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Market consistent bid-ask option pricing under Dempster-Shafer uncertainty

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We refer to the discrete-time market model under ambiguity introduced in Cinfrignini *et al.* (2023a), formed by a frictionless risk-free bond and a non-dividend paying stock with bid-ask spread. For a European derivative, we generalize the classical binomial pricing formula by allowing for bid-ask prices and investigate the properties of the ensuing replicating strategies. Next, for an American derivative, we propose a backward bid-ask pricing procedure and prove that the resulting discounted price processes are the bid-ask Choquet-Snell envelopes of the discounted payoff process, respectively. Moreover, for an American call option, we prove a generalization of the well-known Merton's theorem (Merton 1973) holding for both the bid and the ask price processes. Finally, we introduce a market consistent calibration procedure and show the use of the calibrated model in bid-ask option pricing.

Keywords: Bid-ask prices; DS-multiplicative binomial process; Market data calibration; American option pricing.

JEL Classification: G12; D81; C71; C61.

1. Introduction

The absence of frictions in the market is one of the fundamental assumptions on which the classical no-arbitrage pricing theory is based (Cox *et al.* 1979, Pliska 1997). However, a frictionless market is a suitable mathematical hypothesis that often is not reflected in the reality, since markets show frictions, particularly in the form of bid-ask spreads (Amihud and Mendelson 1986, 1991). The main advantage in assuming a frictionless market is the linearity of the no-arbitrage pricing rule that reduces to a discounted conditional expectation with respect to a risk-neutral probability measure. In order to define pricing models in frictional markets and address bid-ask prices we necessarily need to depart from linearity.

Starting from the works of Chateauneuf *et al.* (1996), Araujo *et al.* (2012), Cerreia-Vioglio *et al.* (2015), and in line with the recent works of Bastianello *et al.* (2022), Chateauneuf and Cornet (2022), Cinfrignini *et al.* (2023b,a), Petturiti and Vantaggi (2023), we consider a dynamic lower (i.e., bid) pricing rule defined as the discounted conditional Choquet integral (Grabisch 2016, Choquet 1954) relying on the framework of belief functions in Dempster-Shafer theory (Dempster 1967, Shafer 1976a).

We refer to the model introduced in Cinfrignini *et al.* (2023a) where we defined a time-homogeneous Markov multiplicative binomial process that we called *DS-multiplicative binomial*

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process (where *DS* stands for *Dempster-Shafer*), which is characterized by a family of transition belief functions with respect to two strictly positive parameters. In a market formed by a frictionless risk-free bond with deterministic price process and a non-dividend paying stock with bid-ask spread, we assumed that the stock bid price was modeled by a DS-multiplicative binomial process. In the same paper, we proved the existence of an equivalent belief function (also referred to as risk-neutral belief function), that further is a *one-step Choquet martingale* and a *global Choquet super-martingale*. Furthermore, for European-type derivatives whose payoff depends on the stock bid price at maturity, we proposed a dynamic lower (i.e., bid) pricing rule as the one-step discounted conditional Choquet expectation with respect to the risk-neutral belief function. On the other hand, the dynamic upper (i.e., ask) pricing rule was defined one-step-wise through duality. Another important feature of the model in Cinfrignini *et al.* (2023a) is its nice parameterization, where the two risk-neutral parameters $\widehat{b_u}, \widehat{b_d}$ can be interpreted as one-step "up" and "down" risk-neutral transition beliefs, and, since their sum may not be 1, we could quantify the ambiguity in terms of the excessive weight to unity $1 - (\widehat{b_u} + \widehat{b_d})$. In turn, this has a direct impact on calibration tasks.

In this paper we go deep into the characterization of the bid and the ask pricing rules for European-type derivatives introduced in Cinfrignini *et al.* (2023a), generalizing the classical binomial pricing formula to obtain bid-ask prices. We prove that, for a non-decreasing (non-increasing) contract function, the bid (ask) pricing rule does not depend on the choice of $\widehat{b_d}$. This allows us to show that for a non-decreasing (non-increasing) contract function, the bid (ask) replicating strategy is self-financing, while the same does not hold for the ask (bid) replicating strategy.

Next, for an American-type derivative whose payoff depends on the stock bid price, we propose a backward bid-ask pricing procedure. Given a suitable notion of bid-ask Snell envelopes with respect to the conditional Choquet expectation, we prove that the discounted bid and the ask price processes are the bid and the ask Choquet-Snell envelopes of the discounted payoff process, respectively.

For an American call option, we prove a generalization of the well-known Merton's theorem (Merton 1973), which states that the early exercise of an American call option is never optimal, with respect to neither the bid price process nor the ask price process. On the other hand, for the American put option we show that early exercise can be optimal both referring to the bid price process and the ask price process of the derivative, therefore, we generally have two distinct exercise strategies.

The interpretable risk-neutral parameterization of our model allows us to derive a market consistent calibration procedure. Such calibration relies on the bid-ask prices of European call and put options on a non-dividend paying stock, focusing on a set of strike prices and a fixed maturity, and consists in the minimization of the squared error. It turns out that the minimization procedure requires to solve a constrained optimization problem, where the objective functions contains maximum functions, so, its linearization via binary variables makes the resulting problem too complex, even for a small set of options. For this reason, we carry out the optimization task relying on the *particle swarm optimization (PSO)* (Kennedy and Eberhart 2001), which is a stochastic incomplete method. We show that the suggested procedure can achieve a good fitting in comparison with the market bid-ask prices. Finally, we use the calibrated parameters to determine the bid-ask prices of American put options and investigate the theoretical bid-ask spreads.

In the literature there are other proposals for addressing bid-ask prices in a dynamic setting: distinguished models are the axiomatic approach of Jouini (2000), the time consistent pricing procedure of Bion-Nadal (2009), and the conic market model of Carr and Zhu (2018). On the other hand, Driouchi *et al.* (2015) considered dynamic pricing within the Choquet theory, though not coping with bid-ask spreads. The main difference of the model proposed in Cinfrignini *et al.* (2023a) and further analyzed in this paper with the existing literature rests on the newly established DS-multiplicative binomial process and the corresponding conditional Choquet expectation operator. The latter turns out to be a completely monotone conditional operator generated by a global belief function through the notion of conditioning due to Suppes and Zanotti (1977).

The paper is structured as follows. Section 2 recalls the necessary preliminaries. In particular, Subsection 2.1 introduces the Dempster-Shafer theory and the Choquet integral, Subsection 2.2 defines the bid-ask binomial market model under Dempster-Shafer uncertainty, while Subsection 2.3 gives a normative justification of the proposed model in terms of a suitable dynamic no-arbitrage condition. Section 3 reviews the dynamic bid-ask pricing rule for a European-type derivative and generalizes the binomial pricing formula in the non-additive framework, for a non-decreasing and non-increasing contract function, respectively. Moreover, Subsection 3.1 discusses replicating strategies. Section 4 defines a dynamic bid-ask pricing rule for an American-type derivative, proving that the discounted bid and ask price processes are the bid-ask Choquet-Snell envelopes of the discounted payoff process. It is proved that, for an American call option, the early exercise is never optimal. On the contrary, for an American put option, several cases concerning early exercise can occur. In Section 5 we calibrate our model on market data by minimizing the squared error of bid-ask European call and put option prices for a set of strike prices, and use the calibrated model to price American put options. Finally, Section 6 collects our conclusions and draws future perspectives. To improve readability, all proofs are gathered in Appendix A.

2. Preliminaries

We introduce non-additive uncertainty measures in order to encode ambiguity so as to handle partial knowledge like in the celebrated Ellsberg's urn paradox (Ellsberg 1961). In particular, we focus on the uncertainty theory due to Dempster and Shafer. Then, referring to this uncertainty theory we consider the binomial market model with frictions recently introduced in Cinfrignini *et al.* (2023a).

2.1. *Dempster-Shafer uncertainty theory*

Let $\Omega = \{\omega_1, \ldots, \omega_d\}$ be a finite non-empty set of states of the world and $\mathcal{F} = \mathcal{P}(\Omega)$, where $\mathcal{P}(\Omega)$ stands for the power set of Ω. We denote by $\mathbf{R}^\Omega$ the set of all random variables on Ω.

The Dempster-Shafer theory (Dempster 1967, Shafer 1976a) encodes uncertainty through a non-additive measure, called *belief function*, that is a mapping $\nu : \mathcal{F} \to [0,1]$ satisfying:

(i) $\nu(\emptyset) = 0$ and $\nu(\Omega) = 1$;
(ii) for every $k \geq 2$ and every $A_1, \ldots, A_k \in \mathcal{F}$,

$$\nu\left(\bigcup_{i=1}^{k} A_i\right) \geq \sum_{\emptyset \neq I \subseteq \{1,\ldots,k\}} (-1)^{|I|+1} \nu\left(\bigcap_{i \in I} A_i\right).$$

Condition *(ii)* is called *complete monotonicity* and if *(ii)* holds as an equality, ν reduces to a probability measure (customarily denoted by P).

Every belief function is associated with a dual set function called *plausibility function* $\overline{\nu} : \mathcal{F} \to [0,1]$ defined, for all $A \in \mathcal{F}$, as

$$\overline{\nu}(A) = 1 - \nu(A^c).$$

Both ν and $\overline{\nu}$ are completely characterized by the Möbius inverse of ν which is a set function $\mu : \mathcal{F} \to [0,1]$ satisfying $\mu(\emptyset) = 0$ and $\sum_{B \in \mathcal{F}} \mu(B) = 1$, and such that, for all $A \in \mathcal{F}$, (see, e.g., Chateauneuf and Jaffray (1989))

$$\nu(A) = \sum_{B \subseteq A} \mu(B) \quad \text{and} \quad \overline{\nu}(A) = \sum_{B \cap A \neq \emptyset} \mu(B).$$

Every belief function induces a non-empty, closed and convex set of probability measures on $\mathcal{F}$ called *core* (see, e.g., Grabisch (2016)) defined as

$$\mathbf{core}(\nu) = \{P \ : \ P \text{ is a probability measure on } \mathcal{F}, \ P \geq \nu\},$$

for which it holds that, for every $A \in \mathcal{F}$,

$$\nu(A) = \min_{P \in \mathbf{core}(\nu)} P(A) \quad \text{and} \quad \overline{\nu}(A) = \max_{P \in \mathbf{core}(\nu)} P(A),$$

showing that ν and $\overline{\nu}$ are particular *(coherent) lower and upper probabilities* (Walley 1991).

The notion of conditioning in Dempster-Shafer theory is still an open issue and several proposals have been given in the literature (see, e.g., Dempster (1967), Suppes and Zanotti (1977), Shafer (1976b)). In this work we refer to the *product (or geometric) conditioning rule* proposed by Suppes and Zanotti (1977) (see also Coletti and Vantaggi (2008)). For every $E, H \in \mathcal{F}$ with $\nu(H) > 0$, we define

$$\nu(E|H) = \frac{\nu(E \cap H)}{\nu(H)}. \tag{1}$$

We have that, for every $H \in \mathcal{F}$ with $\nu(H) > 0$, $\nu(\cdot|H)$ is still a belief function, so, it induces a core denoted as $\mathbf{core}(\nu(\cdot|H))$.

Given $\nu(\cdot|H)$ and $X \in \mathbf{R}^\Omega$, we can introduce the *conditional Choquet expectation* of X with respect to $\nu(\cdot|H)$, defined through the Choquet integral (Choquet 1954)

$$\oint X(\omega) \mathrm{d}\nu(\omega|H) = \sum_{i=1}^{d} \left(X(\omega_{\sigma(i)}) - X(\omega_{\sigma(i+1)}) \right) \nu(E_i^\sigma|H), \tag{2}$$

where σ is a permutation of Ω such that $X(\omega_{\sigma(1)}) \geq \cdots \geq X(\omega_{\sigma(d)})$, $E_i^\sigma = \{\omega_{\sigma(1)}, \ldots, \omega_{\sigma(i)}\}$ for $i = 1, \ldots, d$ and $X(\omega_{\sigma(d+1)}) = 0$. Moreover, the conditional Choquet expectation can be interpreted as a lower expectation locally on H, by referring to $\mathbf{core}(\nu(\cdot|H))$, since

$$\oint X(\omega) \mathrm{d}\nu(\omega|H) = \min_{P \in \mathbf{core}(\nu(\cdot|H))} \int X(\omega) \mathrm{d}P(\omega). \tag{3}$$

Furthermore, if $\overline{\nu}(\cdot|H)$ is the dual plausibility function of $\nu(\cdot|H)$, it holds that

$$\oint X(\omega) \mathrm{d}\overline{\nu}(\omega|H) = -\oint (-X(\omega)) \mathrm{d}\nu(\omega|H) = \max_{P \in \mathbf{core}(\nu(\cdot|H))} \int X(\omega) \mathrm{d}P(\omega), \tag{4}$$

that can be interpreted as an upper expectation locally on H.

Remark 1 The issue of updating ambiguous beliefs has been investigated, e.g., in Gilboa and Schmeidler (1993), while Araujo *et al.* (2019) deal with updating in the context of pricing rules. In the case of Dempster-Shafer theory, as discussed in Coletti *et al.* (2016), Cinfrignini *et al.* (2023a), two other popular choices are the *Dempster's conditioning rule* (Dempster 1967) and the *Bayesian conditioning rule* (Fagin and Halpern 1991). In our pricing context, the choice of the product conditioning rule is motivated by the smaller dilation that it produces when computing the conditional Choquet integral, with respect to the Bayesian conditioning rule. On the other hand, no dominance relation holds between the product and the Dempster's conditioning rules. Additionally, in our model the product conditioning rule assures that the core of the updated belief function never reduces to a singleton, i.e., ambiguity never vanishes, provided ν is non-additive.

2.2. *Bid-ask binomial market model under Dempster-Shafer uncertainty*

In this paper we refer to the DS-multiplicative binomial process characterized in Cinfrignini *et al.* (2023a), used to model the bid price evolution of a stock over a discrete set of times $\{0,\dots,T\}$, where $T \in \mathbf{N}$ is a finite time horizon.

Consider a discrete-time market model formed by a frictionless risk-free bond and a non-dividend paying stock with frictions, in the form of bid-ask spread. The price of the bond is expressed by the deterministic process $\{B_n\}_{n=0}^T$, where $B_0 = 1$ and, for $n = 1,\dots,T$,

$$B_n = (1+r)B_{n-1}, \tag{5}$$

with $1 + r > 0$, in which r is the risk-free interest rate over each period. On the other hand, the bid price of the stock is expressed by the process $\{S_n\}_{n=0}^T$ such that, for $n = 1,\dots,T$,

$$S_n = \begin{cases} uS_{n-1} \text{ if "up"}, \\ dS_{n-1} \text{ if "down"}, \end{cases} \tag{6}$$

where $S_0 = s_0 > 0$ and $u > d > 0$ are the "up" and "down" coefficients.

All the processes we consider are defined on a filtered measurable space $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n=0}^T)$, where $\Omega = \{1,\dots,2^T\}$ and $\mathcal{F}_n$ is the algebra generated by random variables $\{S_0,\dots,S_n\}$, for $n = 0,\dots,T$, with $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_T = \mathcal{P}(\Omega)$.

The trajectories of $\{S_n\}_{n=0}^T$ can be represented graphically on a recombining binomial tree and every state $\omega \in \Omega$ is identified with the path corresponding to the T-digit binary expansion of number $\omega - 1$. For $n = 1,\dots,T$, denoting $\mathcal{A}_n = \{a_k = u^k d^{n-k} : k = 0,\dots,n\}$, we write $As = \{a_k s : a_k \in A\}$, for every $s > 0$ and $A \in \mathcal{P}(\mathcal{A}_n)$. Therefore, each random variable S_n ranges in $\mathcal{S}_n = \mathcal{A}_n s_0$.

