OPEN ACCESS
Bulletin of Mathematical Sciences
(2025) 2550001 (27 pages)
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DOI: 10.1142/S1664360725500018

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Linear stabilization for a degenerate wave equation in non divergence form with drift

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Received 5 December 2024
Accepted 2 January 2025
Published 4 February 2025

Communicated by Ahmed Alsaedi

We consider a degenerate wave equation in one dimension, with drift and in presence of a leading operator which is not in divergence form. We impose a homogeneous Dirichlet boundary condition where the degeneracy occurs and a boundary damping at the other endpoint. We provide some conditions for the uniform exponential decay of solutions for the associated Cauchy problem.

Keywords: Degenerate wave equation; drift; stabilization; exponential decay.

Mathematics Subject Classification 2020: 35L80, 93D23, 93D15

1. Introduction

It is well known that the displacement of a mass subjected to the action of spring is modeled by a nonlinearly damped oscillator and the displacement u of the mass is described by the scalar equation

$$u'' + h(u') + ku + f(u) = 0.$$

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Here $k > 0$ is a physical parameter, f is continuous and describes some nonlinear phenomenon, while $h(u')$ is the nonlinear damping. Set

$$F(u) = \int_0^u f(t)dt$$

and define the energy associated to a solution u as

$$E_u(t) = \frac{1}{2}(u'(t))^2 + \frac{1}{2}ku^2(t) + F(u(t)).$$

Assuming that

$$sh(s) \geq 0 \quad \forall\, s \in \mathbb{R}, \tag{1.1}$$

then

$$E_u'(t) = -u'(t)h(u'(t)) \leq 0,$$

which is a dissipation relation.

Similarly, consider a vibrating membrane fixed on the boundary. Then the evolution of the displacement of a point x of the membrane at time t is described by the wave equation

$$u_{tt} - \Delta u + h(u_t) + f(u) = 0 \quad \text{in } (0,\infty) \times \Omega,$$

where Ω is a (possibly bounded) domain of $\mathbb{R}^N$. The energy of a solution u is

$$E_u(t) = \int_\Omega \left[\frac{1}{2}(u_t^2 + |\nabla u|^2) + F(u)\right] dx$$

and if everything is smooth enough and h is as before, we get

$$E_u'(t) = -\int_\Omega u_t h(u_t) dx \leq 0.$$

An analogous result can be proved if the wave equation is coupled with a damping in a portion Γ of $\partial\Omega$ as

$$\frac{\partial u}{\partial \nu} + h(u_t) = 0 \quad \text{in } [0,\infty) \times \Gamma.$$

From a monotonicity property of the energy, it is natural to look for conditions which guarantee the stability of even more general problems, namely

$$\|\nabla u\|_{L^2} + \|u_t\|_{L^2} \to 0 \quad \text{as } t \to \infty.$$

In this respect, several conditions have been found, even when (1.1) is not satisfied, also in the presence of nonlinear sources (see, for instance, [13–15, 17, 19] and the two monographs [1, 3]).

However, all the previous results concern nondegenerate problems. On the other hand, the standard linear theory for transverse waves in a string leads to the classical wave equation

$$\rho(x)u_{tt}(t,x) = \frac{\partial \mathcal{T}}{\partial x}(t,x)u_x(t,x) + \mathcal{T}(t,x)u_{xx}(t,x),$$

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where $u(t,x)$ is the vertical displacement of the string from the x-axis at position x and time t, $\rho(x)$ is the mass density of the string at position x, while $\mathcal{T}(t,x)$ denotes the tension in the string at position x and time t. Dividing by $\rho(x)$, assuming $\mathcal{T}$ is independent of t, and setting $a(x) = \mathcal{T}(x)\rho^{-1}(x)$, $b(x) = \mathcal{T}'(x)\rho^{-1}(x)$, we obtain

$$u_{tt}(t,x) = a(x)u_{xx}(t,x) + b(x)u_x(t,x).$$

If density is extremely large at some point, for instance $x = 0$, we can assume $a(0) = 0$. The drift term b may degenerate at $x = 0$, as well.

A related equation in divergence form and without drift is

$$u_{tt}(t,x) = (a(x)u_x)_x(t,x),$$

which has been studied in [2] (see also the arXiv version of 2015) and [20] for a general a and for $a(x) = x^K$, $K \in (0,2)$, respectively. In both cases, boundary controllability was pursued via multiplier methods [2] or spectral methods [20]. Moreover, in [2], stability results were proved, as well. Inspired by such a case, here we consider the problem

$$\begin{cases} y_{tt} - a(x)y_{xx} - b(x)y_x = 0, & (t,x) \in Q, \\ y_t(t,1) + \eta y_x(t,1) + \beta y(t,1) = 0, & t > 0, \\ y(t,0) = 0, & t > 0, \\ y(0,x) = y_0(x), \quad y_t(0,x) = y_1(x), & x \in (0,1), \end{cases} \tag{1.2}$$

where $Q = (0,+\infty) \times (0,1)$ $a, b \in C^0[0,1]$, with $a > 0$ on $(0,1]$, $a(0) = 0$ and $\frac{b}{a} \in L^1(0,1)$: hence, if $a(x) = x^K$, $K > 0$, we can consider $b(x) = x^h$ for any $h > K-1$. In the boundary term, we take $\beta \geq 0$ and η as the well-known absolutely continuous weight function

$$\eta(x) := \exp\left\{\int_{\frac{1}{2}}^{x} \frac{b(s)}{a(s)}ds\right\}, \quad x \in [0,1],$$

introduced by Feller in a related context [10] and used by several authors, see, e.g. [7, 9, 18]. Finally, the initial data u_0 and u_1 belong to suitable weighted spaces.

The main feature in this problem is that a degenerates at $x = 0$ (with b possibly degenerate, as well) and that the leading operator is not in the usual divergence form. As a consequence, classical methods cannot be used directly to study such a problem and a different approach is needed, see the following section.

As for the function a, we consider two cases: a can be weakly degenerate or strongly degenerate. More precisely, we have the following standard definition.

Definition 1.1. A function a is *weakly degenerate* (WD) *at* 0, for short, if $a \in C^0[0,1] \cap C^1(0,1]$ is such that $a(0) = 0$, $a > 0$ on $(0,1]$ and, if

$$K := \sup_{x\in(0,1]} \frac{x|a'(x)|}{a(x)}, \tag{1.3}$$

then $K \in (0,1)$.

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Definition 1.2. A function a is *strongly degenerate* (SD) *at* 0, for short, if $a \in C^1[0,1]$ is such that $a(0)=0$, $a>0$ on $(0,1]$ and in (1.3) we have $K \in [1,2)$.

In the previous definition, we always assume that $K<2$, since it is essential in Hypothesis 4 as follows.

Remark 1. Observe that (1.3) implies that the function

$$x \mapsto \frac{x^\gamma}{a(x)} \tag{1.4}$$

is nondecreasing in $(0,1]$ for all $\gamma \geq K$. In particular, Hypothesis 2 is satisfied. Moreover,

$$\lim_{x\to 0} \frac{x^\gamma}{a(x)} = 0 \tag{1.5}$$

for all $\gamma > K$ and

$$\left|\frac{x^\gamma b(x)}{a(x)}\right| \leq \frac{1}{a(1)}\|b\|_{L^\infty(0,1)} \tag{1.6}$$

for all $\gamma \geq K$, assuming $b \in L^\infty(0,1)$.

2. Preliminary Results and Well-Posedness

In this section, we introduce the functional setting needed to treat our problem. However, our assumptions here are more general than those required to get the desired stability and have an independent interest.

We start assuming a very modest requirement, which will be assumed throughout the paper.

Hypothesis 1. Functions a and b are continuous in $[0,1]$ and such that $\frac{b}{a} \in L^1(0,1)$.

Remark 2. (1) We note that, at this stage, a may not degenerate at $x=0$. However, if it is WD then $\frac{1}{a} \in L^1(0,1)$ and b cannot degenerate. If a is SD then $\frac{1}{a} \notin L^1(0,1)$, hence the assumption $\frac{b}{a} \in L^1(0,1)$ implies $b(0)=0$. In this case, b can be WD or SD.

(2) If a is WD or SD with $K=1$ then (1.6) immediately implies that $\frac{xb}{a}$ is bounded.

If Hypothesis 1 holds, it is clear that the function $\eta : [0,1] \to \mathbb{R}$ introduced before is well defined and we immediately find that $\eta \in C^0[0,1] \cap C^1(0,1]$ is a strictly positive function, which is *bounded above and below by a positive constant.* Note also that η can be extended to a function of class $C^1[0,1]$ when b degenerates at 0 not slower than a, for instance if $a(x)=x^K$ and $b(x)=x^h$ with $K \leq h$.

