{\eta}{k}\big[(\varphi_a+\varphi_\tau g)\frac{w^2}{2}\big]_0^A dxd\tau dt\\
&-s\int_{Q_{T,A,1}}\frac{\eta}{k}\big[(\varphi_a+\varphi_\tau g)g\frac{w^2}{2}\big]_{\tau_1}^{\tau_2} dxdadt+s^2\int_{Q_{T,A,\tau_2}}\frac{\eta}{k}\big[(\varphi_a+\varphi_\tau g)k\varphi_xw^2\big]_0^1 d\tau dadt\\
&+s\int_Q\frac{\eta}{k}(\varphi_{ta}+\varphi_{t\tau}g)\frac{w^2}{2}dxd\tau dadt+s\int_Q\frac{\eta}{k}(\varphi_{aa}+\varphi_{\tau a}g)\frac{w^2}{2}dxd\tau dadt\\
&+s\int_Q\frac{\eta}{k}[(\varphi_a+\varphi_\tau g)]_\tau\frac{w^2}{2}dxd\tau dadt-s^2\int_Q[\eta(\varphi_a+\varphi_\tau g)\varphi_x]_xw^2dxd\tau dadt.
\end{aligned}
\tag{3.17}
$$

Finally,

$$
\begin{aligned}
I_6 = &-\frac{1}{4}\int_Q\frac{1}{\sigma}g_\tau\left((w^2)_t+(w^2)_a+g(w^2)_\tau+g_\tau w^2\right)dxd\tau dadt\\
&+\frac{s}{2}\int_Q\eta g_\tau\varphi_x(w^2)_xdxd\tau dadt+\frac{s}{2}\int_Q g_\tau(\eta\varphi_x)_xw^2dxd\tau dadt\\
= &-\frac{1}{4}\int_{Q_{A,\tau_2,1}}\left[\frac{1}{\sigma}g_\tau w^2\right]_0^T dxd\tau da-\frac{1}{4}\int_{Q_{T,\tau_2,1}}\left[\frac{1}{\sigma}g_\tau w^2\right]_0^A dxd\tau dt\\
&-\frac{1}{2}\int_{Q_{T,A,1}}\left[\frac{1}{\sigma}(g^2)_\tau w^2\right]_{\tau_1}^{\tau_2} dxdadt+\frac{1}{2}\int_Q\frac{1}{\sigma}(g^2)_{\tau\tau}w^2dxd\tau dadt\\
&-\frac{1}{4}\int_Q\frac{1}{\sigma}(g_\tau)^2w^2dxd\tau dadt+\frac{s}{2}\int_{Q_{T,A,\tau_2}}\big[\eta g_\tau\varphi_xw^2\big]_0^1 d\tau dadt.
\end{aligned}
\tag{3.18}
$$

Adding (3.14) - (3.18), (3.13) follows immediately. □

Proceeding as in [9] or in [16], using the definition of φ and the conditions on w, one can prove that the boundary terms in (3.13) satisfy the following equality:

Lemma 3.2. *Assume Hypothesis* 3.1. *The boundary terms* $\{B.T.\}$ *in* (3.13) *become*

$$
\{B.T.\} = -se^R\int_{Q_{T,A,\tau_2}}\eta(1)\Theta(t,a,\tau)w_x^2(t,a,\tau,1)d\tau dadt. \tag{3.19}
$$

On the other hand, the distributed terms $\{D.T.\}$ in Lemma 3.1 satisfy the next inequality:

Lemma 3.3. *Assume Hypotheses 3.1 and 3.3. There exist two positive constants C and s_0 such that, for all $s \geq s_0$, all solutions w of (3.11) satisfy the following estimate*

$$sC\int_Q \eta\Theta w_x^2 dxdadt + s^3C\int_Q \eta\Theta^3\Big(\frac{x}{k}\Big)^2 w^2 dxdadt \leq \{D.T.\}.$$

Proof. Using the definition of φ and setting $\rho(x) := \dfrac{x(b-k_x)}{k}$, the distributed terms of $< L_s^+ w, L_s^- w >_{L^2_{\frac{1}{\sigma}}(Q)}$ take the form

$$\begin{aligned}
\{D.T.\} &= s\int_Q \eta\Theta e^{Rx^2}\left(2+4Rx^2-\frac{k_x}{k}x\right)w_x^2 dxd\tau dadt - \frac{s}{2}\int_Q \eta\Theta e^{Rx^2}(\rho_{xx}-\frac{b}{k}\rho_x)w^2 dxd\tau dadt\\
&- s\int_Q \eta\Theta e^{Rx^2}R((1+2Rx^2)\rho(x)+2x\rho_x(x))w^2 dxd\tau dadt\\
&- s\int_Q \eta\Theta e^{Rx^2}\frac{b}{k}Rx(\rho(x)+3+2Rx^2)w^2 dxd\tau dadt\\
&- s\int_Q \eta\Theta e^{Rx^2}R(3+12Rx^2+4R^2x^4))w^2 dxd\tau dadt\\
&- 2s^2\int_Q \eta\Theta_t\Theta\frac{x^2}{k^2}e^{2Rx^2}w^2 dxd\tau dadt\\
&- s^2\int_Q \eta\Theta_a\Theta\frac{x^2}{k^2}e^{2Rx^2}w^2 dxd\tau dadt - s^2\int_Q \eta\Theta_\tau\Theta\frac{x^2}{k^2}e^{2Rx^2}w^2 dxd\tau dadt\\
&+ \frac{1}{2}\int_Q \eta g_\tau w_x^2 dxd\tau dadt + \frac{1}{2}\int_Q \frac{\eta}{k}(g^2)_{\tau\tau}w^2 dxd\tau dadt - \frac{1}{4}\int_Q \frac{\eta}{k}(g_\tau)^2 w^2 dxd\tau dadt\\
&+ \frac{s}{2}\int_Q \frac{\eta}{k}(p(x)-2\|p\|_{L^\infty(0,1)})w^2(\Theta_{t\tau}+\Theta_{tt}+2\Theta_{ta}+\Theta_{aa}+\Theta_{t\tau}g+2\Theta_{\tau a}g)dxd\tau dadt\\
&+ \frac{s}{2}\int_Q \frac{\eta}{k}(p(x)-2\|p\|_{L^\infty(0,1)})w^2(\Theta_{\tau\tau}g+\Theta_\tau g_\tau)g dxd\tau dadt\\
&- \frac{s}{2}\int_Q \frac{\eta}{k}g_\tau w^2\Theta_t(p(x)-2\|p\|_{L^\infty(0,1)})dxdad\tau dt + \frac{s^2}{2}\int_Q \eta g_\tau w^2\Theta^2\frac{x^2}{k^2}e^{2Rx^2}dxdad\tau dt\\
&- s^2\int_Q \eta\Theta\frac{x^2e^{2Rx^2}}{k^2}(\Theta_a+\Theta_\tau g)w^2 dxd\tau dadt\\
&+ s^3\int_Q \eta\Theta^3\frac{x^2e^{3Rx^2}}{k^2}\big(2+4Rx^2-\frac{k_x x}{k}\big)w^2 dxd\tau dadt
\end{aligned} \tag{3.20}$$

using the fact that

$$\frac{1}{2}s\int_Q(\eta(\mathcal{A}_0\varphi)_x)_x w^2 dxd\tau dadt = \frac{1}{2}s\int_Q \eta\left((\mathcal{A}_0\varphi)_{xx}+\frac{b}{k}(\mathcal{A}_0\varphi)_x\right)w^2 dxd\tau dadt.$$

Thanks to Hypothesis 3.1

$$2-\frac{xk_x}{k} \geq 2-M_1 > 0 \text{ a.e. in } [0,1];$$

thus

$$2-\frac{xk_x}{k}+4Rx^2 \geq 2-M_1 \text{ a.e. in } [0,1].$$

Moreover, for all $x \in (0,1)$ one has that

$$\begin{aligned}
&R\left|xb\left(\rho(x)+3+2Rx^2\right)\right|\\
\leq &R\left\|b\right\|_{L^\infty(0,1)}\left(\left\|\Big(\frac{x(b-k_x)}{k}\Big)\right\|_{L^\infty(J_1)}+3+2R\right) =: C_{R,1}.
\end{aligned}$$

Using Hypothesis 3.1 and Proposition 3.2 (see below), for all $x \in (0,1)$ one has

$$
\begin{aligned}
&R\left|((1+2Rx^2)\rho(x)+2x\rho_x(x))+(3+12Rx^2+4R^2x^4)\right| \leq \\
&R\left(2\left\|x\Big(\frac{x(b-k_x)}{k}\Big)_x\right\|_{L^\infty(0,1)}+(1+2R)\left\|\Big(\frac{x(b-k_x)}{k}\Big)\right\|_{L^\infty(0,1)}+(3+12R+4R^2)\right) \\
&=: C_{R,2}.
\end{aligned}
$$

Now, observe that there exists $c>0$ such that

$$
\begin{gathered}
\Theta^\mu \leq c\Theta^\nu \text{ if } 0<\mu<\nu \\
|\Theta\Theta_t| \leq c\Theta^3, |\Theta\Theta_a| \leq c\Theta^3, |\Theta\Theta_\tau| \leq c\Theta^3 \\
|\Theta_{aa}| \leq c\Theta^{\frac{3}{2}}, |\Theta_{tt}| \leq c\Theta^{\frac{3}{2}}, |\Theta_{t\tau}| \leq c\Theta^{\frac{3}{2}}, |\Theta_{\tau a}| \leq c\Theta^{\frac{3}{2}} \text{ and } |\Theta_{ta}| \leq c\Theta^{\frac{3}{2}}.
\end{gathered}
\tag{3.21}
$$

Then, applying Hypothesis 3.1 and (3.21), one has

$$
\begin{aligned}
\{D.T.\} \geq\; & s(2-M_1)\int_Q \eta\Theta w_x^2 dxd\tau dadt + s^3(2-M_1)\int_Q \eta\Theta^3\Big(\frac{x}{k}\Big)^2 w^2 dxd\tau dadt \\
& - s^2Ce^{2R}\int_Q \eta\Theta^3\Big(\frac{x}{k}\Big)^2 w^2 dxd\tau dadt - sC\|p\|_{L^\infty(0,1)}\int_Q \eta\frac{\Theta^{\frac{3}{2}}}{k} w^2 dxd\tau dadt \\
& - \frac{s}{2}e^R C_1(\varepsilon)\int_Q \eta\frac{\Theta}{x^2} w^2 dxd\tau dadt \\
& - se^R C_{R,1}\int_Q \eta\frac{\Theta}{k} w^2 dxd\tau dadt - se^R C_{R,2}\int_Q \eta\Theta w^2 dxd\tau dadt \\
& + \frac{1}{2}\int_Q \eta g_\tau w_x^2 dxd\tau dadt + \frac{1}{2}\int_Q \frac{\eta}{k}(g^2)_{\tau\tau} w^2 dxd\tau dadt - \frac{1}{4}\int_Q \frac{\eta}{k}(g_\tau)^2 w^2 dxd\tau dadt \\
\geq\; & s(2-M_1)\int_Q \eta\Theta w_x^2 dxd\tau dadt + s^3(2-M_1)\int_Q \eta\Theta^3\Big(\frac{x}{k}\Big)^2 w^2 dxd\tau dadt \\
& - s^2Ce^{2R}\int_Q \eta\Theta^3\Big(\frac{x}{k}\Big)^2 w^2 dxd\tau dadt - \frac{s}{2}e^R C_1(\varepsilon)\int_Q \eta\frac{\Theta}{x^2} w^2 dxd\tau dadt \\
& - s(e^R C_{R,2} + e^R C_{R,1} + C\|k\|_{L^\infty(0,1)}\|p\|_{L^\infty(0,1)})\int_Q \eta\frac{\Theta^{\frac{3}{2}}}{k} w^2 dxd\tau dadt \\
& - \frac{1}{2}C\int_Q \eta\Theta w_x^2 dxd\tau dadt - \frac{1}{2}C\int_Q \frac{\eta}{k}\Theta^{\frac{3}{2}} w^2 dxd\tau dadt - \frac{1}{4}C\int_Q \frac{\eta}{k}\Theta^{\frac{3}{2}} w^2 dxd\tau dadt.
\end{aligned}
\tag{3.22}
$$

By Hardy's inequality, it is possible to estimate the term $\int_Q \eta\frac{\Theta}{x^2} w^2 dxd\tau dadt$ in the following way

$$
\int_Q \eta\frac{\Theta}{x^2} w^2 dxd\tau dadt \leq C_H\frac{\sup_{[0,1]}\{\eta\}}{\inf_{[0,1]}\{\eta\}}\int_Q \eta\Theta w_x^2 dxd\tau dadt.
\tag{3.23}
$$

Here, C_H is a positive constant. Thus,

$$
\begin{aligned}
\{D.T.\} \geq\; & s(2-M_1)\int_Q \eta\Theta w_x^2 dxd\tau dadt - \Big(\frac{s}{2}e^R C_1(\varepsilon)C + \frac{C}{2}\Big)\int_Q \eta\Theta w_x^2 dxd\tau dadt \\
& + s^3(2-M_1)\int_Q \eta\Theta^3\Big(\frac{x}{k}\Big)^2 w^2 dxd\tau dadt - s^2Ce^{2R}\int_Q \eta\Theta^3\Big(\frac{x}{k}\Big)^2 w^2 dxd\tau dadt \\
& - s(e^R C_{R,2} + e^R C_{R,1} + C\|k\|_{L^\infty(0,1)}\|p\|_{L^\infty(0,1)})\int_Q \eta\frac{\Theta^{\frac{3}{2}}}{k} w^2 dxd\tau dadt \\
& - C\int_Q \frac{\eta}{k}\Theta^{\frac{3}{2}} w^2 dxd\tau dadt.
\end{aligned}
\tag{3.24}
$$

Moreover, as in [8] or in [16], one has, for $\gamma > 0$,

$$\int_Q \eta \frac{\Theta^{\frac{3}{2}}}{k} w^2 dxd\tau dadt \le \sup_{[0,1]} \eta \int_Q \left(\frac{1}{\gamma}\Theta^2 \left(\frac{x}{k}\right)^2 w^2\right)^{\frac{1}{2}} \left(\gamma \frac{\Theta}{x^2} w^2\right)^{\frac{1}{2}} dxd\tau dadt$$

$$\le \sup_{[0,1]} \eta \frac{1}{\gamma} \int_Q \Theta^2 \left(\frac{x}{k}\right)^2 w^2 dxd\tau dadt + \gamma \sup_{[0,1]} \eta \int_Q \frac{\Theta}{x^2} w^2 dxd\tau dadt.$$

Again, by Hardy's inequality one has

$$\int_Q \eta \frac{\Theta^{\frac{3}{2}}}{k} w^2 dxd\tau dadt \le \frac{1}{\gamma} C \int_Q \Theta^3 \left(\frac{x}{k}\right)^2 w^2 dxd\tau dadt + \gamma C \int_Q \Theta w_x^2 dxd\tau dadt, \tag{3.25}$$

for a positive constant C.

