Structurally complete finitary extensions of positive Łukasiewicz logic

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Abstract

In this paper we study $\mathcal{MV}^+$, i.e. the positive fragment of Łukasiewicz infinite-valued Logic $\mathcal{MV}$. Using mainly algebraic techniques we characterize all the finitary extensions of $\mathcal{MV}^+$ that are structurally complete and those that are hereditarily structurally complete.

Keywords: Łukasiewicz; logic

1 Introduction

A large class of substructural logics (i.e. logics that lack some structural rules) is given by the extensions of the *Full Lambek Calculus* $\mathcal{FL}$ (see [30] p. 76). All extensions of $\mathcal{FL}$ have a primitive connective **0** that denotes *falsum*; so if $\mathcal{L}$ is an extension of $\mathcal{FL}$ it makes sense to study its *positive* fragment $\mathcal{L}^+$. The language of $\mathcal{L}^+$ is obtained by the language of $\mathcal{L}$ by deleting **0** from the signature; the valid derivations in $\mathcal{L}^+$ are the valid derivations in $\mathcal{L}$, which contain only 0-free formulas.

An important subfamily of substructural logics over $\mathcal{FL}$ consists of logics that satisfy both exchange and weakening but lack contraction; in this case the primitive connectives can be taken as $\vee, \wedge, \to, \mathbf{0}, \mathbf{1}$ where of course **1** represents *truth*. The minimal substructural logic of this kind is denoted by $\mathcal{FL}_{ew}$; among its extensions the most famous is probably Intuitionistic Logic $\mathcal{IL}$ but there are many others such as the monoidal logic $\mathcal{MTL}$ [27], Hajek's Basic Logic $\mathcal{BL}$ [34], Gödel–Dummet's Logic $\mathcal{GL}$ [34], Product Logic $\mathcal{PL}$ [34] and, last but not least, Łukasiewicz's infinite-valued logic $\mathcal{MV}$ [39].

All finitary extensions of $\mathcal{FL}$ are *algebraizable* with an *equivalent algebraic semantics* (in the sense of Blok–Pigozzi [19]) that is at least a quasivariety of algebras; by the same token all positive fragments are algebraizable as well and moreover the machinery of algebraization takes a very transparent form in both cases. The equivalent algebraic semantics of $\mathcal{FL}$ and $\mathcal{FL}^+$ are the variety of FL-algebras [30] and the variety of residuated lattices [36], respectively. Algebraizability in Blok–Pigozzi's fashion entails that the deducibility relation of an algebraizable logic is characterized by means of the algebraic equational consequence of its equivalent algebraic semantics. Therefore, interesting properties of algebraizable logics can be studied via algebraic means.

Hoops are a particular variety of residuated monoids related to logic, which were defined in an unpublished manuscript by Büchi and Owens, inspired by the work on partially ordered monoids

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in [21]. Hoops indeed correspond to residuated commutative monoids whose order is the *inverse divisibility ordering*, i.e. $a \leq b$ if and only if there exists c such that $a = bc$. The first systematic study of hoops was carried out by Ferreirim in her Ph.D. thesis [28] and in her works with Blok [18]; the connection between hoops and substructural logics has been investigated in [8]. A subvariety of hoops, the variety WH of *Wajsberg hoops*, plays a special role in this framework. From the algebraic point of view, they can be used to describe subdirectly irreducible hoops, and the whole variety of hoops can be obtained as the join of iterated powers of the variety of Wajsberg hoops, in the sense defined in [18]. Wajsberg hoops also have a peculiar connection with lattice-ordered abelian groups (abelian ℓ-groups for short). In fact, the variety of Wajsberg hoops is generated by its totally ordered members, which are, in loose terms, either negative cones of abelian ℓ-groups, or intervals in abelian ℓ-groups [10]. However the most relevant property of Wajsberg hoops in this context is that they are the equivalent algebraic semantics of the positive fragment of Łukasiewicz logic $\mathcal{MV}$. This will allow us to reduce logical questions, such as the structural completeness of its finitary extensions, to a purely algebraic matter.

The techniques used in this paper come from two different sources:

- the work of J. Gispert on quasivarieties of Wajsberg algebras [31, 32];
- the investigation on varieties of Wajsberg hoops (mainly the description of the lattices of subvarieties and of the free algebras) carried out in [10].

We also mention that this paper complements and expands [7].

2 Preliminaries

2.1 Universal algebra

Let K be a class of algebras; we denote by $\mathbf{I}, \mathbf{H}, \mathbf{P}, \mathbf{S}, \mathbf{P}_u$ the class operators sending K in the class of all isomorphic copies, homomorphic images, direct products, subalgebras and ultraproducts of members of K. The operators can and will be composed in the obvious way; for instance $\mathbf{SP}(\mathsf{K})$ denotes all algebras that are embeddable in a direct product of members of K; moreover there are relations among the classes resulting from applying operators in a specific order, for instance $\mathbf{PS}(\mathsf{K}) \subseteq \mathbf{SP}(\mathsf{K})$ and $\mathbf{HSP}(\mathsf{K})$ is the largest class we can obtain composing the operators. We will use all the known relations without further notice; for those and for any other unexplained algebraic notion we refer the reader to [22] for a textbook treatment.

If τ is a type of algebras, an **equation** is a pair p, q of τ-terms (i.e. elements of the absolutely free algebra $\mathbf{T}_\tau(\omega)$) that we write suggestively as $p \approx q$; a **universal sentence** in τ is $\Sigma \Rightarrow \Gamma$ where Σ, Γ are finite sets of equations; a universal sentence is a **quasiequation** if $|\Gamma| = 1$ and it is **negative** if $\Gamma = \emptyset$. Clearly an equation is a quasiequation in which $\Sigma = \emptyset$. An equation $p(x_1, \dots, x_n) \approx q(x_1, \dots, x_n)$ is **valid** in $\mathbf{A}$ (and we write $\mathbf{A} \vDash p \approx q$) if for all $a_1, \dots, a_n \in A$, $p(a_1, \dots, a_n) = q(a_1, \dots, a_n)$; if Σ is a set of equations then $\mathbf{A} \vDash \Sigma$ if $\mathbf{A} \vDash \sigma$ for all $\sigma \in \Sigma$. A universal sentence is **valid** in $\mathbf{A}$ (and we write $\mathbf{A} \vDash \Sigma \Rightarrow \Gamma$) if whenever $\mathbf{A} \vDash \Sigma$ there is a $\varepsilon \approx \delta \in \Gamma$ with $\mathbf{A} \vDash \varepsilon \approx \delta$; in other words a universal sentence can be understood as the formula $\forall \mathbf{x}(\bigwedge \Sigma \rightarrow \bigvee \Gamma)$. An equation or a universal sentence is **valid** in a class K if it is valid in all algebras in K.

A class of algebras is a variety if it is closed under $\mathbf{H}, \mathbf{S}$ and $\mathbf{P}$, a quasivariety if it is closed under $\mathbf{I}$,$\mathbf{S}$,$\mathbf{P}$ and $\mathbf{P}_u$ and a universal class if it is closed under $\mathbf{ISP}_u$. The following facts were essentially discovered by A. Tarski, J. Łòs and A. Lyndon in the pioneering phase of model theory; for proof of this and similar statements the reader can consult [23].

LEMMA 2.1
Let K be any class of algebras. Then:

(1) K is a universal class if and only if $\mathbf{ISP}_u(\mathsf{K}) = \mathsf{K}$ if and only if it is the class of all algebras in which a set of universal sentences is valid;
(2) K is a quasivariety if and only if $\mathbf{ISPP}_u(\mathsf{K}) = \mathsf{K}$ if and only if it is the class of all algebras in which a set of quasiequations is valid;
(3) K is a variety if and only if $\mathbf{HSP}(\mathsf{K}) = \mathsf{K}$ if and only if it is the class of all algebras in which a set of equations is valid.

We will often write $\mathbf{V}$ for $\mathbf{HSP}$ and $\mathbf{Q}$ for $\mathbf{ISPP}_u$. It is clear that both $\mathbf{V}$ and $\mathbf{Q}$ are closure operators on classes of algebras of the same type; this implies among other things that for a given variety V the class of subvarieties of V is a complete lattice, which we denote by $\Lambda(\mathsf{V})$. If Q is a quasivariety that is not a variety then $\Lambda(\mathsf{Q})$ is still a lattice but it is not necessarily complete (in particular does not need to have maximum). The class of subquasivarieties of a quasivariety Q is a complete lattice denoted by $\Lambda_q(\mathsf{Q})$.

For the definition of free algebras in a class K on a set X of generators in symbols $\mathbf{F}_{\mathsf{K}}(X)$, we refer again to [22]. We merely observe that every free algebra on a class K belongs to $\mathbf{ISP}(\mathsf{K})$. It follows that every free algebra in K is free in $\mathbf{ISP}(\mathsf{K})$ and therefore for any quasivariety Q, $\mathbf{F}_{\mathsf{Q}}(X) = \mathbf{F}_{\mathbf{V}(\mathsf{Q})}(X)$.

Let $\mathbf{B}, (\mathbf{A}_i)_{i\in I}$ be algebras in the same signature; we say that **B embeds** in $\prod_{i\in I} \mathbf{A}_i$ if $\mathbf{B} \in \mathbf{IS}(\prod_{i\in I} \mathbf{A}_i)$. Let p_i be the i-th projection, or better, the composition of the isomorphism and the i-th projection, from $\mathbf{B}$ to $\mathbf{A}_i$; the embedding is **subdirect** if for all $i \in I$, $p_i(\mathbf{B}) = \mathbf{A}_i$ and in this case we will write

$$\mathbf{B} \leq_{sd} \prod_{i\in I} \mathbf{A}_i.$$

An algebra $\mathbf{B}$ is **subdirectly irreducible** if it is nontrivial and for any subdirect embedding

$$\mathbf{B} \leq_{sd} \prod_{i\in I} \mathbf{A}_i,$$

there is an $i \in I$ such that $\mathbf{B}$ and $\mathbf{A}_i$ are isomorphic. It can be shown that $\mathbf{A}$ is **subdirectly irreducible** if and only if the congruence lattice $\mathrm{Con}(\mathbf{A})$ of $\mathbf{A}$ has a unique minimal element different from the trivial congruence. If V is a variety we denote by V_{si} the class of subdirectly irreducible algebras in V.

THEOREM 2.2

(1) (Birkhoff [17]) Every algebra can be subdirectly embedded in a product of subdirectly irreducible algebras. So if $\mathbf{A} \in \mathsf{V}$, then $\mathbf{A}$ can be subdirectly embedded in a product of members of V_{si}.
(2) (Jónsson's Lemma [37]) Suppose that K is a class of algebras such that $\mathsf{V}(\mathsf{K})$ is congruence distributive; then $\mathsf{V}_{si} \subseteq \mathbf{HSP}_u(\mathsf{K})$.

If Q is a quasivariety and $\mathbf{A} \in \mathsf{Q}$, a **$\mathsf{Q}$-congruence** of $\mathbf{A}$ is a congruence θ such that $\mathbf{A}/\theta \in \mathsf{Q}$; clearly Q-congruences form an algebraic lattice $\mathrm{Con}_{\mathsf{Q}}(\mathbf{A})$. An algebra $\mathbf{A} \in \mathsf{Q}$ is **Q-irreducible** if $\mathrm{Con}_{\mathsf{Q}}(\mathbf{A})$ has a unique minimal element; since clearly $\mathrm{Con}_{\mathsf{Q}}(\mathbf{A})$ is a meet subsemilattice of $\mathrm{Con}(\mathbf{A})$, any subdirectly irreducible algebra is Q-irreducible in any quasivariety Q to which it belongs. For a quasivariety Q we denote by Q_{ir} the class of Q-irreducible algebras in Q. We have the equivalent of Birkhoff's and Jónsson's results for quasivarieties:

THEOREM 2.3
Let Q be any quasivariety.