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Now, we are ready to go back to problem (1.2) and study its well-posedness. To do that, let us define

$$\sigma(x) := \frac{a(x)}{\eta(x)},$$

which is a continuous function in $[0,1]$, independently of the possible degeneracy of a. Moreover, observe that if y is a sufficiently smooth function, e.g. $y \in W^{2,1}_{\text{loc}}(0,1)$, then we can write

$$Ay := ay'' + by' \tag{2.1}$$

as

$$Ay = \sigma(\eta y')'.$$

Following [7] (see [11] and [12] for the case $b = 0$), let us consider the following Hilbert spaces with the related inner products:

$$L^2_{\frac{1}{\sigma}}(0,1) := \left\{u \in L^2(0,1) \;\middle|\; \|u\|_{\frac{1}{\sigma}} < \infty\right\}, \quad \langle u, v\rangle_{\frac{1}{\sigma}} := \int_0^1 uv\frac{1}{\sigma}dx,$$

for every $u, v \in L^2_{\frac{1}{\sigma}}(0,1)$;

$$H^1_{\frac{1}{\sigma}}(0,1) := L^2_{\frac{1}{\sigma}}(0,1) \cap H^1(0,1), \quad \langle u, v\rangle_1 := \langle u, v\rangle_{\frac{1}{\sigma}} + \int_0^1 \eta u'v'dx,$$

for every $u, v \in H^1_{\frac{1}{\sigma}}(0,1)$ and

$$H^2_{\frac{1}{\sigma}}(0,1) := \left\{u \in H^1_{\frac{1}{\sigma}}(0,1) \;\middle|\; Au \in L^2_{\frac{1}{\sigma}}(0,1)\right\}, \quad \langle u, v\rangle_2 := \langle u, v\rangle_1 + \langle Au, Av\rangle_{\frac{1}{\sigma}},$$

for every $u, v \in H^2_{\frac{1}{\sigma}}(0,1)$. The previous inner products obviously induce the related respective norms

$$\|u\|^2_{\frac{1}{\sigma}} = \int_0^1 \frac{u^2}{\sigma}dx, \quad \|u\|^2_{1,\frac{1}{\sigma}} = \|u\|^2_{\frac{1}{\sigma}} + \int_0^1 \eta(u')^2dx$$

and

$$\|u\|^2_{2,\frac{1}{\sigma}} = \|u\|^2_{1,\frac{1}{\sigma}} + \int_0^1 \sigma[(\eta u')']^2dx.$$

Moreover, consider the spaces

$$H^1_{\frac{1}{\sigma},0}(0,1) := \left\{u \in H^1_{\frac{1}{\sigma}}(0,1) : u(0) = 0\right\}$$

and

$$H^2_{\frac{1}{\sigma},0}(0,1) := \left\{u \in H^2_{\frac{1}{\sigma}}(0,1) : u(0) = 0\right\},$$

endowed with the previous inner products and related norms.

Remark 3. Being η bounded *above* and *below* by positive constants, it is clear that the inner product with weight η is equivalent to the standard one without weight.

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However, it will be clear soon that in this way we have a quite better functional setting, as Corollary 2.1 will show as follows.

Hypothesis 2. Hypothesis 1 holds. In addition, a is such that $a(0) = 0$, $a > 0$ on $(0, 1]$ and there exists $K > 0$ such that the function

$$x \mapsto \frac{x^K}{a(x)}$$

is nondecreasing in a right neighborhood of $x = 0$.

Note that here we require only continuity on a (and no differentiability); moreover, the monotonicity property required only near 0 holds globally in $(0, 1]$ if a is WD or SD.

Proceeding as in [5] and using the fact that $v(0) = 0$ for all $v \in H^1_{\frac{1}{\sigma},0}(0, 1)$, one has

Proposition 2.1 (Hardy–Poincaré Inequality). *Assume Hypothesis* 2. *Then, there exists* $C_{\text{HP}} > 0$ *such that*

$$\int_0^1 v^2 \frac{1}{\sigma} dx \le C_{\text{HP}} \int_0^1 (v')^2 dx \quad \forall\, v \in H^1_{\frac{1}{\sigma},0}(0, 1). \tag{2.2}$$

In particular, we have the equivalence as follows.

Corollary 2.1. *Assume Hypothesis* 2. *Then the two norms* $\|u\|^2_{1,\frac{1}{\sigma}}$ *and*

$$\|u\|_1^2 := \int_0^1 (u')^2 dx,$$

are equivalent for all $u \in H^1_{\frac{1}{\sigma},0}(0, 1)$. *In particular*,

$$\|u\|_1^2 \le \frac{1}{\min_{[0,1]} \eta} \|u\|^2_{1,\frac{1}{\sigma}} \quad \text{and} \quad \|u\|^2_{1,\frac{1}{\sigma}} \le \left(C_{\text{HP}} + \max_{[0,1]} \eta\right) \|u\|_1^2,$$

where C_{HP} *is the Hardy–Poincaré constant introduced in Proposition* 2.1.

Now, define the domain $D(A)$ of the operator A given in (2.1) as

$$D(A) = H^2_{\frac{1}{\sigma},0}(0, 1).$$

We start with a set of results, actually of independent interest, which describe the functional setting we shall use.

Lemma 2.1. *Assume Hypothesis* 1. *For all* $(u, v) \in D(A) \times H^1_{\frac{1}{\sigma},0}(0, 1)$ *one has*

$$\langle Au, v\rangle_{\frac{1}{\sigma}} = -\int_0^1 \eta u'v' dx + (\eta u'v)(1). \tag{2.3}$$

Proof. As a first step, we consider the space $H^1_c(0, 1) := \{v \in H^1(0, 1) \mid \operatorname{supp}\{v\} \subset (0, 1]\}$.

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As in the proof of [6, Lemma 2.1], we can see that $H^1_c(0,1)$ is dense in $H^1_{\frac{1}{\sigma},0}(0,1)$. Indeed, fix $v \in H^1_{\frac{1}{\sigma},0}(0,1)$ and consider the sequence $(v_n)_{n\geq 3}$, where $v_n := \xi_n v \in H^1_c(0,1)$ and

$$\xi_n(x) := \begin{cases} 0, & x \in [0, 1/n], \\ 1, & x \in [2/n, 1], \\ nx - 1, & x \in (1/n, 2/n); \end{cases}$$

then $v_n \to v$ in $H^1_{\frac{1}{\sigma},0}(0,1)$.

Now, as in [7], consider

$$\varPhi(v) := \int_0^1 \left((au'' + bu')v\frac{1}{\sigma} + \eta u'v' \right) dx - (\eta u'v)(1)$$

with $u \in H^2_{\frac{1}{\sigma},0}(0,1)$. Then, $\varPhi$ is a bounded linear functional on $H^1_{\frac{1}{\sigma},0}(0,1)$. Moreover, $\varPhi = 0$ on $H^1_c(0,1)$. Indeed, taking $v \in H^1_c(0,1)$, one has that

$$\int_0^1 (au'' + bu')v\frac{1}{\sigma}dx = \int_0^1 \sigma(\eta u')'v\frac{1}{\sigma}dx = -\int_0^1 \eta u'v'dx + (\eta u'v)(1).$$

Thus, $\varPhi = 0$ on $H^1_{\frac{1}{\sigma},0}(0,1)$, that is (2.3) holds. □

To evaluate the boundary terms, the following results are important. Since the proofs are similar to those of [5, Lemma 3.2], we postpone them to Appendix A.

Lemma 2.2. (1) *Assume Hypothesis* 1. *If* $y \in H^2_{\frac{1}{\sigma}}(0,1)$ *and if* $u \in H^1_{\frac{1}{\sigma},0}(0,1)$, *then* $\lim_{x\to 0} u(x)y'(x) = 0$.

(2) *Assume Hypothesis* 2. *If* $u \in D(A)$, *then* $xu'(\eta u')' \in L^1(0,1)$.

(3) *Assume Hypothesis* 2. *If* $u \in D(A)$ *and* $K \leq 1$, *then* $\lim_{x\to 0} x(u'(x))^2 = 0$.

(4) *Assume Hypothesis* 2. *If* $u \in D(A)$, $K > 1$ *and* $\frac{xb}{a} \in L^\infty(0,1)$, *then* $\lim_{x\to 0} x(u'(x))^2 = 0$.

(5) *Assume Hypothesis* 2. *If* $u \in H^1_{\frac{1}{\sigma}}(0,1)$, *then* $\lim_{x\to 0} \frac{x}{a}u^2(x) = 0$.

The last result, which will be crucial to obtain the stabilization of problem (1.2), is given by the following proposition.

Proposition 2.2. *Assume Hypothesis* 2 *and for* $\beta \geq 0$ *define*

$$|||z|||_1^2 := \int_0^1 \eta(z')^2 dx + \beta z^2(1)$$

for all $z \in H^1_{\frac{1}{\sigma},0}(0,1)$. *Then the two norms* $|||\cdot|||_1$ *and* $\|\cdot\|_1$ *are equivalent. Moreover, for every* $\lambda \in \mathbb{R}$, *the variational problem*

$$\int_0^1 \eta z'\phi' dx + \beta z(1)\phi(1) = \lambda\phi(1) \quad \forall\, \phi \in H^1_{\frac{1}{\sigma},0}(0,1) \tag{2.4}$$

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admits a unique solution $z \in H^1_{\frac{1}{\sigma},0}(0,1)$ *which satisfies the estimates*

$$|||z|||_1^2 \le \frac{\lambda^2}{\min_{[0,1]}\eta} \quad \textit{and} \quad \|z\|^2_{\frac{1}{\sigma}} \le \frac{\max_{[0,1]}\eta + C_{\mathrm{HP}}}{\min^2_{[0,1]}\eta}\lambda^2, \tag{2.5}$$

where C_{HP} *is the Hardy–Poincaré constant in Proposition* 2.1. *In addition*, $z \in H^2_{\frac{1}{\sigma},0}(0,1)$ *and solves*

$$\begin{cases} -\sigma(\eta z_x)_x = 0, \\ \eta z_x(t,1) + \beta z(t,1) = \lambda. \end{cases} \tag{2.6}$$

Proof. Observe that

$$|z(1)| = \left|\int_0^1 z'(t)dt\right| \le \|z\|_1, \tag{2.7}$$

for all $z \in H^1_{\frac{1}{\sigma},0}(0,1)$. Thus, $|||\cdot|||_1$ and $\|\cdot\|_1$ are equivalent. Indeed, for all $z \in H^1_{\frac{1}{\sigma},0}(0,1)$

$$\|z\|_1^2 \le \frac{1}{\min_{[0,1]}\eta}|||z|||_1^2; \tag{2.8}$$

moreover, since $\beta z^2(1) \le \beta\|z\|_1^2$ by (2.7), one has

$$|||z|||_1^2 \le \left(\max_{[0,1]}\eta + \beta\right)\|z\|_1^2,$$

and the claim holds.

