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A Game Theory Proof of Optimal Colorings Resilience to Strong Deviations

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Abstract: This paper provides a formal proof of the conjecture stating that optimal colorings in max k-cut games over unweighted and undirected graphs do not allow the formation of any strongly divergent coalition, i.e., a subset of nodes able to increase their own payoffs simultaneously. The result is obtained by means of a new method grounded on game theory, which consists in splitting the nodes of the graph into three subsets: the coalition itself, the coalition boundary and the nodes without relationship with the coalition. Moreover, we find additional results concerning the properties of optimal colorings.

Keywords: max k-cut problem; game theory; optimal colorings; coalitions; Nash equilibrium

MSC: 05C57; 05C15; 90C27

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1. Introduction

The problem of graph coloring is significant and current (see, for example, the recent result [1]). We focus on the resilience of optimal colorings to strong deviations, which is also linked to the max k-cut problem. The max k-cut problem consists in assigning colors to the vertices of a graph with the aim of ensuring the highest heterogeneity of colors in the graph, that is, by partitioning the vertices of the graph in such a way that each of them has the largest possible number of nodes with a different color from its own.

This problem is particularly interesting not only from a theoretical point of view, but also from the applicative perspective. Indeed, it is linked to significant real-life applications with a selfish agent. For instance, consider n agents communicating via radio signals and assume that only two distinct frequencies are available. Such a scenario can be modeled by a symmetric $n \times n$ matrix, which indicates, for each pair of agents, the strength of the interference that they experiment if they select the same frequency. Assuming that each agent chooses their frequency in order to minimize the sum of interference that they experiment with, without worrying about the situation of others, we can use the max k-cut game as a formal framework to study what would be the worst configuration that the selfish agents can reach, compared to a solution where a central entity assigns frequencies optimally. Another possible example is given by a set of companies that choose which good to produce in order to maximize their revenue, and to do so, they have to minimize the number of competitors represented by similar companies in the same geographic area.

Finding an optimal solution for the max k-cut game is NP-complete (see [2]). However, using heuristic-based algorithms (some of them are reported in [3,4]), or approximation ones (for instance, see [5,6]) it is possible to find a good enough solution, trading accuracy for computational time.

A strategic version of the max k-cut problem is the so-called max k-cut game on an undirected and unweighted graph with a set of k colors, where vertices represent players and the edges indicate their mutual relations. Each player chooses one of the available colors as its own strategy, and the corresponding payoff is the number of its neighbors that have chosen a different color.

One of the main problems for the max k-cut game is to prove the existence of strong Nash equilibria (briefly, *SE*) [7], i.e., a refinement of the Nash equilibrium. A strong equilibrium corresponds to assigned colorings in which no coalition, assuming the actions of its complements as given, can cooperatively deviate in a way that benefits all of its members, in other words, each player of the coalition strictly improves its utility. In fact, in this direction of research, the most significant focus is on equilibrium concepts, which are resilient to deviations of groups. Few results are known concerning the existence of strong equilibria in max k-cut games. The authors of [8] proved that an optimal coloring, i.e., a coloring that maximizes the sum of the players' payoffs, is an SE for the max 2-cut game. Later, they extended their results, showing that it is also a 3-SE, for the max k-cut game, for any $k \geq 2$, but an optimal strategy profile is not necessarily a 4-SE, for any $k \geq 3$.

In [9], it was proved that, if the number of colors is at least equal to the number of players minus two, then an optimal strategy profile is an SE, and it was conjectured that a SE always exists for the max k-cut game. In particular, it is conjectured that any optimal coloring is a SE.

The most important existing result was provided by the authors of [10,11], who showed that on undirected unweighted graphs, optimal colorings are 5-strong equilibria (5-SE), i.e., colorings in which no coalition of at most five vertices can profitably deviate. These results were extended to 7-SE by the same authors of this work [12].

In this paper, we show that optimal colorings are strong equilibria by using a completely new approach based on the distinction between sets of nodes with optimal or not optimal payoffs, where only the last ones are potentially interested to deviate. This makes our approach fully different from the existing ones. Indeed, the results in the literature use the relation between the number of available colors and the size of the coalition. This approach is easy to use in the case of small coalitions, but it is difficult to generalize due to the rapidly increasing number of cases to be considered. A step forward in this direction has been made by us, since our reasoning is independent of the number of colors available and the size of the coalition; in fact, it is based only on the topological proprieties of the graph, such the degrees of the nodes.

Our approach considers a max k-cut game played by n individuals or players. The individuals are assumed to be arranged on an undirected and unweighted graph. Specifically, nodes of the graph represent the individuals, while the edges describe the connections among them. The strategy space of each player is composed by a set of k available colors, $K = \{1, \ldots, k\}$. For the sake of simplicity, we assume that the color set is the same for each player. Given a strategy profile or a coloring, represented by the sequence of colors chosen by players, the payoff of a player v is the sum of the weights of all edges $\{v, w\}$ incident to w, such that the color chosen by w is different from the one chosen by v. When all weights associated to links are equal to 1, then the payoff corresponds to the number of neighbors with a color different from its color. Each player is selfish, and then its objective is to maximize its own utility. The main problem concerning a max k-cut game is related to the possibility for players to achieve autonomously a social optimum (i.e., maximize the cut value by themselves) rather than forcing individuals by an external regulator. Indeed, in such games on graphs, it is beneficial for players to anti-coordinate their choices with the ones of their neighbors by selecting different colors. Therefore, the players may attempt to increase their utility by coordinating their choices in groups, called coalitions.

Our Results

In our work, we extend the main results of [10,12] since we show important properties of minimal subsets and strong deviation. Moreover, we prove that there does not exist any subsets of nodes able to increase their own payoffs simultaneously. This means that optimal colorings in max k-cut games over unweighted and undirected graphs do not allow the presence of any strongly divergent coalition. These results were recently conjectured in [12].

These results have many relevant conceptual consequences and applications. Indeed, we extend previous results by showing that optimal colorings are robust and resilient against coalitions of any dimension aimed at forcing groups of players to selfishly diverge from the optimal equilibrium.