(1) (Mal'cev [40]) Every $\mathbf{A} \in \mathsf{Q}$ is subdirectly embeddable in a product of algebras in Q_{ir}.
(2) (Czelakowski–Dziobiak [25]) If $\mathsf{Q} = \mathbf{Q}(\mathsf{K})$, then $\mathsf{Q}_{ir} \subseteq \mathbf{ISP}_u(\mathsf{K})$.

2.2 The Blok–Pigozzi connection

A *consequence relation* on the set of terms $\mathbf{T}_\rho(\omega)$ (also called *algebra of formulas*) of some algebraic language ρ is a relation $\vdash \subseteq \mathcal{P}(\mathbf{T}_\rho(\omega)) \times \mathbf{T}_\rho(\omega)$ (and we write $\Sigma \vdash \gamma$ for $(\Sigma, \gamma) \in \vdash$) such that:

(1) if $\alpha \in \Gamma$ then $\Gamma \vdash \alpha$;
(2) if $\Gamma \vdash \delta$ for all $\delta \in \Delta$ and $\Delta \vdash \beta$, then $\Gamma \vdash \beta$.

We call *substitution* any endomorphism of $\mathbf{T}_\rho(\omega)$; $\vdash$ is *substitution invariant* (also called *structural*) if $\Gamma \vdash \alpha$ implies $\{\sigma(\gamma) : \gamma \in \Gamma\} \vdash \sigma(\alpha)$ for each substitution σ. Finally, $\vdash$ is *finitary* if $\Gamma \vdash \alpha$ implies that there is a finite $\Gamma' \subseteq \Gamma$ such that $\Gamma' \vdash \alpha$. By a *logic* $\mathcal{L}$ in what follows we mean a substitution-invariant finitary consequence relation $\vdash_\mathcal{L}$ on $\mathbf{T}_\rho(\omega)$ for some algebraic language ρ, $\vdash_\mathcal{L} \subseteq \mathcal{P}(\mathbf{T}_\rho(\omega)) \times \mathbf{T}_\rho(\omega)$.

A **clause** in $\mathcal{L}$ is a formal expression $\Sigma \Rightarrow \Delta$ where Σ, Δ are finite set of formulas of $\mathcal{L}$; a clause is a **rule** if $\Delta = \{\delta\}$. A rule is an **axiom** if $\Sigma = \emptyset$.

If the equivalence between formulas (i.e. provable equivalence according to $\vdash$) is a congruence on the algebra of formulas, then we can form the *Lindenbaum–Tarski algebra* as the quotient of provably equivalent formulas of the algebra of formulas. If the quasivariety $\mathsf{Q}_\mathcal{L}$ generated by the Lindenbaum–Tarski algebra satisfies further conditions, then it is the class of algebraic models of $\mathcal{L}$ and $\mathcal{L}$ is **algebraizable** with **equivalent algebraic semantics** $\mathsf{Q}_\mathcal{L}$; Blok and Pigozzi [19] formalized this connection and gave necessary and sufficient conditions for a logic to be algebraizable. Essentially, one needs a finite set of equations $\tau = \{\delta_i \approx \varepsilon_i : i = 1, \dots, n\}$ in the language of $\mathsf{Q}_\mathcal{L}$ and a finite set of formulas of $\mathcal{L}$ in two variables $\Delta(x,y) = \{\varphi_1(x,y), \dots, \varphi_m(x,y)\}$ that allow to transform equations, quasiequations and universal sentences in $\mathsf{Q}_\mathcal{L}$ into formulas, rules and clauses of $\mathcal{L}$ and vice versa; moreover, this transformation must respect both the consequence relation and the semantical consequence. That is to say, for all sets of formulas Γ of $\mathcal{L}$ and formulas $\varphi \in \mathbf{T}_\tau(\omega)$

$$\Gamma \vdash \varphi \quad \text{iff} \quad \{\delta_i(\Gamma) \approx \varepsilon_i(\Gamma) : i = 1, \dots, n\} \vDash_{\mathsf{Q}_\mathcal{L}} \{\delta_i(\varphi) \approx \varepsilon_i(\varphi) : i = 1, \dots, n\}$$

where of course $\delta_i(\Gamma) \approx \varepsilon_i(\Gamma)$ is a shorthand for $\delta_i(\psi) \approx \varepsilon_i(\psi)$ for all $\psi \in \Gamma$ and also

$$\mathsf{Q}_\mathcal{L} \vDash (x \approx y) \Leftrightarrow (\bigwedge_{i=1}^{n} \bigwedge_{j=1}^{m} (\delta_i(\varphi_j(x,y)) \approx \varepsilon_i(\varphi_j(x,y))).$$

A quasivariety Q is a *quasivariety of logic* if it is the equivalent quasivariety semantics for some logic L_Q; the Galois connection between algebraizable logics and quasivarieties of logic is given by

$$\mathcal{L}_{\mathsf{Q}_\mathcal{L}} = \mathcal{L} \qquad\qquad \mathsf{Q}_{\mathcal{L}_\mathsf{Q}} = \mathsf{Q}.$$

2.3 Structural completeness in algebra and logic

An **extension** of $\mathcal{L}$ over the language τ is a logic $\mathcal{L}'$ over the same language such that $\Sigma \vdash_\mathcal{L} \delta$ implies $\Sigma \vdash_{\mathcal{L}'} \delta$. A **finitary extension** of $\mathcal{L}$ is an extension of $\mathcal{L}$ that is finitary; clearly any such extension can be obtained by adding a set of rules to the calculus of $\mathcal{L}$. An **axiomatic extension** of $\mathcal{L}$ is an extension of $\mathcal{L}$ obtained by adding a set of axioms to the calculus of $\mathcal{L}$. The class of the (finitary, axiomatic) extension of the logic $\mathcal{L}$ is never empty, since $\mathcal{L}$ clearly belongs to it. Therefore we can define closure operators in a standard way and hence complete lattices; so $\mathrm{Th}(\mathcal{L})$, $\mathrm{Th}_f(\mathcal{L})$ and $\mathrm{Th}_a(\mathcal{L})$ will be the lattice of all the extensions, finitary extensions and axiomatic extensions of $\mathcal{L}$, respectively. Via the Blok–Pigozzi connection we get:

THEOREM 2.4
Let $\mathcal{L}$ be an algebraizable logic with equivalent algebraic semantics $\mathsf{Q}_{\mathcal{L}}$. Then

(1) the lattice $\mathrm{Th}_a(\mathcal{L})$ of the axiomatic extensions of $\mathcal{L}$ is dually isomorphic with $\Lambda(\mathsf{Q})$;
(2) the lattice $\mathrm{Th}_f(\mathcal{L})$ of the finitary extensions of $\mathcal{L}$ is dually isomorphic with $\Lambda_q(\mathsf{Q})$.

An algebraizable logic $\mathcal{L}$ is **tabular** if it is the logic of a finite algebra; in other words $\mathsf{Q}_{\mathcal{L}}$ is a finitely generated quasivariety, i.e. $\mathsf{Q}_{\mathcal{L}} = \mathbf{Q}(\mathbf{A})$ for some finite algebra $\mathbf{A}$. A logic is **locally tabular** if, for any finite k, there exist only finitely many pairwise nonequivalent formulas in $\mathcal{L}$ built from the variables $x_1, \dots, x_k$. It is clear that an algebraizable logic is locally tabular if and only if $\mathsf{Q}_{\mathcal{L}}$ is a locally finite quasivariety.

A clause $\Sigma \Rightarrow \Delta$ is **admissible** in $\mathcal{L}$ if every substitution that makes the premises into a theorem of $\mathcal{L}$, also makes at least one of the conclusions in Δ a theorem of $\mathcal{L}$. In particular a rule is admissible in $\mathcal{L}$ if, when added to its calculus, it does not produce any new theorem. A clause $\Sigma \Rightarrow \Delta$ is **derivable** in $\mathcal{L}$ if $\Sigma \vdash \delta$ for some $\delta \in \Delta$. A logic $\mathcal{L}$ is **structurally complete** if every admissible rule of $\mathcal{L}$ is derivable in $\mathcal{L}$; a logic is **hereditarily structurally complete** if every finitary extension of $\mathcal{L}$ is structurally complete. Determining structural completeness of a logic is in general a very deep and challenging problem; here we will use only the parts of the theory that are necessary but for an extensive treatment of the subject we direct the reader to [12].

A quasivariety Q is **structural** if for every subquasivariety $\mathsf{Q}' \subseteq \mathsf{Q}$, $\mathbf{H}(\mathsf{Q}') = \mathbf{H}(\mathsf{Q})$ implies $\mathsf{Q}' = \mathsf{Q}$.

THEOREM 2.5
[15] For a quasivariety Q the following are equivalent:

(1) Q is structural;
(2) $\mathbf{Q}(\mathbf{F}_{\mathsf{Q}}(\omega)) = \mathsf{Q}$.

For any quasivariety Q, we define the **structural core of Q** as the smallest $\mathsf{Q}' \subseteq \mathsf{Q}$ such that $\mathbf{H}(\mathsf{Q}) = \mathbf{H}(\mathsf{Q}')$. The structural core of a quasivariety always exists:

COROLLARY 2.6
For any quasivariety Q, $\mathbf{Q}(\mathsf{F}_{\mathsf{Q}}(\omega))$ is structural and it is the structural core of Q.

PROOF. $\mathbf{Q}(\mathsf{F}_{\mathsf{Q}}(\omega))$ is structural by Theorem 2.5; if $\mathsf{Q}' \subseteq \mathsf{Q}$ is such that $\mathbf{H}(\mathsf{Q}') = \mathbf{H}(\mathsf{Q})$, then clearly $\mathbf{F}_{\mathsf{Q}}(\omega) \in \mathsf{Q}'$ from which the thesis follows. □

It follows at once that a quasivariety Q is structural if and only if it coincides with its structural core. As a consequence the structural subquasivarieties of a quasivariety Q are exactly those that coincide with the structural cores of Q' for some $\mathsf{Q}' \subseteq \mathsf{Q}$; even more, since $\mathbf{H}(\mathsf{Q})$ is a variety, the structural subquasivarieties of a variety V are exactly the structural cores of V' for some subvariety V' of V. As we will see in the sequel this observation is particularly useful when the free countably generated algebra in V has a reasonable description.

If Q is a quasivariety and Q' is a subquasivariety of Q we say that Q' is **equational** in Q if $\mathsf{Q}' = \mathbf{H}(\mathsf{Q}') \cap \mathsf{Q}$; this is clearly equivalent to saying that Q' is axiomatized modulo Q by a set of equations. A quasivariety Q is **primitive** if each subquasivariety of Q' is equational in Q. The following lemma is straightforward:

LEMMA 2.7
For a quasivariety Q the following are equivalent;

(1) Q is primitive;

(2) every subquasivariety of Q is structural.

From the Blok–Pigozzi connection we get at once:

THEOREM 2.8
Let $\mathcal{L}$ be an algebraizable logic with equivalent algebraic semantics $\mathsf{Q}_{\mathcal{L}}$. Then

(1) $\mathcal{L}$ is structurally complete if and only if $\mathsf{Q}_{\mathcal{L}}$ is structural;
(2) $\mathcal{L}$ is hereditarily structurally complete if and only if $\mathsf{Q}_{\mathcal{L}}$ is primitive.

Combining Theorems 2.4 and 2.8 it follows at once that if $\mathcal{L}$ is algebraizable and $\mathsf{Q}_{\mathcal{L}}$ is primitive, then every finitary extension of $\mathcal{L}$ is axiomatic.