Now, consider the bilinear and symmetric form $\Lambda : H^1_{\frac{1}{\sigma},0}(0,1) \times H^1_{\frac{1}{\sigma},0}(0,1) \to \mathbb{R}$, given by

$$\Lambda(z,\phi) := \int_0^1 \eta z'\phi' dx + \beta z(1)\phi(1).$$

for all $z, \phi \in H^1_{\frac{1}{\sigma},0}(0,1)$. Clearly, Λ is also coercive and continuous. Indeed, by Corollary 2.1

$$\Lambda(z,z) = \int_0^1 \eta(z')^2dx + \beta z^2(1) \ge \int_0^1 \eta(z')^2dx \ge \frac{\min_{[0,1]}\eta}{C_{\mathrm{HP}} + \max_{[0,1]}\eta}\|z\|^2_{1,\frac{1}{\sigma}}.$$

Moreover,

$$|\Lambda(z,\phi)| \le \max_{[0,1]}\eta\|z'\|_{L^2(0,1)}\|\phi'\|_{L^2(0,1)} + \beta|z(1)||\phi(1)|.$$

With (2.7) applied to z and ϕ, one has

$$|\Lambda(z,\phi)| \le \left(\max_{[0,1]}\eta + \beta\right)\|z\|_1\|\phi\|_1.$$

Now, consider the linear functional

$$\mathcal{L}(\phi) := \lambda\phi(1),$$

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with $\phi \in H^1_{\frac{1}{\sigma},0}(0,1)$. Clearly, $\mathcal{L}$ is continuous and linear. Thus, by the Lax–Milgram Theorem, there exists a unique solution $z \in H^1_{\frac{1}{\sigma},0}(0,1)$ of

$$\Lambda(z,\phi) = \mathcal{L}(\phi) \tag{2.9}$$

for all $\phi \in H^1_{\frac{1}{\sigma},0}(0,1)$. In particular,

$$\Lambda(z,z) = \int_0^1 \eta(z')^2 dx + \beta z^2(1) = \mathcal{L}(z) = \lambda z(1). \tag{2.10}$$

By (2.7), (2.8) and (2.10) we have

$$|||z|||_1^2 = \lambda z(1) \leq \frac{|\lambda|}{\sqrt{\min_{[0,1]}\eta}} |||z|||_1;$$

thus

$$|||z|||_1 \leq \frac{|\lambda|}{\sqrt{\min_{[0,1]}\eta}} \quad \text{and} \quad |||z|||_1^2 \leq \frac{\lambda^2}{\min_{[0,1]}\eta}.$$

Moreover, by Corollary 2.1, we know that in $H^1_{\frac{1}{\sigma},0}(0,1)$ the two norms $\|\cdot\|_1$ and $\|\cdot\|_{1,\frac{1}{\sigma}}$ are equivalent. Thus

$$\begin{aligned} |||z|||_1^2 &\geq \min_{[0,1]}\eta\|z\|_1^2 + \beta z^2(1) \geq \min_{[0,1]}\eta\|z\|_1^2 \\ &\geq \frac{\min_{[0,1]}\eta}{\max_{[0,1]}\eta + C_{\mathrm{HP}}}\|z\|^2_{1,\frac{1}{\sigma}} \geq \frac{\min_{[0,1]}\eta}{\max_{[0,1]}\eta + C_{\mathrm{HP}}}\|z\|^2_{L^2_{\frac{1}{\sigma}}(0,1)}. \end{aligned}$$

Thus, by (2.5)

$$\|z\|^2_{\frac{1}{\sigma}} \leq \frac{\max_{[0,1]}\eta + C_{\mathrm{HP}}}{\min_{[0,1]}\eta}|||z|||_1^2 \leq \frac{\max_{[0,1]}\eta + C_{\mathrm{HP}}}{\min^2_{[0,1]}\eta}\lambda^2.$$

Now, we will prove that $z \in H^2_{\frac{1}{\sigma},0}(0,1)$ solves (2.6). With this aim, we consider again (2.9). Since it holds for every $\phi \in H^1_{\frac{1}{\sigma},0}(0,1)$, it holds in particular for every $\phi \in C_c^\infty(0,1)$, so that

$$\int_0^1 \eta z'\phi' = 0 \quad \text{for all } \phi \in C_c^\infty(0,1).$$

By the fundamental lemma of the calculus of variations (for instance, see [16, Lemma 1.2.1]), we get that $\eta z'$ is constant a.e. in $(0,1)$ and so $(\eta z')' = 0$ a.e. in $(0,1)$; in particular

$$\sigma(\eta z')' = 0 \quad \text{a.e. in } (0,1)$$

and so $Az = \sigma(\eta z')' \in L^2_{\frac{1}{\sigma}}(0,1)$.

Now, coming back to (2.9), we have

$$\int_0^1 \eta z'\phi' dx + \beta z(1)\phi(1) = \lambda\phi(1) \Leftrightarrow [\eta z'\phi]_{x=0}^{x=1} + \beta z(1)\phi(1) = \lambda\phi(1)$$

for all $\phi \in H^1_{\frac{1}{\sigma},0}(0,1)$. Thus, since $\phi(0) = 0$, we obtain

$$(\eta z')(1) + \beta z(1) = \lambda,$$

that is, z solves (2.6). □

We are now ready to study the well-posedness of problem (1.2). For this, we introduce the Hilbert space

$$\mathcal{H}_0 := H^1_{\frac{1}{\sigma},0}(0,1) \times L^2_{\frac{1}{\sigma}}(0,1),$$

with the inner product

$$\langle (u,v),(\tilde{u},\tilde{v})\rangle_{\mathcal{H}_0} := \int_0^1 u'\tilde{u}'dx + \int_0^1 v\tilde{v}\frac{1}{\sigma}dx + \beta u(1)\tilde{u}(1)$$

for every $(u,v),(\tilde{u},\tilde{v}) \in \mathcal{H}_0$, and the induced norm

$$\|(u,v)\|^2_{\mathcal{H}_0} := \int_0^1 (u')^2 dx + \int_0^1 v^2\frac{1}{\sigma}dx + \beta u^2(1).$$

Observe that if $u \in H^1_{\frac{1}{\sigma},0}(0,1)$, then u is continuous, so that $u(1)$ is well defined. Moreover, being $\eta \in C^0[0,1] \cap C^1(0,1]$ far away from 0, for every $(u,v),(\tilde{u},\tilde{v}) \in \mathcal{H}_0$, the norm $\|(u,v)\|^2_{\mathcal{H}_0}$ is equivalent to

$$\|(u,v)\|^2_1 := \int_0^1 \eta(u')^2 dx + \int_0^1 v^2\frac{1}{\sigma}dx + \beta u^2(1).$$

Obviously, to such a norm, we associate the inner product

$$\langle (u,v),(\tilde{u},\tilde{v})\rangle_1 := \int_0^1 \eta u'\tilde{u}'dx + \int_0^1 v\tilde{v}\frac{1}{\sigma}dx + \beta u(1)\tilde{u}(1),$$

which we will use from now on, being more convenient for our treatment.

Now, consider the matrix operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H}_0 \to \mathcal{H}_0$, given by

$$\mathcal{A} := \begin{pmatrix} 0 & Id \\ A & 0 \end{pmatrix},$$

and

$$D(\mathcal{A}) := \left\{(u,v) \in H^2_{\frac{1}{\sigma},0}(0,1) \times H^1_{\frac{1}{\sigma},0}(0,1) : (\eta u')(1) + v(1) + \beta u(1) = 0\right\}.$$

Thus, by using the operator $(\mathcal{A}, D(\mathcal{A}))$, we rewrite (1.2) as a Cauchy problem. Indeed, setting, as usual,

$$\mathcal{Y}(t) := \begin{pmatrix} y \\ y_t \end{pmatrix} \quad \text{and} \quad \mathcal{Y}_0 := \begin{pmatrix} y_0 \\ y_1 \end{pmatrix},$$

one has that (1.2) can be rewritten as

$$\begin{cases} \dot{\mathcal{Y}}(t) = \mathcal{A}\mathcal{Y}(t), & t \geq 0, \\ \mathcal{Y}(0) = \mathcal{Y}_0. \end{cases} \tag{2.11}$$

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If we prove that $(\mathcal{A}, D(\mathcal{A}))$ generates a contraction semigroup $(S(t))_{t\geq 0}$ and $\mathcal{Y}_0 \in \mathcal{H}_0$, then $\mathcal{Y}(t) = S(t)\mathcal{Y}_0$ gives the mild solution of (2.11). The following theorem holds.

Theorem 2.1. *Assume Hypothesis* 2. *Then the operator* $(\mathcal{A}, D(\mathcal{A}))$ *is non positive with dense domain and generates a contraction semigroup* $(S(t))_{t\geq 0}$.

For the proof of this theorem, we use the following result.

Theorem 2.2 ([8], Corollary 3.20). *Let* $(\mathcal{A}, D(\mathcal{A}))$ *be a dissipative operator on a reflexive Banach space such that* $\lambda I - \mathcal{A}$ *is surjective for some* $\lambda > 0$. *Then* $\mathcal{A}$ *is densely defined and generates a contraction semigroup.*

Proof of Theorem 2.1. According to the previous theorem, it is sufficient to prove that $\mathcal{A} : D(\mathcal{A}) \to \mathcal{H}_0$ is dissipative and that $I - \mathcal{A}$ is surjective.