We assume that uncertainty is captured by a belief function $\nu : \mathcal{F} \to [0,1]$ singled out by a reference family of t-step transition belief functions $\{\beta_t\}_{t=1}^T$ determined by two parameters, b_u and b_d, such that $b_u, b_d \in (0,1)$ and $b_u + b_d \le 1$, that can be interpreted as one-step "up" and "down" conditional beliefs. We refer to the family of t-step transition belief functions given, for $t = 1,\dots,T$ and for all $A \in \mathcal{P}(\mathcal{A}_t)$, by

$$\beta_t(A) = \sum_{a_k \in A} \binom{t}{k} b_u^k b_d^{t-k} + \sum_{\substack{[a_k, a_{k+j}] \subseteq A \\ j \ge 1}} \binom{t-j}{k} b_u^k b_d^{t-j-k} (1 - (b_u + b_d)), \tag{7}$$

where, for $i \le j$, $[a_i, a_j] = \{a_k \in \mathcal{A}_t \ : a_i \le a_k \le a_j\}$.

As proved in Cinfrignini *et al.* (2023a), there exists a strictly positive belief function ν on $\mathcal{F}$ such that the process $\{S_n\}_{n=0}^T$ is Markov and time-homogeneous with transitions given by (7), namely a *DS-multiplicative binomial process*. Explicitly, this means that for every $0 \le n \le T-1$ and $1 \le t \le T-n$, $A \in \mathcal{P}(\mathcal{A}_t)$, and $s_0 \in \mathcal{S}_0,\dots,s_n \in \mathcal{S}_n$ it holds that

$$\nu(S_{n+t} \in As_n | S_0 = s_0,\dots,S_n = s_n) = \nu(S_{n+t} \in As_n | S_n = s_n) = \beta_t(A). \tag{8}$$

We notice that, there can be infinitely many belief functions that make the process a DS-multiplicative binomial process, and the entire family $\{\beta_t\}_{t=1}^T$ must be fixed to constrain a global

ν (see Cinfrignini *et al.* (2022) for a related discussion). Our choice of (7) is due to its nice interpretation and parameterization. Indeed, the belief function β_t in (7) generalizes the binomial distribution by taking into account the contribution of intervals contained in A with a binomial-like weighting, deflated by the excessive weight to unity $1-(b_u+b_d)$. In detail, $\beta_t(A)$ is the sum of binomial-like weights of all partial trajectories with decreasing length that support the evidence of having a final state of the process belonging to As_n. Furthermore, if $b_u+b_d=1$, β_t reduces to the classical binomial distribution with parameters b_u and t. As will be shown in Section 5, the parameterization in (7) will play a crucial role in the model calibration.

From now on we assume the "real-world" filtered belief space $(\Omega,\mathcal{F},\{\mathcal{F}_n\}_{n=0}^T,\nu)$ to be fixed. This allows us to define (see Cinfrignini *et al.* (2023a)), for every $X\in\mathbf{R}^\Omega$, the random variables $\mathbf{C}[X|\mathcal{F}_n]$ and $\mathbf{C}[X|S_0,\ldots,S_n]$ by setting, for all $\omega\in\{S_0=s_0,\ldots,S_n=s_n\}$,

$$\mathbf{C}[X|\mathcal{F}_n](\omega):=\mathbf{C}[X|S_0,\ldots,S_n](\omega):=\oint X\mathrm{d}\nu(\cdot|S_0=s_0,\ldots,S_n=s_n), \tag{9}$$

while the random variable $\mathbf{C}[X|S_n]$ is defined analogously by referring to $\nu(\cdot|S_n=s_n)$, for all $\omega\in\{S_n=s_n\}$. In what follows we refer to the operator $\mathbf{C}[\cdot|\mathcal{F}_n]$ as *conditional Choquet expectation* which, for every $0\le n\le T-1$, $1\le t\le T-n$, and every real-valued function of one real variable $\varphi(x)$ defined on the range of S_{n+t}, by (8) satisfies

$$\mathbf{C}[\varphi(S_{n+t})|\mathcal{F}_n]=\mathbf{C}[\varphi(S_{n+t})|S_n]. \tag{10}$$

As usual, taking the process $\{B_n\}_{n=0}^T$ as *numeraire*, we can define the discounted process $\{S_n^*\}_{n=0}^T$ setting, for $n=0,\ldots,T$,

$$S_n^*=\frac{S_n}{B_n}=\frac{S_n}{(1+r)^n}. \tag{11}$$

Theorem 2 in Cinfrignini *et al.* (2023a) shows that the condition $u>1+r>d>0$ is necessary and sufficient to the existence of an *equivalent one-step Choquet martingale belief function*, where we define a belief function $\widehat{\nu}$ to be *equivalent* to ν if $\nu(A)=0\iff\widehat{\nu}(A)=0$, for every $A\in\mathcal{F}$. Such a belief function $\widehat{\nu}:\mathcal{F}\to[0,1]$ is strictly positive and makes the process $\{S_n^*\}_{n=0}^T$ on the filtered belief space $(\Omega,\mathcal{F},\{\mathcal{F}_n\}_{n=0}^T,\widehat{\nu})$ a DS-multiplicative binomial process and a *one-step Choquet martingale*, while the process is globally only a *Choquet super-martingale*. Denoting by $\widehat{\mathbf{C}}[\cdot|\mathcal{F}_n]$ the conditional Choquet expectation with respect to $\widehat{\nu}$, the last two properties mean that, for every $0\le n\le T-1$ and $1\le t\le T-n$, it holds that

$$\widehat{\mathbf{C}}[S_{n+1}^*|\mathcal{F}_n]=S_n^*, \tag{12}$$

$$\widehat{\mathbf{C}}[S_{n+t}^*|\mathcal{F}_n]\le S_n^*. \tag{13}$$

We stress that if $\widehat{\nu}$ is not additive, then (12) does not imply that (13) holds as an equality.

The belief function $\widehat{\nu}$ is completely determined by the parameters

$$\widehat{b_u}=\frac{1+r-d}{u-d}\quad\text{and}\quad\widehat{b_d}\in(0,1-\widehat{b_u}], \tag{14}$$

and will also be referred to as *risk-neutral belief function* in the following: varying $\widehat{b_d}\in(0,1-\widehat{b_u}]$ we actually have infinitely many risk-neutral belief functions. For fixed $\widehat{b_u}$ and $\widehat{b_d}$, $\widehat{\nu}$ determines through the product conditioning rule a family of risk-neutral t-step transitions belief functions $\{\widehat{\beta_t}\}_{t=1}^T$ of the form (7) with parameters $\widehat{b_u}$ and $\widehat{b_d}$.

2.3. *Normative justification of the model*

The choice of parameters in (14) can be justified through a one-step dynamic generalized notion of no-arbitrage that refers to the one for the single period case given in Cinfrignini *et al.* (2023b) and is based on the *partially resolving uncertainty* principle proposed by Jaffray (1989).

At this aim, we consider a one-step no-arbitrage condition where, given the history $\{S_0 = s_0, \ldots, S_n = s_n\}$, at time $n+1$ we define the two events $U(s_n) = \{S_{n+1} = us_n\}$ and $D(s_n) = \{S_{n+1} = ds_n\}$ that correspond to the two price movements "up" and "down", respectively. Therefore, every random quantity X depending on S_{n+1} can be seen as a function on $\mathcal{W}(s_n) = \{U(s_n), D(s_n)\}$ from time n. The one-period market can be augmented by adding two artificial Arrow-Debreu securities with payoffs and prices, respectively,

$$A^u_{n+1} = \mathbf{1}_{U(s_n)}, \quad \text{and} \quad A^d_{n+1} = \mathbf{1}_{D(s_n)}, \tag{15}$$

$$A^u_n = \frac{\widehat{b_u}}{1+r}, \quad \text{and} \quad A^d_n = \frac{\widehat{b_d}}{1+r}. \tag{16}$$

Under the partially resolving uncertainty principle, given the history $\{S_0 = s_0, \ldots, S_n = s_n\}$, at time $n+1$ we may not be able to determine which one between $U(s_n)$ and $D(s_n)$ has occurred. For this reason we consider the set $\mathcal{U}(s_n) = \{U(s_n), D(s_n), U(s_n) \cup D(s_n)\}$ and, for any function X defined on $\mathcal{W}(s_n)$, we adopt a *systematically pessimistic behavior*, i.e., we consider $[X]^{\mathbf{L}}$ in place of X, where, for every $E \in \mathcal{U}(s_n)$,

$$[X]^{\mathbf{L}}(E) = \min\{X(F) \,:\, F \subseteq E, F \in \mathcal{W}(s_n)\}. \tag{17}$$

In the augmented one-period market over $[n, n+1]$, a portfolio is a vector $\boldsymbol{\delta}_n = (\delta^0_n, \delta^1_n, \delta^2_n, \delta^3_n)$, where the δ^i_n's are $\mathcal{F}_n$-measurable random variables expressing, respectively, the number of units of bond, stock and Arrow-Debreu's securities to buy (if positive) or short-sell (if negative) at time n up to time $n+1$.

Following Cinfrignini *et al.* (2023b), we define a generalized one-step arbitrage opportunity as a portfolio $\boldsymbol{\delta}_n = (\delta^0_n, \delta^1_n, \delta^2_n, \delta^3_n)$ that satisfies one of the following two conditions:

(a) $\tilde{\pi}^{\boldsymbol{\delta}}_n < 0$ and $\tilde{\pi}^{\boldsymbol{\delta}}_{n+1} \geq 0$ with $\tilde{\pi}^{\boldsymbol{\delta}}_{n+1} = 0$ over $\mathcal{W}(s_n)$;
(b) $\tilde{\pi}^{\boldsymbol{\delta}}_n \leq 0$ and $\tilde{\pi}^{\boldsymbol{\delta}}_{n+1} \geq 0$ with $\tilde{\pi}^{\boldsymbol{\delta}}_{n+1} \neq 0$ over $\mathcal{W}(s_n)$;

where $\tilde{\pi}^{\boldsymbol{\delta}}_n = \delta^0_n[B_n]^{\mathbf{L}} + \delta^1_n[S_n]^{\mathbf{L}} + \delta^2_n[A^u_n]^{\mathbf{L}} + \delta^3_n[A^d_n]^{\mathbf{L}}$ and $\tilde{\pi}^{\boldsymbol{\delta}}_{n+1} = \delta^0_n[B_{n+1}]^{\mathbf{L}} + \delta^1_n[S_{n+1}]^{\mathbf{L}} + \delta^2_n[A^u_{n+1}]^{\mathbf{L}} + \delta^3_n[A^d_{n+1}]^{\mathbf{L}}$ are the price and the payoff of the portfolio $\boldsymbol{\delta}_n$, under partial resolving uncertainty and systematic pessimism. Notice that $\tilde{\pi}^{\boldsymbol{\delta}}_n$ turns out to be a constant.

In *(a)* we have a portfolio for which we are paid at time n, that produces a non-negative payoff under partially resolving uncertainty at time $n+1$, with no losses on those events where we have completely resolving uncertainty. Similarly, in *(b)* we have a portfolio for which we are paid or we pay nothing at time n, that produces a non-negative payoff under partially resolving uncertainty at time $n+1$, with at least a gain on those events where we have completely resolving uncertainty. So, behaviorally, it seems natural to avoid such generalized one-step arbitrage opportunities.

As proved in Cinfrignini *et al.* (2023b), avoiding generalized one-step arbitrage opportunities is equivalent to the existence of a conditional belief function $\widehat{\nu}$ defined on the ring generated by $\mathcal{W}(s_n)$ such that

(i) $\widehat{\nu}(S_{n+1} = us_n | S_0 = s_0, \ldots, S_n = s_n) = \widehat{b_u}$,
(ii) $\widehat{\nu}(S_{n+1} = ds_n | S_0 = s_0, \ldots, S_n = s_n) = \widehat{b_d}$,
(iii) $\frac{1}{1+r}\widehat{\mathbf{C}}[S_{n+1} | S_0 = s_0, \ldots, S_n = s_n] = s_n$,

where $\widehat{b_u}$ and $\widehat{b_d}$ are as in (14) Notice that, if we assume completely resolving uncertainty, that is we work on $\mathcal{W}(s_n)$ in place of $\mathcal{U}(s_n)$, then *(a)* and *(b)* reduce to two standard one-step no-arbitrage opportunities (see Černý (2009)). In this case, the only possible choice is to take $\widehat{b_d} = 1 - \widehat{b_u}$ and conditions *(i)–(iii)* entirely characterize a global additive belief function $\widehat{\nu}$, that coincides with the usual equivalent martingale measure in the classical model by Cox *et al.* (1979).

3. Bid-ask pricing of European-type derivatives

We refer to the market model introduced in the previous section, formed by the processes $\{B_n\}_{n=0}^T$ and $\{S_n\}_{n=0}^T$. We consider a European-type derivative contract with maturity T, whose payoff is

$$Y_T = \varphi(S_T), \tag{18}$$

where $\varphi(x)$ is a suitable contract function defined on the range of S_T.

In agreement with Cinfrignini *et al.* (2023a), for a fixed $\widehat{b_d} \in (0, 1 - \widehat{b_u}]$, we take the risk-neutral belief space $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n=0}^T, \widehat{\nu})$ an we define the bid price process $\{Y_n\}_{n=0}^T$ of the European-type derivative contract as

$$Y_n = \frac{1}{1+r}\widehat{\mathbf{C}}[Y_{n+1}|\mathcal{F}_n], \tag{19}$$

for $n = 0, \ldots, T-1$.

In turn, assuming that $\overline{Y}_T = Y_T = \varphi(S_T)$, the ask price process $\{\overline{Y}_n\}_{n=0}^T$, is defined proceeding one-step-wise by duality setting, for $n = 0, \ldots, T-1$,

$$\overline{Y}_n = -\frac{1}{1+r}\widehat{\mathbf{C}}[-\overline{Y}_{n+1}|\mathcal{F}_n]. \tag{20}$$

By the time-homogeneity and Markov properties of $\{S_n\}_{n=0}^T$, we have that, under $\widehat{\nu}$,

$$Y_n = \frac{1}{1+r}\widehat{\mathbf{C}}[Y_{n+1}|S_n] \quad \text{and} \quad \overline{Y}_n = -\frac{1}{1+r}\widehat{\mathbf{C}}[-\overline{Y}_{n+1}|S_n]. \tag{21}$$

Theorem 3 in Cinfrignini *et al.* (2023a) shows that the discounted bid price process $\{Y_n^*\}_{n=0}^T$ is a one-step Choquet martingale while globally is only a Choquet super-martingale, i.e., for every $0 \leq n \leq T-1$ and $1 \leq t \leq T-n$,

$$\widehat{\mathbf{C}}[Y_{n+1}^*|\mathcal{F}_n] = Y_n^* \tag{22}$$

$$\widehat{\mathbf{C}}[Y_{n+t}^*|\mathcal{F}_n] \leq Y_n^*. \tag{23}$$

We have that $\widehat{\nu}$ is additive if and only if $\widehat{b_u} + \widehat{b_d} = 1$: in such case $\widehat{\mathbf{C}}[\cdot|\mathcal{F}_n]$ reduces to a classical linear conditional expectation, so the bid price process $\{Y_n\}_{n=0}^T$ and the ask price process $\{\overline{Y}_n\}_{n=0}^T$ coincide. Therefore, bid-ask spreads are possible in the model if and only if $\widehat{b_u} + \widehat{b_d} < 1$, which is equivalent to the fact that $\widehat{\nu}(E) + \widehat{\nu}(E^c) \leq 1$, for every $E \in \mathcal{F}$, with at least a strict inequality. In turn, the last condition is a dual version of that appearing in Lécuyer and Lefort (2021) for the single period case.

Remark 2 In the single period case, a non-linear pricing rule having a discounted Choquet expectation representation has been studied in Cerreia-Vioglio *et al.* (2015), Lécuyer and Lefort (2021), Bastianello *et al.* (2022). Such a functional is usually interpreted as an upper (i.e., ask) pricing rule and has been characterized by a form of put-call parity relation. In Cinfrignini *et al.* (2023b) we studied a single period lower (i.e., bid) pricing rule, which has a discounted Choquet expectation representation with respect to a belief function and is shown to satisfy the put-call parity relation given in Cerreia-Vioglio *et al.* (2015): the related upper pricing rule is obtained by duality. Notice that, another non-equivalent form of put-call parity relation has been introduced in Chateauneuf *et al.* (1996) (see also Bastianello *et al.* (2022)). The dynamic bid pricing rule given by (19) is a multi-period extension of Cinfrignini *et al.* (2023b), but is not the only possibility. In Section 5 of Cinfrignini *et al.* (2023a) we presented another dynamic bid pricing rule that can be generated by $\widehat{\mathbf{C}}[\cdot|\mathcal{F}_n]$: set $\underline{Y}_T = \varphi(S_T)$ and, for $n = 0, \dots, T-1$, define

$$\underline{Y}_n = \frac{1}{(1+r)^{T-n}} \widehat{\mathbf{C}}[\underline{Y}_T | \mathcal{F}_n]. \tag{24}$$

There, we also showed that the resulting bid price process $\{\underline{Y}_n\}_{n=0}^T$ satisfies a dynamic version of the put-call parity relation of Cerreia-Vioglio *et al.* (2015). Nevertheless, in Section 6 of Cinfrignini *et al.* (2023a) we highlighted that the dynamic bid pricing rule in (24) can produce price dilation and may violate the time-consistency property in the sense of Cheridito and Stadje (2009). On the other hand, the bid pricing rule in (19) produces less dilation and is time-consistent, therefore, we limit the discussion to it.