Some investigations using (part of) the machinery we have described in this section has already been used to investigate structural completeness in (quasi)varieties of fuzzy logics (see for instance [42] or [24]). In this note however we will get more into the details for a specific variety and this will allow us to characterize all the structurally complete finitary extensions of positive Łukasiewicz logic.

2.4 *Hoops as quasivarieties of logics*

A commutative integral residuated lattice is an algebra $\langle A, \vee, \wedge, \cdot, \rightarrow, 1\rangle$ such that

(1) $\langle A, \vee, \wedge, 1\rangle$ is a lattice with greatest element 1;
(2) $\langle A, \cdot, 1\rangle$ is a commutative monoid;
(3) $(\cdot, \rightarrow)$ form a residuated pair w.r.t. the lattice ordering, i.e. for all $a, b, c \in A$

$$a \cdot b \leq c \qquad \text{if and only if} \qquad a \leq b \rightarrow c.$$

In what follows, we will often write xy for $x \cdot y$. If we augment the signature with an extra constant 0 that is the least element in the lattice order, then we get FL_{ew}-algebras. Commutative integral residuated lattices and FL_{ew}-algebras form varieties that have a very rich structure; we denote by FL_{ew} the variety of FL_{ew}-algebras. For equational axiomatizations and a list of valid identities we refer the reader to [20] (for commutative integral residuated lattices) and [34] (for FL_{ew}-algebras).

Varieties of FL_{ew}-algebras and commutative integral residuated lattices are *ideal determined* (w.r.t. 1) in the sense of [13]; this means that there is a one-to-one correspondence (which is in fact a lattice isomorphism) between the congruences of an algebra $\mathbf{A}$ and certain special subsets of $\mathbf{A}$. In the present case if $\mathbf{A}$ is an FL_{ew}-algebra or a commutative integral residuated lattice, then a **filter** of $\mathbf{A}$ is a filter F of the lattice structure that is also closed under the monoidal operation. If $\theta \in \mathrm{Con}(\mathbf{A})$ then $1/\theta$ is clearly a filter of $\mathbf{A}$ and it is easily checked that if F is a filter then $\theta_F = \{(a, b) : a \rightarrow b, b \rightarrow a \in F\} \in \mathrm{Con}(\mathbf{A})$ and the correspondence is

$$\theta \longmapsto 1/\theta \qquad F \longmapsto \theta_F.$$

If F is a filter we will write $\mathbf{A}/F$ for $\mathbf{A}/\theta_F$.

Let's focus on two equations that bear interesting consequences, i.e. prelinearity and divisibility:

$$(x \rightarrow y) \vee (y \rightarrow x) \approx 1. \tag{prel}$$

$$x(x \rightarrow y) \approx y(y \rightarrow x); \tag{div}$$

It can be shown (see [20] and [36]) that a subvariety of FL_{ew} satisfies the prelinearity equation (prel) if and only if any algebra therein is a subdirect product of totally ordered algebras, and this implies via Birkhoff's Theorem that all the subdirectly irreducible algebras are totally ordered. Such varieties are called *representable* (or *semilinear*) and the subvariety axiomatized by (prel) is the largest subvariety of FL_{ew} that is representable; such variety is usually denoted by MTL, since it is the equivalent algebraic semantics of Esteva–Godo's *Monoidal t-norm based logic* [27].

If an algebra in FL_{ew} satisfies both (prel) and (div) then it is called a BL-algebra and the variety of all BL-algebras is denoted by BL. Again the name comes from logic: the variety of BL-algebras is the equivalent algebraic semantics of *Hájek's Basic Logic* $\mathcal{BL}$ [34]. A systematic investigation of varieties of BL-algebras started with [9] and it is still ongoing (see [4] and the bibliography therein).

It follows from the definition that given a variety of bounded commutative integral residuated lattices, the class of its 0-*free subreducts* is a class of residuated lattices; we have a very general result.

LEMMA 2.9
Let V be any subvariety of FL_{ew}; then the class $\mathbf{S}^0(\mathsf{V})$ of the zero-free subreducts of algebras in V is a variety.

PROOF. The proof is as in Proposition 1.10 of [8]; it is stated for varieties of BL-algebras but it uses only the description of the congruence filters, which is applicable in any subvariety of FL_{ew} (as the reader can easily check). □

This implies at once that if a variety of FL_{ew}-algebras is the equivalent algebraic semantics of a logic $\mathcal{L}$, then the variety of its zero-free subreducts is the equivalent algebraic semantics of the positive fragment $\mathcal{L}^+$. A **basic hoop** is a zero-free subreduct of a divisible and prelinear FL_{ew}-algebra. Note that in any FL_{ew}-algebras the prelinearity equation makes the join definable using $\wedge$ and $\to$ (see for instance [5]):

$$((x \to y) \to y) \wedge ((y \to x) \to x) \approx x \vee y.$$

So basic hoops are often presented in the signature $\wedge, \to, 1$; as noted in [8] the variety BH of basic hoops is the equivalent algebraic semantics of the positive fragment of the logic $\mathcal{BL}$.

A **Wajsberg hoop** is a basic hoop satisfying the so-called *Tanaka's equation*

$$(x \to y) \to y \approx (y \to x) \to x.$$

If we add a constant 0 to the signature that is the least element in the lattice order then we have **Wajsberg algebras**; Wajsberg algebras are term equivalent to MV-algebras (see [10] p. 354 for a detailed explanation) and the variety of MV-algebras is usually presented as the equivalent algebraic semantics of Łukasiewicz logic $\mathcal{MV}$. It follows that the variety WH of Wajsberg hoops, which is the variety of zero-free subreducts of Wajsberg algebras, is the equivalent algebraic semantics of $\mathcal{MV}^+$, i.e. the positive fragment of Łukasiewicz logic.

In FL_{ew}-algebras it is customary to introduce the derived operation $\neg x := x \to 0$; now in a bounded (i.e. with a minimum element a) Wajsberg hoop we can still introduce a negation $\neg x = x \to a$ that is of course not a term but rather a polynomial. Now it is easy to verify that any bounded Wajsberg hoop is polynomially equivalent to a Wajsberg algebra and we will freely use the expression $\neg x$ letting the context clear the meaning.

A commutative integral residuated lattice is **cancellative** if the underlying monoid is cancellative in the usual sense.

LEMMA 2.10
[18] Every cancellative basic hoop is a Wajsberg hoop. A totally ordered Wajsberg hoop is either cancellative or bounded.

This allows us to show that the connection between Wajsberg hoops and Wajsberg algebras is even stricter. For instance the operator $\mathbf{ISP}_u$ on Wajsberg hoops has been studied in [9] using the results about Wajsberg algebras appearing in [31]; while we maintain that it should be clear why we can do this (and in [9] no explanation was given), some clarification may be useful. Wajsberg algebras are polynomially equivalent to bounded Wajsberg hoops; it is easy to see that if $\mathbf{O}$ is a class operator that is a composition of $\mathbf{I}, \mathbf{H}, \mathbf{S}, \mathbf{P}, \mathbf{P}_u$, $\mathbf{A}, \mathbf{B}$ are Wajsberg algebras and $\mathbf{A}_0, \mathbf{B}_0$ are their Wajsberg hoop reducts, then $\mathbf{O}(\mathbf{A}) \subseteq \mathbf{O}(\mathbf{B})$ if and only if $\mathbf{O}(\mathbf{A}_0) = \mathbf{O}(\mathbf{B}_0)$. This allows us to consider bounded Wajsberg hoops *as if they were* Wajsberg algebras. Since a totally ordered Wajsberg hoop is either bounded or cancellative we can use results about Wajsberg algebras and integrate them with the cancellative case.

3 Some useful tools

3.1 Wajsberg chains

Bounded Wajsberg hoops have a *canonical representation*. Let $\mathbf{G}$ be a lattice ordered abelian group; by [41], if u is a strong unit of $\mathbf{G}$ we can construct a bounded Wajsberg hoop $\Gamma(\mathbf{G}, u) = \langle [0, u], \rightarrow, \cdot, 0, u \rangle$ where $ab = \max\{a + b - u, 0\}$ and $a \rightarrow b = \min\{u - a + b, u\}$. The main result of [41] is that any bounded Wajsberg hoop can be presented in this way (really there is a categorical equivalence between the category abelian ℓ-groups with strong unit and the category of bounded Wajsberg hoops). Let now $\mathbb{Z} \times_l \mathbb{Z}$ denote the lexicographic product of two copies of $\mathbb{Z}$. In other words, the universe is the Cartesian product and the group operations are defined componentwise and the ordering is the lexicographic ordering (w.r.t. the natural ordering of $\mathbb{Z}$); then $\mathbb{Z} \times_l \mathbb{Z}$ is a totally ordered abelian group and we can apply Γ to it. A **Wajsberg chain** is a totally ordered Wajsberg hoop. Let's define some useful Wajsberg chains:

- the finite Wajsberg chain with $n + 1$ elements $\mathbf{Ł}_n = \Gamma(\mathbb{Z}, n)$;
- the infinite finitely generated Wajsberg chain $\mathbf{Ł}_n^\infty = \Gamma(\mathbb{Z} \times_l \mathbb{Z}, (n, 0))$;
- the infinite finitely generated Wajsberg chain $\mathbf{Ł}_{n,k} = \Gamma(\mathbb{Z} \times_l \mathbb{Z}, (n, k))$;
- the unbounded Wajsberg chain $\mathbf{C}_\omega$; this can be regarded either as the free monoid on one generator, where the product is the monoid product and $a^l \rightarrow a^m = a^{\max(l-m,0)}$ or, equivalently, as the negative cone of $\mathbb{Z}$ with the operations defined in the obvious way.

We observe that $\mathbf{Ł}_n^\infty = \mathbf{Ł}_{n,0}$; moreover the proof of the following is a simple verification:

LEMMA 3.1

(1) For $n, m \in \mathbb{N}$, $\mathbf{Ł}_n \in \mathbf{IS}(\mathbf{Ł}_m)$ if and only if $\mathbf{Ł}_n \in \mathbf{IS}(\mathbf{Ł}_m^\infty)$ if and only if $n \mid m$.
(2) For $n, r, j \in \mathbb{N}$, $\mathbf{Ł}_n \in \mathbf{IS}(\mathbf{Ł}_{r,j})$ if and only if $n \mid \gcd\{r, j\}$.
(3) If $\mathbf{A}$ is a cancellative Wajsberg chain and $a \in A \setminus \{1\}$, then a generates a subalgebra of $\mathbf{A}$ isomorphic with $\mathbf{C}_\omega$.

In [10] it has been shown that every proper variety of Wajsberg hoops is generated by a finite number of chains of the type described above. More precisely a **presentation** P is a triple (I, J, K), where I, J are finite subsets of $\mathbb{N} \setminus \{0\}$ and $K \subseteq \{\omega\}$. We say that the presentation $P = (I, J, K)$ is **reduced** if

- $I \cup J \cup K \neq \emptyset$;
- if $J \neq \emptyset$ then $K = \emptyset$;
- no $m \in I$ divides any $m' \in (I \setminus \{m\}) \cup J$;
- no $n \in J$ divides any $n' \in J \setminus \{n\}$.

For any reduced presentation $P = (I, J, K)$ we define a set of Wajsberg hoops K_P in the following way:

- if $P = (I, J, \emptyset)$ then $\mathsf{K}_P = \{\mathbf{Ł}_i : i \in I\} \cup \{\mathbf{Ł}_j^\infty : j \in J\}$,
- if $P = (I, \emptyset, \{\omega\})$ then $\mathsf{K}_P = \{\mathbf{Ł}_i : i \in I\} \cup \{\mathbf{C}_\omega\}$.

Then we set $\mathsf{V}(P) = \mathbf{V}(\mathsf{K}_P)$ and $\mathsf{Q}(P) = \mathbf{Q}(\mathsf{K}_P)$.