The article is structured as follows. First, in the next section, we give the definitions useful for our study and we introduce the main problem. In Section 3, we provide some results concerning the properties of optimal colorings. Then in Section 4, we prove that, in the monochromatic case, optimal colorings in max k-cut games over unweighted and undirected graphs do not allow the formation of any strongly divergent coalition. We conclude the article dealing with issues stemming from the approach used to prove our main results and pointing out some open problems, in Section 5.

2. Preliminaries

In this section, we introduce the main concepts we use below. The notions are divided into subsections.

2.1. The Graph

We investigate the max k-cut game, that is played on an undirected unweighted graph $G = (\mathcal{V}, E)$ where the elements of the set of nodes $\mathcal{V} = \{1, \ldots, N\}$, with $N \geq 2$, correspond to the players, and the edges indicates their mutual relations. We are assuming that no self-loops are present, and that, given two nodes, there is at most one edge connecting them. The graph G can be represented by the undirected adjacency matrix $A = \{a_{v,w}\} \in \{0,1\}^{N \times N}$, where $a_{v,w} = 1$ if there is an edge connecting v and w. The adjacency matrix of such a graph is symmetric, $\mathbf{A} = \mathbf{A}^\top$. For $v \in \mathcal{V}$, the *degree* of the vertex v is $\delta_v = \sum\limits_{w \in \mathcal{V}} a_{v,w}$. Additionally, we introduce $\delta_v(S) = \sum\limits_{w \in S} a_{v,w}$, with $S \subset \mathcal{V}$ as the number of connections of v restricted to the set S. Notice that, given $S \subseteq T$, then

$$\delta_v(T \setminus S) = \sum_{w \in T \setminus S} a_{v,w} = \sum_{w \in T} a_{v,w} - \sum_{w \in S} a_{v,w} = \delta_v(T) - \delta_v(S). \tag{1}$$

2.2. The Colorings

Each player chooses its own color in the set $\mathcal{K} = \{1, \ldots, M\}$ of $M \geq 2$ colors as strategy space. For the sake of simplicity, we assume that the color set is the same for each player. Precisely, σ_v will indicate the color of v in the coloring σ. Given a set $S \subseteq \mathcal{V}$, the set of colors in S is $\mathcal{K}(S) \subseteq \mathcal{K}$. A coloring $\sigma \in \mathcal{K}^N$ is an assignment of colors to each node of the graph. Given such σ and a color $b \in \mathcal{K}$, we define the monochromatic set $S \subseteq \mathcal{V}$ of nodes of $\mathcal{V}$ having color b as

$$S_b(\sigma) = \{v \in S : \sigma_v = b\}. \tag{2}$$

2.3. The Payoff

Given a coloring, the payoff of a player is the number of neighbors that have a color different from its own. In symbols, the payoff of node $v \in \mathcal{V}$ is

$$\mu_v(\sigma) = \sum_{\substack{w \in \mathcal{V} \\ \sigma_w \neq \sigma_v}} a_{v,w}. \tag{3}$$

Moreover, $\mu_v(S, \sigma) = \sum\limits_{\substack{w \in S \\ \sigma_w \neq \sigma_v}} a_{v,w}$ is the payoff of node v gained with players in $S \subseteq \mathcal{V}$.

Notice that

$$\mu_v(S, \sigma) = \sum_{\substack{w \in S \\ \sigma_w \neq \sigma_v}} a_{v,w} = \sum_{w \in S} a_{v,w} - \sum_{\substack{w \in S \\ \sigma_w = \sigma_v}} a_{v,w} = \delta_v(S) - \delta_v(S_{\sigma_v}(\sigma)). \tag{4}$$

Considering a partition of $\mathcal{V}$ in m subsets $S_1, \ldots, S_m$ and a coloring σ, we have that

$$\mu_v(\mathcal{V}, \sigma) = \sum_{i=1,\ldots,m} \mu_v(S_i, \sigma). \tag{5}$$

In addition, given $S \subseteq T$ and $D = T \setminus S$, following Equations (1) and (4), we obtain

$$\begin{aligned} \mu_v(D,\sigma) &= \delta_v(D) - \delta_v(D_{\sigma_v}(\sigma)) \\ &= \delta_v(T) - \delta_v(S) - [\delta_v(T_{\sigma_v}(\sigma)) - \delta_v(S_{\sigma_v}(\sigma))] \\ &= \delta_v(T) - \delta_v(T_{\sigma_v}(\sigma)) - [\delta_v(S) - \delta_v(S_{\sigma_v}(\sigma))] \\ &= \mu_v(T,\sigma) - \mu_v(S,\sigma). \end{aligned} \tag{6}$$

Additionally, $\mu(S,\sigma) = \sum_{v \in S} \mu_v(\sigma)$ is the payoff of the set $S \subseteq \mathcal{V}$. As a consequence, $\mu(\mathcal{V},\sigma)$ is the payoff of the whole population $\mathcal{V}$ which is using the coloring σ.

The *payoff variation* of a node v on a set $S \subseteq \mathcal{V}$ will be denoted by $\Delta\mu_v(S,\gamma,\sigma) = \mu_v(S,\gamma) - \mu_v(S,\sigma)$. If $S = \mathcal{V}$, then the payoff variation is indicated by $\Delta\mu_v(\gamma,\sigma) = \mu_v(\gamma) - \mu_v(\sigma)$.

Using Equation (4), the following holds:

$$\begin{aligned} \Delta\mu_v(S,\gamma,\sigma) &= \mu_v(S,\gamma) - \mu_v(S,\sigma) \\ &= \delta_v(S) - \delta_v(S_{\gamma_v}(\gamma)) - (\delta_v(S) - \delta_v(S_{\sigma_v}(\sigma))) \\ &= \delta_v(S_{\sigma_v}(\sigma)) - \delta_v(S_{\gamma_v}(\gamma)). \end{aligned} \tag{7}$$

The cut difference between the coloring σ and another coloring γ is

$$\Delta\mu(S,\gamma,\sigma) = \sum_{v \in S} \Delta\mu_v(S,\gamma,\sigma) = \mu(S,\gamma) - \mu(S,\sigma). \tag{8}$$

It expresses the variation of payoff of players in $S \subseteq \mathcal{V}$ when the coloring changes from σ to γ. In particular, the global variation of payoff is $\Delta\mu(\mathcal{V},\gamma,\sigma)$. Moreover, if $\mathcal{V}$ is partitioned in m subsets $S_1, \ldots S_m$, we have

$$\Delta\mu(\mathcal{V},\gamma,\sigma) = \sum_{i=1,\ldots,m} \Delta\mu(S_i,\gamma,\sigma). \tag{9}$$

A coloring σ is *optimal* if and only if $\mu(\mathcal{V},\sigma)$ is maximum, or equivalently

$$\Delta\mu(\mathcal{V},\gamma,\sigma) \leq 0 \; \forall \gamma \in \mathcal{K}^N.$$

The equal sign holds if and only if γ is also an optimal coloring.