Let $Y_T = \overline{Y}_T = \varphi(S_T)$ be the payoff of a European derivative, where $\varphi(x)$ is a non-decreasing contract function. A typical example is the contract function of a European call option $\varphi(x) = \max\{x-K, 0\}$ with strike price K, for which the bid-ask price processes are customarily denoted as $\{C_n\}_{n=0}^T$ and $\{\overline{C}_n\}_{n=0}^T$.

The following proposition provides a closed form expression for both $\{Y_n\}_{n=0}^T$ and $\{\overline{Y}_n\}_{n=0}^T$, when $\varphi(x)$ is non-decreasing.

Proposition 3.1 *Let $\varphi(x)$ be a non-decreasing contract function, $Y_T = \overline{Y}_T = \varphi(S_T)$ be the payoff of a European derivative, and let $\widehat{\nu}$ be a risk-neutral belief function with parameters $\widehat{b_u}, \widehat{b_d}$ satisfying (14). Let $\{Y_n\}_{n=0}^T$ and $\{\overline{Y}_n\}_{n=0}^T$ be, respectively, the bid and the ask price processes according to (19) and (20). Then, for every $n = 0, \dots, T-1$, for every $s_n \in \mathcal{S}_n$, and for every $\omega \in \{S_n = s_n\}$ it holds that*

$$Y_n(\omega) = \frac{1}{(1+r)^{T-n}} \sum_{k=0}^{T-n} \binom{T-n}{k} \varphi(u^k d^{T-n-k} s_n) \widehat{b_u}^k (1-\widehat{b_u})^{T-n-k}, \tag{25}$$

$$\overline{Y}_n(\omega) = \frac{1}{(1+r)^{T-n}} \sum_{k=0}^{T-n} \binom{T-n}{k} \varphi(u^k d^{T-n-k} s_n) (1-\widehat{b_d})^k \widehat{b_d}^{T-n-k}. \tag{26}$$

Another distinguished case is given by the payoff $Y_T = \overline{Y}_T = \varphi(S_T)$ of a European derivative, where $\varphi(x)$ is a non-increasing contract function. A typical example is the contract function of a European put option $\varphi(x) = \max\{K-x, 0\}$ with strike price K, for which the bid-ask price processes are customarily denoted as $\{P_n\}_{n=0}^T$ and $\{\overline{P}_n\}_{n=0}^T$.

The following proposition provides a closed form expression for both $\{Y_n\}_{n=0}^T$ and $\{\overline{Y}_n\}_{n=0}^T$, when $\varphi(x)$ is non-increasing.

Proposition 3.2 *Let $\varphi(x)$ be a non-increasing contract function, $Y_T = \overline{Y}_T = \varphi(S_T)$ be the payoff of a European derivative, and let $\widehat{\nu}$ be a risk-neutral belief function with parameters $\widehat{b_u}, \widehat{b_d}$ satisfying (14). Let $\{Y_n\}_{n=0}^T$ and $\{\overline{Y}_n\}_{n=0}^T$ be, respectively, the bid and the ask price processes according to (19) and (20). Then, for every $n = 0, \dots, T-1$, for every $s_n \in \mathcal{S}_n$, and for every $\omega \in \{S_n = s_n\}$*

it holds that

$$Y_n(\omega) = \frac{1}{(1+r)^{T-n}} \sum_{k=0}^{T-n} \binom{T-n}{k} \varphi(u^k d^{T-n-k} s_n)(1-\widehat{b_d})^k \widehat{b_d}^{T-n-k}, \tag{27}$$

$$\overline{Y}_n(\omega) = \frac{1}{(1+r)^{T-n}} \sum_{k=0}^{T-n} \binom{T-n}{k} \varphi(u^k d^{T-n-k} s_n)\widehat{b_u}^k (1-\widehat{b_u})^{T-n-k}. \tag{28}$$

Propositions 3.1 and 3.2 further determine the evolution of the bid-ask spread $\{\overline{Y}_n - Y_n\}_{n=0}^T$, when the contract function φ is monotonic. The following proposition shows that, in this case, the initial bid-ask spread $\overline{Y}_0 - Y_0$ reduces to a polynomial, that depends on the choice of the particular risk-neutral belief function.

Proposition 3.3 *Let $\varphi(x)$ be a monotonic contract function, $Y_T = \overline{Y}_T = \varphi(S_T)$ be the payoff of a European derivative, and let $\widehat{\nu}$ be a risk-neutral belief function with parameters $\widehat{b_u}$ satisfying* (14) *and $\widehat{b_d} = \epsilon(1-\widehat{b_u})$, for $\epsilon \in (0,1]$. Then the bid-ask spread $\overline{Y}_0 - Y_0$ at time* 0 *is a polynomial of degree T in ϵ.*

Figure 1 shows the graphs of the bid-ask spreads of European call and put options according to Proposition 3.3, assuming $S_0 = K = \$100$, $r = 0.01$, $u = 1.05$, $d = 0.95$, and T varying in $\{5, 10, 15, 20\}$. The two graphs highlight that, for this choice of parameters and a fixed time horizon T, the spread achievable by the call is much larger than that achievable by the put, for small values of ϵ. In general, no dominance relation may hold between the spread curves corresponding to different T's and realistic values of spread correspond to values of ϵ close to 1.

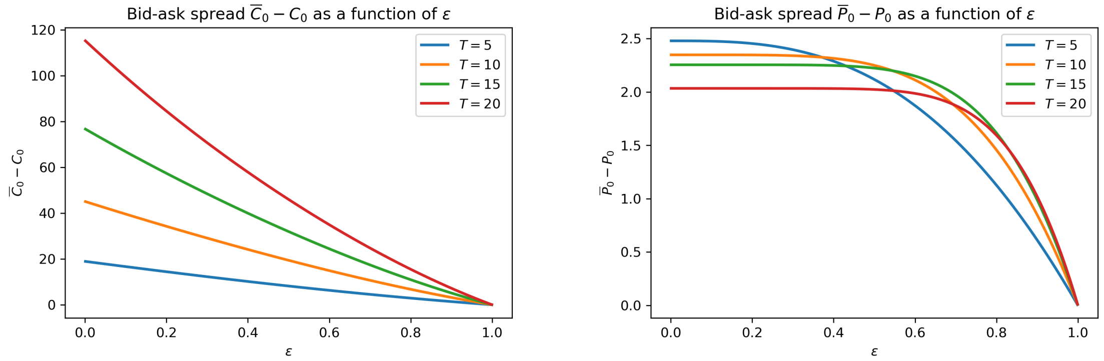

Figure 1. Bid-ask spreads $\overline{C}_0 - C_0$ and $\overline{P}_0 - P_0$ as functions of $\epsilon \in (0,1]$, for $S_0 = K = \$100$, $r = 0.01$, $u = 1.05$, $d = 0.95$, and T varying in $\{5, 10, 15, 20\}$.

In case of a non-monotonic contract function, we cannot derive a closed form expression of bid and ask price processes $\{Y_n\}_{n=0}^T$ and $\{\overline{Y}_n\}_{n=0}^T$, nor of the corresponding bid-ask spread process $\{\overline{Y}_n - Y_n\}_{n=0}^T$. In this latter case, prices can be computed through the one-step backward procedure singled out by (21).

Example 1 Let $T = 3$, $r = 0.04$, $S_0 = \$100$, $u = 1.2$ and $d = 0.8$. It holds that $\widehat{b_u} = 0.6$ and $\widehat{b_d} \in (0, 0.4]$. Figure 2 shows the evolution of $\{S_n\}_{n=0}^3$.

Let $Y_3 = \max\{K - S_3, 0\} + \max\{S_3 - K, 0\}$ be the payoff of a straddle derivative that consists in buying both a put and a call option with the same stock and strike price, which is taken as $K = \$100$. For every choice of $\widehat{b_d} \in (0, 0.4]$, the bid price Y_0 and the ask price $\overline{Y}_0$ have the following

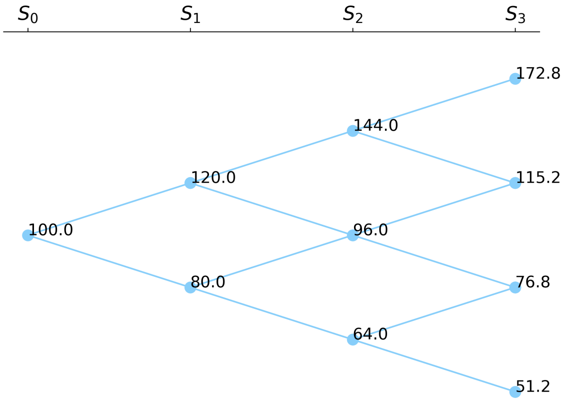

Figure 2. Bid price process of the stock $\{S_n\}_{n=0}^{3}$ with $S_0 = 100$, $u = 1.2$ and $d = 0.8$.

expressions

$$\underline{Y}_0 = \frac{\widehat{b_u}(57.6\widehat{b_u}^2 - 8\widehat{b_u}\widehat{b_d} - 8\widehat{b_d}) + \widehat{b_d}(16 + 17.6\widehat{b_d} - 17.6\widehat{b_u}\widehat{b_d}) + 15.2}{1.04^3},$$

$$\overline{Y}_0 = \frac{\widehat{b_u}(17.6\widehat{b_u}\widehat{b_d} + 8\widehat{b_d}^2 - 59.2\widehat{b_d}) + \widehat{b_d}(-57.6\widehat{b_d}^2 + 164.8\widehat{b_d} - 131.2) + 72.8}{1.04^3}.$$

Figure 3 shows the bid and the ask price processes $\{\underline{Y}_n\}_{n=0}^{3}$ and $\{\overline{Y}_n\}_{n=0}^{3}$ for $\widehat{b_d} = 0.4 \cdot 0.999$.

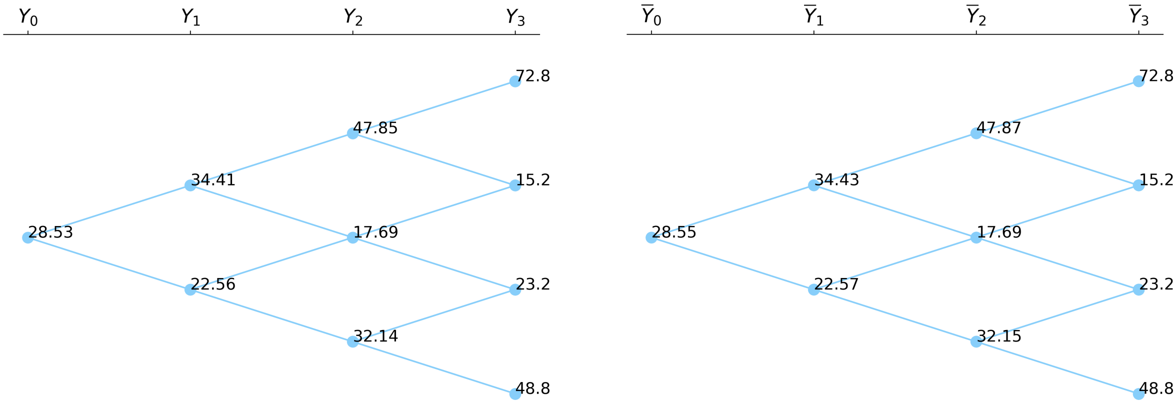

Figure 3. Bid and ask price processes $\{\underline{Y}_n\}_{n=0}^{3}$ and $\{\overline{Y}_n\}_{n=0}^{3}$ of a straddle derivative with payoff $Y_3 = \max\{K - S_3, 0\} + \max\{S_3 - K, 0\}$, with $\widehat{b_u} = 0.6$ and $\widehat{b_d} = 0.4 \cdot 0.999$.

3.1. *Replicating strategies*

In this context, since we have the bid and the ask price processes $\{\underline{Y}_n\}_{n=0}^{T}$ and $\{\overline{Y}_n\}_{n=0}^{T}$, replication translates in finding two bivariate processes $\{\boldsymbol{\lambda}_n\}_{n=0}^{T-1}$ and $\{\boldsymbol{\delta}_n\}_{n=0}^{T-1}$ where $\boldsymbol{\lambda}_n = (\lambda_n^0, \lambda_n^1)$ and $\boldsymbol{\delta}_n = (\delta_n^0, \delta_n^1)$ with λ_n^0, δ_n^0 and λ_n^1, δ_n^1 the units of bond and stock to buy or short-sell, respectively.

As shown in Cinfrignini *et al.* (2023a), in general, we cannot represent the bid and the ask price processes through self-financing strategies $\{\boldsymbol{\lambda}_n\}_{n=0}^{T-1}$ and $\{\boldsymbol{\delta}_n\}_{n=0}^{T-1}$. Indeed, in order to have a replicating strategy, in every period $[n, n+1]$, working conditionally on the history of the stock bid price process up to time n, the random vectors $\boldsymbol{\lambda}_n$ and $\boldsymbol{\delta}_n$ must be chosen so as to satisfy

$$\lambda_n^0 B_{n+1} + \lambda_n^1 S_{n+1} = \underline{Y}_{n+1}, \tag{29}$$

$$\delta_n^0 B_{n+1} + \delta_n^1 S_{n+1} = \overline{Y}_{n+1}. \tag{30}$$

The lack of linearity of $\widehat{\mathbf{C}}[\cdot|\mathcal{F}_n]$ implies that the resulting replicating strategies $\{\boldsymbol{\lambda}_n\}_{n=0}^{T-1}$ and $\{\boldsymbol{\delta}_n\}_{n=0}^{T-1}$ are generally not self-financing as we may have

$$Y_n \neq \lambda_n^0 B_n + \lambda_n^1 S_n, \tag{31}$$
$$\overline{Y}_n \neq \delta_n^0 B_n + \delta_n^1 S_n, \tag{32}$$

unless $\lambda_n^1 \geq 0$ and $\delta_n^1 \leq 0$.

Proposition 3.4 *Let $\varphi(x)$ be a monotonic contract function, $Y_T = \overline{Y}_T = \varphi(S_T)$ be the payoff of a European derivative, and let $\widehat{\nu}$ be a risk-neutral belief function with parameters $\widehat{b}_u, \widehat{b}_d$ satisfying* (14)*. For the processes $\{\boldsymbol{\lambda}_n\}_{n=0}^{T-1}$ and $\{\boldsymbol{\delta}_n\}_{n=0}^{T-1}$ defined as* (29) *and* (30)*, the following statements hold:*

(i) if $\varphi(x)$ is non-decreasing then $\{\boldsymbol{\lambda}_n\}_{n=0}^{T-1}$ is self-financing;
(ii) if $\varphi(x)$ is non-increasing then $\{\boldsymbol{\delta}_n\}_{n=0}^{T-1}$ is self-financing.

The following example shows that the self-financing property of both $\{\boldsymbol{\lambda}_n\}_{n=0}^{T-1}$ and $\{\boldsymbol{\delta}_n\}_{n=0}^{T-1}$ may fail if $\varphi(x)$ is non-monotonic.

Example 2 Let T, r, S_0, u, d as in Example 1 and consider the same straddle derivative, whose contract function $\varphi(x) = \max\{K - x, 0\} + \max\{x - K, 0\}$ is non-monotonic. The bid-ask price processes $\{Y_n\}_{n=0}^{3}$ and $\{\overline{Y}_n\}_{n=0}^{3}$ are reported in Figure 3.