THEOREM 3.2
[10] The proper subvarieties of Wajsberg hoops are in one-to-one correspondence with the reduced presentations via

$$P \longmapsto \mathsf{V}(P).$$

There are two more observations we would like to make:

(1) not every quasivariety of Wajsberg hoops comes from a presentation of the type described above (see [7] Section 5); however as far as structural completeness is concerned those are the only ones we need to deal with;
(2) it is easy to check that any variety generated by Wajsberg chains in generated as a quasivariety by those chains (see again [7]); we will use this fact without further mention.

3.2 The construction of $\mathbf{B}_\Delta$

Any proper subvariety of Wajsberg hoops is axiomatizable (modulo Wajsberg hoops) by an equation in a single variable; this is essentially Theorem 4.4 in [10] and its proof uses the functional characterization of free Wajsberg hoops, which we will be using later in this paper. From that, it follows that for any proper subvariety V of Wajsberg hoops $\mathbf{Q}(\mathbf{F}_\mathsf{V}(\omega)) = \mathbf{Q}(\mathbf{F}_\mathsf{V}(x))$. Indeed, $\mathsf{V} = \mathbf{HQ}(\mathbf{F}_\mathsf{V}(\omega)) = \mathbf{HQ}(\mathbf{F}_\mathsf{V}(x))$ because V can be axiomatized in one variable and, since $\mathbf{Q}(\mathbf{F}_\mathsf{V}(\omega))$ is structural, this means that it has to be the smallest quasivariety that generates V, so $\mathbf{Q}(\mathbf{F}_\mathsf{V}(\omega)) \subseteq \mathbf{Q}(\mathbf{F}_\mathsf{V}(x))$; the other inclusion is trivial.

In this section we will give an alternative description of $\mathbf{F}_\mathsf{V}(x)$ where V is a proper subvariety of Wajsberg hoops.

LEMMA 3.3
Let $\mathbf{A}$ be a totally ordered Wajsberg hoop, assume that $\mathbf{A}$ is one-generated, then $\mathbf{A} \in \mathbf{V}(\mathbf{Ł}_n^\infty)$ if and only if one of the following holds:

(1) there exists $1 \leq k|n$ such that $\mathbf{A} \cong \mathbf{L}_k$;
(2) there exist $1 \leq k|n$ and $0 \leq h < k$ with k, h relatively prime such that $\mathbf{A} \cong \mathbf{Ł}_{k,h}$;
(3) $\mathbf{A} \cong C_\omega$.

PROOF. As $\mathbf{A}$ is totally ordered, it is either bounded or cancellative. If it is cancellative then since it is one-generated, it must be $\mathbf{A} \cong C_\omega$; if it is bounded, then it can be seen as an MV-algebra, so, by [26] (Theorem 1.8), either (1) or (2) holds.

Conversely if $\mathbf{A} \cong C_\omega$, then clearly $\mathbf{A} \in \mathbf{V}(\mathbf{Ł}_n^\infty)$; in the other cases we appeal again to Theorem 1.8 in [26]. □

From now on, given a finite subset X of $\mathbb{N}$, we will denote by $X{\downarrow}$ the set of all the divisors of elements of X.

LEMMA 3.4
Let $\mathbf{A}$ be a totally ordered Wajsberg hoop, assume that $\mathbf{A}$ is one generated and let $P = (I, J, \emptyset)$ be a reduced presentation with $J \neq \emptyset$. Then $\mathbf{A} \in \mathsf{V}(P)$ if and only if one of the following holds:

(1) there exists $k \in I{\downarrow} \cup J{\downarrow}$ such that $\mathbf{A} \cong \mathbf{Ł}_k$;
(2) there exist $k \in J{\downarrow}$ and $0 \leq h < k$ with k, h relatively prime such that $\mathbf{A} \cong \mathbf{Ł}_{k,h}$;
(3) $\mathbf{A} \cong C_\omega$.

PROOF. The 'if' part is exactly as in Lemma 3.3.

If $\mathbf{A} \cong \mathbf{Ł}_k$ for some $k \in I{\downarrow} \cup J{\downarrow}$, then, if $k \in I{\downarrow}$ there exists $i \in I$ such that $\mathbf{A} \in \mathbf{V}(\mathbf{Ł}_i) \subseteq \mathsf{V}(I, J, \emptyset)$; if $k \in J{\downarrow}$, then by Lemma 3.3 there exists a $j \in J$ such that $\mathbf{A} \in \mathbf{V}(\mathbf{Ł}_j^\infty) \subseteq \mathsf{V}(I, J, \emptyset)$.

If $\mathbf{A} \cong \mathbf{Ł}_{k,h}$ for some $k \in J{\downarrow}$ and $0 \leq h < k$ with h, k relatively prime, then by Lemma 3.3 there exists $j \in J$ such that $\mathbf{A} \in \mathbf{V}(\mathbf{Ł}_j^\infty) \subseteq \mathsf{V}(I, J, \emptyset)$. Finally, if $\mathbf{A} \cong C^\omega$ clearly $\mathbf{A} \in \mathbf{V}(\mathbf{Ł}_j^\infty)$ for every $j \in J{\downarrow}$, so, since $J \neq \emptyset$, $\mathbf{A} \in \mathsf{V}(I, J, \emptyset)$. ☐

REMARK 3.5

If $\mathbf{A} \cong \mathbf{Ł}_k$, then h generates $\mathbf{A}$ if and only if h, k are relatively prime.

If $\mathbf{A} \cong \mathbf{Ł}_{k,h}$ with $k \neq 1$ and $h < k$ then there exists a unique $g_{k,h} \in \mathbf{A}$ with $g_{k,h} \leq \neg g_{k,h}$ such that a generates $\mathbf{A}$ if and only if $a = g_{k,h}$ or $a = \neg g_{k,h}$. Moreover $g_{k,1} = (1, 0), g_{k,k-1} = (1, 1)$ and, if $h \neq 1, h \neq k - 1$, then $g_{k,h} = (r, s)$ with $1 < r < \frac{k}{2}$. If $k = 1$ we get that $\mathbf{Ł}_{1,0}$ is generated by $g_{1,0} = (0, 1)$, but this time $\neg g_{1,0} = (1, -1)$ generates a subalgebra of $\mathbf{Ł}_{1,0}$ isomorphic to C_ω.

Note that in all these cases, we can use the operation $\neg$ because all the algebras are bounded; in particular, since the generator of $\mathbf{Ł}_{k,h}$ has always order 2, we can write 0 as $g_{k,h}^2$, so for every $a \in \mathbf{Ł}_{k,h}$ we get $\neg a = a \to g_{k,h}^2$.

Now we fix a reduced triple $P = (I, J, K)$ and let

$$\Delta_I = \{(k, h, 2) : 0 \leq h < k \in I{\downarrow}, k, h \text{ relatively prime}\}$$

$$\Delta_J = \{(k, h, i) : i \in \{0, 1\}, h < k \in J{\downarrow}, k, h \text{ relatively prime}\}$$

$$\Delta_K = \begin{cases} \emptyset, & \text{if } J \neq \emptyset; \\ \{(0, 0, 3)\}, & \text{otherwise.} \end{cases}$$

Moreover for any k, h we define

$$\mathbf{A}_{k,h}^0 = \mathbf{A}_{k,h}^1 := \mathbf{Ł}_{k,h}$$

$$\mathbf{A}_{k,h}^2 = \mathbf{Ł}_k$$

$$\mathbf{A}_{k,h}^3 = \mathbf{C}_\omega.$$

If $\Delta = \Delta_I \cup \Delta_J \cup \Delta_K$ we let $\mathbf{A}_\Delta = \prod_{(k,h,i) \in \Delta} \mathbf{A}_{k,h}^i$; moreover we denote by c the generator of $\mathbf{C}_\omega$. We want to define a $\overline{g} \in \mathbf{A}_\Delta$ by cases; for any $(k, h, i) \in \Delta$

$$\text{if } J = \emptyset, \ \ \overline{g}(k, h, i) = \begin{cases} h, & \text{if } i = 2; \\ c, & \text{if } i = 3. \end{cases}$$

$$\text{if } J \neq \emptyset, \ \ \overline{g}(k, h, i) = \begin{cases} g_{k,h}, & \text{if } i = 0; \\ \neg g_{k,h}, & \text{if } i = 1; \\ h, & \text{if } i = 2. \end{cases}$$

THEOREM 3.6

Let $P = (I, J, K)$ be any reduced triple and let $\overline{g}$ and $\mathbf{A}_\Delta$ be as above. If $\mathbf{B}_\Delta$ is the subalgebra of $\mathbf{A}$ generated by $\overline{g}$, then $\mathbf{B}_\Delta \cong \mathbf{F}_{\mathsf{V}(P)}(x)$.

PROOF. First we observe that $\mathbf{A}_\Delta \in \mathsf{V}(P)$ and so does $\mathbf{B}_\Delta$. Suppose that $p(x) \approx q(x)$ is an equation that fails in $\mathsf{V}(P)$; then it must fail in some one-generated totally ordered algebra $\mathbf{C} \in \mathsf{V}(P)$ and such algebra is either bounded or cancellative.

First, let us show that we only need to discuss the case in which $p(x) \approx q(x)$ fails in a generator of $\mathbf{C}$. Suppose that the equation fails in some $x \in \mathbf{C}$, then, if we call $\mathbf{C}'$ the subalgebra of $\mathbf{C}$ generated by x, we have that $p(x) \approx q(x)$ fails in the generator of $\mathbf{C}'$, which is still an algebra in $\mathsf{V}(P)$.

Suppose that $J = \emptyset$. If $\mathbf{C}$ is bounded, then it cannot be infinite, as $\mathbf{Ł}_n^\infty \notin \mathsf{V}(P)$ for any $n \in \mathbb{N}$. Hence it must be equal to $\mathbf{Ł}_k$ for some $k \in I\downarrow$; this implies that $p(x) \approx q(x)$ fails in $g(k, h, 2)$ for some h and, as above, fails in $\mathbf{B}_\Delta$. This covers the case $K = \emptyset$ and half of the case $K \neq \emptyset$. To conclude, if $\mathbf{C}$ is cancellative, then the equation must fail in $\mathbf{C}_\omega$ and hence (if c is the generator of $\mathbf{C}_\omega$), $p(c) \neq q(c)$; this implies that $p(\overline{g}(0,0,3)) \neq p(\overline{g}(0,0,3))$, so $p(\overline{g}) \neq q(\overline{g})$ and $p(x) \approx q(x)$ fails in $\mathbf{B}_\Delta$.

Suppose now that $J \neq \emptyset$ (and thus $K = \emptyset$). By Lemma 3.4 we have only three possibilities.

If $\mathbf{C} \cong \mathbf{Ł}_k$ for some $k \in I\downarrow \cup J\downarrow$; if $k \in I\downarrow$ then $p(x) \approx q(x)$ fails in some generator of $\mathbf{Ł}_k$, so it fails in some $\overline{g}(k, h, 2)$ for some h and eventually fails in $\mathbf{B}_\Delta$. If $k \in J\downarrow$ again $p(x) \approx q(x)$ fails in some generator h of $\mathbf{Ł}_k$; now h and k must be relatively prime and thus $p(x) \approx q(x)$ fails in $\mathbf{Ł}_{k,h}$. By Remark 3.5 it fails either in $g_{h,k}$ or $\neg g_{h,k}$, hence $p(x) \approx q(x)$ fails either in $\overline{g}(h, k, 0)$ or $\overline{g}(h, k, 1)$. In any case $p(x) \approx q(x)$ fails in $\mathbf{B}_\Delta$.