2.4. The Max k-Cut Problem

The max k-cut problem consists of partitioning the vertices of G into M subsets of different colors, denoted by $V_1, \ldots, V_M$, such that the number of nodes having neighbors in different sets is maximized. The max k-cut game constitutes a strategic version of the max k-cut problem.

The next two definitions refer to the notion of coalition, which is a subset of the set of vertices, $\mathcal{C} \subseteq \mathcal{V}$.

Definition 1 (Deviating coalition)**.** *Given two colorings σ and γ and a coalition $\mathcal{C}$, we say that $\mathcal{C}$ deviates from σ to γ if and only if $\sigma_v = \gamma_v \; \forall v \notin \mathcal{C}$ and $\sigma_v \neq \gamma_v \; \forall v \in \mathcal{C}$.*

Definition 2 (Strong deviation)**.** *Given two colorings σ and γ and a coalition $\mathcal{C}$, we say that $\mathcal{C}$ strongly deviates from σ to γ if and only if $\mathcal{C}$ deviates from σ to γ and*

$$\Delta\mu_v(\gamma,\sigma) > 0 \; \forall v \in \mathcal{C}.$$

3. Results on Optimal Colorings

We remark that if σ is an optimal coloring, according to the definition of profit reported in Equation (3), two situations are feasible: $\mu_v(\sigma) = \delta_v$ or $\mu_v(\sigma) < \delta_v$. In the first case, each player w connected to v is such that $\sigma_w \neq \sigma_v$. In the second case, v is connected to at least one player w such that $\sigma_w = \sigma_v$. No node with $\mu_v(\sigma) = \delta_v$ can belong to a strongly deviating coalition.

Starting from this observation, we consider the following partition of $\mathcal{V}$ with respect to the optimal coloring σ:

- $C(\sigma) = \{v \in \mathcal{V} : \mu_v(\sigma) < \delta_v\}$ is the set of the nodes candidate to belong to a strong deviation;
- $B(\sigma) = \{v \in \mathcal{V} : \mu_v(\sigma) = \delta_v \wedge \exists w \in C(\sigma) : a_{v,w} = 1\}$ is the boundary set of $C(\sigma)$, i.e., it contains all nodes not in $C(\sigma)$ which are connected to some node in $C(\sigma)$;
- $E(\sigma) = \{v \in \mathcal{V} : \mu_v(\sigma) = \delta_v \wedge a_{v,w} = 0 \; \forall w \in C(\sigma)\}$ is set of the nodes which are not connected to $C(\sigma)$ (external set).

Clearly, $\mathcal{V} = C(\sigma) \cup B(\sigma) \cup E(\sigma)$.

For example, consider a graph $G = (\mathcal{V}, E)$ as depicted in Figure 1, referred to an optimal coloring σ. Note that vertex v_1 belongs to the set $C(\sigma)$, since its profit, 5, is less than its degree, 6. Similarly we reason for the vertices v_2, v_3 and v_7. Instead, vertex v_5 belongs to set $B(\sigma)$ since its profit is exactly equal to its degree, i.e., 3, and v_5 is the neighbor of at least one vertex in $C(\sigma)$, e.g., v_1. Similarly we reason for the vertices v_4 and v_8. Finally, v_9 belongs to $E(\sigma)$, as its profit is exactly equal to its degree, i.e., 1 and has no neighbors in $C(\sigma)$.

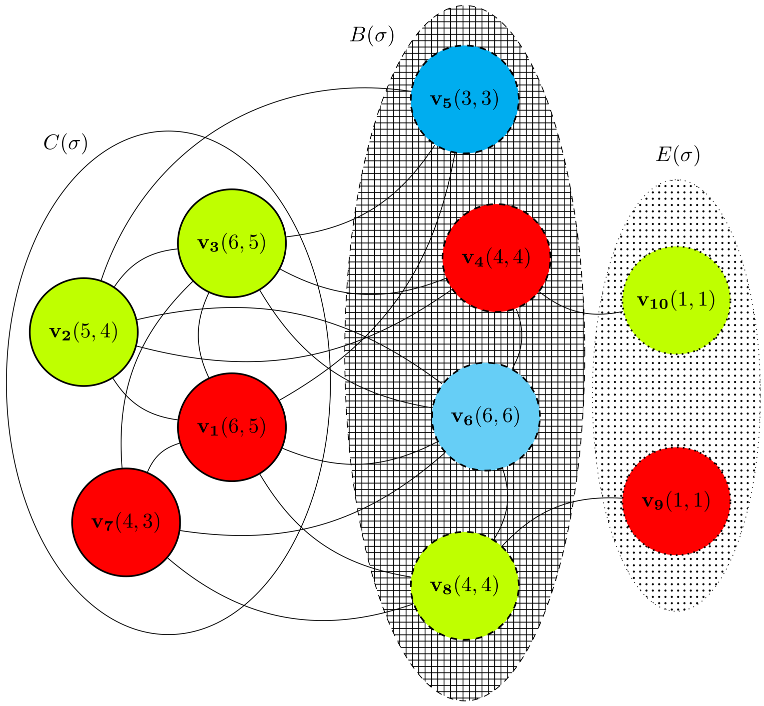

Figure 1. An example of a graph with an optimal coloring partitioned in $C(\sigma) = \{v_1, v_2, v_3, v_7\}$ (bold line nodes), $B(\sigma) = \{v_4, v_5, v_6, v_8\}$ (dashed line nodes) and $E(\sigma) = \{v_9, v_{10}\}$ (dotted line nodes). The numbers in bold are the label, while the ones in the brackets indicate the degrees and the payoffs of nodes, respectively.