$\mathcal{A}$ is dissipative: Take $(u, v) \in D(\mathcal{A})$. Then $(u, v) \in H^2_{\frac{1}{\sigma},0}(0,1) \times H^1_{\frac{1}{\sigma},0}(0,1)$ and so (2.3) holds. Hence, by Lemma 2.1

$$\begin{aligned}
\langle \mathcal{A}(u,v), (u,v)\rangle_1 &= \langle (v, Au), (u,v)\rangle_1 \\
&= \int_0^1 \eta u'v'dx + \int_0^1 vAu\frac{1}{\sigma}dx + \beta v(1)u(1) \\
&= \int_0^1 \eta u'v'dx - \int_0^1 \eta u'v'dx + (\eta u'v)(1) + \beta v(1)u(1) \\
&= v(1)((\eta u')(1) + \beta u(1)) \\
&= -v^2(1) \leq 0.
\end{aligned}$$

$I - \mathcal{A}$ is surjective: Take $(f, g) \in \mathcal{H}_0 = H^1_{\frac{1}{\sigma},0}(0,1) \times L^2_{\frac{1}{\sigma}}(0,1)$. We have to prove that there exists $(u, v) \in D(\mathcal{A})$ such that

$$(I - \mathcal{A})\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} \Leftrightarrow \begin{cases} v = u - f, \\ -Au + u = f + g. \end{cases} \tag{2.12}$$

Thus, define $F : H^1_{\frac{1}{\sigma},0}(0,1) \to \mathbb{R}$ as

$$F(z) = \int_0^1 (f+g)z\frac{1}{\sigma}dx + z(1)f(1).$$

Obviously, $F \in H^{-1}_{\frac{1}{\sigma},0}(0,1)$, the dual space of $H^1_{\frac{1}{\sigma},0}(0,1)$ with respect to the pivot space $L^2_{\frac{1}{\sigma}}(0,1)$: indeed, $f \in H^1_{\frac{1}{\sigma},0}(0,1)$, and so also $f(1)$ is well defined, and $g \in L^2_{\frac{1}{\sigma}}(0,1)$. Now, introduce the bilinear form $L : H^1_{\frac{1}{\sigma},0}(0,1) \times H^1_{\frac{1}{\sigma},0}(0,1) \to \mathbb{R}$ given by

$$L(u,z) := \int_0^1 uz\frac{1}{\sigma}dx + \int_0^1 \eta u'z'dx + (\beta+1)u(1)z(1)$$

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for all $u, z \in H^1_{\frac{1}{\sigma},0}(0,1)$. Clearly, since $\beta \geq 0$, $L(u,z)$ is coercive. Moreover $L(u,z)$ is continuous: indeed, for all $u \in H^1_{\frac{1}{\sigma},0}(0,1)$, as in (2.7)

$$|u(1)| \leq \int_0^1 |u'(t)|dt = \|u'\|_{L^1(0,1)} \leq \|u'\|_{L^2(0,1)};$$

thus, for all $u, z \in H^1_{\frac{1}{\sigma},0}(0,1)$,

$$|L(u,z)| \leq \|u\|_{L^2_{\frac{1}{\sigma}}(0,1)}\|z\|_{L^2_{\frac{1}{\sigma}}(0,1)} + (\|\eta\|_{L^\infty(0,1)} + \beta + 1)\|u'\|_{L^2(0,1)}\|z'\|_{L^2(0,1)},$$

and the conclusion follows from Corollary 2.1.

As a consequence, by the Lax–Milgram Theorem, there exists a unique solution $u \in H^1_{\frac{1}{\sigma},0}(0,1)$ of

$$L(u,z) = F(z) \quad \text{for all } z \in H^1_{\frac{1}{\sigma},0}(0,1),$$

namely

$$\int_0^1 uz\frac{1}{\sigma}dx + \int_0^1 \eta u'z'dx + (\beta+1)u(1)z(1) = \int_0^1 (f+g)z\frac{1}{\sigma}dx + z(1)f(1) \tag{2.13}$$

for all $z \in H^1_{\frac{1}{\sigma},0}(0,1)$.

Now, take $v := u - f$; then $v \in H^1_{\frac{1}{\sigma},0}(0,1)$. We will prove that $(u,v) \in D(\mathcal{A})$ and solves (2.12). To begin with, (2.13) holds for every $z \in C_c^\infty(0,1)$. Thus, we have

$$\int_0^1 \eta u'z'dx = \int_0^1 (f+g-u)z\frac{1}{\sigma}dx$$

for every $z \in C_c^\infty(0,1)$. Hence $-(\eta u')' = (f+g-u)\frac{1}{\sigma}$ a.e. in $(0,1)$. This implies that $-\sigma(\eta u')' = (f+g-u) \in L^2_{\frac{1}{\sigma}}(0,1)$, i.e. $Au \in L^2_{\frac{1}{\sigma}}(0,1)$; thus $u \in D(A)$. Moreover, coming back to (2.13) and thanks to (2.3)

$$-\int_0^1 \sigma(\eta u')'z\frac{1}{\sigma}dx + (\eta u'z)(1) + (\beta+1)u(1)z(1)$$
$$= \int_0^1 (f+g-u)z\frac{1}{\sigma}dx + z(1)f(1).$$

Using the fact that $-\sigma(\eta u')' = (f+g-u)$ a.e. in $(0,1)$, we obtain

$$(\eta u'z)(1) + (\beta+1)u(1)z(1) = z(1)f(1)$$

for all $z \in H^1_{\frac{1}{\sigma},0}(0,1)$. Hence, $\eta(1)u'(1) + (\beta+1)u(1) - f(1) = 0$. Recalling that $v = u - f$, one has

$$\eta(1)u'(1) + \beta u(1) + v(1) = 0.$$

In conclusion, $(u,v) \in D(\mathcal{A})$, $u - Au = f + g$ and $v = u - f$, i.e. (u,v) solves (2.12). □

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As usual in the semigroup theory, the mild solution of (2.11) obtained above can be more regular: if $\mathcal{Y}_0 \in D(\mathcal{A})$, then the solution is classical, in the sense that $\mathcal{Y} \in C^1([0,+\infty);\mathcal{H}_0) \cap C([0,+\infty);D(\mathcal{A}))$ and the equation in (1.2) holds for all $t \geq 0$. Hence, as in [2, Corollary 4.2] or in [4, Proposition 3.15], one has the following theorem.

Theorem 2.3. *Assume Hypothesis* 2.

If $(y_0, y_1) \in \mathcal{H}_0$, *then there exists a unique mild solution*

$$y \in C^1([0,+\infty); L^2_{\frac{1}{\sigma}}(0,1)) \cap C([0,+\infty); H^1_{\frac{1}{\sigma},0}(0,1))$$

of (1.2) *which depends continuously on the initial data* $(y_0, y_1) \in \mathcal{H}_0$. *Moreover, if* $(y_0, y_1) \in D(\mathcal{A})$, *then the solution* y *is classical, in the sense that*

$$y \in C^2([0,+\infty); L^2_{\frac{1}{\sigma}}(0,1)) \cap C^1([0,+\infty); H^1_{\frac{1}{\sigma},0}(0,1)) \cap C([0,+\infty); H^2_{\frac{1}{\sigma},0}(0,1))$$

and the equation of (1.2) *holds for all* $t \geq 0$.

3. The Stability Result

In this section, we prove the main result of the paper when a is WD or SD. Actually, we will prove that the energy associated to the initial problem is nonincreasing and, in particular, it decreases exponentially under suitable assumptions.

With this aim, let y be a mild solution of (1.2) and consider its energy, given by

$$E_y(t) = \frac{1}{2}\left[\int_0^1 \left(\frac{1}{\sigma} y_t^2(t,x) + \eta y_x^2(t,x)\right) dx + \beta y^2(t,1)\right], \quad t \geq 0. \tag{3.1}$$

With this definition in hand, one can prove that the energy is nonincreasing.

Theorem 3.1. *Assume Hypothesis* 2 *and let* y *be a classical solution of* (1.2) (*for instance, if* $(y_0, y_1) \in D(\mathcal{A})$). *Then the energy is nonincreasing and*

$$\frac{dE_y(t)}{dt} = -y_t(t,1)^2, \quad t \geq 0.$$

Proof. By multiplying the equation by $\frac{y_t}{\sigma}$, integrating over $(0,1)$ and using the boundary conditions, one has

$$\begin{aligned} 0 &= \frac{1}{2}\int_0^1 \frac{d}{dt}\left(\frac{y_t^2}{\sigma}\right) dx - [\eta y_x y_t]_{x=0}^{x=1} + \int_0^1 \eta y_x y_{tx} dx \\ &= \frac{1}{2}\frac{d}{dt}\left[\int_0^1 \left(\frac{y_t^2}{\sigma} + \eta y_x^2\right) dx + \beta y^2(t,1)\right] + y_t^2(t,1) \\ &= \frac{1}{2}\frac{d}{dt} E_y(t) + y_t^2(t,1). \end{aligned}$$

Indeed, taking $u = y_t$ in Lemma 2.2(1), one has $\lim_{x\to 0}(\eta y_x y_t)(t,x) = 0$. Hence,

$$\frac{1}{2}\frac{d}{dt} E_y(t) = -y_t^2(t,1) \leq 0$$

for all $t \geq 0$. □

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Now, our aim is to estimate the energy $E_y(t)$ with the value of the energy E_y at $t=0$. To do that, we need to restrict Hypothesis 2, requiring the crucial assumption as follows.