Figure 4 depicts the bid replicating strategy $\{\boldsymbol{\lambda}_n\}_{n=0}^{2}$ with $\boldsymbol{\lambda}_n = (\lambda_n^0, \lambda_n^1)$. In the right tree, negative values of λ_n^1 appear in red and are those responsible for (31). It actually holds that $Y_n \leq \lambda_n^0 B_n + \lambda_n^1 S_n$, for $n = 0, 1, 2$. In particular, the failure of the self-financing property is implied by $\lambda_n^0 B_n + \lambda_n^1 S_n \neq \lambda_{n-1}^0 B_n + \lambda_{n-1}^1 S_n$, for $n = 1, 2$.

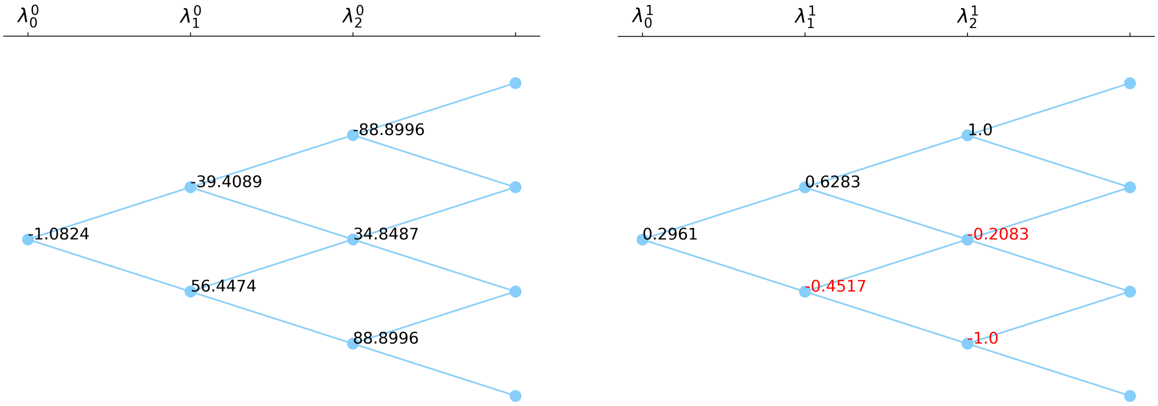

Figure 4. Bid replicating strategy $\{\boldsymbol{\lambda}_n\}_{n=0}^{2}$ with $\boldsymbol{\lambda}_n = (\lambda_n^0, \lambda_n^1)$.

Analogously, Figure 5 shows the ask replicating strategy $\{\boldsymbol{\delta}_n\}_{n=0}^{2}$ with $\boldsymbol{\delta}_n = (\delta_n^0, \delta_n^1)$. In the right tree, positive values of δ_n^1 appear in red and are those responsible for (32). It actually holds that $\overline{Y}_n \geq \delta_n^0 B_n + \delta_n^1 S_n$, for $n = 0, 1, 2$. In particular, the failure of the self-financing property is implied by $\delta_n^0 B_n + \delta_n^1 S_n \neq \delta_{n-1}^0 B_n + \delta_{n-1}^1 S_n$, for $n = 1, 2$.

4. Bid-ask pricing of American-type derivatives

In this section we still refer to the market described in Subsection 2.2, where we consider an American-type derivative contract on the bid price process of the stock $\{S_n\}_{n=0}^{T}$ characterized by a non-negative contract function $\varphi(x)$, defined on the range of S_n, for $n = 0, \ldots, T$, giving rise to a payoff process $\{\Phi_n\}_{n=0}^{T}$, where $\Phi_n = \varphi(S_n)$.

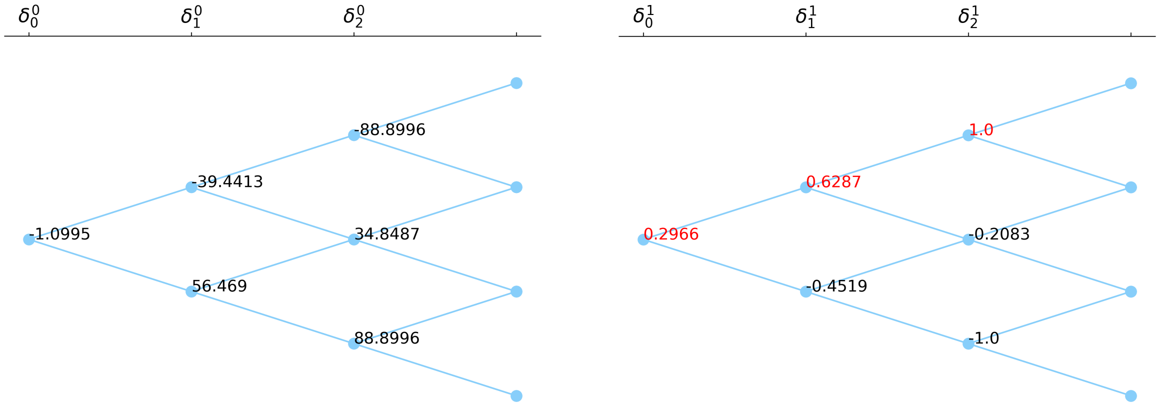

Figure 5. Ask replicating strategy $\{\boldsymbol{\delta}_n\}_{n=0}^2$ with $\boldsymbol{\delta}_n = (\delta_n^0, \delta_n^1)$.

The American-type derivative will be associated with a bid price process $\{Y_n\}_{n=0}^T$ and an ask price process $\{\overline{Y}_n\}_{n=0}^T$. As usual, an American-type derivative gives the holder the right to exercise it at each time $n = 0, \dots, T$, by getting the quantity Φ_n. We assume that the payoff Φ_n is the same we get both in computing the bid and the ask price, when performing early exercise.

We define the bid price process of the derivative through the following one-step backward procedure

$$Y_n = \begin{cases} \Phi_T & \text{if } n = T, \\ \max\left\{\Phi_n, \frac{1}{1+r}\widehat{\mathbf{C}}[Y_{n+1}|\mathcal{F}_n]\right\} & \text{if } n = 0, \dots, T-1, \end{cases} \tag{33}$$

where

- Φ_n is the *intrinsic value* of the derivative, that is the payoff of the derivative if it is exercised at time n (*termination value*);
- $\frac{1}{1+r}\widehat{\mathbf{C}}[Y_{n+1}|\mathcal{F}_n]$ is the discounted one-step Choquet expectation, that is the bid price of the derivative if it is not exercised (*bid continuation value*).

We interpret the bid continuation value as the maximum price that an agent is willing to pay to buy the derivative, if it is not exercised today.

Starting from time $T-1$ and proceeding backward, at each step n the opportunity of early exercise is studied comparing the termination value and the bid continuation value at each node: the derivative contract is exercised at the first occurrence n^*_{bid} such that the termination value is greater than the bid continuation value, i.e.,

$$n^*_{bid} = \min\left\{ n = 0, \dots, T \;\middle|\; \Phi_n > \frac{1}{1+r}\widehat{\mathbf{C}}[Y_{n+1}|\mathcal{F}_n] \right\}, \tag{34}$$

where we set $\frac{1}{1+r}\widehat{\mathbf{C}}[Y_{T+1}|\mathcal{F}_T] := -1$. Notice that n^*_{bid} is a random variable that corresponds to the strategy in which we wait the most to exercise the derivative, by looking for a strict convenience and referring to the bid price process of the American derivative. We point out that an anticipated early exercise strategy with respect to the bid price process is obtained by asking a weak inequality to hold in (34).

The bid price process of an American-type derivative can be associated with the corresponding ask price process $\{\overline{Y}_n\}_{n=0}^T$ still having the same payoff process $\{\Phi_n\}_{n=0}^T$, given by the following one-step backward procedure

$$\overline{Y}_n = \begin{cases} \Phi_n & \text{if } n = T \\ \max\left\{\Phi_n, -\frac{1}{1+r}\widehat{\mathbf{C}}[-\overline{Y}_{n+1}|\mathcal{F}_n]\right\} & \text{if } n = 0, \dots, T-1, \end{cases} \tag{35}$$

where Φ_n is still interpreted as *termination value* and $-\frac{1}{1+r}\widehat{\mathbf{C}}[-\overline{Y}_{n+1}|\mathcal{F}_n]$ is the *ask continuation value*. The latter is interpreted as the minimum value that an agent is willing to accept to sell the derivative, if it is not exercised today. We stress that the latter can be equivalently written as

$$-\frac{1}{1+r}\widehat{\mathbf{C}}[-\overline{Y}_{n+1}|\mathcal{F}_n] = -\frac{1}{1+r}\widehat{\mathbf{C}}[-\overline{Y}_{n+1}|S_n] = \frac{1}{1+r}\oint \overline{Y}_{n+1}\mathrm{d}\widehat{\overline{\nu}}(\cdot|S_n), \tag{36}$$

where $\widehat{\overline{\nu}}(\cdot|S_n) = 1 - \widehat{\nu}((\cdot)^c|S_n)$ is the dual risk-neutral conditional plausibility function of $\widehat{\nu}(\cdot|S_n)$, where both are seen as function of S_n.

In analogy with (34), the derivative contract is exercised at the first occurrence n^*_{ask} such that the termination value is greater than the ask continuation value, i.e.,

$$n^*_{ask} = \min\left\{n = 0, \ldots, T \;\middle|\; \Phi_n > -\frac{1}{1+r}\widehat{\mathbf{C}}[-\overline{Y}_{n+1}|\mathcal{F}_n]\right\}, \tag{37}$$

where we set $-\frac{1}{1+r}\widehat{\mathbf{C}}[-\overline{Y}_{T+1}|\mathcal{F}_T] := -1$. Notice that n^*_{ask} is a random variable that corresponds to the strategy in which we wait the most to exercise the derivative, by looking for a strict convenience and referring to the ask price process of the American derivative. Also in this case, an anticipated early exercise strategy with respect to the ask price process is obtained by asking a weak inequality to hold in (37).

PROPOSITION 4.1 *Let* $\{Y_n\}_{n=0}^T$ *and* $\{\overline{Y}_n\}_{n=0}^T$ *be the bid and the ask price processes defined as in* (33) *and* (35)*. Then, for all* $n = 0, \ldots, T$*, it holds that* $Y_n \leq \overline{Y}_n$*.*

Let us consider the discounted bid and ask price processes $\{Y^*_n\}_{n=0}^T$ and $\{\overline{Y}^*_n\}_{n=0}^T$, and the discounted payoff process $\{\Phi^*_n\}_{n=0}^T$, where

$$Y^*_n = \frac{Y_n}{B_n}, \quad \overline{Y}^*_n = \frac{\overline{Y}_n}{B_n}, \quad \text{and} \quad \Phi^*_n = \frac{\Phi_n}{B_n}. \tag{38}$$

DEFINITION 1 The discounted bid and ask price processes $\{Y^*_n\}_{n=0}^T$ and $\{\overline{Y}^*_n\}_{n=0}^T$ are called the *bid-ask Choquet-Snell envelopes* of the discounted payoff process $\{\Phi^*_n\}_{n=0}^T$ if the following conditions are satisfied:

- *(i)* $\{Y^*_n\}_{n=0}^T$ is the smallest $\widehat{\nu}$-Choquet one-step super-martingale dominating $\{\Phi^*_n\}_{n=0}^T$, that is
  - *(i.a)* $Y^*_n \geq \widehat{\mathbf{C}}[Y^*_{n+1}|\mathcal{F}_n]$, for $n = 0, \ldots, T-1$,
  - *(i.b)* $Y^*_n \geq \Phi^*_n$, for $n = 0, \ldots, T$,
  - *(i.c)* there is no other process $\{V^*_n\}_{n=0}^T$ satisfying *(i.a)* and *(i.b)* such that for some n, $Y^*_n \geq V^*_n$ and $Y^*_n \neq V^*_n$.
- *(ii)* $\{\overline{Y}^*_n\}_{n=0}^T$ is the smallest $\widehat{\overline{\nu}}$-Choquet one-step super-martingale dominating $\{\Phi^*_n\}_{n=0}^T$, that is
  - *(ii.a)* $\overline{Y}^*_n \geq -\widehat{\mathbf{C}}[-\overline{Y}^*_{n+1}|\mathcal{F}_n]$, for $n = 0, \ldots, T-1$,
  - *(ii.b)* $\overline{Y}^*_n \geq \Phi^*_n$, for $n = 0, \ldots, T$,
  - *(ii.c)* there is no other process $\{\overline{V}^*_n\}_{n=0}^T$ satisfying *(ii.a)* and *(ii.b)* such that for some n, $\overline{Y}^*_n \geq \overline{V}^*_n$ and $\overline{Y}^*_n \neq \overline{V}^*_n$.

The following proposition shows that for a fixed risk-neutral belief function $\widehat{\nu}$, the processes $\{Y_n\}_{n=0}^T$ and $\{\overline{Y}_n\}_{n=0}^T$ are indeed the bid-ask Choquet-Snell envelopes of $\{\Phi^*_n\}_{n=0}^T$.

PROPOSITION 4.2 *Let* $\{Y_n\}_{n=0}^T$ *and* $\{\overline{Y}_n\}_{n=0}^T$ *be the bid and the ask price processes of an American-type derivative with payoff process* $\{\Phi_n\}_{n=0}^T$*, defined in* (33) *and* (35)*, respectively. Then,* $\{Y^*_n\}_{n=0}^T$ *and* $\{\overline{Y}^*_n\}_{n=0}^T$ *are the bid-ask Choquet-Snell envelopes of* $\{\Phi^*_n\}_{n=0}^T$*.*

In the following theorem we prove that, in analogy with the well-known theorem due to Merton

(Merton 1973) formulated for the classical probabilistic pricing models assuming no frictions, the early exercise of an American call option on a non-dividend paying stock when $r \geq 0$ is never optimal, neither looking at the bid nor at the ask price process.

THEOREM 4.3 *Let* $\{\Phi_n^C\}_{n=0}^T$ *with* $\Phi_n^C = \max\{S_n - K, 0\}$ *be the payoff process of an American call option, where* $\{S_n\}_{n=0}^T$ *is the bid price process of a non-dividend paying stock, modeled as a DS-multiplicative binomial process with respect to a risk-neutral belief function* $\widehat{\nu}$ *with* $\widehat{b_u}, \widehat{b_d}$ *as in* (14)*, and* $r \geq 0$*. Let* $\{C_n\}_{n=0}^T$ *and* $\{\overline{C}_n\}_{n=0}^T$ *be the bid and the ask price processes of the American call option, defined as* (33) *and* (35)*, respectively. Then, the early exercise is never optimal, neither with respect to the bid price process nor to the ask price process, that is* $n^*_{bid} = n^*_{ask} \equiv T$.

Let us consider an American put option with payoff process $\Phi_n^P = \max\{K - S_n, 0\}$, for $n = 0, \ldots, T$, and consider the corresponding bid and ask price processes $\{P_n\}_{n=0}^T$ and $\{\overline{P}_n\}_{n=0}^T$. The following proposition shows that $\{\overline{P}_n\}_{n=0}^T$ does not depend on the choice of $\widehat{b_d} \in (0, 1 - \widehat{b_u}]$.

PROPOSITION 4.4 *Let* $\{\Phi_n^P\}_{n=0}^T$ *with* $\Phi_n^P = \max\{K - S_n, 0\}$ *be the payoff process of an American put option, where* $\{S_n\}_{n=0}^T$ *is the bid price process of a non-dividend paying stock, modeled as a DS-multiplicative binomial process with respect to a risk-neutral belief function* $\widehat{\nu}$ *with* $\widehat{b_u}, \widehat{b_d}$ *as in* (14)*, and* $r \geq 0$*. Let* $\{\overline{P}_n\}_{n=0}^T$ *be the ask price process of the American put option, defined as* (35)*. Then,* $\{\overline{P}_n\}_{n=0}^T$ *does not depend on the choice of* $\widehat{b_d} \in (0, 1 - \widehat{b_u}]$.

Given an American put option with payoff process $\{\Phi_n^P\}_{n=0}^T$, the following cases can occur:

(1) The ask continuation value is less than the termination value, thus

$$\frac{1}{1+r}\widehat{\mathbf{C}}[P_{n+1}|\mathcal{F}_n] \leq -\frac{1}{1+r}\widehat{\mathbf{C}}[-\overline{P}_{n+1}|\mathcal{F}_n] < \Phi_n^P. \tag{39}$$

The early exercise is optimal at time n, for both the bid and the ask price processes.