Thus, for s_0 large enough and γ small enough, by (3.24) and (3.25), the lemma follows. □

Proposition 3.2. *Assume Hypothesis* 3.1. *Then there exists $l \in \mathbb{R}$ such that*

$$\lim_{x \to 0^+} x \left(\frac{x(b(x) - k_x(x))}{k(x)}\right)_x = l.$$

As a consequence of Lemmas 3.2 and 3.3, we have

Proposition 3.3. *Assume Hypotheses* 3.1 *and* 3.3. *There exist two positive constants C and s_0 such that, for all $s \ge s_0$, all solutions w of* (3.11) *in $\mathcal{V}$ satisfy*

$$\int_Q \left(s\eta\Theta w_x^2 + s^3\eta\Theta^3 \left(\frac{x}{k}\right)^2 w^2\right) dxd\tau dadt \le C \int_Q f^2 \frac{e^{2s\varphi}}{k} dxd\tau dadt$$
$$+ sC \int_{Q_{T,A,\tau_2}} \Theta(t,a,\tau) w_x^2(t,a,\tau,1) d\tau dadt.$$

Recalling the definition of w, we have $v = e^{-s\varphi}w$ and $v_x = (w_x - s\varphi_x w)e^{-s\varphi}$. Thus, Theorem 3.1 follows immediately by Proposition 3.3 when $\mu \equiv 0$.

Now, we assume that $\mu \not\equiv 0$.

To complete the proof of Theorem 3.1, we proceed as in [16] and we consider the function $\overline{f} = f + \mu v$. Hence, there are two positive constants C and s_0 such that, for all $s \ge s_0$, the following inequality holds

$$\begin{aligned} &\int_Q \left(s\eta\Theta v_x^2 + s^3\eta\Theta^3 \left(\frac{x}{k}\right)^2 v^2\right) e^{2s\varphi} dxd\tau dadt \le C \int_Q \overline{f}^2 \frac{e^{2s\varphi}}{\sigma} dxd\tau dadt \\ &+ s\, C \int_{Q_{T,A,\tau_2}} \Theta(t,a,\tau_2) \left[v_x^2 e^{2s\varphi}\right](t,a,1) d\tau dadt. \end{aligned} \tag{3.26}$$

On the other hand, we have

$$\int_Q |\,\overline{f}\,|^2 \frac{e^{2s\varphi}}{\sigma} dxd\tau dadt \le 2\Big(\int_Q |f|^2 \frac{e^{2s\varphi}}{\sigma} dxd\tau dadt + \|\mu\|^2_{L^\infty(Q)} \int_Q |v|^2 \frac{e^{2s\varphi}}{\sigma} dxd\tau dadt\Big). \tag{3.27}$$

Now, applying Hardy-Poincaré inequality to the function $\nu := e^{s\varphi}v$, we obtain

$$\int_Q |v|^2 \frac{e^{2s\varphi}}{\sigma} dxd\tau dadt = \int_Q \frac{\nu^2}{\sigma} dxd\tau dadt \le \sup_{[0,1]} \eta \int_Q \frac{x^2}{k} \frac{\nu^2}{x^2} dxd\tau dadt \le C \int_Q \frac{\nu^2}{x^2} dxd\tau dadt$$

$$\le C \int_Q (e^{s\varphi}v)_x^2 dxd\tau dadt \le C \int_Q e^{2s\varphi} v_x^2 dxd\tau dadt + Cs^2 \int_Q \Theta^2 e^{2s\varphi} \left(\frac{x}{k}\right)^2 v^2 dxd\tau dadt.$$

Using this last inequality in (3.27), it follows

$$\begin{aligned} \int_Q |\overline{f}|^2 \frac{e^{2s\varphi}}{\sigma} dxd\tau dadt &\le 2 \int_Q |f|^2 \frac{e^{2s\varphi}}{\sigma} dxd\tau dadt + C \int_Q e^{2s\varphi} v_x^2 dxd\tau dadt \\ &+ Cs^2 \int_Q \Theta^2 e^{2s\varphi} \left(\frac{x}{k}\right)^2 v^2 dxd\tau dadt. \end{aligned} \tag{3.28}$$

Substituting in (3.26), one can conclude

$$\int_Q \left(s\eta\Theta v_x^2 + s^3\eta\Theta^3 \left(\frac{x}{k}\right)^2 v^2 \right) e^{2s\varphi} dxd\tau dadt \leq C\Big(\int_Q |f|^2 \frac{e^{2s\varphi}}{\sigma} \; dxd\tau dadt$$
$$+ \int_Q e^{2s\varphi} v_x^2 dxd\tau dadt + s^2 \int_Q \Theta^2 e^{2s\varphi} \left(\frac{x}{k}\right)^2 v^2 dxd\tau dadt + sC \int_{Q_{T,A,\tau_2}} \Theta(t,a,\tau) \Big[v_x^2 e^{2s\varphi} \Big](t,a,1) d\tau dadt.$$

This completes the proof of Theorem 3.1 when $\mu \not\equiv 0$.

4 Observability and controllability of linear equations

In this section we will prove, as a consequence of the Carleman estimates established in Section 3, observability inequalities for the associated homogeneous adjoint problem of (1.1). To this aim, we assume that the control set ω is such that

$$\omega = (\alpha, \rho) \subset\subset (0,1). \tag{4.29}$$

Hypothesis 4.1. On the birth rate β we assume that there exists $\bar{a} < \min\{T, A\}$ such that

$$\beta(a,\tau,x) = 0 \text{ for all } (a,\tau,x) \in [0,\bar{a}] \times [\tau_1,\tau_2] \times [0,1]. \tag{4.30}$$

Observe that Hypothesis 4.1 is the biological meaningful one. Indeed, $\bar{a}$ is the minimal age in which the female of the population become fertile, thus it is natural that before $\bar{a}$ there are no newborns. For other comments on this assumptions we refer to [15] or to [16].

Under the previous hypotheses, the following observability inequality holds:

Proposition 4.1. *Assume Hypotheses* 3.1 *or* 3.2*,* 3.3 *and* 4.1*. Moreover, suppose that* ω *satisfies* (4.29)*. Then,* $\forall\ n \in \mathbb{N}$ *and* $\forall\ \vartheta \in \left(0, \frac{1}{n}\right)$*, there exists a positive constant* $C = C(\vartheta)$ *such that every solution* $v \in \mathcal{U}$ *of* (3.4) *satisfies*

$$\int_{Q_{A,\tau_2,1}} \frac{1}{\sigma} v^2(T-\bar{a},a,\tau,x) dxd\tau da \leq C \int_0^{\Xi} \int_{\tau_1}^{\tau_2} \int_0^1 \frac{v_T^2(\gamma,\tau,x)}{\sigma} dxd\tau d\gamma + C \int_{Q_{T,A,\tau_2}} \int_\omega \frac{v^2}{\sigma} dxd\tau dadt,$$

where $\Xi := \max\{A - \vartheta\bar{a}, \bar{a}\}$*. Here* $v_T(a,\tau,x)$ *is such that* $v_T(A,\tau,x) = 0$ *in* $(\tau_1,\tau_2) \times (0,1)$*.*

As a first step we will prove Proposition 4.1 for more regular solutions. To this aim, we introduce the following class of functions

$$\mathcal{W} := \Big\{ v \text{ solution of } (3.4) \;\big|\; v_T \in D(\mathcal{A}^2) \Big\},$$

where

$$D(\mathcal{A}^2) = \Big\{ u \in D(\mathcal{A}) \;\big|\; \mathcal{A}u \in D(\mathcal{A}) \Big\}.$$

Obviously,

$$\mathcal{W} \subset \mathcal{V} \subset \mathcal{U}.$$

In order to prove the $\omega-$local Carleman estimate given in Theorem 4.1 (see below), we need the following Caccioppoli's inequality, whose proof is postponed to the Appendix.

Proposition 4.2 (Caccioppoli's inequality)**.** *Assume Hypothesis* 3.1 *or* 3.2*. Let* ω' *and* ω *be two open subintervals of* $(0,1)$ *such that* $\omega' \subset\subset \omega \subset\subset (0,1)$*. Let* $\psi(t,a,\tau,x) := \Theta(t,a,\tau)\Psi(x)$*, where* Θ *is defined in* (3.10) *and* $\Psi \in C^1(0,1)$ *is a strictly negative function. Then, there exist two strictly positive constants* C *and* s_0 *such that, for all* $s \geq s_0$*,*

$$\begin{aligned} \int_0^T \int_0^A \int_{\omega'} v_x^2 e^{2s\psi} dxdadt \;&\leq\; C \left(\int_0^T \int_0^A \int_\omega v^2 dxdadt + \int_Q f^2 e^{2s\psi} dxdadt \right) \\ &\leq\; C \left(\int_0^T \int_0^A \int_\omega \frac{v^2}{\sigma} dxdadt + \int_Q f^2 \frac{e^{2s\psi}}{\sigma} dxdadt \right), \end{aligned} \tag{4.31}$$

for every solution v *of* (3.5)*.*

With the aid of Theorems 3.1, 3.2 and Propositions 4.2, we can now show the following $\omega-$local Carleman estimate for (3.5). To this aim, we consider the function Φ defined as follows

$$\Phi(a,t,x) = \Theta(t,a,\tau)\Psi(x), \quad \Psi(x) = e^{\kappa\gamma(x)} - e^{2\kappa\|\gamma\|_\infty}, \tag{4.32}$$

for all $(t,a,\tau,x) \in Q$. Here Θ is defined as in (3.10), $\kappa > 0$ and $\gamma(x) := \mathfrak{d}\int_x^1 \frac{1}{\sigma(y)}dy$, where $\mathfrak{d} = \|\sigma'\|_{L^\infty(0,1)}$.

Theorem 4.1. *Assume Hypotheses* 3.1, 3.3 *and* (4.29)*. Then, there exist two positive constants C and s_0 such that every solution v of* (3.5) *in $\mathcal{V}$ satisfies, for all $s \geq s_0$,*

$$\int_Q \left(s\eta\Theta v_x^2 + s^3\eta\Theta^3\left(\frac{x}{k}\right)^2 v^2\right)e^{2s\varphi}dxd\tau dadt \leq C\left(\int_Q f^2\frac{e^{2s\Phi}}{\sigma}\, dxd\tau dadt + \int_{Q_{T,A,\tau_2}}\int_\omega \frac{v^2}{\sigma}dxd\tau dadt\right).$$

The proof of Theorem 4.1 is analogous to the one of [16, Theorem 4.1], but we repeat it for the reader's convenience. First of all, we need the following Carleman estimate that holds for the non degenerate problem. We omit its proof, but we refer the reader to the proof of [16, Theorem 3.1].

Theorem 4.2. *Let $z \in \mathcal{V}$ be the solution of* (3.5) *where $f \in L^2(Q)$ and $k \in C^1[0,1]$ is a strictly positive function. Then, there exist two positive constants C and s_0, such that, for any $s \geq s_0$, z satisfies the estimate*

$$\int_Q (s^3\phi^3 z^2 + s\phi z_x^2)e^{2s\Phi}dxd\tau dadt \leq C\Big(\int_Q f^2 e^{2s\Phi}dxd\tau dadt - s\kappa\int_{Q_{T,A,\tau_2}}\left[ke^{2s\Phi}\phi(z_x)^2\right]_{x=0}^{x=1}d\tau dadt\Big). \tag{4.33}$$

Here $\phi(t,a,\tau,x) = \Theta(t,a,\tau)e^{\kappa\gamma(x)}$, Φ and γ are as in (4.32) *and Θ is as in* (3.10).

Proof of Theorem 4.1. Let us consider a smooth function $\xi : [0,1] \to \mathbb{R}$ such that

$$\begin{cases} 0 \leq \xi(x) \leq 1, & \text{for all } x \in [0,1], \\ \xi(x) = 1, & x \in [0,(2\alpha+\rho)/3], \\ \xi(x) = 0, & x \in [(\alpha+2\rho)/3, 1]. \end{cases}$$

We define $w(t,a,\tau,x) := \xi(x)v(t,a,\tau,x)$ where $v \in \mathcal{V}$ satisfies (3.5). Then w satisfies

$$\begin{cases} w_t + w_a + g(\tau)\frac{\partial w}{\partial \tau} + kw_{xx} + b(x)w_x - \mu w = \xi f + k(\xi_{xx}v + 2\xi_x v_x) =: h, & (t,a,x) \in Q, \\ w(t,a,\tau,0) = w(t,a,\tau,1) = 0, & (t,a,\tau) \in Q_{T,A,\tau_2} \\ w(t,A,\tau,x) = 0, & (t,\tau,x) \in Q_{T,\tau_2,1}, \\ w(t,a,\tau_2,x) = 0, & (t,a,x) \in Q_{T,A,1}. \end{cases}$$

Thus, proceeding as in [16, Theorem 4.1], via Theorem 3.1 and Proposition 4.2 one can prove

$$\begin{aligned} &\int_0^T\int_0^A\int_0^{\frac{2\alpha+\rho}{3}} \left(s\eta\Theta v_x^2 + s^3\eta\Theta^3\left(\frac{x}{k}\right)^2 v^2\right)e^{2s\varphi}dxd\tau dadt \\ &= \int_0^T\int_0^A\int_0^{\frac{2\alpha+\rho}{3}} \left(s\eta\Theta w_x^2 + s^3\eta\Theta^3\left(\frac{x}{k}\right)^2 w^2\right)e^{2s\varphi}dxd\tau dadt \\ &\leq C\left(\int_Q f^2\frac{e^{2s\varphi}}{\sigma}dxd\tau dadt + \int_0^T\int_0^A\int_\omega \frac{v^2}{\sigma}dxd\tau dadt\right). \end{aligned} \tag{4.34}$$