Using the definition of the monochromatic set given in Equation (2), the next proposition states that nodes in $C(\sigma)$ are not connected to nodes of $B(\sigma)$, having their own color.

Proposition 1. *Let $\sigma \in \mathcal{K}^N$ be an optimal coloring. Then, for all $v \in C(\sigma)$ the following properties hold:*

$$\delta_v(B_{\sigma_v}(\sigma)) = 0 \tag{10}$$

$$\delta_v(C_{\sigma_v}(\sigma)) \geq 1. \tag{11}$$

Proof. Suppose that for a player $v \in C(\sigma)$, there exists $w \in B_{\sigma_v}(\sigma)$ connected to v. However, this means that $\mu_w(\sigma) < \delta_w$, since it is connected to v, which has the same color. This is in contradiction with the membership of w in the set $B(\sigma)$. Hence,

$$\delta_v(B_{\sigma_v}(\sigma)) = 0 \; \forall v \in C(\sigma).$$

Similarly, from the definition of $C(\sigma)$, inequality (11) follows. □

Below, we report a remark where the case $C(\sigma) = \emptyset$ is considered.

Remark 1. *Concerning the case $C(\sigma) = \emptyset$, we can make the following topological observations.*

- *If a graph is a star, then two colors are enough to have $C(\sigma) = \emptyset$. In fact, it is sufficient to color the central vertex of one color and the remaining nodes of the other.*
- *If a graph is bipartite, then two colors are enough to have $C(\sigma) = \emptyset$. Indeed, the vertices of such a graph can be partitioned into two sets, and it will be sufficient to color the nodes in one of the two sets of one color and the other one with the remaining color.*
- *If a graph is complete, since each node is connected to all the others, the only possibility to have $C(\sigma) = \emptyset$ is that the number of colors is greater than or equal to the number of nodes.*
- *If a graph is such that each node has a degree less than the number of colors, then there exists an optimal coloring such that $C(\sigma) = \emptyset$.*

Since $C(\sigma)$ contains all the players that are incentivized to change their colors in order to increase their payoff, in the rest of this work, we will not consider the cases where $C(\sigma) = \emptyset$, since it is trivially impossible to form strong deviating coalition in such situations.

Theorem 1. *Let $\sigma \in \mathcal{K}^N$ be an optimal coloring. Then*

$$\forall v \in C(\sigma), \forall b \in \mathcal{K} \setminus \{\sigma_v\}, \delta_v(C_b(\sigma)) + \delta_v(B_b(\sigma)) \geq 1.$$

Proof. Suppose that there exists a node v in $C(\sigma)$ such that it has no neighbors in $C(\sigma)$ and in $B(\sigma)$ with color $b \in \mathcal{K} \setminus \{\sigma_v\}$, i.e., $\delta_v(C_b(\sigma)) = 0$ and $\delta_v(B_b(\sigma)) = 0$. Consider a coloring γ, where only the node v in $C(\sigma)$ changes its color to b. Then,

- v in γ increases its payoff of $\delta_v(C_{\sigma_v}(\sigma))$, since now its color is different from the colors of all other connected members of $C(\sigma)$;
- Each node $w \in C_{\sigma_v}(\sigma) \setminus \{v\}$ connected to v increases its payoff of exactly 1, since v changed its color.
- Nodes in $B(\sigma)$ connected to v do not vary their payoff. Indeed, for Equation (10), none of them have color σ_v, i.e., $\delta_v(B_{\sigma_v}(\sigma)) = 0$, nor the color b by hypothesis.

Summarizing, the global payoff variation from σ to γ is

$$\begin{aligned} \Delta\mu(\mathcal{V}, \gamma, \sigma) &= \underbrace{\delta_v(C_{\sigma_v}(\sigma))}_{\substack{\text{Contribution} \\ \text{of } v}} + \underbrace{1 \cdot \delta_v(C_{\sigma_v}(\sigma))}_{\substack{\text{Contribution} \\ \text{of neighbors of} \\ v \text{ in } C(\sigma)}} = \\ &= 2\delta_v(C_{\sigma_v}(\sigma)) > 0, \end{aligned}$$

since from (11), $\delta_v(C_{\sigma_v}(\sigma)) \geq 1$.

However, $\Delta\mu(\mathcal{V}, \gamma, \sigma) > 0$ contradicts the optimality of σ. Then, for each color $b \in \mathcal{K} \setminus \{\sigma_v\}$, each node v in $C(\sigma)$ has at least one link connecting it to a node with color b in $C(\sigma)$ and/or in $B(\sigma)$ in the coloring σ. □

Theorem 2. *Let $\sigma \in \mathcal{K}^N$ be an optimal coloring. Then,*

$$\forall v \in C(\sigma), \forall b \in \mathcal{K} \setminus \{\sigma_v\}, \delta_v(C_{\sigma_v}(\sigma)) \leq \delta_v(C_b(\sigma)) + \delta_v(B_b(\sigma)).$$

Proof. Suppose that there exists a color $b \in \mathcal{K} \setminus \{\sigma_v\}$, and a node $v \in C(\sigma)$, such that $\delta_v(C_{\sigma_v}(\sigma)) > \delta_v(C_b(\sigma)) + \delta_v(B_b(\sigma))$. Consider a new coloring γ where only a node $v \in C(\sigma)$ changes its color to b.

In γ, we observe a change of payoff for the node v, for all nodes $w \in C_{\sigma_v}(\sigma) \setminus \{v\}$ connected to v, and for all nodes in $C_b(\sigma)$ and $B_b(\sigma)$ connected to v. In particular, we have the following:

- v increases its payoff of $\delta_v(C_{\sigma_v}(\sigma))$ and it loses $\delta_v(C_b(\sigma)) + \delta_v(B_b(\sigma))$;
- Each node $w \in C_{\sigma_v}(\sigma) \setminus \{v\}$ connected to v gains exactly 1;
- Each node $w \in C_b(\sigma) \cup B_b(\sigma)$ connected to v loses 1.