(2) The bid continuation value is greater than or equal to the termination value, thus

$$-\frac{1}{1+r}\widehat{\mathbf{C}}[-\overline{P}_{n+1}|\mathcal{F}_n] \geq \frac{1}{1+r}\widehat{\mathbf{C}}[P_{n+1}|\mathcal{F}_n] \geq \Phi_n^P. \tag{40}$$

The early exercise at time n is not optimal, for both the bid and the ask price processes.

(3) The ask continuation value is greater than or equal to the termination value while the bid continuation value is less than the termination value, thus the following condition holds

$$-\frac{1}{1+r}\widehat{\mathbf{C}}[-\overline{P}_{n+1}|\mathcal{F}_n] \geq \Phi_n^P > \frac{1}{1+r}\widehat{\mathbf{C}}[P_{n+1}|\mathcal{F}_n]. \tag{41}$$

The early exercise is not optimal at time n with respect to the ask price process, but it is for the bid price process.

The following example shows the computation of the bid-ask price processes for an American put option, singling out also the early exercise region.

Example 3 Let $T = 3$, $S_0 = \$100$, $u = 1.1$, $d = 0.9$, and $r = 0.05$. We consider an American put option with strike price $K = \$102$. Figure 6 shows the bid price process of the stock $\{S_n\}_{n=0}^3$ and the payoff process $\{\Phi_n^P\}_{n=0}^3$, respectively.

It results that $\widehat{b_u} = \frac{3}{4}$, $\widehat{b_d} \in (0, \frac{1}{4}]$ and we parameterize $\widehat{b_d} = \frac{1}{4}\epsilon$ with $\epsilon \in (0, 1]$. Figure 7 shows the bid and the ask price processes of the put option for $\epsilon = \frac{9}{10}$, where nodes and edges in orange point out when early exercise is optimal. We stress that early exercise is checked only for times $n < T$.

Figure 7 shows that for this American put, the bid and ask early exercise strategies differ. Indeed, for the bid price process (left tree) early exercise is optimal in the second node of time $n = 2$, while

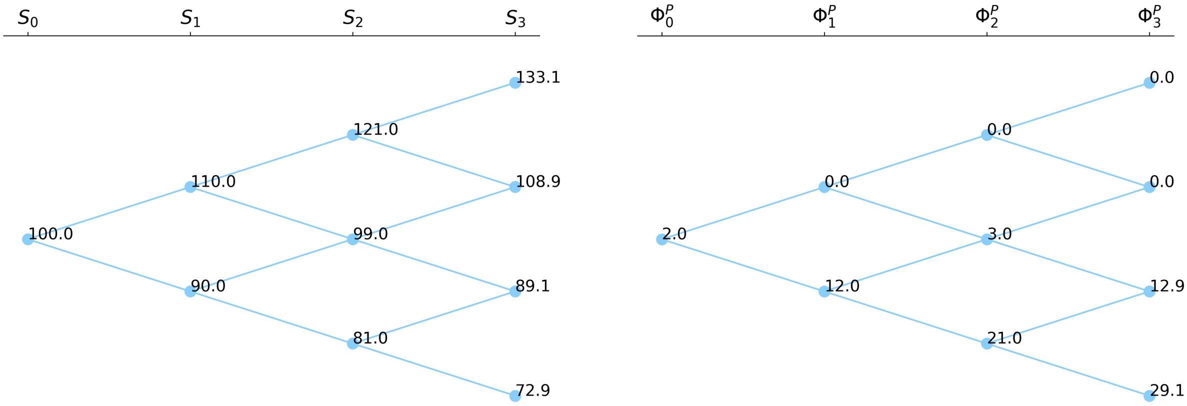

Figure 6. Bid price process of the stock $\{S_n\}_{n=0}^3$ with $S_0 = \$100$, $u = 1.1$, $d = 0.9$, and $r = 0.05$, and payoff process $\{\Phi^P_n\}_{n=0}^3$ of an American put option with strike $K = \$102$.

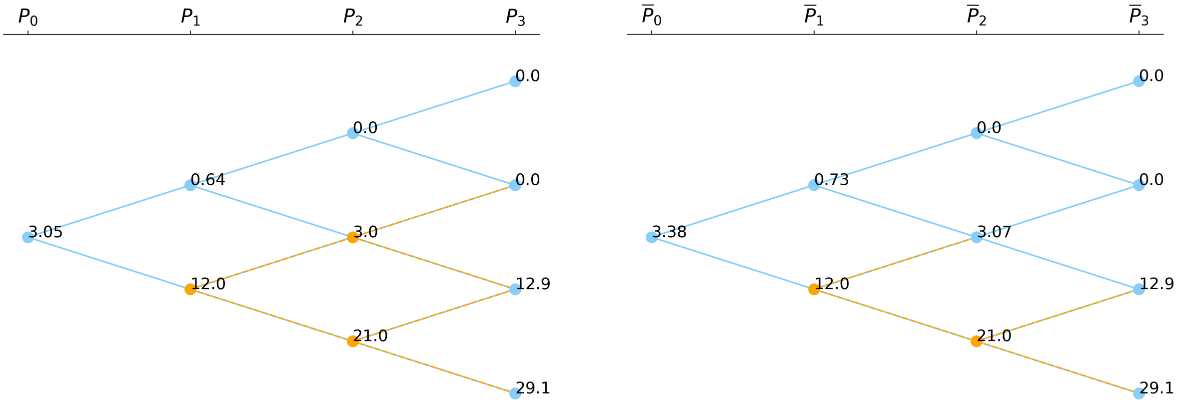

Figure 7. Bid and ask price processes $\{P_n\}_{n=0}^3$ and $\{\overline{P}_n\}_{n=0}^3$ of the American put option with strike $K = \$102$, with $\epsilon = \frac{9}{10}$, i.e., $\widehat{b_d} = \frac{9}{40}$. Nodes and edges in orange point out when early exercise is optimal.

it is not for the corresponding node and time in the ask price process (right tree).

Keeping all the other parameters fixed, Figure 8 shows the bid-ask price processes $\{P'_n\}_{n=0}^3$ and $\{\overline{P}'_n\}_{n=0}^3$ of an American put with strike price $K' = \$100$. In this case, the bid-ask early exercise strategies coincide.

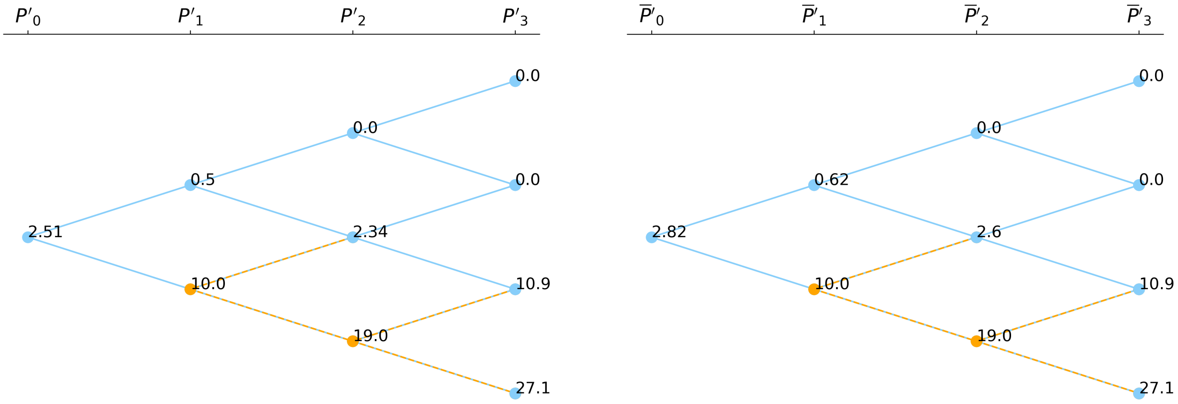

Figure 8. Bid and ask price processes $\{P'_n\}_{n=0}^3$ and $\{\overline{P}'_n\}_{n=0}^3$ for the American put with strike $K' = \$100$. Nodes and edges in orange point out when early exercise is optimal.

5. Calibration on market data

In this section we aim to calibrate the market model described in Subsection 2.2 to real market data. Therefore, we refer to a frictionless risk-free bond and a non-dividend paying risky stock with frictions in the form of bid-ask spreads. All the calibration procedure is carried out in `Python` and the reference code is available on `GitHub`[1]. Data are retrieved from `Yahoo! finance`, relying on the `yfinance` library.

Our calibration procedure assumes that the stock bid price process $\{S_n\}_{n=0}^T$ is a DS-multiplicative binomial process under an equivalent one-step Choquet martingale belief function $\widehat{\nu}$. In turn, this implies that $\widehat{\nu}$ gives rise to t-step transition belief functions $\{\widehat{\beta}_t\}_{t=1}^T$ of the form (7), with parameters $\widehat{b_u}$ and $\widehat{b_d}$. From a behavioral point of view, this amounts in assuming that the discounted bid price process $\{Y_n^*\}_{n=0}^T$ of any European derivative with payoff $Y_T = \varphi(S_T)$, built through the one-step backward procedure shown in (19), is a one-step Choquet martingale, and globally only a Choquet super-martingale, under $\widehat{\nu}$.

The parameters we need to estimate are r, u, d and $\widehat{b_d} \in (0, 1 - \widehat{b_u}]$, where $\widehat{b_u}$ is automatically determined by r, u, d as in (14). We point out that the classical historical estimation approach (see, e.g., Hull (2018)) is not applicable tout-court in our setting, due to the non-additivity of uncertainty measures we consider. Therefore, here we propose a least square calibration procedure where r is fixed through a sovereign zero-coupon bond, while u, d, $\widehat{b_d}$ are implied by bid-ask prices of European options on the stock.

Let T be a fixed maturity, and $\mathcal{K}_{call}$ and $\mathcal{K}_{put}$ be the available sets of strike prices for European options on the stock. For every $K \in \mathcal{K}_{call}$, denote by $C_{0,M}^K$ and $\overline{C}_{0,M}^K$ the market bid-ask prices of the call with strike K, while C_0^K and $\overline{C}_0^K$ denote the theoretical bid-ask prices computed as in Proposition 3.1. Analogously, for every $K \in \mathcal{K}_{put}$, denote by $P_{0,M}^K$ and $\overline{P}_{0,M}^K$ the market bid-ask prices of the put with strike K, while P_0^K and $\overline{P}_0^K$ denote the theoretical bid-ask prices computed as in Proposition 3.2. It is easily seen that, once r has been fixed, the theoretical bid-ask prices C_0^K, $\overline{C}_0^K$, P_0^K, $\overline{P}_0^K$ are actually functions of u, d, $\widehat{b_d}$.

We define the *squared error* as a function of u, d, $\widehat{b_d}$, by setting

$$\begin{aligned} E(u, d, \widehat{b_d}) = &\sum_{K \in \mathcal{K}_{call}} [(C_{0,M}^K - C_0^K)^2 + (\overline{C}_{0,M}^K - \overline{C}_0^K)^2] \\ &+ \sum_{K \in \mathcal{K}_{put}} [(P_{0,M}^K - P_0^K)^2 + (\overline{P}_{0,M}^K - \overline{P}_0^K)^2]. \end{aligned} \tag{42}$$

Our goal is to solve the following optimization problem

$$\begin{gathered} \text{minimize } E(u, d, \widehat{b_d}) \\ \text{subject to:} \\ \begin{cases} 0 < d < 1 + r < u, \\ 0 < \widehat{b_d} \leq 1 - \frac{1+r-d}{u-d}. \end{cases} \end{gathered} \tag{43}$$

Problem (43) is a constrained optimization problem where $E(u, d, \widehat{b_d})$ is a non-linear objective function that contains maxima, due to the call and put contract functions appearing in C_0^K, $\overline{C}_0^K$, P_0^K, $\overline{P}_0^K$. A possible solution to this issue is to move the computation of C_0^K, $\overline{C}_0^K$, P_0^K, $\overline{P}_0^K$ in the constraint section, by linearizing the maxima in the payoffs, through the introduction of binary

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[1] Public `GitHub` repository: `https://github.com/itsdavide/bid-ask-DS`.

variables. Nevertheless, such an approach makes the problem very difficult to solve, even for small values of T and a small number of options.

Therefore, here we face the problem by relying on the *particle swarm optimization (PSO)* technique, which is a stochastic incomplete method operating on a fixed number of candidate solutions (Kennedy and Eberhart 2001). For the PSO implementation we refer to the `PySwarm` library (Lee 2023).

Remark 3 The candidate solutions in the PSO method are dubbed “particles” forming a “swarm”. The method proceeds iteratively by randomly moving these particles in the search-space, taking care of their position and velocity, the latter being hyper-parameters. Each particle moves keeping track of its local best and of the global best in the search-space. PSO is a metaheuristic and does not require that the objective function is differentiable, nevertheless, it does not guarantee that the optimal solution is found at the end of the iterations.

We identify the initial time $n = 0$ with the date 2023-09-29 and consider call and put options on the `META` stock with maturity 2023-10-27, that corresponds to $T = 20$ trading days. We notice that `META` was a non-dividend paying stock up to the beginning of year 2024, as the company announced the payment of a quarterly dividend starting from February 2024.

Figure 9 shows the market bid-ask prices for the options on `META` with the given maturity, while Figure 10 depicts the market bid-ask spread for the options on `META` with the given maturity. We point out that, for increasing K, the bid-ask spread tends to vanish for European call options, while it tends to increase for European put options.

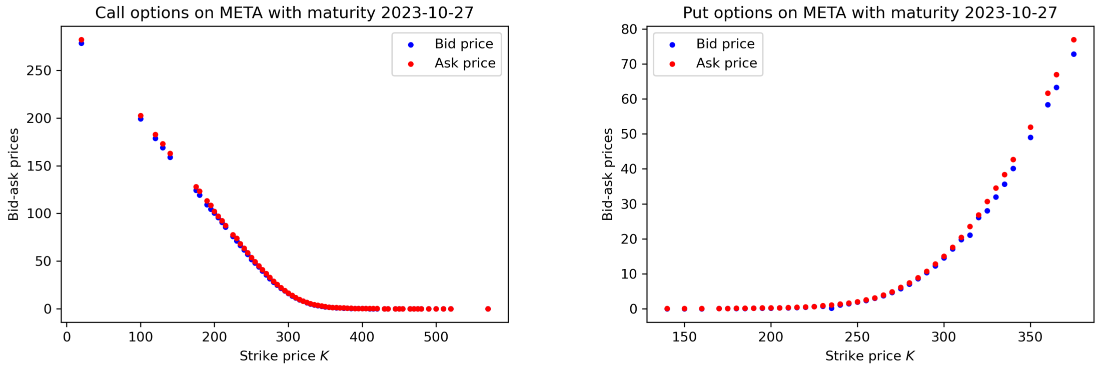

Figure 9. Bid-ask prices of call and put options on `META` with maturity 2023-10-27 retrieved on 2023-09-29.

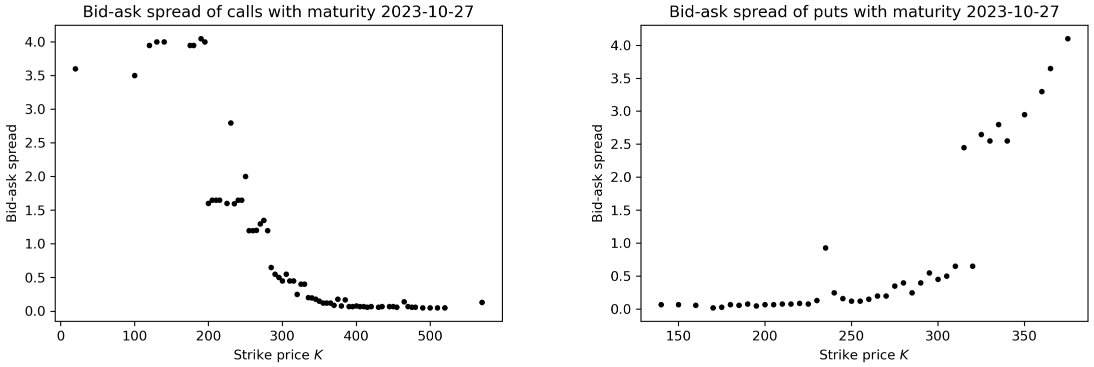

Figure 10. Bid-ask spreads of call and put options on `META` with maturity 2023-10-27 retrieved on 2023-09-29.