Now, consider $z = \eta v$, where $\eta = 1 - \xi$ and take $\bar{\alpha} \in (0,\alpha)$. Then z satisfies

$$\begin{cases} z_t + z_a + g(\tau)\dfrac{\partial z}{\partial \tau} + kz_{xx} + b(x)z_x - \mu z = \eta f + k(\eta_{xx}v + 2\eta_x v_x) =: h, & (t,a,x) \in Q_{T,A,\tau_2} \times (\bar{\alpha},1) =: \bar{Q}, \\ z(t,a,\tau,\bar{\alpha}) = z(t,a,\tau,1) = 0, & (t,a,\tau) \in Q_{T,A,\tau_2}, \\ z(t,A,\tau,x) = 0, & (t,\tau,x) \in Q_{T,\tau_2,1}, \\ z(t,a,\tau_2,x) = 0, & (t,a,x) \in Q_{T,A,1}. \end{cases} \tag{4.35}$$

Clearly the equation satisfied by z is not degenerate, thus applying Theorem 4.2 and Proposition 4.2, one has

$$\int_{Q_{T,A,\tau_2}}\int_{\frac{\alpha+2\rho}{3}}^1 (s^3\phi^3v^2+s\phi v_x^2)e^{2s\Phi}dxd\tau dadt = \int_{Q_{T,A,\tau_2}}\int_{\frac{\alpha+2\rho}{3}}^1 (s^3\phi^3z^2+s\phi z_x^2)e^{2s\Phi}dxd\tau dadt$$
$$\le \int_{\bar{Q}}(s^3\phi^3z^2+s\phi z_x^2)e^{2s\Phi}dxd\tau dadt \le C\left(\int_Q \frac{f^2}{\sigma}e^{2s\Phi}dxd\tau dadt + \int_{Q_{T,A,\tau_2}}\int_\omega \frac{v^2}{\sigma}dxd\tau dadt\right),$$

for a positive constant C. As in [16, Theorem 4.1], for a strictly positive constant C,

$$\begin{aligned}
&\int_{Q_{T,A,\tau_2}}\int_{\frac{\alpha+2\rho}{3}}^1 \left(s\eta\Theta v_x^2+s^3\eta\Theta^3\left(\frac{x}{k}\right)^2v^2\right)e^{2s\varphi}dxd\tau dadt\\
&\le C\left(\int_{Q_{T,A,\tau_2}}\int_{\frac{\alpha+2\rho}{3}}^1 (s^3\phi^3v^2+s\phi v_x^2)e^{2s\Phi}dxd\tau dadt\right)\\
&\le C\left(\int_Q f^2\frac{e^{2s\Phi}}{\sigma}dxd\tau dadt+\int_{Q_{T,A,\tau_2}}\int_\omega \frac{v^2}{\sigma}dxd\tau dadt\right).
\end{aligned}\tag{4.36}$$

Now, consider $\tilde{\alpha}\in(\alpha,(2\alpha+\rho)/3)$, $\tilde{\rho}\in((\alpha+2\rho)/3,\rho)$ and a smooth function $\gamma:[0,1]\to\mathbb{R}$ such that

$$\begin{cases} 0\le\gamma(x)\le1, & \text{for all } x\in[0,1],\\ \gamma(x)=1, & x\in[(2\alpha+\rho)/3,(\alpha+2\rho)/3],\\ \gamma(x)=0, & x\in[0,\tilde{\alpha}]\cup[\tilde{\rho},1],\end{cases}$$

and define $\zeta(t,a,\tau,x):=\gamma(x)v(t,a,x)$. Clearly, ζ satisfies (4.35) with $h:=\gamma f+k(\gamma_{xx}v+2\gamma_xv_x)$. Observe that in this case $\gamma_x,\gamma_{xx}\not\equiv0$ in $\bar{\omega}:=\left(\tilde{\alpha},\dfrac{2\alpha+\rho}{3}\right)\cup\left(\dfrac{\alpha+2\rho}{3},\tilde{\rho}\right)$. Again

$$\begin{aligned}
&\int_{Q_{T,A,\tau_2}}\int_{\frac{2\alpha+\rho}{3}}^{\frac{\alpha+2\rho}{3}} \left(s\eta\Theta v_x^2+s^3\eta\Theta^3\left(\frac{x}{k}\right)^2v^2\right)e^{2s\varphi}dxd\tau dadt\\
&\le C\left(\int_{Q_{T,A,\tau_2}}\int_{\frac{2\alpha+\rho}{3}}^{\frac{\alpha+2\rho}{3}} (s^3\phi^3v^2+s\phi v_x^2)e^{2s\Phi}dxd\tau dadt\right)\\
&\le C\left(\int_Q f^2\frac{e^{2s\Phi}}{\sigma}dxd\tau dadt+\int_{Q_{T,A,\tau_2}}\int_\omega \frac{v^2}{\sigma}dxd\tau dadt\right).
\end{aligned}\tag{4.37}$$

Adding (4.34), (4.36) and (4.37), the theorem follows. □

Proceeding as before one can prove

Theorem 4.3. *Assume Hypotheses* 3.2, 3.3 *and* (4.29)*. Then, there exist two positive constants C and s_0 such that every solution v of* (3.5) *in $\mathcal{V}$ satisfies, for all $s\ge s_0$,*

$$\int_Q\left(s\eta\Theta v_x^2+s^3\eta\Theta^3\left(\frac{1-x}{k}\right)^2v^2\right)e^{2s\varphi}dxd\tau dadt\le C\left(\int_Q f^2\frac{e^{2s\Phi}}{\sigma}dxd\tau dadt+\int_{Q_{T,A,\tau_2}}\int_\omega\frac{v^2}{\sigma}dxdadt\right).$$

Remark 1. Observe that the results of Theorems 4.1 and 4.3 still hold true if we substitute the interval $(0,T)$ with a general interval (T_1,T_2), provided that μ and β satisfy the required assumptions. In this case, in place of the function Θ defined in (3.10), we have to consider the weight function

$$\tilde{\Theta}(t,a,\tau):=\frac{1}{(t-T_1)^4(T_2-t)^4a^4(\tau-\tau_1)^4}.\tag{4.38}$$

Using the previous local Carleman estimates one can prove the next observability inequalities.

Theorem 4.4. *Assume Hypotheses* 3.1 *or* 3.2, 3.3 *and* 4.1. *Suppose that* ω *satisfies* (4.29). *Then, for every* $\delta \in (0, A)$ *and* $\iota \in (\tau_1, \tau_2)$, *there exists a positive constant* $C = C(\delta, \iota)$ *such that every solution* v *of* (3.4) *in* $\mathcal{V}$ *satisfies*

$$\int_{Q_{A,\tau_2,1}} \frac{1}{\sigma} v^2(\tilde{T}, a, \tau, x) dx d\tau da \leq C \int_0^T \int_0^\delta \int_{\tau_1}^{\tau_2} \int_0^1 \frac{1}{\sigma} v^2(t, a, \tau, x) dx d\tau da dt$$
$$+ C \int_0^T \int_\delta^A \int_{\tau_1}^{\iota} \int_0^1 \frac{1}{\sigma} v^2(t, a, \tau, x) dx d\tau da dt + C\|\beta\|^2_{L^\infty(Q)} \int_{Q_{\bar{a},\tau_2,1}} \frac{v_T^2(s, \tau, x)}{\sigma} dx d\tau ds$$
$$+ C \int_{Q_{T,A,\tau_2}} \int_\omega \frac{v^2}{\sigma} dx d\tau da dt + C \int_Q \frac{1}{\sigma} v^2(s, a, \tau, x) dx d\tau da ds.$$

Here

$$\tilde{T} := T - \bar{a}. \tag{4.39}$$

Moreover, if $v_T(s, \tau, x) = 0$ *for all* $(s, \tau, x) \in (0, \bar{a}) \times (\tau_1, \tau_2) \times (0, 1)$, *one has*

$$\int_{Q_{A,\tau_2,1}} \frac{1}{\sigma} v^2(\tilde{T}, a, \tau, x) dx d\tau da \leq C \int_0^T \int_0^\delta \int_{\tau_1}^{\tau_2} \int_0^1 \frac{1}{\sigma} v^2(t, a, \tau, x) dx d\tau da dt$$
$$+ C \int_0^T \int_\delta^A \int_{\tau_1}^{\iota} \int_0^1 \frac{1}{\sigma} v^2(t, a, \tau, x) dx d\tau da dt + C \int_{Q_{T,A,\tau_2}} \int_\omega \frac{v^2}{\sigma} dx d\tau da dt$$
$$+ C \int_Q \frac{1}{\sigma} v^2(s, a, \tau, x) dx d\tau da ds.$$

Proof. The operator $(\mathcal{B}, D(\mathcal{B}))$, where $\mathcal{B} := \mathcal{A}_0 + g\frac{\partial}{\partial \tau}$ and

$$D(\mathcal{B}) := \left\{ v \in L^2(\tau_1, \tau_2; D(\mathcal{A}_0)); \frac{\partial v}{\partial \tau} \in L^2(\tau_1, \tau_2; H^1_{\frac{1}{\sigma}}(0, 1)) \right\},$$

generates a strongly continuous semigroup on $L^2_{\frac{1}{\sigma}}(Q_{\tau_2,1}) := L^2((\tau_1, \tau_2); L^2_{\frac{1}{\sigma}}(0, 1))$ (we recall that the operator $\mathcal{A}_0$ is defined in Lemma 2.1). As is stated after Lemma 2.1, the operator $B(t)$ defined as

$$B(t)u := \mu(t, a, \tau, x)u,$$

for $u \in D(\mathcal{B})$, can be seen as a bounded perturbation of $\mathcal{B}$; thus also $(\mathcal{B} + B(t), D(\mathcal{B}))$ generates an evolution family. Let $(S(t))_{t \geq 0}$ be the associated semigroup. Then, using the method of characteristic lines, the assumption on β and $v(t, A, \tau, x) = 0$ for all $(t, \tau, x) \in Q_{T,\tau_2,1}$, one can obtain the same implicit formula for v solution of (3.4) given in [16]:

$$S(T - t)v_T(T + a - t, \cdot, \cdot), \tag{4.40}$$

if $t \geq \tilde{T} + a$ (observe that in this case $T + a - t \leq \bar{a}$) and

$$v(t, a, \cdot, \cdot) = \begin{cases} S(T - t)v_T(T + a - t, \cdot, \cdot) + \int_a^{T+a-t} S(s - a)\beta(s, \cdot, \cdot)v(s + t - a, 0, \cdot, \cdot)ds, & \Gamma = \bar{a} \\ \int_a^A S(s - a)\beta(s, \cdot, \cdot)v(s + t - a, 0, \cdot, \cdot)ds, & \Gamma = \Gamma_{A,T}, \end{cases} \tag{4.41}$$

where $\Gamma_{A,T} := A - a + t - \tilde{T}$ and

$$\Gamma := \min\{\bar{a}, \Gamma_{A,T}\}. \tag{4.42}$$

In particular, it results that

$$v(t, 0, \cdot, \cdot) := S(T - t)v_T(T - t, \cdot, \cdot), \tag{4.43}$$

if $t \geq T - \bar{a}$.
Now, define, for $\varsigma > 0\varrho$, the function $w = e^{\varsigma t}v$, where v solves (3.4). Then w satisfies

$$\begin{cases} \frac{\partial w}{\partial t} + \frac{\partial w}{\partial a} + g(\tau)\frac{\partial w}{\partial \tau} + k(x)w_{xx} + b(x)w_x - (\mu(t, a, , \tau, x) + \varsigma)w = -\beta(a, \tau, x)w(t, 0, \tau, x), & (t, a, \tau, x) \in \tilde{Q}, \\ w(t, a, \tau, 0) = w(t, a, \tau, 1) = 0, & (t, a, \tau) \in \tilde{Q}_{T,A,\tau_2}, \\ w(T, a, \tau, x) = e^{\varsigma T}v_T(a, \tau, x), & (a, \tau, x) \in Q_{A,\tau_2,1}, \\ w(t, A, \tau, x) = 0, & (t, \tau, x) \in \tilde{Q}_{T,\tau_2,1}, \\ w(t, a, \tau_2, x) = 0, & (t, a, x) \in \tilde{Q}_{T,A,1}, \end{cases} \tag{4.44}$$

where $\tilde{Q} := (\tilde{T}, T) \times Q_{A,\tau_2,1}$, $\tilde{Q}_{T,A,1} := (\tilde{T}, T) \times (0, A) \times (0, 1)$, $\tilde{Q}_{T,A,\tau_2} := (\tilde{T}, T) \times (0, A) \times (\tau_1, \tau_2)$ and $\tilde{Q}_{T,\tau_2,1} := (\tilde{T}, T) \times (\tau_1, \tau_2) \times (0, 1)$. Multiplying the equation of (4.44) by $-\frac{w}{\sigma}$ and integrating by parts on $Q_t := (\tilde{T}, t) \times Q_{A,\tau_2,1}$, it results that

$$\begin{aligned}
&-\frac{1}{2}\int_{Q_{A,\tau_2,1}} \frac{1}{\sigma} w^2(t,a,\tau,x)dxd\tau da + \frac{e^{\varsigma \tilde{T}}}{2}\int_{Q_{A,\tau_2,1}} \frac{1}{\sigma} v^2(\tilde{T},a,\tau,x)dxd\tau da \\
&+\frac{1}{2}\int_{\tilde{T}}^{t}\int_{\tau_1}^{\tau_2}\int_0^1 \frac{1}{\sigma} w^2(s,0,\tau,x)dxd\tau ds + \varsigma\int_{Q_t}\frac{1}{\sigma} w^2(s,a,\tau,x)dxd\tau dads \\
&+\int_{Q_t}\frac{g_\tau(\tau)}{\sigma} w^2(s,a,\tau,x)dxd\tau dads \le \int_{Q_t}\frac{1}{\sigma}\beta w(s,0,\tau,x)wdxd\tau dads \\
&\le \|\beta\|_{C(\bar{Q}_{A,\tau_2,1})}\frac{1}{\epsilon}\int_{Q_t}\frac{1}{\sigma} w^2 dxd\tau dads + \epsilon A\|\beta\|_{C(\bar{Q}_{A,\tau_2,1})}\int_{\tilde{T}}^{t}\int_{\tau_1}^{\tau_2}\int_0^1 \frac{1}{\sigma} w^2(s,0,\tau,x)dxd\tau ds,
\end{aligned} \tag{4.45}$$

for $\epsilon > 0$. Choosing $\epsilon = \dfrac{1}{2\|\beta\|_{C(\bar{Q}_{A,\tau_2,1})}A}$ and $\varsigma = \dfrac{\|\beta\|_{C(\bar{Q}_{A,\tau_2,1})}}{\epsilon}$, we have

$$\begin{aligned}
\frac{e^{\varsigma \tilde{T}}}{2}\int_{Q_{A,\tau_2,1}} \frac{1}{\sigma} v^2(\tilde{T},a,\tau,x)dxd\tau da \le \frac{1}{2}\int_{Q_{A,\tau_2,1}} \frac{1}{\sigma} w^2(t,a,\tau,x)dxd\tau da \\
+ \|g_\tau\|_{C[\tau_1,\tau_2]}\int_{Q_t}\frac{1}{\sigma} w^2(s,a,\tau,x)dxd\tau dads.
\end{aligned}$$