Summarizing, the global payoff variation is

$$\begin{aligned} \Delta\mu(\mathcal{V}, \gamma, \sigma) &= \underbrace{\delta_v(C_{\sigma_v}(\sigma)) - \delta_v(C_b(\sigma)) - \delta_v(B_b(\sigma))}_{\substack{\text{Contribution} \\ \text{of } v}} \\ &\quad + \underbrace{1 \cdot \delta_v(C_{\sigma_v}(\sigma))}_{\substack{\text{Contribution} \\ \text{of neighbors of} \\ v \text{ in } C_{\sigma_v}(\sigma)}} \\ &\quad + \underbrace{(-1 \cdot \delta_v(C_b(\sigma)) - 1 \cdot \delta_v(B_b(\sigma)))}_{\substack{\text{Contribution} \\ \text{of nodes in} \\ C_b(\sigma) \cup B_b(\sigma)}} \\ &= 2(\delta_v(C_{\sigma_v}(\sigma)) - \delta_v(C_b(\sigma)) - \delta_v(B_b(\sigma))). \end{aligned}$$

For Theorem 1 it holds that $\delta_v(C_b(\sigma)) + \delta_v(B_b(\sigma)) \geq 1$. Additionally, thanks to Equation (11), $\delta_v(C_{\sigma_v}(\sigma)) \geq 1$. By the hypothesis that $\delta_v(C_{\sigma_v}(\sigma)) > \delta_v(C_b(\sigma)) + \delta_v(B_b(\sigma))$, we can conclude that $\Delta\mu(\mathcal{V}, \gamma, \sigma) > 0$. However, this contradicts the optimality of σ, and hence this concludes the proof. □

4. Results on Strong Deviations

Theorem 2 asserts that in the neighborhood of each node $v \in C(\sigma)$ (i.e., the set of all the vertices connected to v), the number of nodes having the same color σ_v of v is lower than the number of neighbors with colors different from σ_v. As a consequence, any color different from σ_v will give v a worse or equal payoff, thus preventing it from changing unilaterally its own strategy to improve its profit. This is in agreement with the fact that an optimal coloring is also a Nash equilibrium, as stated by the following proposition.

Proposition 2. *Optimal colorings are Nash equilibria.*

Proof. First, we recall that, if σ is a Nash equilibrium, then for all $v \in \mathcal{V}$ and $\gamma \in \mathcal{K}^N$, we have

$$\mu_v(\sigma) \geq \mu_v(\gamma), \tag{12}$$

where γ is such that for all $w \neq v$, $\gamma_w = \sigma_w$ and $\gamma_v \neq \sigma_v$.

Equation (12) is obviously true for any $v \in B(\sigma)$ or $v \in E(\sigma)$, since these nodes have already their maximum payoff in σ. Consider a generic node $v \in C(\sigma)$ with $\gamma_v \neq \sigma_v$ and $\gamma_v = b$, from the definition of payoff variation and Equation (9), we obtain

$$\Delta\mu_v(\gamma, \sigma) \quad = \quad \Delta\mu_v(C(\sigma), \gamma, \sigma) + \Delta\mu_v(B(\sigma), \gamma, \sigma) + \Delta\mu_v(E(\sigma), \gamma, \sigma)$$

Furthermore, from Equation (7) we obtain

- $\Delta\mu_v(C(\sigma),\gamma,\sigma) = \delta_v(C_{\sigma_v}(\sigma)) - \delta_v(C_b(\gamma)) = \delta_v(C_{\sigma_v}(\sigma)) - \delta_v(C_b(\sigma))$, since only v changed color moving from σ to γ;
- $\Delta\mu_v(B(\sigma),\gamma,\sigma) = \delta_v(B_{\sigma_v}(\sigma)) - \delta_v(B_b(\gamma)) = -\delta_v(B_b(\gamma)) = -\delta_v(B_b(\sigma))$, since no node in $B(\sigma)$ connected to v has the same color of v, then $\delta_v(B_{\sigma_v}(\sigma)) = 0$. The last equality holds since no node in $B(\sigma)$ changed color from σ to γ;
- $\Delta\mu_v(E(\sigma),\gamma,\sigma) = 0$ since v has no connection in $E(\sigma)$.

  Thus for Theorem 2 it follows that

$$\Delta\mu_v(\gamma,\sigma) = \delta_v(C_{\sigma_v}(\sigma)) - \delta_v(C_b(\gamma)) - \delta_v(B_b(\sigma)) \leq 0.$$

in accordance with the fact that σ is a Nash equilibrium (12). □

In the following proposition, we evaluate the degree of nodes belonging to the sets $C(\sigma)$, $B(\sigma)$ and $E(\sigma)$.

Proposition 3. *Let $\sigma \in \mathcal{K}^N$ be an optimal coloring. Then, we have*

(i) $\forall v \in C(\sigma), \delta_v \geq |\mathcal{K}|$.
(ii) $\forall v \in B(\sigma), \delta_v \geq |\mathcal{K}(C(\sigma))|$.
(iii) $\forall v \in E(\sigma), \delta_v \geq 0$.

Proof. The proof acts as follows:

(i) Suppose that there is a node $v \in C(\sigma)$ such that $\delta_v < |\mathcal{K}|$. By definition of $C(\sigma)$, there exists a color b different from σ_v and σ_w for all $w \in \mathcal{V}$ connected to v. Let γ be a coloring such that $\gamma_w = \sigma_w$, for all $w \neq v$ and $\gamma_v = b$. In this case, v is able to improve unilaterally its own payoff, contradicting the fact that the optimal colorings are Nash equilibria (see Proposition 2).

(ii) From the definition of the set $C(\sigma)$, it is clear that each vertex v in $B(\sigma)$ must have at least one neighbor in $C(\sigma)$ for each color present in $C(\sigma)$; otherwise a node in $C(\sigma)$ not connected to v could take the color of v and increase its profit.

(iii) A vertex in $E(\sigma)$ could be an isolated point, which means having a degree equal to zero.

□

We now give a technical lemma, which will be useful in the sequel.