Hypothesis 3. Hypothesis 1 holds and a is WD or SD; if $K>1$, then also assume that $\frac{xb}{a}\in L^\infty(0,1)$.

Remark 4. It is clear that, by definition of WD or SD, if Hypothesis 3 holds, then Hypothesis 2 holds, as well.

The following preliminary result holds.

Proposition 3.1. *Assume Hypothesis* 3 *and let* y *be a classical solution of* (1.2). *Then*

$$0 = 2\int_0^1 \left[\frac{xy_xy_t}{\sigma}\right]_{t=s}^{t=T} dx - \frac{1}{\sigma(1)}\int_s^T y_t^2(t,1)dt - \eta(1)\int_s^T y_x^2(t,1)dt \\ - \int_{Q_s} x\eta\frac{b}{a}y_x^2 dxdt + \int_{Q_s}\left(1-\frac{x(a'-b)}{a}\right)\frac{1}{\sigma}y_t^2 dxdt + \int_{Q_s}\eta y_x^2 dxdt, \tag{3.2}$$

for every $T>s>0$.

Proof. Take $s\in(0,T)$; then, multiplying the equation of (1.2) by $\frac{xy_x}{\sigma}$, integrating over $Q_s := (s,T)\times(0,1)$ and recalling (2.1), we have

$$\begin{aligned}
0 &= \int_{Q_s}\frac{y_{tt}xy_x}{\sigma}dxdt - \int_{Q_s} x(\eta y_x)_x y_x dxdt \\
&= \int_0^1 \left[\frac{xy_xy_t}{\sigma}\right]_{t=s}^{t=T} dx - \int_{Q_s}\frac{xy_{xt}y_t}{\sigma}dxdt - \int_{Q_s} x(\eta' y_x + \eta y_{xx})y_x dxdt \\
&= \int_0^1 \left[\frac{xy_xy_t}{\sigma}\right]_{t=s}^{t=T} dx - \frac{1}{2}\int_{Q_s}\frac{x}{\sigma}(y_t^2)_x dxdt - \int_{Q_s} x\eta' y_x^2 dxdt \\
&\quad - \frac{1}{2}\int_{Q_s} x\eta(y_x^2)_x dxdt \\
&= \int_0^1 \left[\frac{xy_xy_t}{\sigma}\right]_{t=s}^{t=T} dx - \frac{1}{2}\int_s^T \left[\frac{x}{\sigma}y_t^2\right]_{x=0}^{x=1} dt + \frac{1}{2}\int_{Q_s}\left(\frac{x}{\sigma}\right)' y_t^2 dxdt \\
&\quad - \int_{Q_s} x\eta\frac{b}{a}y_x^2 dxdt \\
&\quad - \frac{1}{2}\int_s^T [x\eta y_x^2]_{x=0}^{x=1} dt + \frac{1}{2}\int_{Q_s}(x\eta)' y_x^2 dxdt.
\end{aligned} \tag{3.3}$$

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Recalling the definition of η and σ, we immediately find

$$\begin{aligned}0 = {} & \int_0^1 \left[\frac{x y_x y_t}{\sigma}\right]_{t=s}^{t=T} dx - \frac{1}{2}\int_s^T \left[\frac{x}{\sigma} y_t^2\right]_{x=0}^{x=1} dt - \frac{1}{2}\int_s^T [x\eta y_x^2]_{x=0}^{x=1} dt \\ & - \frac{1}{2}\int_{Q_s} x\eta \frac{b}{a} y_x^2 dxdt + \frac{1}{2}\int_{Q_s}\left(1 - \frac{x(a'-b)}{a}\right)\frac{1}{\sigma} y_t^2 dxdt \\ & + \frac{1}{2}\int_{Q_s} \eta y_x^2 dxdt \end{aligned} \tag{3.4}$$

for all $s \in (0, T)$.

Now, we consider the boundary terms. Thanks to the boundary conditions of y, (1.5) and Lemma 2.2, we immediately have that

$$\lim_{x\to 0} \frac{x}{\sigma} y_t^2(t,x) = \lim_{x\to 0} \frac{x}{a}\eta y_t^2(t,x) = 0$$

and

$$\lim_{x\to 0} x\eta y_x^2(t,x) = 0.$$

Hence, (3.4) multiplied by 2 gives (3.2). □

Proposition 3.2. *Assume Hypothesis* 3 *and let* y *be a classical solution of* (1.2). *Then, for all* $T > s > 0$ *we have*

$$\int_{Q_s}\left(1 - \frac{x(a'-b)}{a} + \frac{K}{2}\right)\frac{1}{\sigma} y_t^2 dxdt + \int_{Q_s}\left(1 - x\frac{b}{a} - \frac{K}{2}\right)\eta y_x^2 dxdt = (B.T.) \tag{3.5}$$

where

$$(B.T.) = \int_0^1 \left[-2x\frac{y_x y_t}{\sigma} + \frac{K}{2}\frac{y y_t}{\sigma}\right]_{t=s}^{t=T} dx + \int_s^T \left[\frac{1}{\sigma} y_t^2 + \eta y_x^2 - \frac{K}{2}\eta y y_x\right](t,1) dt \tag{3.6}$$

and $Q_s := (s,T)\times(0,1)$.

Proof. By multiplying the equation in (1.2) by $\frac{y}{\sigma}$ and integrating over Q_s, we have

$$\int_{Q_s}\left(-\frac{y_t^2}{\sigma} + \eta y_x^2\right) dxdt + \int_0^1 \left[\frac{y y_t}{\sigma}\right]_{t=s}^{t=T} dx - \int_s^T [\eta y_x y]_{x=0}^{x=1} dt = 0. \tag{3.7}$$

Using the fact that y is a classical solution of (1.2), that $y(t,0) = 0$ and Lemma 2.2, one has

$$\int_s^T [\eta y_x y]_{x=0}^{x=1} dt = \int_s^T [\eta y_x y](t,1) dt;$$

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thus, multiplying (3.7) by $\frac{K}{2}$, one has

$$\frac{K}{2}\int_{Q_s}\left(-\frac{y_t^2}{\sigma}+\eta y_x^2\right)dxdt+\frac{K}{2}\int_0^1\left[\frac{yy_t}{\sigma}\right]_{t=s}^{t=T}dx-\frac{K}{2}\int_s^T[\eta y_x y](t,1)dt=0. \tag{3.8}$$

By summing (3.8) and (3.2), we get the claim. □

Proposition 3.3. *Assume Hypothesis* 3, $\beta \geq 0$ *and let* y *be a classical solution of* (1.2). *Then, for any* $T > s > 0$ *and for every* $\delta > 0$ *we have*

$$\begin{aligned}\int_s^T y^2(t,1)dt \leq &\left(2+\frac{2C_{\rm HP}}{\min_{[0,1]}^3\eta}+\frac{1}{\delta}+\frac{1}{\delta}\frac{\max_{[0,1]}\eta+C_{\rm HP}}{\min_{[0,1]}^2\eta}\right)E_y(s)\\ &+2\delta\left(\frac{1}{\min_{[0,1]}^3\eta}+1\right)\int_s^T E_y(t)dt.\end{aligned} \tag{3.9}$$

Proof. To prove the statement, fix $t \in [s,T]$ and set $\lambda = y(t,1)$ and let $z = z(t,\cdot)$ be the unique solution of

$$\int_0^1 \eta z'\phi' dx + \beta z(1)\phi(1) = \lambda\phi(1) \quad \forall\, \phi \in H^1_{\frac{1}{\sigma},0}(0,1).$$

By Proposition 2.2, $z(t,\cdot) \in H^2_{\frac{1}{\sigma},0}(0,1)$ for all t and solves

$$\begin{cases}-\sigma(\eta z_x)_x = 0,\\ \eta z_x(t,1)+\beta z(t,1)=\lambda.\end{cases} \tag{3.10}$$

Now, multiply the equation in (1.2) by $\frac{z}{\sigma}$ and integrate over Q_s. Then, we have

$$\begin{aligned}0=\int_{Q_s}\left(y_{tt}\frac{z}{\sigma}-(\eta y_x)_x z\right)dxdt &= \int_0^1\left[y_t\frac{z}{\sigma}\right]_{t=s}^{t=T}dx\\ &-\int_{Q_s}y_t\frac{z_t}{\sigma}dxdt-\int_s^T[\eta y_x z]_{x=0}^{x=1}dt+\int_{Q_s}\eta y_x z_x dxdt.\end{aligned}$$

By Lemma 2.2

$$\lim_{\epsilon\to 0}(\eta y_x z)(t,\epsilon)=0,$$

thus we have

$$\int_0^1\left[y_t\frac{z}{\sigma}\right]_{t=s}^{t=T}dx-\int_{Q_s}y_t\frac{z_t}{\sigma}dxdt=\int_s^T\eta y_x(t,1)z(1)dt-\int_{Q_s}\eta y_x z_x dxdt. \tag{3.11}$$

By multiplying the equation in (3.10) by $\frac{y}{\sigma}$ and integrating on Q_s, one has

$$\int_{Q_s}(\eta z_x)_x y dxdt = 0.$$

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Using the fact that $(\eta z_x y)(t,0) = 0$ and $(\eta z_x)(t,1) = \lambda - \beta z(t,1)$, we get

$$\int_s^T [\eta z_x y]_{x=0}^{x=1} dt - \int_{Q_s} \eta z_x y_x dxdt = 0 \Leftrightarrow \int_s^T (\eta z_x y)(t,1)dt = \int_{Q_s} \eta z_x y_x dxdt$$

$$\Leftrightarrow \int_s^T (\lambda - \beta z(t,1))y(t,1)dt = \int_{Q_s} \eta z_x y_x dxdt.$$