Hence

$$\int_{Q_{A,\tau_2,1}} \frac{1}{\sigma} v^2(\tilde{T},a,\tau,x)dxd\tau da \le C\int_{Q_{A,\tau_2,1}} \frac{1}{\sigma} v^2(t,a,\tau,x)dxd\tau da + \|g_\tau\|_{C[\tau_1,\tau_2]}\int_{Q_t}\frac{1}{\sigma} v^2(s,a,\tau,x)dxd\tau dads.$$

Now, integrating over $\left[T - \frac{\bar{a}}{2}, T - \frac{\bar{a}}{4}\right]$, one has

$$\begin{aligned}
\int_{Q_{A,\tau_2,1}} \frac{1}{\sigma} v^2(\tilde{T},a,\tau,x)dxd\tau da &\le C\int_{T-\frac{\bar{a}}{2}}^{T-\frac{\bar{a}}{4}}\int_{Q_{A,\tau_2,1}} \frac{1}{\sigma} v^2(t,a,\tau,x)dxd\tau dadt \\
&+ C\int_{T-\frac{\bar{a}}{2}}^{T-\frac{\bar{a}}{4}}\int_{Q_t}\frac{1}{\sigma} v^2(s,a,\tau,x)dxd\tau dadsdt \\
&= C\int_{T-\frac{\bar{a}}{2}}^{T-\frac{\bar{a}}{4}}\left(\int_0^\delta + \int_\delta^A\right)\int_{\tau_1}^{\tau_2}\int_0^1 \frac{1}{\sigma} v^2(t,a,\tau,x)dxd\tau dadt \\
&+ C\int_{T-\frac{\bar{a}}{2}}^{T-\frac{\bar{a}}{4}}\int_{Q_t}\frac{1}{\sigma} v^2(s,a,\tau,x)dxd\tau dadsdt.
\end{aligned} \tag{4.46}$$

Consider the term

$$\int_{T-\frac{\bar{a}}{2}}^{T-\frac{\bar{a}}{4}}\int_\delta^A\int_{\tau_1}^{\tau_2}\int_0^1 \frac{1}{\sigma} v^2(t,a,\tau,x)dxd\tau dadt = \int_{T-\frac{\bar{a}}{2}}^{T-\frac{\bar{a}}{4}}\int_\delta^A\left(\int_{\tau_1}^{\iota} + \int_\iota^{\tau_2}\right)\int_0^1 \frac{1}{\sigma} v^2(t,a,\tau,x)dxd\tau dadt$$

where $\iota \in (\tau_1, \tau_2)$ is fixed. As in [16, Theorem 4.4], one can prove

$$\int_{T-\frac{\bar{a}}{2}}^{T-\frac{\bar{a}}{4}}\int_\delta^A\int_\iota^{\tau_2}\int_0^1 \frac{1}{\sigma} v^2(t,a,\tau,x)dxd\tau dadt \le C\int_{T-\frac{\bar{a}}{2}}^{T-\frac{\bar{a}}{4}}\int_\delta^A\int_\iota^{\tau_2}\int_0^1 \eta\tilde{\Theta} v_x^2 e^{2s\tilde{\varphi}}dxd\tau dadt$$

where $\tilde{\Theta}$ is defined as in (4.38) with $T_1 := T - \bar{a}$, $T_2 := T$, $\gamma = 0$ and $\tilde{\varphi}$ is the function associated to $\tilde{\Theta}$ according to (3.8). By Theorem 4.1 or 4.3 applied to $\bar{Q} := (T - \bar{a}, T) \times (0, A) \times (\tau_1, \tau_2) \times (0, 1)$,

$$\int_{T-\frac{\bar{a}}{2}}^{T-\frac{\bar{a}}{4}}\int_\delta^A\int_\iota^{\tau_2}\int_0^1 \eta\tilde{\Theta} v_x^2 e^{2s\tilde{\varphi}}dxd\tau dadt \le C\left(\int_{\bar{Q}} f^2\frac{e^{2s\Phi}}{\sigma}\,dxd\tau dadt + \int_{Q_{T,A,\tau_2}}\int_\omega \frac{v^2}{\sigma}dxd\tau dadt\right),$$

where, in this case, $f(t,a,\tau,x) := -\beta(a,\tau,x)v(t,0,\tau,x)$. Thus

$$\int_{T-\frac{\bar a}{2}}^{T-\frac{\bar a}{4}}\int_\delta^A\int_\iota^{\tau_2}\int_0^1 \frac{1}{\sigma}v^2(t,a,\tau,x)dxd\tau dadt \le C\|\beta\|^2_{L^\infty(Q)}\left(\int_{\bar Q}\frac{v^2(t,0,\tau,x)}{\sigma}dxd\tau dadt + \int_{Q_{T,A,\tau_2}}\int_\omega \frac{v^2}{\sigma}dxd\tau dadt\right), \tag{4.47}$$

for a positive constant C. Now, using the fact that the semigroup generated by $\mathcal{B}$ is bounded, by (4.43), one has

$$\int_{\bar Q}\frac{v^2(t,0,\tau,x)}{\sigma}dxd\tau dadt \le C\int_{T-\bar a}^T\int_{\tau_1}^{\tau_2}\int_0^1 \frac{v_T^2(T-t,\tau,x)}{\sigma}dxd\tau dt \le C\int_{Q_{\bar a,\tau_2,1}}\frac{v_T^2(s,\tau,x)}{\sigma}dxd\tau ds, \tag{4.48}$$

where $Q_{\bar a,\tau_2,1} := (0,\bar a)\times(\tau_1,\tau_2)\times(0,1)$. Hence, by (4.47) and (4.48), one has

$$\int_{T-\frac{\bar a}{2}}^{T-\frac{\bar a}{4}}\int_\delta^A\int_\iota^{\tau_2}\int_0^1 \eta\tilde\Theta v_x^2 e^{2s\tilde\varphi}dxd\tau dadt \le C\|\beta\|^2_{L^\infty(Q)}\int_{Q_{\bar a,\tau_2,1}}\frac{v_T^2(s,\tau,x)}{\sigma}dxd\tau ds + C\int_{Q_{T,A,\tau_2}}\int_\omega\frac{v^2}{\sigma}dxd\tau dadt, \tag{4.49}$$

for a positive constant C. From (4.46) and (4.49), it results

$$\begin{aligned}\int_{Q_{A,\tau_2,1}}\frac{1}{\sigma}v^2(\tilde T,a,\tau,x)dxd\tau da &\le C\int_{T-\frac{\bar a}{2}}^{T-\frac{\bar a}{4}}\int_0^\delta\int_{\tau_1}^{\tau_2}\int_0^1\frac{1}{\sigma}v^2(t,a,\tau,x)dxd\tau dadt\\ &+ C\int_{T-\frac{\bar a}{2}}^{T-\frac{\bar a}{4}}\int_\delta^A\int_{\tau_1}^{\iota}\int_0^1\frac{1}{\sigma}v^2(t,a,\tau,x)dxd\tau dadt + C\|\beta\|^2_{C(\bar Q_{A,\tau_2,1})}\int_{Q_{\bar a,\tau_2,1}}\frac{v_T^2(s,\tau,x)}{\sigma}dxd\tau ds\\ &+ C\int_{Q_{T,A,\tau_2}}\int_\omega\frac{v^2}{\sigma}dxd\tau dadt + C\int_{T-\frac{\bar a}{2}}^{T-\frac{\bar a}{4}}\int_{Q_t}\frac{1}{\sigma}v^2(s,a,\tau,x)dxd\tau dadsdt.\end{aligned} \tag{4.50}$$

Hence, the theorem follows. □

Actually, we can improve the previous results in the following way:

Theorem 4.5. *Assume Hypotheses* 3.1 *or* 3.2, 3.3 *and* 4.1. *Moreover, suppose that ω satisfies* (4.29). *Then, for $n\in\mathbb{N}$ and $\forall\,\vartheta\in\left(0,\frac{1}{n}\right)$, there exists a positive constant $C=C(\vartheta)$ such that every solution v of* (3.4) *in $\mathcal{V}$ satisfies*

$$\begin{aligned}\int_{Q_{A,\tau_2,1}}\frac{1}{\sigma}v^2(T-\bar a,a,\tau,x)dxd\tau da &\le C\int_0^{A-\vartheta\bar a}\int_{\tau_1}^{\tau_2}\int_0^1\frac{v_T^2(\gamma,\tau,x)}{\sigma}dxd\tau d\gamma\\ &+ C\int_0^{\bar a}\int_{\tau_1}^{\tau_2}\int_0^1\frac{v_T^2(\gamma,\tau,x)}{\sigma}dxd\tau d\gamma + C\int_{Q_{T,A,\tau_2}}\int_\omega\frac{v^2}{\sigma}dxd\tau dadt\\ &\le C\int_0^{\Xi}\int_{\tau_1}^{\tau_2}\int_0^1\frac{v_T^2(\gamma,\tau,x)}{\sigma}dxd\tau d\gamma + C\int_{Q_{T,A,\tau_2}}\int_\omega\frac{v^2}{\sigma}dxd\tau dadt,\end{aligned}$$

where $\Xi := \max\{A-\vartheta\bar a,\bar a\}$.

Proof. Fixed $n\in\mathbb{N}$, let $\vartheta\in\left(0,\frac{1}{n}\right)$ and set $\delta := (2-n\vartheta)\bar a$; then, as in (4.46), integrating over $[T-n\vartheta\bar a, T-\vartheta\bar a]$,

$$\begin{aligned}\int_{Q_{A,\tau_2,1}}\frac{1}{\sigma}v^2(T-\bar a,a,\tau,x)dxd\tau da &\le C\int_{T-n\vartheta\bar a}^{T-\vartheta\bar a}\left(\int_0^{\delta-\bar a}+\int_{\delta-\bar a}^A\right)\int_{\tau_1}^{\tau_2}\int_0^1\frac{1}{\sigma}v^2(t,a,\tau,x)dxd\tau dadt\\ &+ C\int_{T-n\vartheta\bar a}^{T-\vartheta\bar a}\int_{Q_t}\frac{1}{\sigma}v^2(s,a,\tau,x)dxd\tau dadsdt\\ &\le 2C\int_{T-n\vartheta\bar a}^{T-\vartheta\bar a}\left(\int_0^{\delta-\bar a}+\int_{\delta-\bar a}^A\right)\int_{\tau_1}^{\tau_2}\int_0^1\frac{1}{\sigma}v^2(t,a,\tau,x)dxd\tau dadt.\end{aligned} \tag{4.51}$$

Observe that $\delta - \bar{a} = (1 - n\vartheta)\bar{a} \in (0, \bar{a})$. As in the proof of Theorem 4.4, setting $\iota := \frac{\tau_1+\tau_2}{2}$,

$$\int_{T-n\vartheta\bar{a}}^{T-\vartheta\bar{a}}\int_{\delta-\bar{a}}^{A}\int_{\tau_1}^{\tau_2}\int_0^1 \frac{1}{\sigma}v^2(t,a,\tau,x)dxd\tau dadt \le C\left(\int_{Q_{\bar{a},\tau_2,1}} \frac{v_T^2(\gamma,\tau,x)}{\sigma}dxd\tau d\gamma + \int_{Q_{T,A,\tau_2}}\int_\omega \frac{v^2}{\sigma}dxd\tau dadt\right) + C\int_{T-n\vartheta\bar{a}}^{T-\vartheta\bar{a}}\int_{\delta-\bar{a}}^{A}\int_{\tau_1}^{\iota}\int_0^1 \frac{1}{\sigma}v^2(t,a,\tau,x)dxd\tau dadt. \tag{4.52}$$

Now, consider the term $\int_{T-n\vartheta\bar{a}}^{T-\vartheta\bar{a}}\int_{\delta-\bar{a}}^{A}\int_{\tau_1}^{\iota}\int_0^1 \frac{1}{\sigma}v^2(t,a,\tau,x)dxd\tau dadt$. Using the fact that $t \ge T - n\vartheta\bar{a}$ and the definition of δ, one has $t - T + \bar{a} \ge T - n\vartheta\bar{a} - T + \bar{a} = \delta - \bar{a}$. Moreover, $t - T + \bar{a} \le \bar{a} < A$. Hence,

$$\int_{T-n\vartheta\bar{a}}^{T-\vartheta\bar{a}}\int_{\delta-\bar{a}}^{A}\int_{\tau_1}^{\iota}\int_0^1 \frac{1}{\sigma}v^2(t,a,\tau,x)dxd\tau dadt = \int_{T-n\vartheta\bar{a}}^{T-\vartheta\bar{a}}\int_{\delta-\bar{a}}^{t-T+\bar{a}}\int_{\tau_1}^{\iota}\int_0^1 \frac{1}{\sigma}v^2(t,a,\tau,x)dxd\tau dadt + \int_{T-n\vartheta\bar{a}}^{T-\vartheta\bar{a}}\int_{t-T+\bar{a}}^{A}\int_{\tau_1}^{\iota}\int_0^1 \frac{1}{\sigma}v^2(t,a,\tau,x)dxd\tau dadt.$$

In the first integral we can apply (4.40); thus, using the fact that $\vartheta < \dfrac{1}{n} \le 1$ for all $n \ge 2$, one has

$$\begin{aligned}
&\int_{T-n\vartheta\bar{a}}^{T-\vartheta\bar{a}}\int_{\delta-\bar{a}}^{t-T+\bar{a}}\int_{\tau_1}^{\iota}\int_0^1 \frac{1}{\sigma}v^2(t,a,\tau,x)dxd\tau dadt \le C\int_{T-n\vartheta\bar{a}}^{T-\vartheta\bar{a}}\int_{\delta-\bar{a}}^{t-T+\bar{a}}\int_{\tau_1}^{\iota}\int_0^1 \frac{v_T^2(T+a-t,\tau,x)}{\sigma}dxd\tau dadt\\
&\le C\int_{\vartheta\bar{a}}^{n\vartheta\bar{a}}\int_{\delta-\bar{a}}^{\bar{a}-z}\int_{\tau_1}^{\iota}\int_0^1 \frac{v_T^2(a+z,\tau,x)}{\sigma}dxd\tau dadz \le C\int_{\vartheta\bar{a}}^{n\vartheta\bar{a}}\int_0^{\bar{a}}\int_{\tau_1}^{\iota}\int_0^1 \frac{v_T^2(\gamma,\tau,x)}{\sigma}dxd\tau d\gamma dz\\
&\le C\int_0^{\bar{a}}\int_{\tau_1}^{\tau_2}\int_0^1 \frac{v_T^2(\gamma,\tau,x)}{\sigma}dxd\tau d\gamma.
\end{aligned} \tag{4.53}$$