Lemma 1. *Let σ be an optimal coloring and a set $F \subseteq C(\sigma)$. Moreover, let γ be a coloring with $\gamma_v \neq \sigma_v\ \forall v \in F$ and $\gamma_v = \sigma_v\ \forall v \notin F$. Then the payoff variation of node v when the coloring changes from σ to γ is*

$$\Delta\mu_v(\gamma,\sigma) = \delta_v(F_{\gamma_v}(\sigma)) - \delta_v(F_{\gamma_v}(\gamma)) + \delta_v(C_{\sigma_v}(\sigma)) - \delta_v(C_{\gamma_v}(\sigma)) - \delta_v(B_{\gamma_v}(\sigma)). \tag{13}$$

Proof. The payoff of a generic player $v \in F$ when coloring σ is used is

$$\mu_v(\sigma) = \mu_v(C(\sigma),\sigma) + \mu_v(B(\sigma),\sigma). \tag{14}$$

According to Equation (4), the payoff collected by v in $C(\sigma)$ over coloring σ is

$$\mu_v(C(\sigma),\sigma) = \delta_v(C(\sigma)) - \delta_v(C\sigma_v(\sigma)). \tag{15}$$

Similarly, the payoff collected by v in B^σ over coloring σ is

$$\mu_v(B(\sigma),\sigma) = \delta_v(B(\sigma)) - \delta_v(B_{\sigma_v}(\sigma)) = \delta_v(B(\sigma)). \tag{16}$$

since $\delta_v(B_{\sigma_v}(\sigma)) = 0$ for Equation (10). Joining Equations (14)–(16), we obtain

$$\mu_v(\sigma) = \delta_v(C(\sigma)) - \delta_v(C_{\sigma_v}(\sigma)) + \delta_v(B(\sigma)). \tag{17}$$

On the other hand, the payoff of a generic player $v \in F$ when coloring γ is used is

$$\mu_v(\gamma) = \mu_v(C(\sigma),\gamma) + \mu_v(B(\sigma),\gamma) = \mu_v(F,\gamma) + \mu_v(C(\sigma) \setminus F,\gamma) + \mu_v(B(\sigma),\gamma), \tag{18}$$

where for the last equality we used Equation (6). Notice that, using Equation (1), we obtain

$$\mu_v(F,\gamma) = \delta_v(F) - \delta_v(F_{\gamma_v}(\gamma)). \tag{19}$$

Thanks to Equation (6), the payoff collected by v in γ with respect to the members of $D = C(\sigma) \setminus F$ is

$$\mu_v(C(\sigma) \setminus F,\gamma) = \mu_v(D,\gamma) = \delta_v(D) - \delta_v(D_{\gamma_v}(\gamma)). \tag{20}$$

Since no player in the set D changes its color, $\delta_v(D_{\gamma_v}(\gamma)) = \delta_v(D_{\gamma_v}(\sigma))$. Additionally, using Equation (1), we obtain

$$\delta_v(D_{\gamma_v}(\sigma)) = \delta_v(C_{\gamma_v}(\sigma)) - \delta_v(F_{\gamma_v}(\sigma)). \tag{21}$$

Using (1) and (21), Equation (20) becomes

$$\begin{aligned} \mu_v(C(\sigma) \setminus F,\gamma) &= \delta_v(D) - \delta_v(C_{\gamma_v}(\sigma)) + \delta_v(F_{\gamma_v}(\sigma)) \\ &= \delta_v(C(\sigma)) - \delta_v(F) - \delta_v(C_{\gamma_v}(\sigma)) + \delta_v(F_{\gamma_v}(\sigma)). \end{aligned} \tag{22}$$

Finally, thanks to Equation (4), the payoff of v in $B(\sigma)$ with the coloring γ is

$$\mu_v(B(\sigma),\gamma) = \delta_v(B(\sigma)) - \delta_v(B_{\gamma_v}(\gamma)). \tag{23}$$

Joining Equations (18), (19), (22) and (23), we obtain

$$\begin{aligned} \mu_v(\gamma) &= \delta_v(F) - \delta_v(F_{\gamma_v}(\gamma)) \\ &\quad +\delta_v(C(\sigma)) - \delta_v(F) - \delta_v(C_{\gamma_v}(\sigma)) + \delta_v(F_{\gamma_v}(\sigma)) \\ &\quad +\delta_v(B(\sigma)) - \delta_v(B_{\gamma_v}(\gamma)). \\ &= \delta_v(F_{\gamma_v}(\sigma)) + \delta_v(C(\sigma)) + \delta_v(B(\sigma)) - \delta_v(F_{\gamma_v}(\gamma)) - \delta_v(C_{\gamma_v}(\sigma)) - \delta_v(B_{\gamma_v}(\gamma)). \end{aligned} \tag{24}$$

The payoff difference is obtained upon Equations (17) and (24):

$$\begin{aligned} \Delta\mu_v(\gamma,\sigma) &= \mu_v(\gamma) - \mu_v(\sigma) \\ &= \delta_v(F_{\gamma_v}(\sigma)) + \delta_v(C(\sigma)) + \delta_v(B(\sigma)) - \delta_v(F_{\gamma_v}(\gamma)) - \delta_v(C_{\gamma_v}(\sigma)) \\ &\quad -\delta_v(B_{\gamma_v}(\gamma)) - [\delta_v(C(\sigma)) - \delta_v(C_{\sigma_v}(\sigma)) + \delta_v(B(\sigma))] \\ &= \delta_v(F_{\gamma_v}(\sigma)) + \delta_v(C_{\sigma_v}(\sigma)) - \delta_v(F_{\gamma_v}(\gamma)) - \delta_v(C_{\gamma_v}(\sigma)) - \delta_v(B_{\gamma_v}(\gamma)). \end{aligned}$$

□

Corollary 1. *Suppose $F = C(\sigma)$, then Equation (13) of Lemma 1 simplifies to*

$$\Delta\mu_v(\gamma,\sigma) = \delta_v(C_{\sigma_v}(\sigma)) - \delta_v(C_{\gamma_v}(\gamma)) - \delta_v(B_{\gamma_v}(\sigma)). \tag{25}$$

Moreover, the global payoff variation on $\mathcal{V}$ is

$$\Delta\mu(\mathcal{V},\gamma,\sigma) = \sum_{v\in C(\sigma)} \delta_v(C_{\sigma_v}(\sigma)) - \sum_{v\in C(\sigma)} \delta_v(C_{\gamma_v}(\gamma)) - 2\sum_{v\in C(\sigma)} \delta_v(B_{\gamma_v}(\sigma)). \tag{26}$$