Substituting in (3.11) and recalling that y solves (1.2) and $\lambda = y(t,1)$, we have

$$\begin{aligned}
&\int_0^1 \left[\frac{y_t z}{\sigma}\right]_{t=s}^{t=T} dx - \int_{Q_s} \frac{y_t z_t}{\sigma} dxdt \\
&\quad = \int_s^T (\eta y_x z)(t,1)dt - \int_s^T (\lambda - \beta z(t,1))y(t,1)dt \\
&\quad = \int_s^T (\eta y_x z)(t,1)dt - \int_s^T y^2(t,1)dt + \beta \int_s^T (zy)(t,1)dt \\
&\quad = -\int_s^T y_t(t,1)z(t,1)dt - \int_s^T y^2(t,1)dt.
\end{aligned}$$

Then

$$\int_s^T y^2(t,1)dt = \int_{Q_s} \frac{y_t z_t}{\sigma} dxdt - \int_s^T (y_t z)(t,1)dt - \int_0^1 \left[\frac{y_t z}{\sigma}\right]_{t=s}^{t=T} dx. \tag{3.12}$$

Hence, to bound $\int_s^T y^2(t,1)dt$, we estimate the last three terms in the previous equality. By Proposition 2.1, (2.5) and recalling that $\lambda = y(t,1)$, we have

$$\begin{aligned}
\int_0^1 \left|\frac{y_t z}{\sigma}(\tau,x)\right| dx &\le \frac{1}{2}\int_0^1 \frac{y_t^2(\tau,x)}{\sigma} dx + \frac{1}{2}\int_0^1 \frac{z^2(\tau,x)}{\sigma} dx \\
&\le \frac{1}{2}\int_0^1 \frac{y_t^2(\tau,x)}{\sigma} dx + \frac{1}{2}\frac{C_{\mathrm{HP}}}{\min_{[0,1]}\eta}\int_0^1 (z_x^2\eta)(\tau,x)dx \\
&\le \frac{1}{2}\int_0^1 \frac{y_t^2(\tau,x)}{\sigma} dx + \frac{1}{2}\frac{C_{\mathrm{HP}}}{\min_{[0,1]}^2\eta} y^2(t,1) \\
&\le E_y(\tau) + \frac{1}{2}\frac{C_{\mathrm{HP}}}{\min_{[0,1]}^2\eta} y^2(t,1).
\end{aligned}$$

for all $\tau \in [s,T]$. By (2.7), one has

$$\frac{1}{2}y^2(t,1) \le \frac{1}{2}\frac{1}{\min_{[0,1]}\eta}\int_0^1 (y_x^2\eta)(t,x)dx \le \frac{1}{\min_{[0,1]}\eta}E_y(t), \tag{3.13}$$

for all $t \in [s,T]$. Thus, by Theorem 3.1

$$\int_0^1 \left|\frac{y_t z}{\sigma}(\tau,x)\right| dx \le E_y(\tau) + \frac{C_{\mathrm{HP}}}{\min_{[0,1]}^3\eta}E_y(t) \le \left(1 + \frac{C_{\mathrm{HP}}}{\min_{[0,1]}^3\eta}\right)E_y(s).$$

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Again by Theorem 3.1

$$\left|\int_0^1 \left[\frac{y_t z}{\sigma}\right]_{t=s}^{t=T} dx\right| \leq 2\left(1+\frac{C_{\mathrm{HP}}}{\min^3_{[0,1]}\eta}\right) E_y(s). \tag{3.14}$$

Moreover, for any $\delta > 0$ we have

$$\int_s^T |(y_t z)(t,1)|dt \leq \frac{1}{\delta}\int_s^T y_t^2(t,1)dt + \delta\int_s^T z^2(t,1)dt. \tag{3.15}$$

By (2.7), (2.5) and (3.13), one has

$$\begin{aligned} z^2(t,1) &\leq \frac{1}{\min_{[0,1]}\eta}\int_0^1 (\eta z_x^2)(t,x)dx \leq \frac{1}{\min_{[0,1]}\eta}|||z|||_1^2 \\ &\leq \frac{y^2(t,1)}{\min^2_{[0,1]}\eta} \leq \frac{2}{\min^3_{[0,1]}\eta}E_y(t). \end{aligned}$$

Thus, by (3.15) and Theorem 3.1, we have

$$\begin{aligned} \int_s^T |(y_t z)(t,1)|dt &\leq \frac{1}{\delta}\int_s^T y_t^2(t,1)dt + \delta\frac{2}{\min^3_{[0,1]}\eta}\int_s^T E_y(t)dt \\ &\leq -\frac{1}{\delta}\int_s^T \frac{dE_y(t)}{dt}dt + \delta\frac{2}{\min^3_{[0,1]}\eta}\int_s^T E_y(t)dt \\ &\leq \frac{E_y(s)}{\delta} + \frac{2\delta}{\min^3_{[0,1]}\eta}\int_s^T E_y(t)dt. \end{aligned} \tag{3.16}$$

Finally, we estimate the first integral in (3.12). To this aim, consider again problem (2.6) and differentiate with respect to t. Then

$$\begin{cases} -\sigma(\eta z_{tx})_x = 0, \\ \eta z_{tx}(t,1) + \beta z_t(t,1) = \lambda_t = y_t(t,1). \end{cases}$$

Clearly, z_t satisfies the estimates in (2.5); in particular

$$|||z_t|||_1^2 \leq \frac{y_t^2(t,1)}{\min_{[0,1]}\eta} \quad \text{and} \quad \|z_t\|^2_{\frac{1}{\sigma}} \leq \frac{\max_{[0,1]}\eta + C_{\mathrm{HP}}}{\min^2_{[0,1]}\eta}y_t^2(t,1).$$

Thus, for $\delta > 0$, we find

$$\begin{aligned} \int_{Q_s}\left|\frac{y_t z_t}{\sigma}\right|dxdt &\leq \delta\int_{Q_s}\frac{y_t^2}{\sigma}dxdt + \frac{1}{\delta}\int_{Q_s}\frac{z_t^2}{\sigma}dxdt \\ &\leq 2\delta\int_s^T E_y(t)dt + \frac{1}{\delta}\frac{\max_{[0,1]}\eta + C_{\mathrm{HP}}}{\min^2_{[0,1]}\eta}\int_s^T y_t^2(t,1)dt \\ &= 2\delta\int_s^T E_y(t)dt - \frac{1}{\delta}\frac{\max_{[0,1]}\eta + C_{\mathrm{HP}}}{\min^2_{[0,1]}\eta}\int_s^T \frac{dE_y(t)}{dt}dt \\ &\leq 2\delta\int_s^T E_y(t)dt + \frac{1}{\delta}\frac{\max_{[0,1]}\eta + C_{\mathrm{HP}}}{\min^2_{[0,1]}\eta}E_y(s). \end{aligned} \tag{3.17}$$

Hence, going back to (3.12), by (3.14), (3.16) and (3.17), we get

$$\begin{aligned}\int_s^T y^2(t,1)dt &\le 2\left(1+\frac{C_{\mathrm{HP}}}{\min^3_{[0,1]}\eta}\right)E_y(s)+\frac{E_y(s)}{\delta}+\frac{2\delta}{\min^3_{[0,1]}\eta}\int_s^T E_y(t)dt\\ &\quad +2\delta\int_s^T E_y(t)dt+\frac{1}{\delta}\frac{\max_{[0,1]}\eta+C_{\mathrm{HP}}}{\min^2_{[0,1]}\eta}E_y(s)\\ &=\left(2+\frac{2C_{\mathrm{HP}}}{\min^3_{[0,1]}\eta}+\frac{1}{\delta}+\frac{1}{\delta}\frac{\max_{[0,1]}\eta+C_{\mathrm{HP}}}{\min^2_{[0,1]}\eta}\right)E_y(s)\\ &\quad +2\delta\left(\frac{1}{\min^3_{[0,1]}\eta}+1\right)\int_s^T E_y(t)dt,\end{aligned}$$

as claimed. □

Now, we assume an additional hypothesis on the functions a and b.

Hypothesis 4. Hypothesis 3 holds and there exists $\varepsilon_0>0$ such that

$$(2-K)a-2x|b|\ge\varepsilon_0 a\quad\text{for every } x\in[0,1].$$

Note that the required inequality implies that $|b(x)|\le\frac{(2-K-\varepsilon_0)a}{2x}$. Thus, if $a(x)=x^K$, the condition reads $|b(x)|\le\frac{(2-K-\varepsilon_0)x^{K-1}}{2}$. Of course, the previous condition is automatically satisfied in absence of the drift.