Now, consider the other integral $\int_{T-n\vartheta\bar{a}}^{T-\vartheta\bar{a}}\int_{t-T+\bar{a}}^{A}\int_{\tau_1}^{\iota}\int_0^1 \frac{1}{\sigma}v^2(t,a,\tau,x)dxd\tau dadt$, where (4.41) holds. In this case, we have to distinguish between $\Gamma = \bar{a}$ and $\Gamma = \Gamma_{A,T}$ (we recall that Γ is defined in (4.42)). Observe that $\Gamma = \bar{a}$ if and only if $a \le A - T + t$. Moreover, $t - T + \bar{a} < A - T + t$. Hence,

$$\int_{T-n\vartheta\bar{a}}^{T-\vartheta\bar{a}}\int_{t-T+\bar{a}}^{A}\int_{\tau_1}^{\iota}\int_0^1 \frac{1}{\sigma}v^2(t,a,\tau,x)dxd\tau dadt = \int_{T-n\vartheta\bar{a}}^{T-\vartheta\bar{a}}\left(\int_{t-T+\bar{a}}^{A-T+t} + \int_{A-T+t}^{A}\right)\int_{\tau_1}^{\iota}\int_0^1 \frac{1}{\sigma}v^2(t,a,\tau,x)dxd\tau dadt. \tag{4.54}$$

If $a \in (t-T+\bar{a}, A-T+t)$, we have to apply the first formula of (4.41), on the contrary if $a \in (A-T+t, A)$ the second formula of (4.41) can be applied. Thus,

$$\begin{aligned}
&\int_{T-n\vartheta\bar{a}}^{T-\vartheta\bar{a}}\int_{t-T+\bar{a}}^{A-T+t}\int_{\tau_1}^{\iota}\int_0^1 \frac{1}{\sigma}v^2(t,a,\tau,x)dxd\tau dadt \le C\int_{T-n\vartheta\bar{a}}^{T-\vartheta\bar{a}}\int_{t-T+\bar{a}}^{A-T+t}\int_{\tau_1}^{\iota}\int_0^1 \frac{v_T^2(T+a-t,\tau,x)}{\sigma}dxd\tau dadt\\
&+ C\int_{T-n\vartheta\bar{a}}^{T-\vartheta\bar{a}}\int_{t-T+\bar{a}}^{A-T+t}\int_{\tau_1}^{\iota}\int_0^1 \left(\int_a^{T+a-t} \frac{v_T^2(T-s-t+a,\tau,x)}{\sigma}ds\right)dxd\tau dadt\\
&\le C\int_{\vartheta\bar{a}}^{n\vartheta\bar{a}}\int_{\bar{a}-z}^{A-z}\int_{\tau_1}^{\iota}\int_0^1 \frac{v_T^2(a+z,\tau,x)}{\sigma}dxd\tau dadz\\
&+ C\int_{T-n\vartheta\bar{a}}^{T-\vartheta\bar{a}}\int_{t-T+\bar{a}}^{A-T+t}\int_{\tau_1}^{\iota}\int_0^1 \left(\int_{-a}^{T-t-a} \frac{v_T^2(a+z,\tau,x)}{\sigma}dz\right)dxd\tau dadt\\
&\text{(proceeding as in (4.53) for the first integral)}\\
&\le C\int_0^{A-\vartheta\bar{a}}\int_{\tau_1}^{\tau_2}\int_0^1 \frac{v_T^2(\gamma,\tau,x)}{\sigma}dxd\tau d\gamma + C\int_{T-n\vartheta\bar{a}}^{T-\vartheta\bar{a}}\int_{t-T+\bar{a}}^{A-T+t}\int_{\tau_1}^{\tau_2}\int_0^1 \left(\int_0^{T-t} \frac{v_T^2(\gamma,\tau,x)}{\sigma}d\gamma\right)dxd\tau dadt\\
&\text{(since } T - t < n\vartheta\bar{a} < \bar{a}\text{)}\\
&\le C\int_0^{A-\vartheta\bar{a}}\int_{\tau_1}^{\tau_2}\int_0^1 \frac{v_T^2(\gamma,\tau,x)}{\sigma}dxd\tau d\gamma + C\int_{\tau_1}^{\tau_2}\int_0^1 \left(\int_0^{\bar{a}} \frac{v_T^2(\gamma,\tau,x)}{\sigma}d\gamma\right)dxd\tau.
\end{aligned} \tag{4.55}$$

Moreover,

$$
\begin{aligned}
&\int_{T-n\vartheta\bar{a}}^{T-\vartheta\bar{a}} \int_{A-T+t}^{A} \int_{\tau_1}^{\iota} \int_0^1 \frac{1}{\sigma} v^2(t,a,\tau,x)dxd\tau dadt \\
&\leq C \int_{T-n\vartheta\bar{a}}^{T-\vartheta\bar{a}} \int_{A-T+t}^{A} \int_{\tau_1}^{\iota} \int_0^1 \left(\int_a^A \frac{v_T^2(T-s-t+a,\tau,x)}{\sigma} ds \right) dxd\tau dadt \\
&\leq C \int_{T-n\vartheta\bar{a}}^{T-\vartheta\bar{a}} \int_{A-T+t}^{A} \int_{\tau_1}^{\iota} \int_0^1 \left(\int_{T-A-t}^{T-t-a} \frac{v_T^2(a+z,\tau,x)}{\sigma} dz \right) dxd\tau dadt \\
&\leq C \int_{T-n\vartheta\bar{a}}^{T-\vartheta\bar{a}} \int_{A-T+t}^{A} \int_{\tau_1}^{\tau_2} \int_0^1 \left(\int_{T-t-(A-a)}^{T-t} \frac{v_T^2(\gamma,\tau,x)}{\sigma} d\gamma \right) dxd\tau dadt \\
&\text{(since } T-t < n\vartheta\bar{a}) \\
&\leq C \int_{\tau_1}^{\tau_2} \int_0^1 \left(\int_0^{\bar{a}} \frac{v_T^2(\gamma,\tau,x)}{\sigma} d\gamma \right) dxd\tau.
\end{aligned}
\tag{4.56}
$$

Hence,

$$
\begin{aligned}
&\int_{T-n\vartheta\bar{a}}^{T-\vartheta\bar{a}} \int_{\delta-\bar{a}}^{A} \int_{\tau_1}^{\iota} \int_0^1 \frac{1}{\sigma} v^2(t,a,\tau,x)dxd\tau dadt \leq C \int_0^{A-\vartheta\bar{a}} \int_{\tau_1}^{\tau_2} \int_0^1 \frac{v_T^2(\gamma,\tau,x)}{\sigma} dxd\tau d\gamma \\
&+ C \int_0^{\bar{a}} \int_{\tau_1}^{\tau_2} \int_0^1 \frac{v_T^2(\gamma,\tau,x)}{\sigma} dxd\tau d\gamma.
\end{aligned}
\tag{4.57}
$$

Finally, consider the term $\int_{T-n\vartheta\bar{a}}^{T-\vartheta\bar{a}} \int_0^{\delta-\bar{a}} \int_{\tau_1}^{\tau_2} \int_0^1 \frac{1}{\sigma} v^2(t,a,\tau,x)dxd\tau dadt$ and let us prove that there exists $C>0$ such that

$$
\int_{T-n\vartheta\bar{a}}^{T-\vartheta\bar{a}} \int_0^{\delta-\bar{a}} \int_{\tau_1}^{\tau_2} \int_0^1 \frac{1}{\sigma} v^2(t,a,\tau,x)dxd\tau dadt \leq C \int_0^{\bar{a}} \int_{\tau_1}^{\tau_2} \int_0^1 \frac{v_T^2(\gamma,\tau,x)}{\sigma} dxd\tau d\gamma. \tag{4.58}
$$

Observe that $t \geq T-n\vartheta\bar{a} \geq T-\bar{a}$ and $a \in (0,\delta-\bar{a})$. Thus $T-t \leq \bar{a} \leq \delta - a \leq A-a$. Hence, $\Gamma = \bar{a}$ (recall that Γ is defined in (4.42)). Hence in (4.41) we have to consider the first formula, i.e.

$$
v(t,a,\cdot) = S(T-t)v_T(T+a-t,\cdot) + \int_a^{T+a-t} S(s-a)\beta(s,\cdot,\cdot)v(s+t-a,0,\cdot)ds.
$$

Thus,

$$
\begin{aligned}
&\int_{T-n\vartheta\bar{a}}^{T-\vartheta\bar{a}} \int_0^{\delta-\bar{a}} \int_{\tau_1}^{\tau_2} \int_0^1 \frac{1}{\sigma} v^2(t,a,\tau,x)dxd\tau dadt \\
&\leq C \int_{T-n\vartheta\bar{a}}^{T-\vartheta\bar{a}} \int_0^{\delta-\bar{a}} \int_{\tau_1}^{\tau_2} \int_0^1 \frac{1}{\sigma} v_T^2(T+a-t,\tau,x)dxd\tau dadt \\
&+ C \int_{T-n\vartheta\bar{a}}^{T-\vartheta\bar{a}} \int_0^{\delta-\bar{a}} \int_{\tau_1}^{\tau_2} \int_0^1 \frac{1}{\sigma} \left(\int_a^{T+a-t} v^2(s+t-a,0,\tau,x)ds \right) dxd\tau dadt \\
&= C \int_{\vartheta\bar{a}}^{n\vartheta\bar{a}} \int_0^{\delta-\bar{a}} \int_{\tau_1}^{\tau_2} \int_0^1 \frac{1}{\sigma} v_T^2(a+z,\tau,x)dxd\tau dadz \\
&+ C \int_{T-n\vartheta\bar{a}}^{T-\vartheta\bar{a}} \int_0^{\delta-\bar{a}} \int_{\tau_1}^{\tau_2} \int_0^1 \frac{1}{\sigma} \left(\int_{-a}^{T-a-t} v_T^2(a+z,\tau,x)dz \right) dxd\tau dadt \\
&\leq C \int_0^{(1-n\vartheta)\bar{a}} \int_{\tau_1}^{\tau_2} \int_0^1 \frac{1}{\sigma} v_T^2(\gamma,\tau,x)dxd\tau d\gamma dz \\
&+ C \int_{T-n\vartheta\bar{a}}^{T-\vartheta\bar{a}} \int_0^{\delta-\bar{a}} \int_{\tau_1}^{\tau_2} \int_0^1 \frac{1}{\sigma} \left(\int_0^{T-t} v_T^2(\gamma,\tau,x)d\gamma \right) dxd\tau dadt \\
&\leq C \int_0^{\bar{a}} \int_{\tau_1}^{\tau_2} \int_0^1 \frac{1}{\sigma} v_T^2(\gamma,\tau,x)dxd\tau d\gamma dz
\end{aligned}
\tag{4.59}
$$

$$+ C\int_{T-n\vartheta\bar a}^{T-\vartheta\bar a}\int_0^{\delta-\bar a}\int_{\tau_1}^{\tau_2}\int_0^1 \frac{1}{\sigma}\left(\int_0^{\bar a} v_T^2(\gamma,\tau,x)d\gamma\right)dxd\tau dadt$$
$$\le C\int_0^{\bar a}\int_{\tau_1}^{\tau_2}\int_0^1 \frac{1}{\sigma}v_T^2(\gamma,\tau,x)d\gamma dxd\tau.$$

Hence, (4.58) holds.
By (4.51), (4.52), (4.57) and (4.58), it follows that

$$\int_{Q_{A,\tau_2,1}} \frac{1}{\sigma}v^2(T-\bar a,a,\tau,x)dxd\tau da \le C\int_0^{A-\vartheta\bar a}\int_{\tau_1}^{\tau_2}\int_0^1 \frac{v_T^2(\gamma,\tau,x)}{\sigma}dxd\tau d\gamma + C\int_0^{\bar a}\int_{\tau_1}^{\tau_2}\int_0^1 \frac{v_T^2(\gamma,\tau,x)}{\sigma}dxd\tau d\gamma$$
$$+ C\int_{Q_{T,A,\tau_2}}\int_\omega \frac{v^2}{\sigma}dxd\tau dadt.$$

□

By Theorem 4.5 and using a density argument, one can deduce Proposition 4.1. As a consequence one can prove, as in [16], the following null controllability results:

Theorem 4.6. *Assume Hypotheses* 3.1 *or* 3.2*,* 3.3 *and* 4.1*. Moreover, suppose that* ω *satisfies* (4.29)*. Take* $y_0 \in L^2_{\frac{1}{\sigma}}(Q_{A,\tau_2,1})$ *and* $T>0$*. Then for* $n\in\mathbb{N}$ *and for all* $\vartheta \in \left(0,\frac{1}{n}\right)$ *there exists a control* $f_\vartheta \in L^2_{\frac{1}{\sigma}}(\tilde Q)$ *such that the solution* $y = y_\vartheta \in \mathcal{U}$ *of*

$$\begin{cases} \dfrac{\partial y}{\partial t} + \dfrac{\partial y}{\partial a} + \dfrac{\partial}{\partial \tau}(g(\tau)y) - k(x)y_{xx} - b(x)y_x + \mu(t,a,\tau,x)y = f_\epsilon(t,a,\tau,x) & in\ \tilde Q,\\ y(t,a,\tau,1) = y(t,a,\tau,0) = 0 & on\ \tilde Q_{T,A,\tau_2},\\ y(\tilde T,a,\tau,x) = y_0(a,\tau,x) & in\ Q_{A,\tau_2,1},\\ y(t,a,\tau_1,x) = 0 & in\ \tilde Q_{T,A,1},\\ y(t,0,x,\tau) = \int_0^A \beta(a,\tau,x)y(t,a,\tau,x)da & in\ \tilde Q_{T,\tau_2,1}, \end{cases} \tag{4.60}$$

satisfies

$$y(T,a,\tau,x) = 0 \quad a.e.\ (a,\tau,x) \in (\Xi,A)\times(\tau_1,\tau_2)\times(0,1),$$

where $\Xi := \max\{A-\vartheta\bar a,\bar a\}$*. Moreover, there exists* $C = C(\vartheta) > 0$ *such that*

$$\|f_\vartheta\|_{L^2_{\frac{1}{\sigma}}(\tilde Q)} \le C\|y_0\|_{L^2_{\frac{1}{\sigma}}(Q_{A,\tau_2,1})}. \tag{4.61}$$