Referring to a year composed of 250 trading days, we get r through a US T-bill maturing in 1 month by setting $1 + r = 1.0555^{\frac{1}{250}}$.

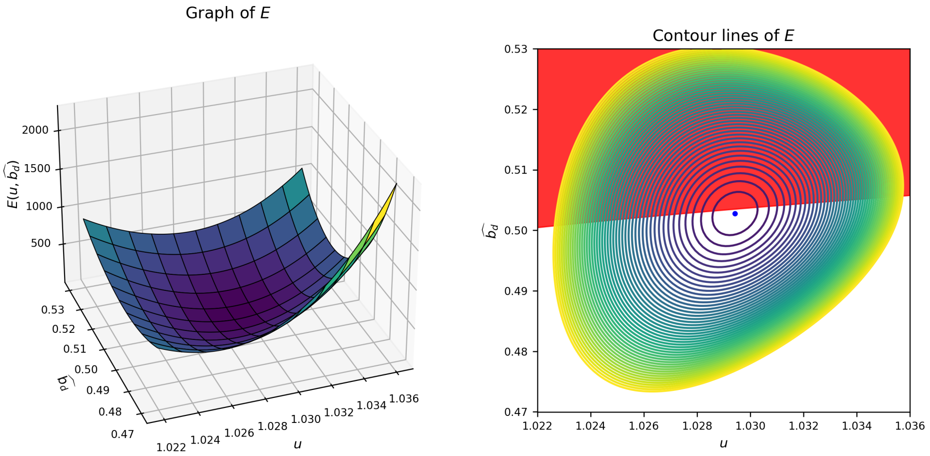

Figure 11. Surface and contour lines of $E(u, \widehat{b}_d)$. The red area denotes pairs $(u, \widehat{b}_d)$ violating the constraint $\widehat{b}_d \in (0, 1 - \widehat{b}_u]$, while the blue point is the minimizer.

To favor fitting, we restrict to call and put options whose bid-ask spread is less than or equal to \$1 and further add the `META` stock market bid-ask prices $\underline{S}_{0,M} = \$300.11$ and $\overline{S}_{0,M} = \$300.30$, as the bid-ask prices of a degenerate call option with strike price $K = 0$. Our restricted dataset consists of 42 call and 33 put options, that are the 62.69% and 78.57% of call and put options from the original dataset, respectively. The call and put options have a spread range of $[\$0.05, \$0.65]$ and $[\$0.02, \$0.93]$, respectively.

We first perform the calibration assuming that $d = \frac{1}{u}$, as is commonly done in the classical binomial calibration scheme: in this way the squared error reduces to a function of two variables $E(u, \widehat{b}_d) := E\left(u, \frac{1}{u}, \widehat{b}_d\right)$. Under this assumption, we perform the PSO technique for 2000 iterations and we get that the minimum is attained at $u = 1.029428$ and $\widehat{b}_d = 0.502787$, for which $E(u, \widehat{b}_d) = 21.950091$. Notice that $\widehat{b}_u = 0.496474$, thus $\widehat{b}_u + \widehat{b}_d = 0.999261 < 1$. Figure 11 shows the surface of error function $E(u, \widehat{b}_d)$, together with the corresponding contour lines and the obtained minimizer.

With the given optimal u and $\widehat{b}_d$ we can compute theoretical stock bid-ask prices that result to be $\underline{S}_0 = \$300.11$ and $\overline{S}_0 = \$300.37$. Figure 12 shows the theoretical bid-ask prices compared to market bid-ask prices for European call and put options highlighting a good fitting, despite the restriction $d = \frac{1}{u}$.

Next, we perform the calibration by removing the restriction on d and calibrating $u, d, \widehat{b}_d$ through the minimization of the squared error $E(u, d, \widehat{b}_d)$. After 2000 iterations of the PSO technique we get that the minimum is attained at $u = 1.023546, d = 0.964259$ and $\widehat{b}_d = 0.392609$, for which $E(u, d, \widehat{b}_d) = 15.441812$. We notice that removing the restriction on d we actually get a lower error, moreover, $\widehat{b}_u = 0.606494$, thus $\widehat{b}_u + \widehat{b}_d = 0.999102 < 1$.

We use the calibrated parameters $u, d, \widehat{b}_d$ to determine the bid-ask prices of an American put with maturity $T = 20$ trading days and strike $K = \$350$, by applying equations (33) and (35) in a backward induction procedure. Figure 13 shows the tree of the bid price process $\{\underline{P}_n\}_{n=0}^{T}$, while Figure 14 shows the tree of the ask price process $\{\overline{P}_n\}_{n=0}^{T}$. In both figures, the orange nodes and edges single out the early exercise region: in this particular example, it is easily seen that n^*_{bid} and n^*_{ask} coincide and take values in $\{2, \ldots, 20\}$, moreover, the bid-ask prices at time 0 result to be $\underline{P}_0 = \$51.24$ and $\overline{P}_0 = \$51.38$.

Still referring to the calibrated parameters and maturity, Figure 15 shows the theoretical bid-ask spread of an American put option as a function of the strike K, ranging in $[200, 400]$, with a step of 0.25, contrasted to the theoretical bid-ask spread of the corresponding European put option.

Notice that the American bid-ask spread is not monotonic as a function of K, while the European bid-ask spread is and actually turns out to be a weighted sum of piecewise linear functions of K,

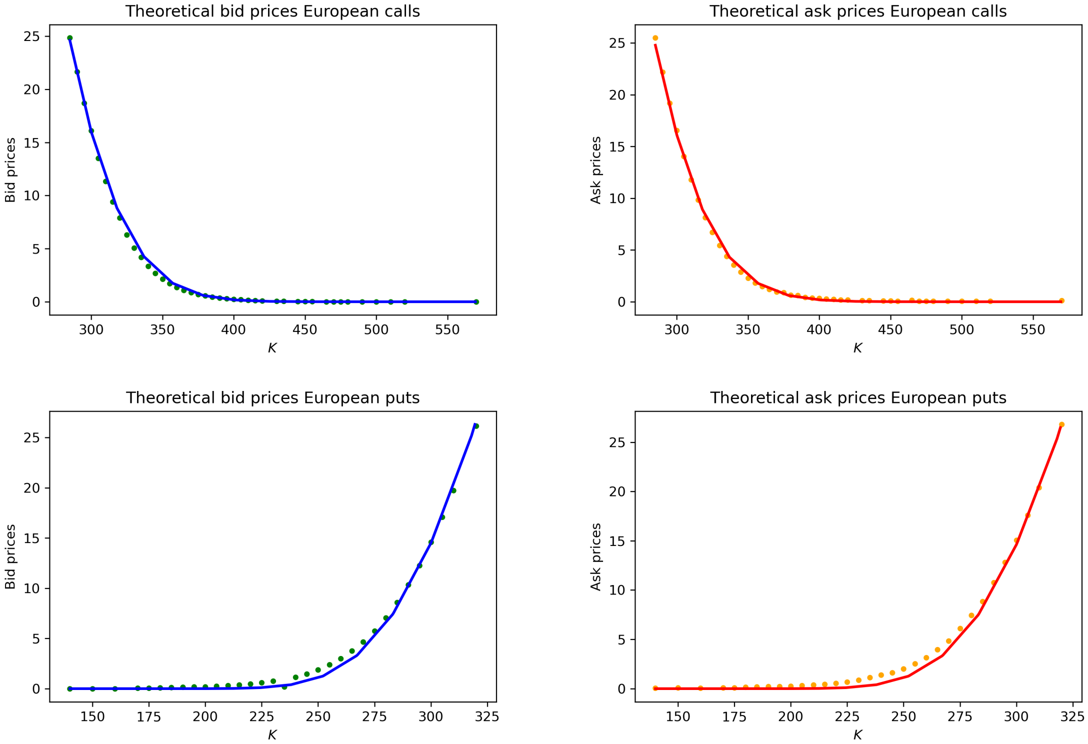

Figure 12. Theoretical bid-ask prices (solid lines) compared to market bid-ask prices (points) for European call and put options.

in agreement with Proposition 3.2. We also notice that the European put option bid-ask spread is an increasing function of K, in agreement with the market bid-ask spread reported in Figure 10. Moreover, the American bid-ask spread graph highlights the presence of a critical value K^* above which the bid-ask spread collapses to 0: this is due to the simultaneous early exercise with respect to both the bid and the ask price processes.

Keeping the calibrated parameters fixed, Figure 16 depicts the theoretical bid-ask spreads of American and European put options as functions of the strike K, ranging in $[200, 400]$, with a step of 0.25, for a maturity $T \in \{20, \dots, 100\}$ measured in trading days.

Figure 16 highlights a clear dominance between both the theoretical American and European put bid-ask spreads, for increasing maturity T. In the case of the American put, it is also possible to observe a non-monotonic behavior, suggesting that the critical value K^* above which the bid-ask spread collapses to 0, tends to increase for increasing maturity T. On the other hand, the European put bid-ask spread appears to be always a monotonic function.

In order to provide a deeper analysis of the proposed model, we repeat the calibration procedure by choosing an initial date in the first quarter of year 2024 and we compare two different underlying stocks on options within the same set of maturities. To this purpose, we identify $n = 0$ with the date 2024-02-09 and take the non-dividend paying stocks `NFLX` and `GOOG`, that experienced quite different market conditions during the year 2023. Figure 17 shows the adjusted close of the quoted stocks, highlighting the presence of two bear market periods (i.e., a decline higher than 20% from the most recent peak) for `NFLX` and no bear market period for `GOOG`. In detail, for `NFLX` the two bear market periods are from 2023-01-25 to 2023-03-10 (decline of 20.44%) and from 2023-07-19 to 2023-10-18 (decline of 27.51%).

We still refer to a year composed of 250 trading days and get r through a US T-bill maturing in 1 month by setting $1 + r = 1.0549^{\frac{1}{250}}$. Both for `NFLX` and `GOOG`, we consider European call and put options maturing in 5, 10, 15, 20, 25, 30, 50, 115 trading days.

Also in this case, we restrict to call and put options whose bid-ask spread is less than or equal

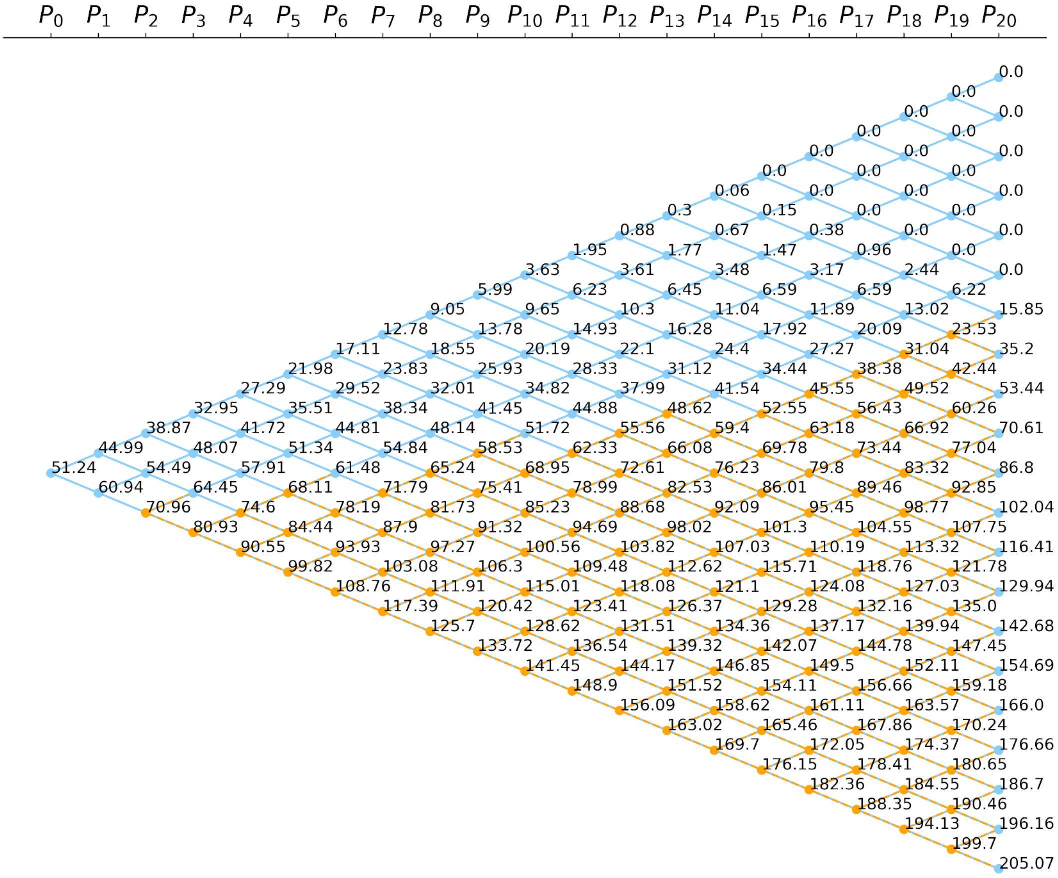

Figure 13. Recombining binomial tree representation of the bid price process $\{P_n\}_{n=0}^{T}$ for the American put option with $T = 20$ trading days and strike $K = \$350$. Nodes and edges in orange point out when early exercise is optimal.

to \$1 and further add the market stock bid-ask prices, as the bid-ask prices of a degenerate call option with strike price $K = 0$. In the case of `NFLX` we have $S_{0,M} = \$561.72$ and $\overline{S}_{0,M} = \$561.88$, while for `GOOG` we have $S_{0,M} = \$150.41$ and $\overline{S}_{0,M} = \$150.42$.

T	Trad. days	N. calls	N. puts	u	d	$\widehat{b_d}$	$\widehat{b_u}$	$E(u, d, \widehat{b_d})$	MSE
2024-02-16	5	43	95	1.017368	0.984052	0.513170	0.485109	10.170075	0.073696
2024-02-23	10	27	63	1.016496	0.984230	0.503668	0.495371	16.964214	0.188491
2024-03-01	15	27	59	1.017521	0.984539	0.524673	0.475248	20.531582	0.238739
2024-03-08	20	22	31	1.017715	0.984324	0.523390	0.475874	15.672492	0.295707
2024-03-15	25	79	119	1.016523	0.982551	0.479785	0.519918	38.341807	0.193645
2024-03-22	30	16	13	1.017858	0.983885	0.519081	0.480648	14.217982	0.490275
2024-04-19	50	59	78	1.018897	0.977071	0.446447	0.553320	116.312505	0.848996
2024-07-19	115	54	42	1.020498	0.978274	0.480394	0.519606	426.402415	4.441692

Table 1. Calibration on European options on `NFLX` on the date 2024-02-09 for different maturities.

Tables 1 and 2 show the calibrated parameters for the different maturity, together with the squared error $E(u, d, \widehat{b_d})$ and the Mean Squared Error (MSE), obtained dividing $E(u, d, \widehat{b_d})$ by the number of calls and puts for the corresponding maturity.

Figure 18 reports the graph of the MSE of the calibration as a function of the maturity T. For the retrieved sets of options we have that the MSE curve of `NFLX` always dominates that of `GOOG` and both curves tend to increase as T increases. The larger MSE obtained for `NFLX` witnesses a lower regularity of the bid-ask prices, compared to `GOOG`. In turn, such aspect could be caused by a bearish feeling hidden in the market, alimented by the two prolonged bear market periods experienced in 2023.