Proposition 3.4. *Assume Hypothesis* 4, $\beta\ge0$ *and let* y *be a classical solution of* (1.2). *Then, for any* $T>s>0$

$$\begin{aligned}&\frac{\varepsilon_0}{2}\int_{Q_s}\left(\frac{y_t^2}{\sigma}+\eta y_x^2\right)dxdt\\ &\quad\le 4\Theta E_y(s)+\left(\frac{1}{\sigma(1)}+\frac{1}{\eta(1)}+\frac{\beta}{\eta(1)}+\frac{K}{4}\right)(E_y(s)-E_y(T))\\ &\qquad+\left(\frac{\beta^2}{\eta(1)}+\frac{K\beta}{2}+\frac{\beta}{\eta(1)}+\frac{K}{4}\right)\int_s^T y^2(t,1)dt, \qquad (3.18)\end{aligned}$$

where

$$\Theta:=\max\left\{\frac{1}{a(1)}+K\frac{C_{\mathrm{HP}}}{\min_{[0,1]}\eta};1+\frac{K}{4}\right\}. \qquad (3.19)$$

Proof. By all the assumptions in force

$$1-\frac{x(a'-b)}{a}+\frac{K}{2}=\frac{(2-K)a+2(Ka-xa')+2xb}{2a}\ge\frac{\varepsilon_0}{2},$$

and

$$\left(1-x\frac{b}{a}-\frac{K}{2}\right)\ge\frac{\varepsilon_0}{2}$$

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as well. Thus, the boundary terms given in (3.6), from (3.5) can be estimated in the following way:

$$(B.T.) \geq \frac{\varepsilon_0}{2} \int_{Q_s} \left(\frac{y_t^2}{\sigma} + \eta y_x^2 \right) dxdt. \tag{3.20}$$

Now, we estimate the boundary terms from above. First of all, consider the term

$$\int_0^1 \left(-2x \frac{y_x y_t}{\sigma} + \frac{K}{2} \frac{y y_t}{\sigma} \right) (\tau, x) dx$$

for all $\tau \in [s, T]$. Using the fact that $\frac{x^2}{a(x)} \leq \frac{1}{a(1)}$ by (1.4), together with Proposition 2.1, one has

$$\begin{aligned}
&\int_0^1 \left(-2x \frac{y_x y_t}{\sigma} + \frac{K}{2} \frac{y y_t}{\sigma} \right) (\tau, x) dx \\
&\quad \leq \int_0^1 \frac{x^2 y_x^2 \eta}{a} (\tau, x) dx + \int_0^1 \frac{y_t^2}{\sigma} (\tau, x) dx + \frac{K}{4} \int_0^1 \frac{y_t^2}{\sigma} (\tau, x) dx + \frac{K}{4} \int_0^1 \frac{y^2}{\sigma} (\tau, x) dx \\
&\quad \leq \frac{1}{a(1)} \int_0^1 \eta y_x^2 (\tau, x) dx + \left(1 + \frac{K}{4} \right) \int_0^1 \frac{y_t^2}{\sigma} (\tau, x) dx \\
&\qquad + K \frac{C_{\text{HP}}}{\min_{[0,1]} \eta} \int_0^1 y_x^2 \eta (\tau, x) dx \\
&\quad \leq \max \left\{ \frac{1}{a(1)} + K \frac{C_{\text{HP}}}{\min_{[0,1]} \eta}; 1 + \frac{K}{4} \right\} (2E_y(\tau) - \beta y^2(\tau, 1)).
\end{aligned}$$

By (3.20) and (3.6), thanks to Theorem 3.1, we get

$$\begin{aligned}
\frac{\varepsilon_0}{2} \int_{Q_s} \left(\frac{y_t^2}{\sigma} + \eta y_x^2 \right) dxdt &\leq 2\Theta E_y(T) + 2\Theta E_y(s) \\
&\quad + \int_s^T \left(\frac{y_t^2}{\sigma} + \eta y_x^2 - \frac{K}{2} \eta y y_x \right) (t, 1) dt \\
&\leq 4\Theta E_y(s) + \int_s^T \left(\frac{y_t^2}{\sigma} + \eta y_x^2 - \frac{K}{2} \eta y y_x \right) (t, 1) dt.
\end{aligned} \tag{3.21}$$

where Θ is defined in (3.19).

Now, we need to estimate $\int_s^T h(t) dt$, where

$$h(t) := \left(\frac{y_t^2}{\sigma} + \eta y_x^2 - \frac{K}{2} \eta y y_x \right) (t, 1).$$

To this aim, recall that $\eta y_x(t, 1) = -\beta y(t, 1) - y_t(t, 1)$. Thus

$$h(t) = \frac{y_t^2}{\sigma}(t, 1) + \frac{1}{\eta(1)} (-\beta y(t, 1) - y_t(t, 1))^2 - \frac{K}{2} (-\beta y(t, 1) - y_t(t, 1)) y(t, 1)$$

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$$\begin{aligned}
&= \frac{y_t^2}{\sigma}(t,1) + \frac{1}{\eta(1)}(\beta^2 y^2(t,1) + y_t^2(t,1) + 2\beta y(t,1)y_t(t,1)) \\
&\quad + \frac{K}{2}\beta y^2(t,1) + \frac{K}{2}y_t(t,1)y(t,1) \\
&= \left(\frac{1}{\sigma(1)} + \frac{1}{\eta(1)}\right) y_t^2(t,1) + \beta\left(\frac{\beta}{\eta(1)} + \frac{K}{2}\right) y^2(t,1) + \left(\frac{2\beta}{\eta(1)} + \frac{K}{2}\right) yy_t(t,1).
\end{aligned}$$

Thus, by Theorem 3.1

$$\begin{aligned}
\int_s^T h(t)dt &= \left(\frac{1}{\sigma(1)} + \frac{1}{\eta(1)}\right)\int_s^T y_t^2(t,1)dt + \beta\left(\frac{\beta}{\eta(1)} + \frac{K}{2}\right)\int_s^T y^2(t,1)dt \\
&\quad + \left(\frac{2\beta}{\eta(1)} + \frac{K}{2}\right)\int_s^T yy_t(t,1)dt \\
&\le \left(\frac{1}{\sigma(1)} + \frac{1}{\eta(1)}\right)\int_s^T -\frac{dE_y}{dt}dt + \beta\left(\frac{\beta}{\eta(1)} + \frac{K}{2}\right)\int_s^T y^2(t,1)dt \\
&\quad + \frac{1}{2}\left(\frac{2\beta}{\eta(1)} + \frac{K}{2}\right)\int_s^T y^2(t,1)dt + \frac{1}{2}\left(\frac{2\beta}{\eta(1)} + \frac{K}{2}\right)\int_s^T -\frac{dE_y}{dt}dt \\
&= \left(\frac{1}{\sigma(1)} + \frac{1}{\eta(1)} + \frac{\beta}{\eta(1)} + \frac{K}{4}\right)(E_y(s) - E_y(T)) \\
&\quad + \left(\frac{\beta^2}{\eta(1)} + \frac{K\beta}{2} + \frac{\beta}{\eta(1)} + \frac{K}{4}\right)\int_s^T y^2(t,1)dt. \qquad (3.22)
\end{aligned}$$

By (3.21) and (3.22), we have

$$\begin{aligned}
\frac{\varepsilon_0}{2}\int_{Q_s}\left(\frac{y_t^2}{\sigma} + \eta y_x^2\right)dxdt \le 4\Theta E_y(s) &+ \left(\frac{1}{\sigma(1)} + \frac{1}{\eta(1)} + \frac{\beta}{\eta(1)} + \frac{K}{4}\right) \\
&\times (E_y(s) - E_y(T)) \\
&+ \left(\frac{\beta^2}{\eta(1)} + \frac{K\beta}{2} + \frac{\beta}{\eta(1)} + \frac{K}{4}\right)\int_s^T y^2(t,1)dt
\end{aligned}$$

and (3.18) holds. □

As a consequence of Propositions 3.3 and 3.4, we can state the main result of the paper. As a first step, define

$$C_1 := \left(\frac{\beta^2}{\eta(1)} + \frac{K\beta}{2} + \frac{\beta}{\eta(1)} + \frac{K}{4} + \frac{\beta\varepsilon_0}{2}\right)\left(\frac{1}{\min_{[0,1]}^3 \eta} + 1\right).$$

Theorem 3.2. *Assume Hypothesis* 4, $\beta \ge 0$ *and let* y *be a mild solution of* (1.2). *Then, for all* $t > 0$ *and for all* $\delta \in \left(0, \frac{\varepsilon_0}{2C_1}\right)$, *we have*

$$E_y(t) \le E_y(0)e^{1-\frac{t}{M}}, \qquad (3.23)$$

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where

$$\begin{aligned} M := {} & \frac{1}{C_{\varepsilon_0,\beta,K,\eta}}\left(4\Theta + \frac{1}{\sigma(1)} + \frac{1}{\eta(1)} + \frac{\beta}{\eta(1)} + \frac{K}{4}\right) \\ & + \frac{1}{C_{\varepsilon_0,\beta,K,\eta}}\left(\frac{\beta^2}{\eta(1)} + \frac{K\beta}{2} + \frac{\beta}{\eta(1)} + \frac{K}{4} + \frac{\beta\varepsilon_0}{2}\right) \\ & \cdot\left(2 + \frac{2C_{\mathrm{HP}}}{\min^3_{[0,1]}\eta} + \frac{1}{\delta} + \frac{1}{\delta}\frac{\max_{[0,1]}\eta + C_{\mathrm{HP}}}{\min^2_{[0,1]}\eta}\right), \qquad (3.24) \end{aligned}$$

being $C_{\varepsilon_0,\beta,K,\eta} := \varepsilon_0 - 2\delta C_1$ *and* Θ *as in* (3.19).

Proof. As usual, let us start assuming that y is a classical solution. If y is the mild solution associated to the initial data $(y_0, y_1) \in \mathcal{H}_0$, consider a sequence $\{(y_0^n, y_1^n)\}_{n\in\mathbb{N}} \in D(\mathcal{A})$ that approximate (y_0, y_1) and let y^n be the classical solution of (1.2) associated to (y_0^n, y_1^n). With standard estimates coming from Theorem 2.1, we can pass to the limit in (3.23) written for y^n and obtain the desired result.