Here, we recall, $\tilde Q = (\tilde T,T)\times(0,A)\times(\tau_1,\tau_2)\times(0,1)$*,* $\tilde Q_{T,A,\tau_2} = (\tilde T,T)\times(0,A)\times(\tau_1,\tau_2)$*,* $\tilde Q_{T,A,1} = (\tilde T,T)\times(0,A)\times(0,1)$ *and* $\tilde Q_{T,\tau_2,1} = (\tilde T,T)\times(\tau_1,\tau_2)\times(0,1)$*.*

Proof. Take $h \in L^2_{\frac{1}{\sigma}}(Q_{A,\tau_2,1})$ such that $h(A,\tau,x) = 0$ in $(\tau_1,\tau_2)\times(0,1)$. Let v be the solution of

$$\begin{cases} \dfrac{\partial v}{\partial t} + \dfrac{\partial v}{\partial a} + g(\tau)\dfrac{\partial v}{\partial \tau} + k(x)v_{xx} + b(x)v_x - \mu(t,a,\tau,x)v = -\beta(a,\tau,x)v(t,0,\tau,x), & (t,a,x)\in\tilde Q,\\ v(t,a,\tau,0) = v(t,a,\tau,1) = 0, & (t,a,\tau)\in\tilde Q_{T,A,\tau_2},\\ v(T,a,\tau,x) = v_T(a,\tau,x) := \begin{cases} h(a,\tau,x), & (a,\tau,x)\in(\Xi,A)\times(\tau_1,\tau_2)\times(0,1),\\ 0, & (a,\tau,x)\in(0,\Xi)\times(\tau_1,\tau_2)\times(0,1),\end{cases} \\ v(t,A,\tau,x) = 0, & (t,\tau,x)\in\tilde Q_{T,\tau_2,1},\\ v(t,a,\tau_2,x) = 0, & (t,a,x)\in\tilde Q_{T,A,1}. \end{cases} \tag{4.62}$$

Now, fixed $y_0 \in L^2_{\frac{1}{\sigma}}(Q_{A,\tau_2,1})$, define

$$J(h) = \frac{1}{2}\int_{\tilde Q_{T,A,\tau_2}}\int_\omega \frac{v^2}{\sigma}dxd\tau dadt + \int_{Q_{A,\tau_2,1}}\frac{1}{\sigma}v(\tilde T,a,\tau,x)y_0(a,\tau,x)dxd\tau da.$$

The functional J is strictly convex, continuous and coercive over the Hilbert space $\mathcal{H}$ defined by the completion of $L^2_{\frac{1}{\sigma}}((\Xi,A)\times(\tau_1,\tau_2)\times(0,1))$ with respect to the norm $\|v\|_{L^2_{\frac{1}{\sigma}}(\tilde Q_{T,A,\tau_2}\times\omega)}$ (see [1] if v is independent of τ). Thus, there exists a unique minimum, $\hat{h}$, of J and $\hat{h}(A,\tau,x)=0$ in $(\tau_1,\tau_2)\times(0,1)$. Let $\hat{v}=\hat{v}_\vartheta$ be the solution of (4.62) associated to $\hat{h}=\hat{h}_\vartheta$. Define $f=f_\vartheta:=\hat{v}\chi_\omega$ and let $y=y_\vartheta$ be the solution of (4.60) in $\tilde{Q}$ associated to f. Since $\hat{h}$ is the minimum of J, it results that

$$0=\left[\frac{d}{d\epsilon}J(\hat{h}+\epsilon h)\right]_{\epsilon=0}=\int_{\tilde{Q}_{T,A,\tau_2}}\int_\omega \frac{1}{\sigma}v\hat{v}dxd\tau dadt+\int_{Q_{A,\tau_2,1}}\frac{1}{\sigma}v(\tilde{T},a,\tau,x)y_0(a,\tau,x)dxd\tau da, \tag{4.63}$$

for all $h\in L^2_{\frac{1}{\sigma}}(Q_{A,\tau_2,1})$ such that $h(A,\tau,x)=0$ in $(\tau_1,\tau_2)\times(0,1)$. In particular, for $h=\hat{h}$, one has

$$0=\int_{\tilde{Q}_{T,A,\tau_2}}\int_\omega \frac{1}{\sigma}\hat{v}^2dxd\tau dadt+\int_{Q_{A,\tau_2,1}}\frac{1}{\sigma}\hat{v}(\tilde{T},a,\tau,x)y_0(a,\tau,x)dxd\tau da.$$

Hence

$$\int_{\tilde{Q}_{T,A,\tau_2}}\int_\omega \frac{1}{\sigma}\hat{v}^2dxd\tau dadt=-\int_{Q_{A,\tau_2,1}}\frac{1}{\sigma}\hat{v}(\tilde{T},a,\tau,x)y_0(a,\tau,x)dxd\tau da, \tag{4.64}$$

and, by Hölder's inequality and Proposition 4.1 applied to $\hat{v}$ in $\tilde{Q}$, one has

$$\begin{aligned}\left|\int_{Q_{A,\tau_2,1}}\frac{1}{\sigma}\hat{v}(\tilde{T},a,\tau,x)y_0(a,\tau,x)dxd\tau da\right| &\le \left(\int_{Q_{A,\tau_2,1}}\frac{1}{\sigma}\hat{v}^2(\tilde{T},a,\tau,x)dxd\tau da\right)^{\frac12}\left(\int_{Q_{A,\tau_2,1}}\frac{1}{\sigma}y_0^2(a,\tau,x)dxd\tau da\right)^{\frac12}\\ &\le C\left(\int_{\tilde{Q}_{T,A,\tau_2}}\int_\omega\frac{\hat{v}^2}{\sigma}dxdadt\right)^{\frac12}\|y_0\|_{L^2_{\frac{1}{\sigma}}(Q_{A,\tau_2,1})}.\end{aligned} \tag{4.65}$$

Thus, by (4.64) and (4.65),

$$\int_{\tilde{Q}_{T,A,\tau_2}}\int_\omega\frac{\hat{v}^2}{\sigma}dxdadt\le C\left(\int_{\tilde{Q}_{T,A,\tau_2}}\int_\omega\frac{\hat{v}^2}{\sigma}dxdadt\right)^{\frac12}\|y_0\|_{L^2_{\frac{1}{\sigma}}(Q_{A,\tau_2,1})}. \tag{4.66}$$

Hence

$$\|f\|_{L^2_{\frac{1}{\sigma}}(\tilde{Q})}=\left(\int_{\tilde{Q}_{T,A,\tau_2}}\int_\omega\frac{\hat{v}^2}{\sigma}dxdadt\right)^{\frac12}\le C\|y_0\|_{L^2_{\frac{1}{k}}(Q_{A,1})}.$$

Now, let y be the solution of (4.60) asssociated to f and y_0.
Multiplying the equation of (4.62) by $\dfrac{y}{\sigma}$ and integrating over $\tilde{Q}$, one has:

$$\begin{aligned}0&=\int_{\tilde{Q}}\left(\frac{\partial v}{\partial t}+\frac{\partial v}{\partial a}+g(\tau)\frac{\partial v}{\partial \tau}+k(x)v_{xx}+b(x)v_x-\mu(t,a,\tau,x)v+\beta(a,\tau,x)v(t,0,\tau,x)\right)\frac{y}{\sigma}dxd\tau dadt \iff\\ 0&=\int_\Xi^A\int_{\tau_1}^{\tau_2}\int_0^1\frac{1}{\sigma}y(T,a,\tau,x)h(a,\tau,x)\ dxd\tau da-\int_{Q_{A,\tau_2,1}}\frac{1}{\sigma}y_0v(\tilde{T},a,\tau,x)dxda\\ &-\int_{\tilde{Q}_{T,\tau_2,1}}\frac{1}{\sigma}y(t,0,\tau,x)v(t,0,\tau,x)dxd\tau dt+\int_{\tilde{Q}}\frac{1}{\sigma}\beta(a,\tau,x)v(t,0,\tau,x)y(t,a,\tau,x)dxd\tau dadt\\ &-\int_{\tilde{Q}}\frac{v}{\sigma}\left(\frac{\partial y}{\partial t}+\frac{\partial y}{\partial a}+\frac{\partial (gy)}{\partial \tau}-\mathcal{A}_0y+\mu(t,a,\tau,x)y\right)dxd\tau dadt\end{aligned}$$

(recall that $y(t,0,\tau,x)=\int_0^A\beta(a,\tau,x)y(t,a,\tau,x)da$). But $\dfrac{\partial y}{\partial t}+\dfrac{\partial y}{\partial a}+\dfrac{\partial (gy)}{\partial \tau}-\mathcal{A}_0y+\mu(t,a,\tau,x)y=f\chi_\omega$; hence

$$0=\int_\Xi^A\int_{\tau_1}^{\tau_2}\int_0^1\frac{1}{\sigma}y(T,a,\tau,x)h(a,\tau,x)\ dxd\tau da-\int_{Q_{A,\tau_2,1}}\frac{1}{\sigma}y_0v(\tilde{T},a,\tau,x)dxd\tau da-\int_{\tilde{Q}_{T,A,\tau_2}}\int_\omega\frac{v\hat{v}}{\sigma}dxd\tau dadt.$$

Thus, being by (4.63)

$$\int_{\tilde{Q}_{T,A,\tau_2}}\int_\omega\frac{1}{\sigma}v\hat{v}dxd\tau dadt=-\int_{Q_{A,\tau_2,1}}\frac{1}{\sigma}v(\tilde{T},a,\tau,x)y_0(a,\tau,x)dxd\tau da,$$

it follows that

$$0=\int_{\Xi}^{A}\int_{\tau_1}^{\tau_2}\int_0^1 \frac{1}{\sigma}y(T,a,\tau,x)h(a,\tau,x)\;dxd\tau da$$

for all $h\in L^2_{\frac{1}{\sigma}}(Q_{A,\tau_2,1})$ with $h(A,\tau,x)=0$ in $(\tau_1,\tau_2)\times(0,1)$. Hence $y(T,a,\tau,x)=0$ a.e. $(a,x)\in(\Xi,A)\times(\tau_1,\tau_2)\times(0,1)$. □

Theorem 4.7. *Assume Hypothesis* 3.1 *or* 3.2, 3.3 *and* 4.1. *Moreover, suppose that ω satisfies* (4.29). *Take $u_0\in L^2(Q_{A,\tau_2,1})$ and $T>0$. Then for $n\in\mathbb{N}$ and for all $\vartheta\in\left(0,\frac{1}{n}\right)$ there exists a control $f_\vartheta\in L^2_{\frac{1}{\sigma}}(Q)$ such that the solution $u=u_\vartheta\in\mathcal{U}$ of* (1.1) *satisfies*

$$u(T,a,\tau,x)=0\quad a.e.\ (a,\tau,x)\in(\Xi,A)\times(\tau_1,\tau_2)\times(0,1).$$

Moreover, there exists $C=C(\vartheta)>0$ such that

$$\|f_\vartheta\|_{L^2_{\frac{1}{\sigma}}(Q)}\le C\|u_0\|_{L^2_{\frac{1}{\sigma}}(Q_{A,\tau_2,1})},\tag{4.67}$$

where Ξ is as in the previous theorem.

Proof. As a first step, set $\tilde T:=T-\bar a\in(0,T)$. By Theorem 2.1, there exists a unique solution y of

$$\begin{cases}\dfrac{\partial y}{\partial t}+\dfrac{\partial y}{\partial a}+\dfrac{\partial}{\partial\tau}(g(\tau)y)-k(x)y_{xx}-b(x)y_x+\mu(t,a,\tau,x)y=0 & \text{in } \bar Q,\\ y(t,a,\tau,1)=y(t,a,\tau,0)=0 & \text{on } \bar Q_{T,A,\tau_2}\\ y(0,a,\tau,x)=u_0(a,\tau,x) & \text{in } Q_{A,\tau_2,1},\\ y(t,a,\tau_1,x)=0 & \text{in } \bar Q_{T,A,1},\\ y(t,0,x,\tau)=\int_0^A\beta(a,\tau,x)y(t,a,\tau,x)da & \text{in } \bar Q_{T,\tau_2,1},\end{cases}\tag{4.68}$$

where $\bar Q=(0,\tilde T)\times(0,A)\times(\tau_1,\tau_2)\times(0,1)$, $\bar Q_{T,A,\tau_2}=(0,\tilde T)\times(0,A)\times(\tau_1,\tau_2)$, $\bar Q_{T,A,1}=(0,\tilde T)\times(0,A)\times(0,1)$ and $\bar Q_{T,\tau_2,1}=(0,\tilde T)\times(\tau_1,\tau_2)\times(0,1)$. Set $\tilde y_0(a,\tau,x):=y(\tilde T,a,\tau,x)$; clearly $\tilde y_0\in L^2_{\frac{1}{\sigma}}(Q_{A,\tau_2,1})$. Now, consider

$$\begin{cases}\dfrac{\partial w}{\partial t}+\dfrac{\partial w}{\partial a}+\dfrac{\partial}{\partial\tau}(g(\tau)w)-k(x)w_{xx}-b(x)w_x+\mu(t,a,\tau,x)w=h(t,a,\tau,x)\chi_\omega & \text{in } \tilde Q,\\ w(t,a,\tau,1)=w(t,a,\tau,0)=0 & \text{on } \tilde Q_{T,A,\tau_2}\\ w(\tilde T,a,\tau,x)=\tilde y_0(a,\tau,x) & \text{in } Q_{A,\tau_2,1},\\ w(t,a,\tau_1,x)=0 & \text{in } \tilde Q_{T,A,1},\\ w(t,0,x,\tau)=\int_0^A\beta(a,\tau,x)w(t,a,\tau,x)da & \text{in } \tilde Q_{T,\tau_2,1}.\end{cases}\tag{4.69}$$

By the previous Theorem, there exists a control $h_\vartheta\in L^2_{\frac{1}{\sigma}}(\tilde Q)$ such that the solution $w_\vartheta\in\mathcal{U}$ of (4.69) associated to h_ϑ satisfies

$$w_\vartheta(T,a,\tau,x)=0 \text{ a.e. } (a,\tau,x)\in(\Xi,A)\times(\tau_1,\tau_2)\times(0,1)$$

and

$$\|h_\vartheta\|_{L^2_{\frac{1}{\sigma}}(\tilde Q)}\le C\|\tilde y_0\|_{L^2_{\frac{1}{\sigma}}(Q_{A,\tau_2,1})},$$

for a positive constant $C=C(\vartheta)$.