If $\mathbf{C} \cong \mathbf{Ł}_{h,k}$ the argument is similar to the one above, but easier. Finally if $\mathbf{C} \cong \mathbf{C}_\omega$, since $\overline{g}(1,0,1) = \neg g_{1,0} = (1,-1)$ generates a subalgebra of $\mathbf{Ł}_{1,0}$ isomorphic to $\mathbf{C}_\omega$, for sure $p(\overline{g}(1,0,1)) \neq q(\overline{g}(1,0,1))$ and so again $p(x) \approx q(x)$ fails in $\mathbf{B}$.

We have thus proved that every equation in one variable that fails in $\mathsf{V}(P)$ fails in $\mathbf{B}_\Delta$. At last, let us show that this is sufficient to say that $\mathbf{B}_\Delta \cong \mathbf{F}_{\mathsf{V}(P)}(x)$. Indeed, since $\mathbf{B}_\Delta$ is in $\mathsf{V}(P)$ and it is one-generated, we know that we have a surjective homomorphism φ from $\mathbf{F}_{\mathsf{V}(P)}(x)$ to $\mathbf{B}_\Delta$; suppose, by way of contradiction, that φ is not injective, this means that $\mathrm{Ker}(\varphi)$ is non-trivial, so there exist two terms $p, q \in \mathbf{F}_{\mathsf{V}(P)}(x)$ such that $p \neq q$ but $\varphi(p) = \varphi(q)$, thus $p(x) \not\approx q(x)$ in $\mathbf{F}_{\mathsf{V}(P)}(x)$ but $p(x) \approx q(x)$ in $\mathbf{B}_\Delta$. Hence φ must be an isomorphism. □

3.3 Wajsberg functions

A **McNaughton function** over the n-cube is a continuous function $f : [0,1]^n \to [0,1]$ such that there exist finitely many linear functions $f_1, \dots, f_k$, where each f_i is of the form $f_i = a_i^0 x_0 + a_i^1 x_1 + \cdots + a_i^n x_n + b_i$ with $a_i^0 \dots a_i^n, b_i$ integers, and such that for any $v \in [0,1]^n$ there exists $i \in \{1, \dots k\}$ with $f(v) = f_i(v)$. A McNaughton function $f(x_1, \dots, x_n)$ is a **Wajsberg function** if $f(1,1,\dots,1) = 1$.

THEOREM 3.7
[10] For each n, the free n-generated Wajsberg hoop $\mathbf{F}_{\mathsf{WH}}(n)$ is isomorphic to the algebra of all Wajsberg functions over the n-cube, where the operations are defined pointwise.

So we can always identify an n-ary term in the language of Wajsberg hoops with a Wajsberg function over the n-cube. Conversely, given a Wajsberg function over the n-cube, we can associate to it an equivalence class of Wajsberg terms (where the equivalence is of course mutual provability in the theory). With the usual abuse of notation we will identify the class with any of its representatives, i.e. given a Wajsberg function f we will denote by $\widehat{f}$ the Wajsberg term that is a representative to the equivalence class corresponding to f.

In [10] the authors used this representation to give an easy way to axiomatize all proper subvarieties of Wajsberg hoops. Let (I, J, K) be a reduced triple, we define two finite subsets $\mathcal{I}, \mathcal{J}$ of rational points of $[0,1]$ (in what follows $\mathrm{den}(u)$ denote the denominator of u in the usual sense) as

- if $K \neq \emptyset$ then $\mathcal{J} = \{1\}$;
- if $K = \emptyset$ then $\mathcal{J} = \{v \in [0,1] : \mathrm{den}(v) \in J{\downarrow}\}$;
- $\mathcal{I} = \{u \in [0,1] : \mathrm{den}(u) \in I{\downarrow}\} \setminus \mathcal{J}$.

Given a reduced triple (I,J,K) an (I,J,K)**-comb** is any $\alpha \in \mathbf{F}_{\mathsf{WH}}(x)$ such that

(1) for every $v \in \mathcal{J}$, there exists a neighborhood V of v such that $\alpha = 1$ on V;
(2) for every $u \in \mathcal{I}$, $\alpha(u) = 1$;
(3) for every $u \in \mathcal{I}$ there exists $v \in \mathcal{I}$ such that $\mathrm{den}(v) | \mathrm{den}(u)$ and α is not identically 1 on any neighborhood of v;
(4) if $d \notin (I \cup J){\downarrow}$, then there exists $0 \leq h < d$ with $\alpha(h/d) \neq 1$.

THEOREM 3.8
[10] Let $P = (I,J,K)$ be a reduced triple and let $\alpha(x) \in F_{\mathsf{WH}}(x)$. Then the identity $\alpha(x) = 1$ axiomatizes $\mathsf{V}(P)$ relative to WH if and only if α is an (I,J,K)-comb.

This is a very powerful result, in that it gives a procedure that allows one to axiomatize every proper subvariety of Wajsberg hoops, and combs are quite easy to construct.

The **radical** of a Wajsberg chain $\mathbf{A}$, in symbols $\mathrm{Rad}(\mathbf{A})$, is the intersection of the maximal filters of $\mathbf{A}$; it is easy to see that $\mathrm{Rad}(\mathbf{A})$ is cancellative and $\mathbf{A}$ is cancellative if and only if $\mathrm{Rad}(\mathbf{A}) = \mathbf{A}$. We say that a bounded Wajsberg hoop $\mathbf{A}$ **has rank** n, if $\mathbf{A}/\mathrm{Rad}(\mathbf{A}) \cong \mathbf{Ł}_n$. For any bounded Wajsberg hoop the **divisibility index** $d_{\mathbf{A}}$ of $\mathbf{A}$, is the maximum k such that $\mathbf{Ł}_k$ is embeddable in $\mathbf{A}$ if any, otherwise $d_{\mathbf{A}} = \infty$.

Next, we have a very useful lemma.

LEMMA 3.9
Let $p(x) \approx q(x)$ be an identity in the language of Wajsberg hoops and let f, g be Wajsberg functions such that $p = \widehat{f}$ and $q = \widehat{g}$. Then for any $n, k \in \mathbb{N}$ with $k \leq n$

(1) if $f(\frac{k}{n}) = g(\frac{k}{n})$, then $p(k) = q(k)$ where $k \in \mathbf{Ł}_n$;
(2) if $f(x) = g(x)$ in a neighborhood of 1, then $\mathbf{C}_\omega \vDash p(x) \approx q(x)$;
(3) if $f(x) = g(x)$ in a neighborhood of $\frac{k}{n}$, then $p(c) = q(c)$ for any $c \in \mathbf{Ł}_{n,h}$ such that $c/\mathrm{Rad}(\mathbf{Ł}_{n,h}) = k$.

The proof of Lemma 3.9 can be extracted from the proof of Theorem 3.3 in [10], by setting $\kappa = 1$.

Next, we want to have an easy way to construct Wajsberg functions and force them to have certain fixed values (see [35]). If $0 = t_0 < t_1 < \cdots < t_k = 1$ and $x_0, \ldots, x_k \in [0,1]$, then we denote by $f = L(t_0, x_0; \ldots; t_k, x_k)$ the continuous piecewise linear function $f : [0,1] \to [0,1]$ such that $f(t_i) = x_i$ and f is linear on each interval $[t_i, t_{i+1}]$; in other words, f is the linear interpolation between the nodes $(t_0, x_0), \ldots, (t_k, x_k)$. Clearly, it is possible to play with the variables in order to make f a Wajsberg function.

Let's see an example. If we want to find a $(2, \emptyset, \emptyset)$-comb we need to take a function f that has value 1 only in $\{0, \frac{1}{2}, 1\}$, so we can take

$$f = L(0,1; \frac{1}{4}, 0; \frac{1}{2}, 1; \frac{3}{4}, 0; 1, 1).$$

This function has integer coefficients in every interval and $f(1) = 1$, so it is a Wajsberg function and the identity $f(x) \approx 1$ axiomatizes the subvariety of Wajsberg hoops generated by $\mathbf{Ł}_2$.

Now we will heavily use Wajsberg functions and Lemma 3.9 to prove a couple of fundamental results. From now on, for any reduced triple $P = (I,J,K)$, $\mathbf{B}_\Delta$ will be the algebra constructed from P following the directions in Section 3.2.

THEOREM 3.10
Let $P = (I, J, K)$ be a reduced triple and let $a \in I$; then $\mathbf{Ł}_a$ is embeddable in $\mathbf{B}_\Delta$.

PROOF. If $1 \in I$, then $P = (\{1\}, \emptyset, K)$ and if $K = \emptyset$, then $\mathbf{B}_\Delta \cong \mathbf{Ł}_1$. So suppose that $K \neq \emptyset$; then by Theorem 3.6 $\mathbf{B}_\Delta$ is isomorphic with the subalgebra of $\mathbf{L}_1 \times \mathbf{C}_\omega$ generated by $(0, c)$. Consider the Wajsberg function

$$f(x) = L(0, 0; \frac{1}{2}, 1; 1, 1);$$

then it is easy to check that $\widehat{f} = (x \to x^2) \to x$. Now $f(0) = 0$ and $f(1) = 1$ in a neighborhood of 1, hence by Lemma 3.9 $f(\overline{g}) = (0, 1)$ and thus it generates a subalgebra of $\mathbf{B}_\Delta$ isomorphic with $\mathbf{L}_1$.

Now suppose that $1 \notin I$; we fix an $a \in I$ and we let m be the product of all the elements of $I \cup J$. Then we consider the Wajsberg function

$$f(x) = L(0, 1; \frac{1}{a} - \frac{1}{2m}, 1; \frac{1}{a}, 0; \frac{1}{a} + \frac{1}{2m}, 1; 1, 1).$$

Now clearly $f(x) = 1$ in any neighborhood of $\frac{n}{m}$ where $n \leq m$ and $\frac{n}{m} \neq \frac{1}{a}$ and moreover $f(\frac{1}{a}) = 0$. Since (I, J, K) is reduced, i does not divide any element of $I \cup J$ and so by Lemma 3.9

$$f(\overline{g}(k, h, i)) = \begin{cases} 0, & \text{if } (k, h, i) = (a, 1, 2); \\ 1, & \text{otherwise.} \end{cases}$$

Hence $\overline{g} \vee f(\overline{g})$ generates a subalgebra of $\mathbf{B}_\Delta$, isomorphic to the one generated by $\overline{g}(a, 1, 2)$. But the latter is a generator of $\mathbf{Ł}_a$ so $\mathbf{Ł}_a$ is embeddable in $\mathbf{B}_\Delta$, as desired. □

THEOREM 3.11
Let $P = (I, \emptyset, \{\omega\})$; then $\mathbf{C}_\omega$ is embeddable in $\mathbf{B}_\Delta$.

PROOF. If $I = \emptyset$, then $\mathbf{B}_\Delta \cong \mathbf{C}_\omega$. Otherwise let m be the product of all elements of I and consider the Wajsberg function

$$f(x) = L(0, 1; \frac{m-1}{m}, 1; \frac{m}{m+1}, \frac{m}{m+1}; 1, 1).$$

Again it is easy to see that $\widehat{f}(x) = x^m \to x^{m+1}$ and that $f(\frac{n}{m}) = 1$ for $\dfrac{n}{m} \neq 1$ so that $f(\overline{g}(k, h, i)) = 1$ whenever $i = 2$. In a neighborhood of 1 we have that $f(x) = x$, so $f(\overline{g}(0, 0, 3)) = c$, which generates $\mathbf{C}_\omega$; thus $f(\overline{g})$ generates a subalgebra of $\mathbf{B}_\Delta$ that is isomorphic with $\mathbf{C}_\omega$ and thus $\mathbf{C}_\omega$ is embeddable in $\mathbf{B}_\Delta$. □

4 Structural and primitive subquasivarieties

By Theorem 2.4 and 2.8 classifying all the structurally complete finitary (axiomatic) extensions of positive Łukasiewicz's Logic amounts to describing all the structural quasivarieties (varieties) of Wajsberg hoops. We will complete the task in this section but first we need some preliminary information. Here is a summary of the main results about the rank and the divisibility index; the proofs are either trivial or can be found in [9] or [31].