So, let y be a classical solution. By (3.18) and (3.9), we have

$$\begin{aligned} \varepsilon_0\int_s^T E_y(t)dt \le {} & 4\Theta E_y(s) + \left(\frac{1}{\sigma(1)} + \frac{1}{\eta(1)} + \frac{\beta}{\eta(1)} + \frac{K}{4}\right)E_y(s) \\ & + \left(\frac{\beta^2}{\eta(1)} + \frac{K\beta}{2} + \frac{\beta}{\eta(1)} + \frac{K}{4} + \frac{\beta\varepsilon_0}{2}\right)\int_s^T y^2(t,1)dt \\ \le {} & \left(4\Theta + \frac{1}{\sigma(1)} + \frac{1}{\eta(1)} + \frac{\beta}{\eta(1)} + \frac{K}{4}\right)E_y(s) \\ & + \left(\frac{\beta^2}{\eta(1)} + \frac{K\beta}{2} + \frac{\beta}{\eta(1)} + \frac{K}{4} + \frac{\beta\varepsilon_0}{2}\right) \\ & \times\left(2 + \frac{2C_{\mathrm{HP}}}{\min^3_{[0,1]}\eta} + \frac{1}{\delta} + \frac{1}{\delta}\frac{\max_{[0,1]}\eta + C_{\mathrm{HP}}}{\min^2_{[0,1]}\eta}\right)E_y(s) \\ & + 2\delta C_1\int_s^T E_y(t)dt. \end{aligned}$$

Hence

$$\begin{aligned} \int_s^T E_y(t)dt \le {} & \left(4\Theta + \frac{1}{\sigma(1)} + \frac{1}{\eta(1)} + \frac{\beta}{\eta(1)} + \frac{K}{4}\right)E_y(s) \\ & + \left(\frac{\beta^2}{\eta(1)} + \frac{K\beta}{2} + \frac{\beta}{\eta(1)} + \frac{K}{4} + \frac{\beta\varepsilon_0}{2}\right) \\ & \times\left(2 + \frac{2C_{\mathrm{HP}}}{\min^3_{[0,1]}\eta} + \frac{1}{\delta} + \frac{1}{\delta}\frac{\max_{[0,1]}\eta + C_{\mathrm{HP}}}{\min^2_{[0,1]}\eta}\right)E_y(s). \end{aligned}$$

Choosing $\delta \in \left(0, \frac{\varepsilon_0}{2C_1}\right)$, we have

$$\varepsilon_0 - 2\delta C_1 > 0.$$

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Hence, we can apply Lemma 3.1 as follows, obtaining

$$E_y(t) \le E_y(0)e^{1-\frac{t}{M}},$$

where M is as in (3.24). □

Lemma 3.1. *Assume that* $E : [0, +\infty) \to [0, +\infty)$ *is a nonincreasing function and that there is a constant* $M > 0$ *such that*

$$\int_t^{\infty} E(s)ds \le ME(t), \quad \forall\, t\ [0, +\infty).$$

Then

$$E(t) \le E(0)e^{1-\frac{t}{M}}, \quad \forall\, t\ [0, +\infty).$$

Appendix A

For the readers' convenience, in this section we prove a technical lemma, used throughout the previous sections.

Proof of Lemma 2.2. (1) We will actually prove an equivalent fact, usually appearing in integrations by parts, namely, that

$$\lim_{x\to 0} \eta(x)u(x)y'(x) = 0.$$

For this, consider the function

$$z(x) := \eta(x)u(x)y'(x), \quad x \in (0, 1].$$

Observe that

$$\int_0^1 |z|dx \le C\|u\|_{L^2(0,1)}\|y'\|_{L^2(0,1)}.$$

Moreover

$$z'(x) = (\eta y')'u + \eta y'u'$$

and, by Hölder's inequality,

$$\int_0^1 |(\eta y')'u|dx = \int_0^1 \sqrt{\sigma}(\eta y')'\frac{u}{\sqrt{\sigma}}dx \le \|\sigma(\eta y')'\|_{\frac{1}{\sigma}}\|u\|_{\frac{1}{\sigma}}$$

and

$$\int_0^1 \eta|y'u'|dx \le C\|y'\|_{L^2(0,1)}\|u'\|_{L^2(0,1)}.$$

Thus z' is summable on $[0, 1]$. Hence $z \in W^{1,1}(0, 1) \hookrightarrow C[0, 1]$ and there exists

$$\lim_{x\to 0} z(x) = \lim_{x\to 0} \eta(x)u(x)y'(x) = L \in \mathbb{R}.$$

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We will prove that $L = 0$. If $L \neq 0$ there would exist a neighborhood $\mathcal{I}$ of 0 such that

$$\frac{|L|}{2} \leq |\eta y' u|,$$

for all $x \in \mathcal{I}$; but, by Hölder's inequality

$$|u(x)| \leq \int_0^x |u'(t)|dt \leq \sqrt{x}\|u'\|_{L^2(0,1)}.$$

Hence

$$\frac{|L|}{2} \leq |\eta y' u| \leq \|\eta\|_\infty |y'| \sqrt{x}\|u'\|_{L^2(0,1)}$$

for all $x \in \mathcal{I}$. This would imply that

$$|y'| \geq \frac{|L|}{2\|u'\|_{L^2(0,1)}\|\eta\|_\infty \sqrt{x}}$$

in contrast to the fact that $y' \in L^2(0,1)$.

Hence, $L = 0$ and the conclusion follows.

(2) It is enough to write

$$xu'(\eta u')' = \frac{xu'}{\sqrt{\sigma}}\sqrt{\sigma}(\eta u')'$$

and note that, by Hypothesis 2

$$\left|\frac{xu'}{\sqrt{\sigma}}\right|^2 \leq \|\eta\|_\infty \frac{x^2}{a(x)}(u')^2 \leq C\|\eta\|_\infty (u')^2 \in L^1(0,1),$$

for a positive constant C, and

$$(\sqrt{\sigma}(\eta u')')^2 = \frac{(Au)^2}{\sigma} \in L^1(0,1).$$

(3) Set $z(x) = x\eta(x)(u'(x))^2$. Of course, $z \in L^1(0,1)$. Moreover,

$$z' = \eta(u')^2 + x\eta'(u')^2 + 2x\eta u' u'' = 2xu'(\eta u')' - x\eta'(u')^2 + \eta(u')^2.$$

By the previous point, $xu'(\eta u')' \in L^1(0,1)$, while

$$|x\eta'(u')^2| = \left|x\eta\frac{b}{a}(u')^2\right| \leq C(u')^2$$

by (1.6). Clearly, $\eta(u')^2 \in L^1(0,1)$, thus $z' \in L^1(0,1)$, so that $z \in W^{1,1}(0,1)$ and there exists $\lim_{x\to 0} z(x) = L \in \mathbb{R}$. If $L \neq 0$, sufficiently close to $x = 0$ we would have that

$$(u'(x))^2 \geq \frac{|L|}{2\eta x} \notin L^1(0,1),$$

while $u \in H^1(0,1)$. The conclusion follows since η is bounded away from 0.

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(4) Proceed as above, observing that

$$|x\eta'(u')^2| = \left|x\eta\frac{b}{a}(u')^2\right| \le \left\|\frac{xb}{a}\right\|_{L^\infty(0,1)} \|\eta\|_{L^\infty(0,1)}(u')^2$$

and that

$$|xu'(\eta u')'| = \left|\frac{x}{\sqrt{a}}\sqrt{\eta}u'\right| \cdot |\sqrt{\sigma}(\eta u')'|.$$

(5) Set $z := \frac{x}{a}u^2(x)$. Then $z \in L^1(0,1)$. Indeed

$$\int_0^1 \frac{x}{a}u^2(x)dx \le \frac{1}{\min_{[0,1]}\eta}\int_0^1 \frac{u^2}{\sigma}dx.$$

Moreover

$$z' = \frac{u^2}{a} + 2\frac{xuu'}{a} - \frac{a'x}{a^2}u^2;$$

thus, for a suitable $\varepsilon > 0$ given by Hypothesis 2

$$\begin{aligned}\int_0^\varepsilon |z'|dx &\le \frac{1}{\min_{[0,1]}\eta}\int_0^1 \frac{u^2}{\sigma}dx \\ &\quad + 2\left(\int_0^\varepsilon \frac{x^2(u')^2}{a}dx\right)^{\frac{1}{2}}\left(\int_0^1 \frac{u^2}{a}dx\right)^{\frac{1}{2}} + K\int_0^1 \frac{u^2}{a}dx \\ &\le \frac{1+K}{\min_{[0,1]}\eta}\int_0^1 \frac{u^2}{\sigma}dx + \frac{2\varepsilon^2}{a(\varepsilon)\min_{[0,1]}\eta}\left(\int_0^1 (u')^2dx\right)^{\frac{1}{2}}\left(\int_0^1 \frac{u^2}{\sigma}dx\right)^{\frac{1}{2}}.\end{aligned}$$

This is enough to conclude that $z \in W^{1,1}(0,1)$ and thus there exists $\lim_{x\to 0} z(x) = L \in \mathbb{R}$. If $L \neq 0$, sufficiently close to $x = 0$ we would have that

$$\frac{u^2(x)}{a} \ge \frac{|L|}{2x} \notin L^1(0,1),$$

while $\frac{u^2}{a} \in L^1(0,1)$ (since $\frac{u^2}{\sigma} \in L^1(0,1)$). □

Acknowledgments

Genni Fragnelli is a member of the *Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA)* of the Istituto Nazionale di Alta Matematica (INdAM). She is supported by the INDAM-GNAMPA Project 2024 "Analysis, control and inverse problems for evolution equations arising in climate science" and by the FFABR *Fondo per il finanziamento delle attività base di ricerca* 2017.

Dimitri Mugnai is a member of the *Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA)* of the Istituto Nazionale di Alta Matematica (INdAM). He is supported by the INdAM-GNAMPA Project 2024 "Nonlinear problems in local and nonlocal settings with applications" and by the FFABR *Fondo per il finanziamento delle attività base di ricerca* 2017.

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