Now, define u_ϑ and f_ϑ by

$$u_\vartheta:=\begin{cases}y, & \text{in } [0,\tilde T],\\ w_\vartheta, & \text{in } [\tilde T,T]\end{cases}\quad\text{and}\quad f_\vartheta:=\begin{cases}0, & \text{in } [0,\tilde T],\\ h_\vartheta, & \text{in } [\tilde T,T].\end{cases}$$

Then u_ϑ satisfies (1.1) and $f_\vartheta\in L^2_{\frac{1}{\sigma}}(Q)$ is such that

$$u_\vartheta(T,a,\tau,x)=0 \text{ a.e. } (a,\tau,x)\in(\Xi,A)\times(\tau_1,\tau_2)\times(0,1).$$

Indeed $u_\vartheta(T,a,\tau,x)=w_\vartheta(T,a,\tau,x)=0$ a.e. $(a,\tau,x)\in(\Xi,A)\times(\tau_1,\tau_2)\times(0,1)$.

Now, we prove (4.67). As a first step, we multiply the equation of (4.68) by y. Then, integrating over $Q_{A,\tau_2,1}$, we obtain:

$$\frac{1}{2}\frac{d}{dt}\int_{Q_{A,\tau_2,1}}\frac{1}{\sigma}y^2dxd\tau da+\frac{1}{2}\int_{\tau_1}^{\tau_2}\int_0^1\frac{1}{\sigma}y^2(t,A,\tau,x)dxd\tau+\int_{Q_{A,\tau_2,1}}\eta y_x^2dxd\tau da+\frac{1}{2}\int_{Q_{A,\tau_2,1}}\frac{1}{\sigma}\frac{d(gy)}{d\tau}ydxd\tau da$$
$$+\int_{Q_{A,\tau_2,1}}\frac{1}{\sigma}\mu y^2dxd\tau da=\frac{1}{2}\int_{\tau_1}^{\tau_2}\int_0^1\frac{1}{\sigma}y^2(t,0,\tau,x)dxd\tau.$$

Clearly,

$$\begin{aligned}\frac{1}{2}\int_{Q_{A,\tau_2,1}}\frac{1}{\sigma}\frac{d(gy)}{d\tau}ydxd\tau da&=\frac{1}{2}\int_0^A\int_0^1\frac{1}{\sigma}(gy^2)(t,a,\tau_2,x)dxd\tau-\frac{1}{4}\int_{Q_{A,\tau_2,1}}\frac{1}{\sigma}g\frac{dy^2}{d\tau}dxd\tau da\\&=\frac{1}{2}\int_0^A\int_0^1\frac{1}{\sigma}(gy^2)(t,a,\tau_2,x)dxd\tau-\frac{1}{4}\int_0^A\int_0^1\frac{1}{\sigma}(gy^2)(t,a,\tau_2,x)dxd\tau\\&+\frac{1}{4}\int_{Q_{A,\tau_2,1}}\frac{1}{\sigma}\frac{dg}{d\tau}y^2dxd\tau da\\&=\frac{1}{4}\int_0^A\int_0^1\frac{1}{\sigma}(gy^2)(t,a,\tau_2,x)dxd\tau+\frac{1}{4}\int_{Q_{A,\tau_2,1}}\frac{1}{\sigma}\frac{dg}{d\tau}y^2dxd\tau da.\end{aligned}$$

It follows that

$$\frac{1}{2}\frac{d}{dt}\int_{Q_{A,\tau_2,1}}\frac{1}{\sigma}y^2dxd\tau da\le\frac{1}{2}\int_{\tau_1}^{\tau_2}\int_0^1\frac{1}{\sigma}y^2(t,0,\tau,x)dxd\tau-\frac{1}{4}\int_{Q_{A,\tau_2,1}}\frac{1}{\sigma}\frac{dg}{d\tau}y^2dxd\tau da.$$

Using the fact that $y(t,0,\tau,x)=\int_0^A\beta(a,\tau,x)y(t,a,\tau,x)da$, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{Q_{A,\tau_2,1}}\frac{1}{\sigma}y^2dxd\tau da&\le\frac{1}{2}\int_{\tau_1}^{\tau_2}\int_0^1\frac{1}{\sigma}\left(\int_0^A\beta(a,\tau,x)y(t,a,x)da\right)^2dxd\tau\\&+\frac{1}{4}\|g_\tau\|_{C[\tau_1,\tau_2]}\int_{Q_{A,\tau_2,1}}\frac{1}{\sigma}y^2dxd\tau da\\&\le\frac{C}{2}\int_{Q_{A,\tau_2,1}}\frac{1}{\sigma}y^2dxd\tau da.\end{aligned}$$

Setting $F(t):=\|y(t)\|^2_{L^2_{\frac{1}{\sigma}}(Q_{A,\tau_2,1})}$ and multiplying the previous inequality by e^{-Ct}, it results that

$$\frac{d}{dt}\left(e^{-Ct}F(t)\right)\le 0.$$

Integrating over $(0,t)$, for all $t\in[0,T]$, we obtain

$$\int_{Q_{A,\tau_2,1}}\frac{1}{\sigma}y^2(t,a,\tau,x)dxd\tau da\le e^{CT}\int_{Q_{A,\tau_2,1}}\frac{1}{\sigma}y^2(0,a,\tau,x)dxd\tau da=e^{CT}\int_{Q_{A,\tau_2,1}}\frac{1}{\sigma}u_0^2(a,\tau,x)dxd\tau da.$$

In particular,

$$\int_{Q_{A,\tau_2,1}}\frac{1}{\sigma}y^2(\tilde{T},a,\tau,x)dxd\tau da\le e^{CT}\int_{Q_{A,\tau_2,1}}\frac{1}{\sigma}u_0^2(a,\tau,x)dxd\tau da.$$

Thus,

$$\begin{aligned}\|f_\vartheta\|^2_{L^2_{\frac{1}{\sigma}}(Q)}&=\int_{\tilde{Q}}\frac{1}{\sigma}h_\vartheta^2dxd\tau dadt\le C\|\tilde{y}_0\|^2_{L^2_{\frac{1}{\sigma}}(Q_{A,\tau_2,1})}\\&=C\int_{Q_{A,\tau_2,1}}\frac{1}{\sigma}u^2(\tilde{T},a,\tau,x)dxd\tau da\le e^{CT}\int_{Q_{A,\tau_2,1}}\frac{1}{\sigma}u_0^2(a,\tau,x)dxd\tau da,\end{aligned}\tag{4.70}$$

for a positive constant C. Hence, (4.67) follows. □

As a consequence of the previous theorem, we obtain the null controllability property if the coefficient k is degenerate at 0 and at 1 at the same time. In particular, we make the following assumption:

Hypothesis 4.2. Hypotheses 2.2, 3.3 and 4.1 are satisfied. Moreover, the function $k \in C^0[0,1]\bigcap C^2(0,1)$ is such that $k(0) = 0 = k(1)$, $k > 0$ on $(0,1)$, the functions $\dfrac{x(b-k_x(x))}{k(x)}$ and $\dfrac{(x-1)(b-k_x(x))}{k(x)}$ belong to $L^\infty(0,1)$ and there exist $M_0, M_1 \in (0,2)$ so that $\dfrac{xk_x(x)}{k(x)} \leq M_0$ a.e. in $[0,1]$ and $\dfrac{(x-1)k_x(x)}{k(x)} \leq M_1$ a.e. in $[0,1]$. Finally, there exist $\varepsilon \in (0,1]$ and two functions $C_i = C_i(\varepsilon) > 0$, $i = 1,2$, defined in $(0,\varepsilon)$, such that

$$C_i(\varepsilon) \to 0 \quad as \quad \varepsilon \to 0^+,$$

and

$$\left|\left(\frac{x(b(x)-k_x(x))}{k(x)}\right)_{xx} - \frac{b(x)}{k(x)}\left(\frac{x(b(x)-k_x(x))}{k(x)}\right)_x\right| \leq C_1(\varepsilon)\frac{1}{x^2},$$

$\forall\, x \in (0,\varepsilon)$ and

$$\left|\left(\frac{(x-1)(b(x)-k_x(x))}{k(x)}\right)_{xx} - \frac{b(x)}{k(x)}\left(\frac{(x-1)(b(x)-k_x(x))}{k(x)}\right)_x\right| \leq C_2(\varepsilon)\frac{1}{(x-1)^2},$$

$\forall\, x \in (1-\varepsilon,1)$.

Theorem 4.8. *Assume Hypotheses 4.2 and (4.29). Take $u_0 \in L^2(Q_{A,\tau_2,1})$ and $T > 0$. Then for $n \in \mathbb{N}$ and for all $\vartheta \in \left(0,\frac{1}{n}\right)$ there exists a control $f_\vartheta \in L^2_{\frac{1}{\sigma}}(Q)$ such that the solution $u = u_\vartheta \in \mathcal{U}$ of (1.1) satisfies*

$$u(T,a,\tau,x) = 0 \quad a.e.\ (a,\tau,x) \in (\Xi, A) \times (\tau_1,\tau_2) \times (0,1).$$

Moreover, there exists $C = C(\vartheta) > 0$ such that

$$\|f_\vartheta\|_{L^2_{\frac{1}{\sigma}}(Q)} \leq C\|u_0\|_{L^2_{\frac{1}{\sigma}}(Q_{A,\tau_2,1})}. \tag{4.71}$$

Here Ξ is as before.

Proof. Fixed $u_0 \in L^2(Q_{A,\tau_2,1})$, consider the two problems

$$(P_1)\quad \begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial a} + \dfrac{\partial}{\partial \tau}(g(\tau)u) - k(x)u_{xx} - b(x)u_x + \mu(t,a,\tau,x)u = f(t,a,\tau,x)\chi_\omega & \text{in } Q_{T,A,\tau_2} \times (0,\bar\beta),\\ u(t,a,\tau,\bar\beta) = u(t,a,\tau,0) = 0 & \text{on } Q_{T,A,\tau_2},\\ u(0,a,\tau,x) = u_0(a,\tau,x) & \text{in } (0,A)\times(\tau_1,\tau_2)\times(0,\bar\beta),\\ u(t,a,\tau_1,x) = 0 & \text{in } (0,T)\times(0,A)\times(0,\bar\beta),\\ u(t,0,\tau,x) = \int_0^A \beta(a,x)u(t,a,\tau,x)da & \text{in } (0,T)\times(\tau_1,\tau_2)\times(0,\bar\beta), \end{cases} \tag{4.72}$$

and

$$(P_2)\quad \begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial a} + \dfrac{\partial}{\partial \tau}(g(\tau)u) - k(x)u_{xx} - b(x)u_x + \mu(t,a,\tau,x)u = f(t,a,\tau,x)\chi_\omega & \text{in } Q_{T,A,\tau_2} \times (\bar\alpha,1),\\ u(t,a,\tau,\bar\alpha) = u(t,a,\tau,1) = 0 & \text{on } Q_{T,A,\tau_2},\\ u(0,a,\tau,x) = u_0(a,\tau,x) & \text{in } (0,A)\times(\tau_1,\tau_2)\times(\bar\alpha,1),\\ u(t,a,\tau_1,x) = 0 & \text{in } (0,T)\times(0,A)\times(\bar\alpha,1),\\ u(t,0,\tau,x) = \int_0^A \beta(a,x)u(t,a,\tau,x)da & \text{in } (0,T)\times(\tau_1,\tau_2)\times(\bar\alpha,1), \end{cases} \tag{4.73}$$

where $\bar\alpha \in (0,\alpha)$ and $\bar\beta \in (\beta,1)$. Thus, by Theorem 4.7, there exist two controls $h_{1,\vartheta}$ and $h_{2,\vartheta}$ such that the solutions $u_{1,\vartheta}$ and $u_{2,\vartheta}$ of (P_1) and (P_2), associated to $h_{1,\vartheta}$ and $h_{2,\vartheta}$, respectively, satisfy

$$u_{1,\vartheta}(T,a,\tau,x) = 0 \quad \text{a.e. } (a,\tau,x) \in (\Xi,A)\times(\tau_1,\tau_2)\times(0,\bar\beta),$$

and

$$u_{2,\vartheta}(T,a,\tau,x) = 0 \quad \text{a.e. } (a,\tau,x) \in (\Xi,A)\times(\tau_1,\tau_2)\times(\bar\alpha,1).$$

Moreover, there exists $C = C(\vartheta) > 0$ such that

$$\int_{Q_{T,A,\tau_2}} \int_0^{\bar{\beta}} \frac{1}{\sigma} h_{1,\vartheta}^2 dx d\tau da dt \le C \|u_0\|^2_{L^2_{\frac{1}{\sigma}}(Q_{A,\tau_2,1})}$$

and

$$\int_{Q_{T,A,\tau_2}} \int_{\bar{\alpha}}^{1} h_{2,\vartheta}^2 dx d\tau da dt \le C \|u_0\|^2_{L^2_{\frac{1}{\sigma}}(Q_{A,\tau_2,1})}.$$

Denote with u_1 and h_1 (respectively u_2 and h_2) the trivial extensions of $u_{1,\vartheta}$ and $h_{1,\vartheta}$ (respectively $u_{2,\vartheta}$ and $h_{2,\vartheta}$) to $[\bar{\beta}, 1]$ (respectively $[0, \bar{\alpha}]$), so that all the functions are defined in the interval $[0, 1]$. Clearly, they depends always on ϑ and

$$\|h_i\|_{L^2_{\frac{1}{\sigma}}(Q)} \le C \|u_0\|_{L^2_{\frac{1}{\sigma}}(Q_{A,\tau_2,1})}, \; i = 1, 2. \tag{4.74}$$

Now, let u_3 be the solution of

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial a} + \dfrac{\partial}{\partial \tau}(g(\tau)u) - k(x)u_{xx} - b(x)u_x + \mu(t, a, \tau, x)u = 0 & \text{in } Q, \\ u(t, a, \tau, 1) = u(t, a, \tau, 0) = 0 & \text{on } Q_{T,A,\tau_2}, \\ u(0, a, \tau, x) = u_0(a, \tau, x) & \text{in } Q_{A,\tau_2,1}, \\ u(t, a, \tau_1, x) = 0 & \text{in } Q_{T,A,1}, \\ u(t, 0, x, \tau) = \int_0^A \beta(a, \tau, x)u(t, a, \tau, x)da & \text{in } Q_{T,\tau_2,1} \end{cases} \tag{4.75}$$

and consider the three smooth cut off functions $\xi, \eta, \phi : [0, 1] \to \mathbb{R}$ defined as

$$\begin{cases} 0 \le \xi(x) \le 1, & \text{for all } x \in [0, 1], \\ \xi(x) = 1, & x \in [0, (2\alpha + \rho)/3], \\ \xi(x) = 0, & x \in [(\alpha + 2\rho)/3, 1], \end{cases}$$

$$\begin{cases} 0 \le \eta(x) \le 1, & \text{for all } x \in [0, 1], \\ \eta(x) = 0, & x \in [0, (2\alpha + \rho)/3], \\ \eta(x) = 1, & x \in [(\alpha + 2\rho)/3, 1] \end{cases}$$

and $\phi := 1 - \xi - \eta$. Finally, take

$$u(t, a, \tau, x) = \xi u_1 + \eta u_2 + F(t)\phi u_3,$$

where $F(t) := \dfrac{T - t}{T}$.