LEMMA 4.1
For any $n, k \geq 1$

(1) $\mathbf{Ł}_n$ is simple and $\mathbf{Ł}_n \in \mathbf{IS}(\mathbf{Ł}_k)$ if and only if $\mathbf{Ł}_n \in \mathbf{IS}(\mathbf{Ł}_k^\infty)$ if and only if $n \mid k$.

(2) Ł_n has rank n and divisibility index n.
(3) For any $k \geq 0$, $\text{Ł}_{n,k}$ is subdirectly irreducible, $\text{Ł}_{n,k}$ has rank n and $d_{\text{Ł}_{n,k}} = \gcd(n,k)$; in particular $d_{\text{Ł}_n^\infty} = n$.
(4) If $\mathbf{A}$ has rank n, then $\mathbf{A} \in \mathbf{ISP}_u(\text{Ł}_{n,k})$ if and only if $d_\mathbf{A}$ divides $\gcd(n,k)$.
(5) If $\mathbf{A}$ has rank n, then $\text{Ł}_{n,k} \in \mathbf{ISP}_u(\mathbf{A})$ if and only if $\gcd(n,k)$ divides $d_\mathbf{A}$.
(6) If $\mathbf{A}$ is a nontrivial totally ordered cancellative hoop then $\mathbf{ISP}_u(\mathbf{A}) = \mathbf{ISP}_u(\mathbf{C}_\omega)$.
(7) If $\mathbf{A}$ is a bounded Wajsberg chain of finite rank k, then $d_\mathbf{A}$ divides k, and $\mathbf{ISP}_u(\mathbf{A}) = \mathbf{ISP}_u(\text{Ł}_{k,d_\mathbf{A}})$.
(8) If $\mathbf{A}$ is a bounded Wajsberg chain of finite rank n, then $\mathbf{ISP}_u(\mathbf{A}) = \mathbf{ISP}_u(\text{Ł}_n^\infty)$ if and only if $d_\mathbf{A} = n$.

4.1 Structural subvarieties

Structural and primitive subvarieties of Wajsberg hoops were completely classified in [7]; the proof therein was more involved and 'ad hoc'. In this section we will show ho to get a better proof using the results in Section 3.

First we observe that every locally finite variety of Wajsberg hoops is primitive (Corollary 6.2 in [7]); since both $\mathbf{C}_\omega$ and Ł_j^∞ contain finitely generated subalgebras that fail to be finite, from Theorem 2.5 in [10] we get that a variety V of Wajsberg hoops is locally finite if and only if it is $\mathsf{V}(P)$ where $P = (I, \emptyset, \emptyset)$, if and only if it is finitely generated. Via the Blok–Pigozzi connection we get:

COROLLARY 4.2
Every locally tabular extension of $\mathcal{MV}^+$ is tabular, axiomatic and hereditarily structurally complete.

PROOF. By Theorem 3.2 every proper subvariety of Wajsberg hoops is of the form $\mathsf{V}(I,J,K)$ for some reduced presentation. It is obvious that $\mathbf{C}_\omega$ is not locally finite and it is easy to see that the same holds for Ł_n^∞ for $n \geq 1$. Therefore if $\mathsf{V}(I,J,K)$ is locally finite, then $J = K = \emptyset$. But I is a finite set, so $\mathsf{V}(I,\emptyset,\emptyset)$ is finitely generated. So every locally finite variety of Wajsberg hoops is finitely generated; so every locally tabular extension of $\mathcal{MV}^+$ is tabular and the rest follows from the previous observations. □

The variety $\mathsf{C} = \mathsf{V}(\emptyset,\emptyset,\{0\})$ is the variety of **cancellative hoops**; now C can be shown to be primitive by a variety of means. The simplest one is probably to observe first that it is an atom in the lattice of subvarieties $\Lambda(\mathsf{WH})$ [14] hence it is equationally complete. Then one can quote [16] where it is stated that any equationally complete congruence modular variety has no proper subquasivarieties. As C is congruence distributive (having a lattice reduct) it has no proper subquasivarieties and hence it is primitive.

Moreover WH itself is not structural; this is well-known and can be shown in several ways. The most direct one is probably to observe that the set $\{\text{Ł}_p : pprime\}$ consists of simple algebras with no proper subalgebras and then invoke Corollary 1 in [1]. Now we can characterize all the structural varieties of Wajsberg hoops

THEOREM 4.3
(see Theorem 6.10 in [7]) Let $P = (I,J,K)$ be a reduced triple such that $\mathsf{V} = \mathsf{V}(I,J,K)$ is a proper subvariety of Wajsberg hoops. Then V is structural if and only if either $J = \emptyset$, or $J = \{1\}$.

PROOF. Suppose that $J \neq \emptyset$ and $J \neq \{1\}$. Then there is an $n \in J$ with $n > 1$. Let $\mathsf{K} = \{\text{Ł}_i : i \in I\} \cup \{\text{Ł}_j : j \in J, j \neq n\} \cup \{\text{Ł}_{n,1}\}$; Clearly $\mathbf{Q}(\mathsf{K}) \subseteq \mathsf{Q}(P) = \mathsf{V}(P)$; if $\text{Ł}_n^\infty \in \mathbf{Q}(\mathsf{K})$ then, by Theorem 2.2, $\text{Ł}_n^\infty \in \mathbf{ISP}_u(\mathsf{K})$, since it is subdirectly irreducible. But all the chains in K are either finite or their divisibility index is not divisible by n; hence, by Lemma 4.1 (3) and (4), $\mathbf{Ł}_n^\infty \notin \mathbf{ISP}_u(\mathsf{K})$. Therefore $\mathbf{Q}(\mathsf{K}) \subsetneq \mathsf{Q}(P)$; however, as $\mathbf{Ł}_{n,1} \in \mathbf{V}(\mathbf{Ł}_n^\infty)$, $\mathbf{H}(\mathbf{Q}(\mathsf{K})) = \mathsf{V}(P)$. So $\mathsf{V}(P)$ is not structural.

For the converse, modulo the results on locally finite varieties above, we need only to prove that $\mathsf{V}(P)$ is structurally complete whenever $P = (I, \emptyset, \{\omega\})$ or $P = (I, \{1\}, \emptyset)$. In either case, by Theorems 3.10 and 3.11, every generator of $\mathsf{V}(P) = \mathsf{Q}(P)$ is embeddable in $\mathbf{B}_\Delta$. So

$$\mathsf{V}(P) = \mathsf{Q}(P) \subseteq \mathbf{Q}(\mathbf{B}_\Delta) = \mathbf{Q}(\mathbf{F}_{\mathsf{V}(P)}(x))$$

and this proves that $\mathsf{V}(P)$ is structural. □

By the description of the proper subvarieties of WH by reduced triples, we get at once:

COROLLARY 4.4
[7] A variety of Wajsberg hoops is structural if and only if it is primitive.

And thus, via the Blok–Pigozzi connection:

COROLLARY 4.5
An axiomatic extension on $\mathcal{MV}^+$ is structurally complete if and only if it is hereditarily structurally complete.

4.2 Structural subquasivarieties

For quasivarieties we need to work a little bit more. First we need a lemma that appears in [32] (Lemma 4.5):

LEMMA 4.6
Let $n > 1$ and let $\mathbf{D}_n$ be the subalgebra of $\mathbf{Ł}_{n,1} \times \mathbf{Ł}_{n,n-1}$ generated by $((1,0),(1,1))$. Then $\mathbf{Ł}_{n,1}$ is embeddable in $\mathbf{D}_n$.

Using Lemma 4.6 we can prove:

LEMMA 4.7
Let $P = (I, J, \emptyset)$ be a reduced triple. Then for any $j \in J$, $\mathbf{Ł}_{j,1}$ is embeddable in $\mathbf{B}_\Delta$.

PROOF. If $J = \emptyset$ we have nothing to prove.

If $1 \in J$, then since P is a reduced triple, $P = (I, \{1\}, \emptyset)$. Let m be the product of all the elements of I (if $I = \emptyset$ take $m = 1$) and consider the Wajsberg function

$$f(x) = L(0, 0; \frac{1}{3m}, 0; \frac{2}{3m}, 1; 1, 1).$$

This function has value 0 in a neighborhood of 0 and has value 1 for every $\frac{n}{m} \neq 0$, so, by Lemma 3.9, $f(\overline{g}(k,h,i)) = 0$ if $(k,h,i) = (1,0,0)$, otherwise $f(\overline{g}(k,h,i)) = 1$. Since $\overline{g}(1,0,0)$ is a generator of $\mathbf{Ł}_1^\infty$, $\overline{g} \vee f(\overline{g})$ generates a subalgebra of $\mathbf{B}_\Delta$ isomorphic to $\mathbf{Ł}_1^\infty$. Now, by Lemma 4.1, $\mathsf{Q}(\mathbf{Ł}_1^\infty) = \mathsf{Q}(\mathbf{Ł}_{1,1})$; hence $\mathbf{Ł}_{1,1}$ is embeddable into $\mathbf{Ł}_1^\infty$ and thus into $\mathbf{B}_\Delta$.

Now suppose $1 \notin J$ and fix $j \in J$. Let m be the product of every element of $I \cup J$ and consider the Wajsberg function

$$f(x) = L(0, 1; \frac{1}{j} - \frac{2}{3m}, 1; \frac{1}{j} - \frac{1}{3m}, 0; \frac{1}{j} + \frac{1}{3m}, 0; \frac{1}{j} + \frac{2}{3m}, 1; 1, 1).$$

This function has value 0 in a neighborhood of $\frac{1}{j}$ and has value 1 in a neighborhood of $\frac{n}{m}$ when $\frac{n}{m} \neq \frac{1}{j}$. By Lemma 3.9, $f(\overline{g}(k,h,i)) = 0$ if $(k,h,i) = (j,1,0)$ or $(k,h,i) = (j, j-1, 0)$. Now let us show that $f(\overline{g}(k,h,i)) = 1$ in all the other cases.

Take $(k,h,i) \neq (j,1,0),(j,j-1,0)$. If $i \in \{0,1\}$, then $\overline{g}(k,h,i) = (r,s)$ for some $(r,s) \in \mathbf{Ł}_{k,h}$; notice that, by construction of f, $f(\overline{g}(k,h,i)) \neq 1$ only if $\frac{r}{k} = \frac{1}{j}$, but this happens only if $jr = k$, but this can not happen because the triple is reduced, so k can not be a multiple of j. If $i = 2$, then $\overline{g}(k,h,i) = h$ for some $0 \leq h < k$ and k,h relatively prime; this time $f(\overline{g}(k,h,i)) \neq 1$ only if $\frac{h}{k} = \frac{1}{j}$, that is only if $j = \frac{k}{h}$, but this means that h divides k, which is not possible because k,h are relatively prime.

Thus, if we consider $\overline{g} \vee f(\overline{g})$, this generates a subalgebra of $\mathbf{B}_\Delta$ isomorphic to $\mathbf{D}_j$ as defined in Lemma 4.6; moreover, by the same lemma, we get that $\mathbf{L}_{j,1}$ is embeddable into $\mathbf{D}_j$ and hence into $\mathbf{B}_\Delta$. □

Let $P = (I,J,K)$ be a triple (not necessarily reduced) and let $\mathsf{Q}[I,J,K]$ be defined in the following way

$$\mathsf{Q}[I,\emptyset,K] = \mathsf{Q}(I,\emptyset,K)$$
$$\mathsf{Q}[I,J,\emptyset] = \mathbf{Q}(\{\mathbf{Ł}_i : i \in I\} \cup \{\mathbf{Ł}_{j,1} : j \in J\}) \quad \text{if } J \neq \emptyset.$$

THEOREM 4.8
Let Q be a quasivariety of Wajsberg hoops; then Q is structural if and only if it is either $\mathsf{Q} = \mathbf{Q}(\mathbf{F}_{\mathsf{WH}}(x))$ or else $\mathsf{Q} = \mathsf{Q}[P]$ for some reduced triple P.