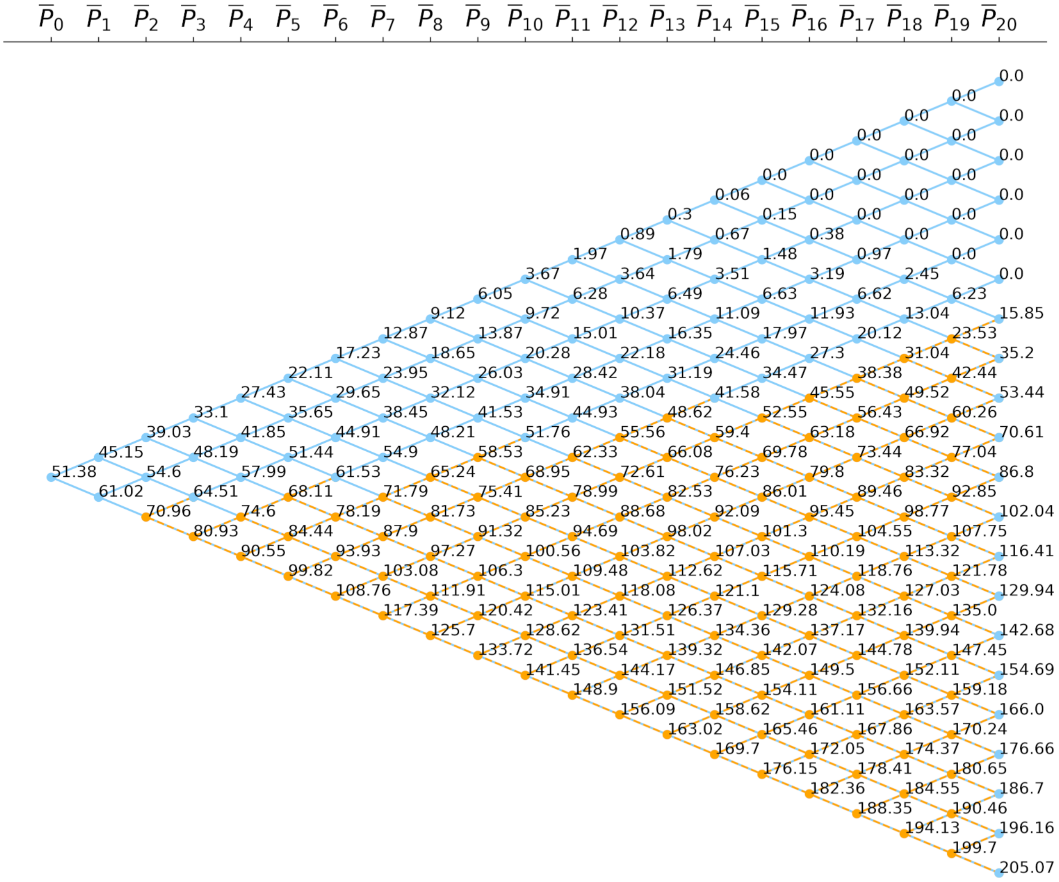

Figure 14. Recombining binomial tree representation of the ask price process $\{\overline{P}_n\}_{n=0}^{T}$ for the American put option with $T = 20$ trading days and strike $K = \$350$. Nodes and edges in orange point out when early exercise is optimal.

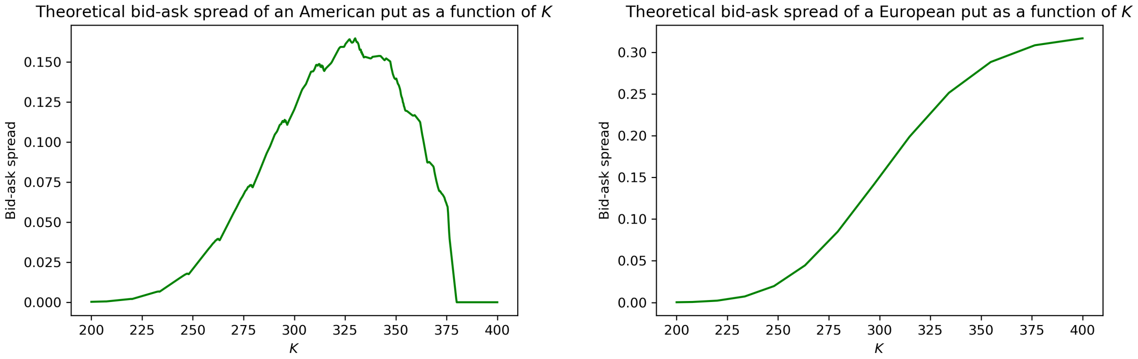

Figure 15. American and European put option bid-ask spreads as functions of K.

T	Trad. days	N. calls	N. puts	u	d	$\widehat{b_d}$	$\widehat{b_u}$	$E(u, d, \widehat{b_d})$	MSE
2024-02-16	5	48	41	1.016157	0.989220	0.587412	0.408141	1.824988	0.020505
2024-02-23	10	45	39	1.016003	0.989467	0.589561	0.404994	5.464529	0.065054
2024-03-01	15	37	39	1.015468	0.988564	0.563680	0.433014	6.757992	0.088921
2024-03-08	20	37	35	1.015554	0.988237	0.559285	0.438437	8.259458	0.114715
2024-03-15	25	33	28	1.015620	0.987226	0.542588	0.457413	9.811651	0.160847
2024-03-22	30	19	21	1.016104	0.987973	0.563912	0.435138	1.581735	0.039543
2024-04-19	50	24	24	1.015796	0.986410	0.530234	0.469744	19.035279	0.396568
2024-07-19	115	24	22	1.015757	0.983097	0.475906	0.524094	128.011714	2.782863

Table 2. Calibration on European options on `GOOG` on the date 2024-02-09 for different maturities.

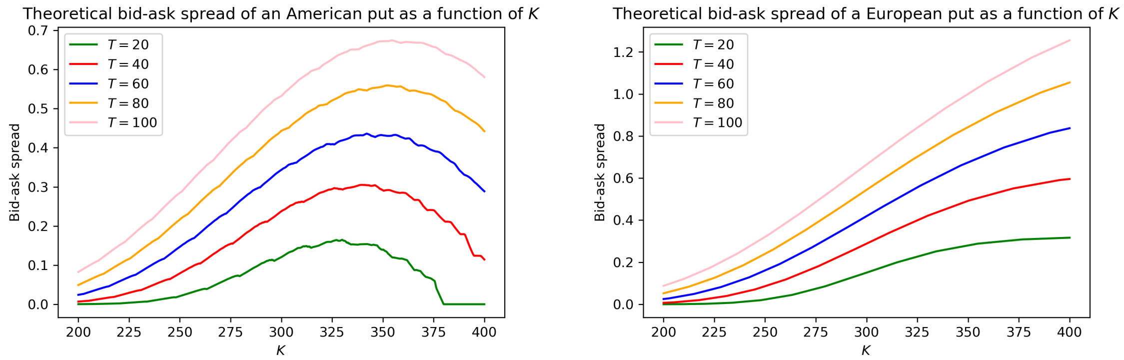

Figure 16. American and European put option bid-ask spreads as functions of K, for several maturities.

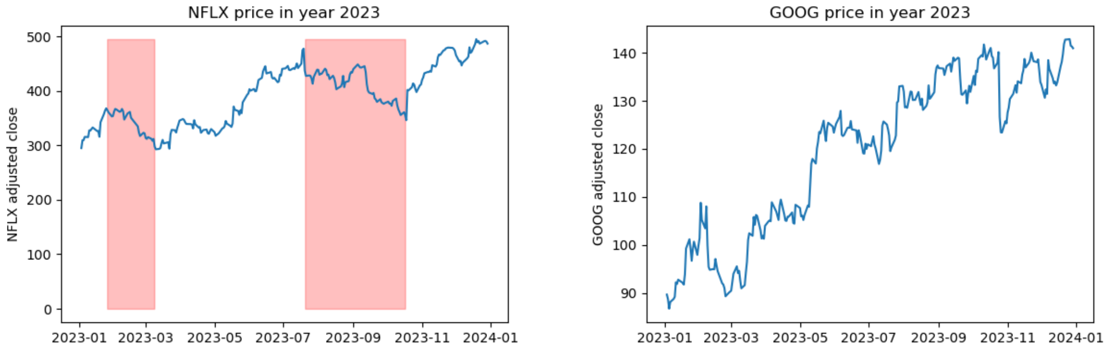

Figure 17. Adjusted close of `NFLX` and `GOOG` stocks in year 2023. The red bands in the `NFLX` graph highlight bear market periods.

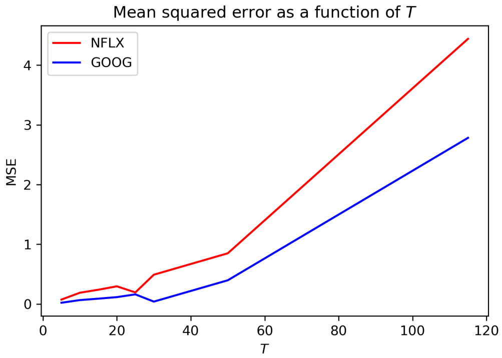

Figure 18. MSE of the calibration as a function of the maturity T.

6. Conclusion

Referring to the market model introduced in Cinfrignini *et al.* (2023a), we provide closed form formulas for the bid-ask prices of European derivatives when the contract function is monotonic. We also consider the problem of pricing American derivatives, proposing a backward bid-ask pricing procedure and prove the validity of an analogue of Merton's theorem in our setting. The bid-ask prices closed formulas for European derivatives allow us to propose a market consistent calibration procedure relying on bid-ask prices of European call and put options.

Proceeding in analogy with the classical binomial model, an open stream of future research is to investigate the convergence of the proposed model in the continuous-time setting, by relying on

Choquet weak convergence (Feng and Nguyen 2007). The resulting theory could shed new light on the calibration or estimation of the parameters in the discrete-time model dealt with in this paper.

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Conflict of interest

The authors declare that they have no conflict of interest.

Appendix A: Proofs

Proof of Proposition 3.1. By Proposition 4 in Cinfrignini *et al.* (2023a), the bid price process of the European derivative $\{Y_n\}_{n=0}^T$ corresponding to a non-decreasing $\varphi(x)$ and defined by (19), does not depend on the choice of $\widehat{b_d} \in (0, 1-\widehat{b_u}]$. On the other hand, the corresponding ask price process $\{\overline{Y}_n\}_{n=0}^T$ defined by (20) depends on the choice of $\widehat{b_d} \in (0, 1-\widehat{b_u}]$.

We prove the statement by backward induction. Define the functions $\varphi_T = \overline{\varphi}_T = \varphi$ on $\mathcal{S}_T$, which are non-decreasing since φ is non-decreasing by hypothesis. For $n = T-1$, define $\varphi_{T-1}, \overline{\varphi}_{T-1} : \mathcal{S}_{T-1} \to \mathbf{R}$ by setting, for every $s_{T-1} \in \mathcal{S}_{T-1}$,

$$\varphi_{T-1}(s_{T-1}) = \frac{1}{1+r}\widehat{\mathbf{C}}[\varphi_T(S_T)|S_{T-1} = s_{T-1}] = \frac{1}{1+r}\left(\widehat{b_u}\varphi_T(us_{T-1}) + (1-\widehat{b_u})\varphi_T(ds_{T-1})\right),$$

$$\overline{\varphi}_T(s_{T-1}) = -\frac{1}{1+r}\widehat{\mathbf{C}}[-\overline{\varphi}_T(S_T)|S_{T-1} = s_{T-1}] = \frac{1}{1+r}\left((1-\widehat{b_d})\overline{\varphi}_T(us_{T-1}) + \widehat{b_d}\overline{\varphi}_T(ds_{T-1})\right),$$

where the last equality of both equations follows from Proposition 3 in Cinfrignini *et al.* (2023a). We have that both $\varphi_{T-1}, \overline{\varphi}_{T-1}$ are non-decreasing since, for every $s^i_{T-1}, s^j_{T-1} \in \mathcal{S}_{T-1}$ with $s^i_{T-1} < s^j_{T-1}$ it holds $ds^i_{T-1} < us^i_{T-1} \leq ds^j_{T-1} < us^j_{T-1}$, which implies $\varphi_{T-1}(s^i_{T-1}) \leq \varphi_{T-1}(s^j_{T-1})$ and $\overline{\varphi}_{T-1}(s^i_{T-1}) \leq \overline{\varphi}_{T-1}(s^j_{T-1})$.

Now, for every $n = 0, \ldots, T-1$, assuming that $\varphi_{n+1}, \overline{\varphi}_{n+1} : \mathcal{S}_{n+1} \to \mathbf{R}$ are non-decreasing, define $\varphi_n, \overline{\varphi}_n : \mathcal{S}_n \to \mathbf{R}$ by setting, for every $s_n \in \mathcal{S}_n$,

$$\varphi_n(s_n) = \frac{1}{1+r}\widehat{\mathbf{C}}[\varphi_{n+1}(S_{n+1})|S_n = s_n] = \frac{1}{1+r}\left(\widehat{b_u}\varphi_{n+1}(us_n) + (1-\widehat{b_u})\varphi_{n+1}(ds_n)\right),$$

$$\overline{\varphi}_n(s_n) = -\frac{1}{1+r}\widehat{\mathbf{C}}[-\overline{\varphi}_{n+1}(S_{n+1})|S_n = s_n] = \frac{1}{1+r}\left((1-\widehat{b_d})\overline{\varphi}_{n+1}(us_n) + \widehat{b_d}\overline{\varphi}_{n+1}(ds_n)\right),$$

where the last equality of both equations follows again from Proposition 3 in Cinfrignini *et al.* (2023a). We have that both $\varphi_n, \overline{\varphi}_n$ are non-decreasing since, for every $s^i_n, s^j_n \in \mathcal{S}_n$ with $s^i_n < s^j_n$ it holds $ds^i_n < us^i_n \leq ds^j_n < us^j_n$, which implies $\varphi_n(s^i_n) \leq \varphi_n(s^j_n)$ and $\overline{\varphi}_n(s^i_n) \leq \overline{\varphi}_n(s^j_n)$.

Finally, substituting the expression of φ_{n+1} in that of φ_n and proceeding backward from $n = T-1$, we get the claim. $\square$

Proof of Proposition 3.2. Proposition 4 in Cinfrignini *et al.* (2023a) implies that the ask price process of the European derivative $\{\overline{Y}_n\}_{n=0}^T$ corresponding to a non-increasing $\varphi(x)$ and defined by (20), does not depend on the choice of $\widehat{b_d} \in (0, 1-\widehat{b_u}]$. On the other hand, the corresponding bid price process $\{Y_n\}_{n=0}^T$ defined by (19) depends on the choice of $\widehat{b_d} \in (0, 1-\widehat{b_u}]$.

We prove the statement by backward induction. Define the functions $\varphi_T = \overline{\varphi}_T = \varphi$ on $\mathcal{S}_T$, which are non-increasing since φ is non-increasing by hypothesis. For $n = T-1$, define $\varphi_{T-1}, \overline{\varphi}_{T-1} : \mathcal{S}_{T-1} \to \mathbf{R}$ by setting, for every $s_{T-1} \in \mathcal{S}_{T-1}$,

$$\begin{aligned}\varphi_{T-1}(s_{T-1}) &= \frac{1}{1+r}\widehat{\mathbf{C}}[\varphi_T(S_T)|S_{T-1} = s_{T-1}] = \frac{1}{1+r}\left((1-\widehat{b_d})\varphi_T(us_{T-1}) + \widehat{b_d}\varphi_T(ds_{T-1})\right),\\ \overline{\varphi}_T(s_{T-1}) &= -\frac{1}{1+r}\widehat{\mathbf{C}}[-\overline{\varphi}_T(S_T)|S_{T-1} = s_{T-1}] = \frac{1}{1+r}\left(\widehat{b_u}\overline{\varphi}_T(us_{T-1}) + (1-\widehat{b_u})\overline{\varphi}_T(ds_{T-1})\right),\end{aligned}$$

where the last equality of both equations follows from Proposition 3 in Cinfrignini *et al.* (2023a). We have that both $\varphi_{T-1}, \overline{\varphi}_{T-1}$ are non-increasing since, for every $s^i_{T-1}, s^j_{T-1} \in \mathcal{S}_{T-1}$ with $s^i_{T-1} < s^j_{T-1}$ it holds $ds^i_{T-1} < us^i_{T-1} \leq ds^j_{T-1} < us^j_{T-1}$, which implies $\varphi_{T-1}(s^i_{T-1}) \geq \varphi_{T-1}(s^j_{T-1})$ and $\overline{\varphi}_{T-1}(s^i_{T-1}) \geq \overline{\varphi}_{T-1}(s^j_{T-1})$.