It is easy to verify that $u(t, a, \tau, 0) = u(t, a, \tau, 1) = 0$, $u(0, a, \tau, x) = u_0(a, \tau, x)$ (since $F(0) = 1$) and $u(t, 0, \tau, x) = \int_0^A \beta(a, \tau, x)u(t, a, \tau, x)da$. Moreover,

$$u(T, a, \tau, x) = 0 \quad \text{a.e. } (a, \tau, x) \in (\Xi, A) \times (\tau_1, \tau_2) \times (0, 1)$$

and u satisfies the equation of (1.1) with

$$\begin{aligned} f_\vartheta = {} & \xi h_1 \chi_\omega + \eta h_2 \chi_\omega - \frac{1}{T}\phi u_3 - b(\xi' u_1 + \eta' u_2 + F(t)\phi' u_3) \\ & - k(\xi' u_{1,x} + (\xi' u_1)_x + \eta' u_{2,x} + (\eta' u_2)_x) \\ & - F(t)k\phi' u_{3,x} - F(t)k(\phi' u_3)_x. \end{aligned}$$

Obviously, the support of f_ϑ is contained in ω and the terms $(\phi' u_3)_x$, $(\xi' u_1)_x$ and $(\eta' u_2)_x$ are $L^2_{\frac{1}{\sigma}}(0, 1)$ (recall that $\phi'(x) = \xi'(x) = \eta'(x) = 0$ for all $x \in (0, 1) \setminus \omega$); thus $f_\vartheta \in L^2_{\frac{1}{\sigma}}(Q)$ as required.

Now, we will prove (4.71). To this aim, consider the equation satisfied by u_1 and multiply it by $\dfrac{u_1}{\sigma}$.

Then, integrating over $Q_{A,\tau_2,1}$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u_1(t)\|^2_{L^2_{\frac{1}{\sigma}}(Q_{A,\tau_2,1})} + \frac{1}{2}\int_{\tau_1}^{\tau_2}\int_0^1 \frac{1}{\sigma}u_1^2(t,A,\tau,x)dxd\tau \\
&- \frac{1}{2}\int_{\tau_1}^{\tau_2}\int_0^1 \frac{1}{\sigma}{u_1}^2(t,0,\tau,x)dxd\tau + \int_{Q_{A,\tau_2,1}} \eta {u_{1,x}}^2 dxd\tau da \\
&= -\int_{Q_{A,\tau_2,1}} \mu\frac{1}{\sigma}u_1^2 dxd\tau da - \frac{1}{2}\int_{Q_{A,\tau_2,1}} g_\tau \frac{1}{\sigma}u_1^2 dxd\tau da \\
&- \frac{1}{2}\int_0^A\int_0^1 \frac{1}{\sigma}g(\tau_2)u_1^2(t,a,\tau_2,x)dxda + \int_0^A\int_{\tau_1}^{\tau_2}\int_\omega \frac{1}{\sigma}fu_1 dxd\tau da.
\end{aligned}$$

Hence, using the initial condition $u_1(t,0,\tau,x) = \int_0^A \beta(a,\tau,x)u_1(t,a,\tau,x)da$, the assumptions on β, μ and g and the Jensen's inequality, one has

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u_1(t)\|^2_{L^2_{\frac{1}{\sigma}}(Q_{A,\tau_2,1})} &\le \frac{1}{2}\frac{d}{dt}\|u_1(t)\|^2_{L^2_{\frac{1}{\sigma}}(Q_{A,\tau_2,1})} + \frac{1}{2}\int_{\tau_1}^{\tau_2}\int_0^1 \frac{1}{\sigma}u_1^2(t,A,\tau,x)dxd\tau + \int_{Q_{A,\tau_2,1}} \eta u_{1,x}^2 dxd\tau da \\
&\le \frac{C}{2}\int_{Q_{A,\tau_2,1}} \frac{1}{\sigma}u_1^2 dxd\tau da - \frac{1}{2}\int_0^A\int_0^1 g(\tau_2)u_1^2(t,a,\tau_2,x)dxda \\
&+ \int_0^A\int_{\tau_1}^{\tau_2}\int_\omega \frac{1}{\sigma}h_1u_1 dxd\tau da \\
&\le \frac{C}{2}\int_{Q_{A,\tau_2,1}} \frac{1}{\sigma}u_1^2 dxd\tau da + \frac{1}{2}\int_0^A\int_{\tau_1}^{\tau_2}\int_\omega \frac{1}{\sigma}h_1^2 dxd\tau da,
\end{aligned} \tag{4.76}$$

where C is a positive constant. Hence,

$$\frac{d}{dt}\|u_1(t)\|^2_{L^2_{\frac{1}{\sigma}}(Q_{A,\tau,1})} \le C\|u_1(t)\|^2_{L^2(Q_{A,\tau,1})} + \|h_1(t)\|^2_{L^2_{\frac{1}{\sigma}}(Q_{A,\tau,1})}.$$

Setting $F(t) := \|u_1(t)\|^2_{L^2_{\frac{1}{\sigma}}(Q_{A,\tau,1})}$ and multiplying the previous inequality by e^{-Ct}, one has

$$\frac{d}{dt}(e^{-Ct}F(t)) \le e^{-Ct}\|h_1(t)\|^2_{L^2_{\frac{1}{\sigma}}(Q_{A,\tau,1})}. \tag{4.77}$$

Integrating (4.77) over $(0,t)$, for all $t \in [0,T]$ it follows that

$$e^{-Ct}F(t) \le F(0) + \int_0^t e^{-C\tau}\|h_1(\tau)\|^2_{L^2_{\frac{1}{\sigma}}(Q_{A,\tau,1})}d\tau.$$

Hence, for all $t \in [0,T]$,

$$F(t) \le e^{CT}\left(F(0) + \int_0^T \|h_1(\tau)\|^2_{L^2_{\frac{1}{\sigma}}(Q_{A,\tau,1})}d\tau\right)$$

and

$$\sup_{t\in[0,T]} \|u_1(t)\|^2_{L^2_{\frac{1}{\sigma}}(Q_{A,\tau,1})} \le C\left(\|u_0\|^2_{L^2_{\frac{1}{\sigma}}(Q_{A,\tau,1})} + \|h_1\|^2_{L^2_{\frac{1}{\sigma}}(Q)}\right). \tag{4.78}$$

Therefore, by (4.76), it follows that

$$\frac{1}{2}\frac{d}{dt}\|u_1(t)\|^2_{L^2_{\frac{1}{\sigma}}(Q_{A,\tau_2,1})} + \int_{Q_{A,\tau_2,1}} \eta u_{1,x}^2 dxd\tau da \le \frac{C}{2}\int_{Q_{A,\tau_2,1}} \frac{1}{\sigma}u_1^2 dxd\tau da + \frac{1}{2}\int_0^A\int_{\tau_1}^{\tau_2}\int_\omega \frac{1}{\sigma}h_1^2 dxd\tau da.$$

Integrating over $(0,T)$, we have

$$\frac{1}{2}\|u_1(T)\|^2_{L^2_{\frac{1}{\sigma}}(Q_{A,\tau_2,1})} + \int_Q \eta u_{1,x}^2 dxd\tau dadt \le \frac{1}{2}\|u_0\|^2_{L^2_{\frac{1}{\sigma}}(Q_{A,\tau,1})} + \frac{C}{2}\int_Q \frac{1}{\sigma}u_1^2(t,a,\tau,x)dxd\tau dadt + \frac{1}{2}\|h_1\|^2_{L^2_{\frac{1}{\sigma}}(Q)}.$$

Hence, by (4.78),

$$\int_0^T \|\sqrt{\eta}u_{1,x}\|_{L^2_{\frac{1}{\sigma}}(Q_{A,\tau,1})}dxd\tau dadt \le \|u_0\|^2_{L^2_{\frac{1}{\sigma}}(Q_{A,\tau,1})} + C\int_Q \frac{1}{\sigma}u_1^2(t,a,\tau,x)dxd\tau dadtt + \frac{1}{2}\|h_1\|^2_{L^2_{\frac{1}{\sigma}}(Q)}$$
$$\le C\left(\|u_0\|^2_{L^2_{\frac{1}{\sigma}}(Q_{A,\tau,1})} + \|h_1\|^2_{L^2_{\frac{1}{\sigma}}(Q)}\right). \tag{4.79}$$

Estimates (4.78) and (4.79) hold also for u_2 and u_3, where h_1 is replaced by h_2 and 0, respectively. By the definition of f_ϑ and (4.74), the theorem follows. □

5 Appendix

5.1 Proof of Proposition 4.2:

Let us consider a smooth function $\xi : [0,1] \to \mathbb{R}$ such that

$$\begin{cases} 0 \le \xi(x) \le 1, & \text{for all } x \in [0,1], \\ \xi(x) = 1, & x \in \omega', \\ \xi(x) = 0, & x \in (0,1) \setminus \omega. \end{cases}$$

Then, integrating by parts one has

$$\begin{aligned} 0 = &\int_0^T \frac{d}{dt}\left(\int_{Q_{A,\tau_2,1}} (\xi e^{s\psi})^2 v^2 dxd\tau da\right) dt \\ = &\int_Q 2s\psi_t(\xi e^{s\psi})^2v^2 + 2(\xi e^{s\psi})^2 v(-v_a - gv_\tau - \mathcal{A}_0 v + \mu v + f)\, dxd\tau dadt \\ = &\ 2s\int_Q \psi_t(\xi e^{s\psi})^2 v^2 dxd\tau dadt + 2s\int_Q \psi_a(\xi e^{s\psi})^2v^2dxd\tau dadt + 2\int_Q \left(\xi^2 e^{2s\psi}\sigma\right)_x \eta v v_x dxd\tau dadt \\ &+ 2\int_Q (\xi^2 e^{2s\psi}k)v_x^2 dxd\tau dadt + 2\int_Q \xi^2 e^{2s\psi}\mu v^2 dxd\tau dadt + 2\int_Q \xi^2 e^{2s\psi} f v dxd\tau dadt \\ &+ 2s\int_Q \psi_\tau(\xi e^{s\psi})^2 g v^2 dxd\tau dadt + \int_Q \xi^2 e^{2s\psi} g_\tau v^2 dxd\tau dadt. \end{aligned}$$

Hence, using Young's inequality

$$\begin{aligned} 2\int_Q (\xi^2 e^{2s\psi}k)v_x^2 dxd\tau dadt = &-2s\int_Q \psi_t\left(\xi e^{s\psi}\right)^2 v^2 dxd\tau dadt - 2s\int_Q \psi_a(\xi e^{s\psi})^2 v^2 dxd\tau dadt \\ &- 2\int_Q \frac{(\xi^2 e^{2s\psi}\sigma)_x}{\xi e^{s\psi}\sqrt{\sigma}}\xi e^{s\psi}\sqrt{\sigma}\eta v v_x\, dxd\tau dadt - 2\int_Q \xi^2 e^{2s\psi}\mu v^2 dxd\tau dadt \\ &- 2\int_Q \xi^2 e^{2s\psi} f v dxd\tau dadtt \\ &- 2s\int_Q \psi_\tau(\xi e^{s\psi})^2 g v^2 dxd\tau dadt - \int_Q \xi^2 e^{2s\psi} g_\tau v^2 dxd\tau dadt \\ \le &-2s\int_Q \psi_t(\xi e^{s\psi})^2 v^2 dxd\tau dadt - 2s\int_Q \psi_a(\xi e^{s\psi})^2 v^2 dxd\tau dadt \\ &- 2s\int_Q \psi_\tau(\xi e^{s\psi})^2 g v^2 dxd\tau dadt - \int_Q \xi^2 e^{2s\psi} g_\tau v^2 dxd\tau dadt \\ &+ 4\int_Q \left(\xi e^{s\psi}\sqrt{\sigma}\right)_x^2 \eta v^2 dxd\tau dadt + \int_Q (\xi^2 e^{2s\psi}k)v_x^2 dxd\tau dadt \\ &+ (2\|\mu\|_{L^\infty(Q)} + 1)\int_Q \xi^2 v^2 dxd\tau dadt + \int_Q \xi^2 e^{2s\psi} f^2 dxd\tau dadt. \end{aligned}$$

It follows that,

$$\begin{aligned}\int_Q (\xi^2 e^{2s\psi} k) v_x^2 dxd\tau dadt \leq & -2s \int_Q \psi_t (\xi e^{s\psi})^2 v^2 dxd\tau dadt - 2s \int_Q \psi_a (\xi e^{s\psi})^2 v^2 dxd\tau dadt \\ & - 2s \int_Q \psi_\tau (\xi e^{s\psi})^2 g v^2 dxd\tau dadt - \int_Q \xi^2 e^{2s\psi} g_\tau v^2 dxd\tau dadt \\ & + 4 \int_Q \left(\xi e^{s\psi} \sqrt{\sigma}\right)_x^2 \eta v^2 dxd\tau dadt \\ & + (2\|\mu\|_{L^\infty(Q)} + 1) \int_Q \xi^2 v^2 dxd\tau dadt + \int_Q \xi^2 e^{2s\psi} f^2 dxd\tau dadt.\end{aligned}$$

Thus,

$$\begin{aligned}& \inf_{\omega'} \{k\} \int_{Q_{T,A,\tau_2}} \int_{\omega'} e^{2s\psi} v_x^2 dxd\tau dadt \\ & \leq \left(\sup_{\omega \times (0,T)} \left\{ \left| 4 \left(\xi e^{s\psi} \sqrt{\sigma}\right)_x^2 - 2s(\psi_t + \psi_a + \psi_\tau g + g_\tau)(\xi e^{s\psi})^2 \right| \right\} + 2\|\mu\|_{L^\infty(Q)} + 1 \right) \int_{Q_{T,A,\tau_2}} \int_\omega v^2 dxd\tau dadt \\ & + \int_Q f^2 e^{2s\psi} dxd\tau dadt.\end{aligned}$$

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