Proof. Equation (25) follows directly from Equation (13) when $F = C(\sigma)$:

$$\begin{aligned} \Delta\mu_v(\gamma,\sigma) &= \delta_v(C_{\gamma_v}(\sigma)) + \delta_v(C_{\sigma_v}(\sigma)) - \delta_v(C_{\gamma_v}(\gamma)) - \delta_v(C_{\gamma_v}(\sigma)) - \delta_v(B_{\gamma_v}(\sigma)) \\ &= \delta_v(C_{\sigma_v}(\sigma)) - \delta_v(C_{\gamma_v}(\gamma)) - \delta_v(B_{\gamma_v}(\sigma)). \end{aligned}$$

The global payoff variation is given by

$$\begin{aligned}\Delta\mu(\mathcal{V},\gamma,\sigma) &= \sum_{v\in\mathcal{V}}\Delta\mu_v(\gamma,\sigma)\\ &= \sum_{v\in C(\sigma)}\delta_v(C_{\sigma_v}(\sigma)) - \sum_{v\in C(\sigma)}\delta_v(C_{\gamma_v}(\gamma))\\ &\quad -2\sum_{v\in C(\sigma)}\delta_v(B_{\gamma_v}(\sigma)).\end{aligned} \tag{27}$$

□

The next theorem is one of the main results of this paper. Indeed, it proves that when the nodes of $C(\sigma)$ have all the same color, i.e., $|\mathcal{K}(C(\sigma))| = 1$, then strongly deviating coalitions do not exist.

Theorem 3. *Let $\sigma \in \mathcal{K}^N$ be an optimal coloring. Then, any set $F \subseteq C(\sigma)$ such that $\sigma_v = f$, $\forall v \in F$ does not strongly deviate.*

Proof. Let $v \in F$ and let γ be a coloring such that $\gamma_v \neq \sigma_v = f$. From Lemma 1, it follows that the payoff of node v when the coloring changes from σ to γ is

$$\begin{aligned}\Delta\mu_v(\gamma,\sigma) &= \delta_v(F_{\gamma_v}(\sigma)) + \delta_v(C_{\sigma_v}(\sigma)) - \delta_v(F_{\gamma_v}(\gamma))\\ &- \delta_v(C_{\gamma_v}(\sigma)) - \delta_v(B_{\gamma_v}(\sigma)).\end{aligned} \tag{28}$$

According to Theorem 2, we have that

$$\delta_v(C_{\sigma_v}(\sigma)) \le \delta_v(C_{\gamma_v}(\sigma)) + \delta_v(B_{\gamma_v}(\sigma)),$$

and hence from Equation (28), we obtain

$$\Delta\mu_v(\gamma,\sigma) \le \delta_v(F_{\gamma_v}(\sigma)) - \delta_v(F_{\gamma_v}(\gamma)).$$

Since no node in F has color $\gamma_v \neq f$ by hypothesis, we have that $\delta_v(F_{\gamma_v}(\sigma)) = 0$, it follows that the global payoff variation on F is

$$\Delta\mu_v(\gamma,\sigma) = -\delta_v(F_{\gamma_v}(\gamma)) \le 0.$$

For the generality of v, it follows that no member of F can improve its own payoff from σ to γ. Then F does not strongly deviate from σ to γ. □

Remark 2. *Theorem 3 can be proved under the weaker hypothesis on the colors, which can be assumed by the nodes of set $F \in C(\sigma)$ in the deviating coloring γ. Indeed, it is sufficient that any generic node v in F will not assume in γ any color already present in F in the coloring σ. This weakens the hypothesis of monochrome nodes of F.*

It is worth noticing the following.

Corollary 2. *Let $\sigma \in \mathcal{K}^N$ be an optimal coloring. Then, $F = C(\sigma)$ such that $\sigma_v = f$, $\forall v \in C(\sigma)$ does not strongly deviate.*

Now, we are able to prove a theorem which constitutes a first step in proving the conjecture for the multi-color case. In particular, we show that if the sum of the degrees of the nodes $v \in C(\sigma)$ has colors σ_v and γ_v, which is a property fulfilled by permutations of colors within $C(\sigma)$, then $C(\sigma)$ is not able to form any strong deviations.

A mapping on colors in $C(\sigma)$ is defined as $\pi : K(C^\sigma) \to K(C^\sigma)$. Starting from π, we build a new coloring γ such that

1. $\gamma_v = \sigma_v, \forall v \notin C(\sigma)$,
2. $\gamma_v = \pi(\sigma_v), \forall v \in C(\sigma)$,

3. given $v, w \in C(\sigma)$ such that $\sigma_v = \sigma_w$, then $\pi(\sigma_v) = \pi(\sigma_w)$.
   Since π is a bijective function, it holds

$$\pi(x) = \pi(y) \iff x = y. \tag{29}$$

We remark that π is a permutation of colors, but constrained by the fact that nodes of the same color must keep the same color in the image.

Moreover,

$$\begin{aligned}\sum_{v \in C(\sigma)} \delta_v(C_{\sigma_v}(\gamma)) &= \sum_{v \in C(\sigma)} \sum_{\substack{w \in C(\sigma) \\ \gamma_w = \gamma_v}} a_{v,w} = \sum_{v \in C(\sigma)} \sum_{\substack{w \in C(\sigma) \\ \pi(\sigma_w) = \pi(\sigma_v)}} a_{v,w} \\ &= \sum_{v \in C(\sigma)} \sum_{\substack{w \in C(\sigma) \\ \sigma_w = \sigma_v}} a_{v,w} = \sum_{v \in C(\sigma)} \delta_v(C_{\sigma_v}(\sigma)), \end{aligned} \tag{30}$$

where the penultimate equality derives from the application of Equation (29).

Proposition 4. *Let $\sigma \in \mathcal{K}^N$ be an optimal coloring and γ a deviating coloring. If γ satisfies the property*

$$\sum_{v \in C(\sigma)} \delta_v(C_{\gamma_v}(\gamma)) = \sum_{v \in C(\sigma)} \delta_v(C_{\sigma_v}(\sigma)), \tag{31}$$

then $C(\sigma)$ is not a strong deviation.