Now, for every $n = 0, \ldots, T-1$, assuming that $\varphi_{n+1}, \overline{\varphi}_{n+1} : \mathcal{S}_{n+1} \to \mathbf{R}$ are non-increasing, define $\varphi_n, \overline{\varphi}_n : \mathcal{S}_n \to \mathbf{R}$ by setting, for every $s_n \in \mathcal{S}_n$,

$$\begin{aligned}\varphi_n(s_n) &= \frac{1}{1+r}\widehat{\mathbf{C}}[\varphi_{n+1}(S_{n+1})|S_n = s_n] = \frac{1}{1+r}\left((1-\widehat{b_d})\varphi_{n+1}(us_n) + \widehat{b_d}\varphi_{n+1}(ds_n)\right),\\ \overline{\varphi}_n(s_n) &= -\frac{1}{1+r}\widehat{\mathbf{C}}[-\overline{\varphi}_{n+1}(S_{n+1})|S_n = s_n] = \frac{1}{1+r}\left(\widehat{b_u}\overline{\varphi}_{n+1}(us_n) + (1-\widehat{b_u})\overline{\varphi}_{n+1}(ds_n)\right),\end{aligned}$$

where the last equality of both equations follows again from Proposition 3 in Cinfrignini *et al.* (2023a). We have that both $\varphi_n, \overline{\varphi}_n$ are non-increasing since, for every $s^i_n, s^j_n \in \mathcal{S}_n$ with $s^i_n < s^j_n$ it holds $ds^i_n < us^i_n \leq ds^j_n < us^j_n$, which implies $\varphi_n(s^i_n) \geq \varphi_n(s^j_n)$ and $\overline{\varphi}_n(s^i_n) \geq \overline{\varphi}_n(s^j_n)$.

Finally, substituting the expression of φ_{n+1} in that of φ_n and proceeding backward from $n = T-1$, we get the claim. □

Proof of Proposition 3.3. The proof is an immediate consequence of Propositions 3.1 and 3.2. □

Proof of Proposition 3.4. For statement *(i)*, by the proof of Proposition 3.1, equation (29) can be rewritten as

$$\begin{cases}\lambda^0_n(1+r)B_n + \lambda^1_n uS_n = \varphi_{n+1}(uS_n),\\ \lambda^0_n(1+r)B_n + \lambda^1_n dS_n = \varphi_{n+1}(dS_n),\end{cases}$$

where φ_{n+1} is a non-decreasing function defined on the range of S_{n+1}. Therefore, we get that

$$\lambda^1_n = \frac{\varphi_{n+1}(uS_n) - \varphi_{n+1}(dS_n)}{uS_n - dS_n} \geq 0,$$

which implies that

$$\begin{aligned}
Y_n &= \frac{1}{1+r}\widehat{\mathbf{C}}[Y_{n+1}|\mathcal{F}_n] \\
&= \frac{1}{1+r}\widehat{\mathbf{C}}[\lambda_n^0 B_{n+1} + \lambda_n^1 S_{n+1}|\mathcal{F}_n] \\
&= \frac{1}{1+r}\left(\lambda_n^0(1+r)B_n + \widehat{\mathbf{C}}[\lambda_n^1 S_{n+1}|\mathcal{F}_n]\right) \\
&= \lambda_n^0 B_n + \frac{1}{1+r}\widehat{\mathbf{C}}[\lambda_n^1 S_{n+1}|\mathcal{F}_n] \\
&= \lambda_n^0 B_n + \lambda_n^1 S_n,
\end{aligned}$$

where the last equality is due to Proposition 2 in Cinfrignini *et al.* (2023a) and equation (19). Indeed, it holds $\frac{1}{1+r}\widehat{\mathbf{C}}[\lambda_n^1 S_{n+1}|\mathcal{F}_n] = \lambda_n^1 \frac{1}{1+r}\widehat{\mathbf{C}}[S_{n+1}|\mathcal{F}_n] = \lambda_n^1 S_n$.

Analogously, for statement *(ii)* by the proof of Proposition 3.2, equation (30) can be rewritten as

$$\begin{cases} \delta_n^0(1+r)B_n + \delta_n^1 u S_n = \overline{\varphi}_{n+1}(uS_n), \\ \delta_n^0(1+r)B_n + \delta_n^1 d S_n = \overline{\varphi}_{n+1}(dS_n), \end{cases}$$

where $\overline{\varphi}_{n+1}$ is a non-increasing function defined on the range of S_{n+1}. Therefore, we get that

$$\delta_n^1 = \frac{\overline{\varphi}_{n+1}(uS_n) - \overline{\varphi}_{n+1}(dS_n)}{uS_n - dS_n} \leq 0,$$

which implies that

$$\begin{aligned}
\overline{Y}_n &= -\frac{1}{1+r}\widehat{\mathbf{C}}[-\overline{Y}_{n+1}|\mathcal{F}_n] \\
&= -\frac{1}{1+r}\widehat{\mathbf{C}}[-\delta_n^0 B_{n+1} - \delta_n^1 S_{n+1}|\mathcal{F}_n] \\
&= -\frac{1}{1+r}\left(-\delta_n^0(1+r)B_n + \widehat{\mathbf{C}}[-\delta_n^1 S_{n+1}|\mathcal{F}_n]\right) \\
&= \delta_n^0 B_n - \frac{1}{1+r}\widehat{\mathbf{C}}[-\delta_n^1 S_{n+1}|\mathcal{F}_n] \\
&= \delta_n^0 B_n + \delta_n^1 S_n,
\end{aligned}$$

where, again, the last equality is due to Proposition 2 in Cinfrignini *et al.* (2023a) and equation (19). Indeed, it holds $-\frac{1}{1+r}\widehat{\mathbf{C}}[-\delta_n^1 S_{n+1}|\mathcal{F}_n] = -(-\delta_n^1)\frac{1}{1+r}\widehat{\mathbf{C}}[S_{n+1}|\mathcal{F}_n] = \delta_n^1 S_n$. □

Proof of Proposition 4.1. The proof is by backward induction. For $n = T$ we have that $Y_T = \overline{Y}_T = \Phi_T$.

For all $n = 0, \ldots, T-1$, supposing that $Y_{n+1} \leq \overline{Y}_{n+1}$, by the monotonicity with respect to the integrated function and the monotonicity with respect to the integrating capacity of the conditional Choquet expectation operator (Grabisch 2016), it holds that

$$\widehat{\mathbf{C}}[Y_{n+1}|\mathcal{F}_n] \leq \widehat{\mathbf{C}}[\overline{Y}_{n+1}|\mathcal{F}_n] \leq -\widehat{\mathbf{C}}[-\overline{Y}_{n+1}|\mathcal{F}_n]$$

which implies

$$Y_n = \max\left\{\Phi_n, \frac{1}{1+r}\widehat{\mathbf{C}}[Y_{n+1}|\mathcal{F}_n]\right\} \leq \overline{Y}_n = \max\left\{\Phi_n, -\frac{1}{1+r}\widehat{\mathbf{C}}[-\overline{Y}_{n+1}|\mathcal{F}_n]\right\}.$$

□

Proof of Proposition 4.2. The proof goes along the same line of that of Lemma 3.8 in Pascucci and Runggaldier (2012), working with the conditional Choquet expectation with respect to $\widehat{\nu}$.

We firstly prove it for the discounted bid price process $\{Y^*_n\}_{n=0}^T$. From (33) it follows that $Y^*_T = \Phi^*_T$ and, for every $n = 0, \dots, T-1$,

$$Y^*_n \geq \Phi^*_n \quad \text{and} \quad Y^*_n \geq \widehat{\mathbf{C}}[Y^*_{n+1}|\mathcal{F}_n],$$

hence the process satisfies *(i.a)* and *(i.b)* in Definition 1.

Now we prove that $\{Y^*_n\}_{n=0}^T$ satisfies *(i.c)*. Let us consider another discounted $\widehat{\nu}$-Choquet one-step super-martingale $V^*_n \geq \Phi^*_n$ and prove that $V^*_n \geq Y^*_n$ for all $n = 0, \dots, T-1$. Starting from $n = T$, we have that

$$V^*_T \geq \Phi^*_T = Y^*_T.$$

At time $n = T-1$ we have

$$V^*_{T-1} \geq \widehat{\mathbf{C}}[V^*_T|\mathcal{F}_{T-1}] \geq \widehat{\mathbf{C}}[Y^*_T|\mathcal{F}_{T-1}],$$

where the first inequality is due to the $\widehat{\nu}$-Choquet one-step super-martingale property of $\{V^*_n\}_{n=0}^T$ and the second inequality is due to the monotonicity property of the conditional Choquet expectation operator. Since $\{V^*_n\}_{n=0}^T$ dominates $\{\Phi^*_n\}_{n=0}^T$, it holds that

$$V^*_{T-1} \geq \Phi^*_{T-1},$$

hence

$$V^*_{T-1} \geq \max\left\{\Phi^*_{T-1}, \widehat{\mathbf{C}}[Y^*_T|\mathcal{F}_{T-1}]\right\} = Y^*_{T-1}.$$

Finally, proceeding by backward induction it is easily shown that the statement holds for every $n = 0, \dots, T-1$, i.e., *(i.c)* in Definition 1 is verified.

Let us prove the proposition for the discounted ask price process $\{\overline{Y}^*_n\}_{n=0}^T$. From (35) it follows that, for every $n = 0, \dots, T-1$,

$$\overline{Y}^*_n \geq \Phi^*_n \quad \text{and} \quad \overline{Y}^*_n \geq -\widehat{\mathbf{C}}[-\overline{Y}^*_{n+1}|\mathcal{F}_n],$$

hence it satisfies *(ii.a)* and *(ii.b)* in Definition 1.

As before, let us consider another discounted $\widehat{\overline{\nu}}$-Choquet one-step super-martingale $\overline{V}^*_n \geq \Phi^*_n$ and prove that $\overline{V}^*_n \geq \overline{Y}^*_n$ for all $n = 0, \dots, T-1$. Starting from $n = T$, we have that

$$\overline{V}^*_T \geq \Phi^*_T = \overline{Y}^*_T.$$

At time $n = T-1$ we have

$$\overline{V}^*_{T-1} \geq -\widehat{\mathbf{C}}[-\overline{V}^*_T|\mathcal{F}_{T-1}] \geq -\widehat{\mathbf{C}}[-\overline{Y}^*_T|\mathcal{F}_{T-1}],$$

where the first inequality is due to the $\widehat{\nu}$-Choquet one-step super-martingale property of $\{\overline{V}^*_n\}_{n=0}^T$ and the second inequality is due to the monotonicity property of the conditional Choquet expectation operator. Since $\{\overline{V}^*_n\}_{n=0}^T$ dominates $\{\Phi^*_n\}_{n=0}^T$, it also holds that

$$\overline{V}^*_{T-1} \geq \Phi^*_{T-1},$$

hence

$$\overline{V}^*_{T-1} \geq \max\left\{\Phi^*_{T-1}, -\widehat{\mathbf{C}}[-\overline{Y}^*_T|\mathcal{F}_{T-1}]\right\} = \overline{Y}^*_{T-1}.$$

Finally, proceeding by backward induction it is easily shown that the statement holds for every $n = 0, \dots, T-1$, i.e., *(ii.c)* in Definition 1 is verified. □

Proof of Theorem 4.3. Let $\{C^E_n\}_{n=0}^T$ and $\{C_n\}_{n=0}^T$ be the bid price processes of a European and an American call on $\{S_n\}_{n=0}^T$ with maturity T, respectively. By Proposition 3.1, $\{C^E_n\}_{n=0}^T$ is such that $C^E_T = \max\{S_T - K, 0\}$ and for $n = 0, \dots, T-1$,

$$\frac{1}{1+r}\widehat{\mathbf{C}}[C^E_{n+1}|\mathcal{F}_n] = \frac{1}{1+r}\widehat{\mathbf{E}}[C^E_{n+1}|\mathcal{F}_n],$$

where $\widehat{\mathbf{E}}[\cdot|\mathcal{F}_n]$ is the conditional expectation operator induced by the additive risk-neutral belief function $\widehat{\nu}$ with $\widehat{b_u}$ and $\widehat{b_d} = 1 - \widehat{b_u}$, that reduces to the classical risk-neutral probability measure $\widehat{P}$ in the binomial model (Černý 2009, Pliska 1997). By Merton's theorem (see Merton (1973)) in the classical binomial model we have that, for $n = 0, \dots, T$,

$$C^E_n \geq \Phi^C_n,$$

therefore early exercise is never optimal with respect to the bid price process, and it follows that $\{C_n\}_{n=0}^T$ coincides with $\{C^E_n\}_{n=0}^T$.

Let $\{\overline{C}^E_n\}_{n=0}^T$ and $\{\overline{C}_n\}_{n=0}^T$ be the ask price processes of a European and an American call on $\{S_n\}_{n=0}^T$ with maturity T, respectively. By the monotonicity of property of the Choquet integral with respect to the integrating capacity (Grabisch 2016), the ask continuation value dominates the bid continuation value for all $n = 0, \dots, T-1$

$$-\frac{1}{1+r}\widehat{\mathbf{C}}[-\overline{C}_{n+1}|\mathcal{F}_n] \geq \frac{1}{1+r}\widehat{\mathbf{C}}[C_{n+1}|\mathcal{F}_n] \geq \Phi^C_n$$

therefore early exercise is never optimal with respect to the ask price process, and it follows that $\{\overline{C}_n\}_{n=0}^T$ coincides with $\{\overline{C}^E_n\}_{n=0}^T$. □

Proof of Proposition 4.4. We prove the statement by backward induction. Define the put ask price process as in (35), therefore we have that $\overline{P}_n = \overline{\varphi}_n(S_n)$, where $\overline{\varphi}_n : \mathcal{S}_n \to \mathbf{R}$ is defined as

$$\overline{\varphi}_n(s_n) = \begin{cases} \max\{K - s_T\} & \text{if } n = T \\ \max\left\{\max\{K - s_n\}, \psi_n(s_n)\right\} & \text{if } n = 0, \dots, T-1, \end{cases}$$

and $\psi_n(s_n) = -\frac{1}{1+r}\widehat{\mathbf{C}}[-\overline{P}_{n+1}|S_n = s_n]$. We have that $\overline{\varphi}_T$ is non-increasing and the same holds for $\overline{\varphi}_{T-1}$ since

$$\psi_{T-1}(s_{T-1}) = \frac{1}{1+r}(\widehat{b_u}\overline{\varphi}_T(us_{T-1}) + (1 - \widehat{b_u})\overline{\varphi}_T(ds_{T-1}))$$

where the last equality follows from Proposition 3 in Cinfrignini *et al.* (2023a). Indeed, for every $s^i_{T-1}, s^j_{T-1} \in \mathcal{S}_{T-1}$ with $s^i_{T-1} < s^j_{T-1}$ it holds $ds^i_{T-1} < us^i_{T-1} \leq ds^j_{T-1} < us^j_{T-1}$, which implies $\psi_{T-1}(s^i_{T-1}) \geq \psi_{T-1}(s^j_{T-1})$ and $\overline{\varphi}_{T-1}(s^i_{T-1}) \geq \overline{\varphi}_{T-1}(s^j_{T-1})$, where the last inequality holds since $\overline{\varphi}_{T-1}$ is the maximum of two non-increasing functions.

Now, for every $n = 0, \ldots, T-1$, assuming that $\overline{\varphi}_{n+1} : \mathcal{S}_{n+1} \to \mathbf{R}$ we have that the same holds for $\overline{\varphi}_n$ since

$$\psi_n(s_n) = \frac{1}{1+r}(\widehat{b_u}\overline{\varphi}_{n+1}(us_n) + (1-\widehat{b_u})\overline{\varphi}_{n+1}(ds_n))$$

where the last equality follows from Proposition 3 in Cinfrignini *et al.* (2023a). Indeed, for every $s^i_n, s^j_n \in \mathcal{S}_n$, with $s^i_n < s^j_n$, it holds $ds^i_n < us^i_n \leq ds^j_n < us^j_n$, which implies $\psi_n(s^i_n) \geq \psi_n(s^j_n)$ and $\overline{\varphi}_n(s^i_n) \geq \overline{\varphi}_n(s^j_n)$, where the last inequality holds again since $\overline{\varphi}_n$ is the maximum of two non-increasing functions.

Therefore, $\widehat{b_d}$ does not play any role in the computation of $\{\overline{P}_n\}_{n=0}^T$, and this completes the proof. □

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