PROOF. As the structurally complete subquasivarieties are exactly $\mathbf{Q}(\mathbf{F}_{\mathsf{V}}(x))$ for $\mathsf{V} \subseteq \mathsf{WH}$, it is enough to show that for every reduced triple $P = (I,J,K)$, $\mathsf{Q}[P] = \mathbf{Q}(\mathbf{F}_{\mathsf{V}(P)}(x))$. Moreover if V is a subvariety of WH, then $\mathbf{Q}(\mathbf{F}_{\mathsf{V}}(x))$ is the structural core of V, i.e. the smallest subquasivariety Q of V such that $\mathbf{V}(\mathsf{Q}) = \mathsf{V}$. Now for any reduced presentation P, we clearly have that $\mathbf{V}(\mathsf{Q}[P]) = \mathbf{V}(P)$; so to get the conclusion it is enough to prove that $\mathsf{Q}[P] \subseteq \mathbf{Q}(\mathbf{F}_{\mathsf{V}(P)}(x))$.

But Theorem 3.10, 3.11 and Lemma 4.7 show that any generator of $\mathsf{Q}[P]$ is embeddable in $\mathbf{B}_\Delta$; so

$$\mathsf{Q}[P] \subseteq \mathsf{Q}(\mathbf{B}_\Delta) = \mathbf{Q}(\mathbf{F}_{\mathsf{V}(P)}(x))$$

and the conclusion holds. □

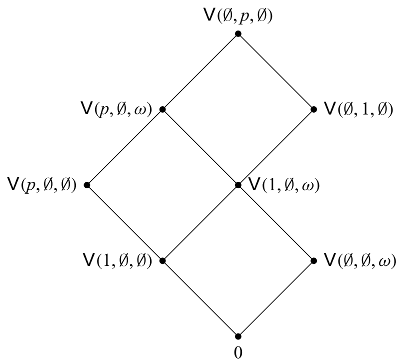
Observe that $\mathbf{Q}(\mathbf{Ł}_{1,1}) = \mathbf{Q}(\mathbf{Ł}_1^\infty)$; so if $P = (I,J,K)$ is such that either $J = \emptyset$ or $J = \{1\}$, then $\mathsf{Q}[P] = \mathsf{Q}(P)$ and they are all in fact primitive varieties, by Corollary 4.4. We can make another observation of some relevance based on the results in [25]. It is well known (and easy to prove) that the Wajsberg chains coincide with the finitely subdirectly irreducible Wajsberg hoops and the variety of Wajsberg hoops is congruence distributive. Then:

- every structural subquasivariety Q is generated by finitely subdirectly irreducible Wajsberg hoops and hence it is relatively congruence distributive;
- in any structural subquasivariety Q the finitely Q-irreducible algebras are finitely subdirectly irreducible in the absolute sense, i.e. they are all Wajsberg chains;
- hence any algebra $\mathbf{A} \in \mathsf{Q}$ is subdirectly embeddable in a product of Wajsberg chains that belong to Q.

4.3 Primitive subquasivarieties

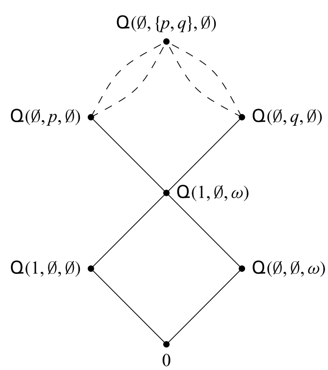
We consider only the quasivarieties $\mathsf{Q}[I,J,K]$ where $K = \emptyset$ and $J \neq \emptyset, \{1\}$, the reason being that in any other case $\mathsf{Q}[I,J,K]$ is either a primitive variety or it is not structurally complete. In this case we will simply write $\mathsf{Q}[I,J]$. We start with an example of the 'easy' part.

EXAMPLE 4.9
Let $\mathsf{Q} = \mathsf{Q}[\emptyset, p]$, where p is a prime number. We know that $\mathsf{V}(\mathsf{Q}) = \mathsf{V}(\emptyset, p, \emptyset)$ and, since Q is structural, this means that every quasivariety strictly contained in Q generates a variety that is strictly contained in $\mathsf{V}(\mathsf{Q})$. So now let's consider all the subvarieties of $\mathsf{V}(\emptyset, p, \emptyset)$.



If we consider a quasivariety Q' that generates a variety strictly contained in $\mathsf{V}(\emptyset, p, \emptyset)$, then it has to be contained in the coatoms of the lattice, which are $\mathsf{V}(p, \emptyset, \omega)$ and $\mathsf{V}(\emptyset, 1, \emptyset)$. But these two varieties are primitive, so every quasivariety contained in them is a (necessarily structural) variety. Therefore Q is primitive.

However it is not always the case that all the quasivarieties strictly contained in Q are varieties. Consider for example the quasivariety $\mathsf{Q} = \mathsf{Q}[\emptyset, \{p, q\}]$. Using Lemma 4.10 below, we may sketch what the lattice of the subquasivarieties of Q looks like:



Now, $\mathsf{Q}(\emptyset, p, \emptyset)$ and $\mathsf{Q}(\emptyset, q, \emptyset)$ are primitive by Lemma 4.9; however, we cannot conclude that there is no quasivariety in the intervals $[\mathsf{Q}(\emptyset, p, \emptyset), \mathsf{Q}(\emptyset, \{p, q\}, \emptyset)]$ and $[\mathsf{Q}(\emptyset, q, \emptyset), \mathsf{Q}(\emptyset, \{p, q\}, \emptyset)]$. Note that, if such a quasivariety exists, then it cannot be generated by chains; this would immediately imply that Q is not primitive, because by Theorem 4.8 this quasivariety would not be structural. We will see that also in this case there are no such quasivarieties, but it will take some doing. First we observe:

LEMMA 4.10
Let $(I, J, \emptyset)$ and $(I', J', \emptyset)$ be two triples (not necessarily reduced), then $\mathsf{Q}[I, J] \subseteq \mathsf{Q}[I', J']$ if and only if for every $i \in I$ s.t. $i \neq 1$ and for every $j \in J$, there are $i' \in I'$ and $j' \in J'$ with $i|i'$ and $j|j'$. In other words $\mathsf{Q}[I, J] \subseteq \mathsf{Q}[I', J']$ if and only if $I \subseteq I'\!\downarrow$ and $J \subseteq J'\!\downarrow$.

PROOF. If $i = 1$, then $\mathbf{Ł}_1 \in \mathbf{IS}(\mathbf{Ł}_n)$ and $\mathbf{Ł}_1 \in \mathbf{IS}(\mathbf{Ł}_{n,1})$ for every n. Take now $i \neq 1$ and suppose that $\mathbf{Ł}_i \in \mathbf{ISP}_u(\{\mathbf{Ł}_{i'} : i \in I'\} \cup \{\mathbf{Ł}_{j',1} : j' \in J'\})$; then $\mathbf{Ł}_i \notin \mathbf{ISP}_u(\{\mathbf{Ł}_{j',1} : j' \in J'\})$ as $\mathbf{Ł}_i$ has divisibility index $i \neq 1$ and each $\mathbf{Ł}_{j',1}$ has divisibility index equal to 1. On the other hand no $\mathbf{Ł}_{j,1}$ can belong to $\mathbf{ISP}_u(\{\mathbf{Ł}_{i'} : i \in I'\})$ for obvious reasons. From here it is straightforward to check that the conclusion holds. □

Next we have to refine our presentation via triples. Given a set of natural numbers I, we define $\overline{I}$ as the set given by $I \setminus \{i \in I \mid \exists i' \in I, i \neq i' \text{ and } i \text{ divides } i'\}$. Notice that $\overline{I} \subseteq I$ and, if $I \neq \{1\}$, then $1 \notin \overline{I}$. Moreover, for every set I we always have that $(\overline{I}, \emptyset, \emptyset)$ is a reduced triple because no $i \in \overline{I}$ can divide any $i' \in \overline{I}$ with $i' \neq i$; for the same reason $(\emptyset, \overline{J}, \emptyset)$ is a reduced triple for any set J.

LEMMA 4.11
$\mathsf{Q}[I, J] = \mathsf{Q}[\overline{I}, \overline{J}]$.

PROOF. Clearly $\mathsf{Q}[\overline{I}, \overline{J}] \subseteq \mathsf{Q}[I, J]$ because $\overline{I} \subseteq I$ and $\overline{J} \subseteq J$.

Let $i \in I \setminus \overline{I}$, then there exists $i' \in \overline{I}$ such that i divides i', so $L_i \in \mathsf{Q}[\overline{I}, \overline{J}]$ by lemma 4.12. Similarly, let $j \in J \setminus \overline{J}$, then there exists $j' \in \overline{J}$ such that j divides j', so $L_{j,1} \in \mathsf{Q}[\overline{I}, \overline{J}]$ by lemma 4.12. □

As an easy corollary to that, we get the following.

COROLLARY 4.12
If $\mathsf{Q}[I, J] = \mathsf{Q}[I', J']$ and $I, I' \neq \{1\}$, then $\overline{I} = \overline{I'}$ and $\overline{J} = \overline{J'}$. In particular $\overline{I}$ and $\overline{J}$ are the smallest (with respect to inclusion) I', J' such that $\mathsf{Q}[I, J] = \mathsf{Q}[I', J']$.

In a certain sense, given a triple not necessarily reduced $(I, J, \emptyset)$, $(\overline{I}, \overline{J}, \emptyset)$ is the 'most reduced triple' that doesn't lose any information w.r.t. the original one. This remark gives us a way to check if, given a quasivariety $\mathsf{Q}[I, J]$ with $(I, J, \emptyset)$ not necessarily reduced, there exists a reduced triple $(I', J', \emptyset)$ such that $\mathsf{Q}[I, J] = \mathsf{Q}[I', J']$.

LEMMA 4.13
Let $(I, J, \emptyset)$ be a triple not necessarily reduced with $I \neq \{1\}$, then there exists a reduced triple $(I', J', \emptyset)$ such that $\mathsf{Q}[I, J] = \mathsf{Q}[I', J']$ if and only if $(\overline{I}, \overline{J}, \emptyset)$ is a reduced triple.

PROOF. One direction is obvious. If there exists a reduced triple $(I', J', \emptyset)$ such that $\mathsf{Q}[I, J] = \mathsf{Q}[I', J']$, by the previous corollary we know that $\overline{I} \subseteq I'$ and $\overline{J} \subseteq J'$, so if $(\overline{I}, \overline{J}, \emptyset)$ is not reduced neither is $(I', J', \emptyset)$. □

Notice that in the previous lemmas we had to put an annoying $I \neq \{1\}$. This is only because $\overline{\{1\}} = \{1\}$ so, if $J \neq \emptyset$ (like in the cases where we want to apply this result), $(\overline{\{1\}}, \overline{J}, \emptyset)$ would never be a reduced triple. However we know that $\mathsf{Q}[\{1\}, J] = \mathsf{Q}[\emptyset, J]$ (Lemma 4.10); in particular there exists a reduced triple $(I', J', \emptyset) = (\emptyset, \overline{J}, \emptyset)$ such that $\mathsf{Q}[\{1\}, J] = \mathsf{Q}[I', J']$, and that's why we had to exclude this case.

Combining Corollary 4.12 with Theorem 4.8 we immediately get that:

COROLLARY 4.14
Let $(I, J, \emptyset)$ be a triple not necessarily reduced with $I \neq \{1\}$, then $\mathsf{Q}[I, J]$ is structural if and only if $(\overline{I}, \overline{J}, \emptyset)$ is a reduced triple.

The second ingredient we need is the following observation:

LEMMA 4.15
The quasivarieties of Wajsberg hoops generated by chains form a *distributive sublattice* of the lattice $\Lambda_q(\mathsf{WH})$ of all subquasivarieties of Wajsberg hoops.

This fact (crucial in our proofs) was stated in [31] and [32] for MV-algebras without a formal proof and, from the truth of it, one can easily deduce the same statement for Wajsberg hoops. However we believe that the argument given in those papers is not entirely correct or clear, so we provide a more transparent argument. The following lemma is an easy (and well-known) exercise in lattice theory, whose proof is left to the reader.

LEMMA 4.16
Let $\mathbf{L}, \mathbf{M}$ be two posets and let $\varphi : L \longrightarrow M$ be a bijective map such that φ and φ^{-1} are order preserving. Then if $\mathbf{L}$ is a (meet/join) semilattice or a lattice, then so is $\mathbf{M}$ and they are in fact isomorphic.

Let now WH_t be the class of all Wajsberg chains; for any quasivariety Q of Wajsberg hoops let $C_{\mathsf{Q}} = \mathsf{WH}_t \cap \mathsf{Q}$. If $\mathsf{Q} = \mathbf{Q}(M)$ is generated by chains, then since any other chain in Q is in $\mathbf{ISP}_u(M)$, we may assume that $\mathsf{Q} = \mathbf{Q}(C_{\mathsf{Q}})$. Now the operator $\mathbf{ISP}_u$ is clearly a closure operator on WH_t so the closed sets, i.e. the universal subclasses of WH_t, form a complete lattice where the meet is the intersection and $N \vee M = \mathbf{ISP}_u(N \cup M)$. Now let $\mathbf{U}$ be this lattice and let L_c be the class of all quasivarieties of Wajsberg hoops generated by chains, which is a poset under inclusion.

Let's define for $N \in U$, $\varphi(N) = \mathbf{Q}(N)$ and for $\mathsf{Q} = \mathbf{Q}(C_{\mathsf{Q}}) \in \mathsf{L}_c$, $\psi(\mathsf{Q}) = C_{\mathsf{Q}}$, then

$$\varphi\psi(\mathsf{Q}) = \varphi(C_{\mathsf{Q}}) = \mathbf{Q}(C_{\mathsf{Q}}) = \mathsf{Q} \qquad \psi\varphi(N) = \psi(\mathbf{Q}(N)) = \mathbf{Q}(N) \cap \mathsf{WH}_t = N$$

as N is universal. So φ, ψ compose to the identity on the respective domains and hence they are both bijections and $\varphi^{-1} = \psi$. That they both respect the inclusion is obvious. From Lemma 4.16 it follows that L_c is a lattice where

$$\mathsf{Q} \vee \mathsf{R} = \mathbf{Q}(C_{\mathsf{Q}} \vee C_{\mathsf{R}}) = \mathbf{Q}(\mathbf{ISP}_u(C_{\mathsf{Q}} \cup C_{\mathsf{R}})) = \mathbf{Q}(C_{\mathsf{Q}} \cup C_{\mathsf{R}})$$
$$\mathsf{Q} \wedge \mathsf{R} = \mathbf{Q}(C_{\mathsf{Q}} \cap C_{\mathsf{R}}).$$

So the join and the meet (in $\Lambda_q(\mathsf{WH})$) of two quasivarieties generated by chains are still quasivarieties generated by chains. The proof that L_c is distributive is left to the reader.

This fact combined with the results in [25] imply that the quasivarieties of Wajsberg hoops generated by chains are exactly those that are relatively congruence distributive (for MV-algebras this was observed long ago by H. Gaitan [29]). So the join of two relative congruence distributive quasivarieties of Wajsberg hoops is again relatively congruence distributive, another entirely nontrivial fact.

The last piece we need is a sufficient condition for primitivity, which holds in general. From Lemma 2.7 we know that, for any quasivariety Q, every subquasivariety is equational if and only if every subquasivariety is structural. The following lemma gives a sufficient condition for primitivity, which will be very useful.

LEMMA 4.17
Let Q be any quasivariety of algebras. If every equational subquasivariety of Q is structural, then Q is primitive.

PROOF. Let $\mathsf{Q}' \subseteq \mathsf{Q}$; then it is easy to check that $\mathsf{Q}'' = \mathbf{H}(\mathsf{Q}') \cap \mathsf{Q}$ is an equational subquasivariety of Q. So by hypothesis, Q'' is structural and also $\mathsf{Q}' \subseteq \mathsf{Q}''$. Now

$$\begin{aligned}\mathbf{H}(\mathsf{Q}'') &= \mathbf{H}(\mathbf{H}(\mathsf{Q}') \cap \mathsf{Q}) \\ &= \mathbf{H}(\mathsf{Q}') \cap \mathbf{H}(\mathsf{Q}) = \mathbf{H}(\mathsf{Q}').\end{aligned}$$

As Q'' is structural we must have $\mathsf{Q}' = \mathsf{Q}''$; hence Q' is equational and thus Q is primitive. □

We are now ready to prove the main theorem of this section.

THEOREM 4.18
$\mathsf{Q}[I,J]$ is primitive if and only if $I{\downarrow} \cap J{\downarrow} \subseteq \{1\}$.

PROOF. If $I{\downarrow} \cap J{\downarrow} \supsetneq \{1\}$, then there exists $n \neq 1$ in $I{\downarrow} \cap J{\downarrow}$ and, by Lemma 4.10, $\mathsf{Q}[n,n] \subseteq \mathsf{Q}[I,J]$. Now, notice that $\mathsf{V}(\mathsf{Q}[n,n]) = \mathsf{V}(\emptyset,n,\emptyset) = \mathsf{V}(\mathsf{Q}[\emptyset,n])$, so, by Theorem 4.8, the structural core of $\mathsf{Q}[n,n]$ is $\mathsf{Q}[\emptyset,n]$. But, by Lemma 4.10 again, $\mathsf{Q}[\emptyset,n] \subsetneq \mathsf{Q}[n,n]$ so, in particular, $\mathsf{Q}[n,n]$ is different from its structural core; hence it is not structural. Therefore Q contains a quasivariety that is not structural and this means that it is not primitive.

For the converse implication, suppose that $\mathsf{Q} = \mathsf{Q}[I,J]$ is not primitive. We want to show that $I{\downarrow} \cap J{\downarrow} \supsetneq \{1\}$.

If Q is not primitive, there by Lemma 4.17 there exists a $\mathsf{Q}' \leq \mathsf{Q}$ that is not structural and is such that $\mathsf{Q}' = \mathbf{H}(\mathsf{Q}') \cap \mathsf{Q}$. We know that $\mathbf{H}(\mathsf{Q}) = \mathsf{V}(I,J,\emptyset)$, and, since $\mathsf{Q}' \leq \mathsf{Q}$ we have $\mathbf{H}(\mathsf{Q}') \leq \mathsf{V}(I,J,\emptyset)$. In particular, $\mathbf{H}(\mathsf{Q}')$ can be seen as a quasivariety generated by chains, so $\mathsf{Q}' = \mathbf{H}(\mathsf{Q}') \cap \mathsf{Q}$ is still a quasivariety generated by chains (Lemma 4.15) and it is contained in Q. Hence $\mathsf{Q}' = \mathsf{Q}[I',J',K']$ for some triple (I',J',K') not necessarily reduced with $I' \subseteq I{\downarrow}$ and $J' \subseteq J{\downarrow}$.

First, notice that if $J' = \emptyset$, then $\mathsf{Q}' = \mathsf{Q}[I',\emptyset,K']$ would be structural, therefore we must have $J' \neq \emptyset$ and $K' = \emptyset$, so $\mathsf{Q}' = \mathsf{Q}[I',J']$. If $I' = \emptyset$ then, by Lemma 4.10, $\mathsf{Q}[I',J'] = \mathsf{Q}[\emptyset,\overline{J'}]$ and also $(\emptyset,\overline{J},\emptyset)$ is a reduced triple; so Q' would be structural by Corollary 4.14, again a contradiction. In the same fashion, if $I' = \{1\}$, since $J' \neq \emptyset$, we have $\mathsf{Q}[\{1\},J'] = \mathsf{Q}[\emptyset,\overline{J'}]$ that is structural. In conclusion we must have $I \neq \emptyset, \{1\}$.

Since Q' is not structural, by Corollary 4.14 we have that $(\overline{I'},\overline{J'},\emptyset)$ is not a reduced triple and this means $\exists i \in \overline{I'}$ and $\exists j \in \overline{J'}$ such that i divides j. Notice that $i \neq 1$ because $I' \neq \{1\}$ implies $1 \notin \overline{I'}$; but now $\overline{I'} \subseteq I' \subseteq I{\downarrow}$ and $\overline{J'} \subseteq J' \subseteq J{\downarrow}$. Since m $J{\downarrow}$ is closed under divisors of its elements, $i \in I{\downarrow} \cap J{\downarrow}$, which concludes the proof. □

5 Conclusions and future work

What can we say about the fragments of $\mathcal{MV}^+$? We will consider only the fragments containing $\rightarrow$, as they are the algebraizable ones. The $\{\rightarrow\}$-fragment has been studied first in [38]; its equivalent algebraic semantics is the variety LBCK of **Łukasiewicz BCK algebras**. We have that:

- every locally finite quasivariety of LBCK-algebras is a primitive variety [11];
- the only non-locally finite subvariety is the entire variety LBCK [38];
- LBCK is generated as a quasivariety by its finite chains [8];
- every infinite chain contains all the finite chains as subalgebras [38];
- so if Q is a quasivariety that contains only finitely many chains, then $\mathbf{V}(\mathsf{Q})$ is locally finite, hence primitive;
- otherwise Q contains infinitely many chains and so $\mathbf{V}(\mathsf{Q}) = \mathsf{Q} = \mathsf{LBCK}$.

Hence every subquasivariety of LBCK is a variety and LBCK is primitive.

For the other algebraizable fragments, observe that if $\rightarrow$ is present then $\vee$ is definable and if $\rightarrow$ and $\cdot$ are present, then $\wedge$ is definable. So the only remaining interesting fragment is the $\{\rightarrow,\wedge,1\}$-fragment that has been considered in [6]. Its equivalent algebraic semantics is the variety $\mathsf{LBCK}^{\wedge}$ of LBCK-semilattices; from the results in [6] (and some straightforward calculations) one can prove that $\mathsf{LBCK}^{\wedge}$ is primitive.

As far as the future work is concerned, there is a very natural path to follow. In this paper we have characterized all the structurally complete finitary extensions of $\mathcal{MV}^+$ (all structural and/or primitive quasivarieties of Wajsberg hoops). A sensible next step is to investigate the positive fragment $\mathcal{BL}^+$ of Hajek's Basic Logic $\mathcal{BL}$ [34], whose equivalent algebraic semantics is the variety of *basic hoops* [8]. This because in [9] (Theorem 3.7) it has been shown that there is a very deep algebraic connection between basic hoops and and Wajsberg hoops.

We observe also that in [32] the author characterized the structurally complete finitary extensions of $\mathcal{MV}$, which corresponds to structural quasivarieties of MV-algebras. He did not characterize the primitive subquasivarieties though (see also [7], end of Section 8, for some information on the problem). The main drawback seems to be the fact that structural quasivarieties of MV-algebras are not always generated by chains and therefore our arguments cannot be applied directly.

However, using the knowledge we have accumulated so far (and using also more general techniques introduced in [11]), we believe we can tackle these problems with some degree of success.

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