Proof. From the definition of deviating coloring, we have that $\gamma_v \neq \sigma_v, \forall v \in C(\sigma)$, $\gamma_v = \sigma_v$, $\forall v \notin C(\sigma)$. If $C(\sigma)$ would be a strong deviation then for all of its nodes v

$$\Delta\mu_v(\gamma, \sigma) \geq 1,$$

and this implies that

$$\sum_{v \in C(\sigma)} \Delta\mu_v(\gamma, \sigma) \geq |C(\sigma)|. \tag{32}$$

Using (25) and (32), we obtain

$$\sum_{v \in C(\sigma)} \delta(C_{\sigma_v}(\sigma)) - \sum_{v \in C(\sigma)} \delta(C_\gamma(\gamma)) - \sum_{v \in C(\sigma)} \delta(B_\gamma(\sigma)) \geq |C(\sigma)| \tag{33}$$

For the Hypothesis (31), we have that the previous becomes

$$-\sum_{v \in C(\sigma)} \delta(B_\gamma(\sigma)) \geq |C(\sigma)| \Rightarrow |C(\sigma)| + \sum_{v \in C(\sigma)} \delta_v(B_\gamma(\sigma)) \leq 0,$$

and this is absurd. Therefore, the generic deviating coloring γ does not allow $C(\sigma)$ to form a strong deviation. □

The case $F \subseteq C^\sigma$ can be proved similarly to Theorem 4, starting from Equation (28).

Theorem 4. *Let $\sigma \in \mathcal{K}^N$ be an optimal coloring. If*

$$\sum_{v \in C(\sigma)} \delta_v(C_{\sigma_v}(\sigma)) < 2|C(\sigma)|, \tag{34}$$

then $C(\sigma)$ is not a strong deviation.

Proof. Suppose that $C(\sigma)$ is a strong deviation, thanks to the deviating coloring $\gamma \in \mathcal{K}^N$. In this case

$$\Delta\mu_v(\gamma, \sigma) \geq 1 \; \forall v \in C(\sigma) \Rightarrow \sum_{v \in C(\sigma)} \Delta\mu_v(\gamma, \sigma) \geq |C(\sigma)|. \tag{35}$$

According to (25), we have that

$$\sum_{v\in C(\sigma)} \Delta\mu_v(\gamma,\sigma) = \sum_{v\in C(\sigma)} \delta_v(C_{\sigma_v}(\sigma)) - \sum_{v\in C(\sigma)} \delta_v(C_{\gamma_v}(\gamma)) - \sum_{v\in C(\sigma)} \delta_v(B_{\gamma_v}(\sigma)) \geq |C(\sigma)|,$$

which yields

$$\sum_{v\in C(\sigma)} \delta_v(C_{\gamma_v}(\gamma)) \leq \sum_{v\in C(\sigma)} \delta_v(C_{\sigma_v}(\sigma)) - \sum_{v\in C(\sigma)} \delta_v(B_{\gamma_v}(\sigma)) - |C(\sigma)|. \tag{36}$$

On the other hand, according to Equation (26), the global payoff variation is

$$\Delta\mu(\mathcal{V},\gamma,\sigma) = \sum_{v\in C(\sigma)} \delta_v(C_{\sigma_v}(\sigma)) - \sum_{v\in C(\sigma)} \delta_v(C_{\gamma_v}(\gamma)) - 2\sum_{v\in C(\sigma)} \delta_v(B_{\gamma_v}(\sigma)).$$

For the optimality of σ, we have that $\Delta\mu(\mathcal{V},\gamma,\sigma) \leq 0$, and hence

$$\sum_{v\in C(\sigma)} \delta_v(C_{\sigma_v}(\sigma)) - \sum_{v\in C(\sigma)} \delta_v(C_{\gamma_v}(\gamma)) - 2\sum_{v\in C(\sigma)} \delta_v(B_{\gamma_v}(\sigma)) \leq 0,$$

or alternatively

$$\sum_{v\in C(\sigma)} \delta_v(C_{\gamma_v}(\gamma)) \geq \sum_{v\in C(\sigma)} \delta_v(C_{\sigma_v}(\sigma)) - 2\sum_{v\in C(\sigma)} \delta_v(B_{\gamma_v}(\sigma)). \tag{37}$$

Joining the inequalities (36) and (37), we get that:

$$\begin{aligned}&\sum_{v\in C(\sigma)} \delta_v(C_{\sigma_v}(\sigma)) - 2\sum_{v\in C(\sigma)} \delta_v(B_{\gamma_v}(\sigma))\\ &\leq \sum_{v\in C(\sigma)} \delta_v(C_{\sigma_v}(\sigma)) - \sum_{v\in C(\sigma)} \delta_v(B_{\gamma_v}(\sigma)) - |C(\sigma)|,\end{aligned}$$

then, we have that

$$\sum_{v\in C(\sigma)} \delta_v(B_{\gamma_v}(\sigma)) \geq |C(\sigma)|. \tag{38}$$

Using Hypothesis (34) and Equation (38) and plugging them into the Equation (36), we obtain

$$\begin{aligned}\sum_{v\in C(\sigma)} \delta_v(C_{\gamma_v}(\gamma)) &\leq \sum_{v\in C(\sigma)} \delta_v(C_{\sigma_v}(\sigma)) - \sum_{v\in C(\sigma)} \delta_v(B_{\gamma_v}(\sigma)) - |C(\sigma)|\\ &< 2|C(\sigma)| - |C(\sigma)| - |C(\sigma)| = 0,\end{aligned}$$

which is a contradiction. Hence, $C(\sigma)$ does not strongly deviate with any deviation γ. □

Summarizing, Theorems 3 and 4 state that in general, undirected, unweighted graphs, strong deviations do not exist in the monochromatic case, i.e., when all nodes of the coalition have the same color (see Theorem 3). On the other hand, when the coalition is polychromatic, Theorem 4 states that strong deviations do not exist, provided that the connectivity of the nodes of $C(\sigma)$ is small enough. A general result on the polychromatic case with unbounded connectivity is under development by the authors.

5. Conclusions and Future Developments

In this paper, a formal proof of the well-known conjecture stating that optimal colorings in max k-cut games over unweighted and undirected graphs do not allow the existence of any strongly divergent coalition is proposed.

In particular, we prove that in undirected unweighted graphs, strong deviations do not exist in the monochromatic case. Moreover, when the coalition is polychromatic, we show that strong deviations do not exist, provided that the connectivity of the nodes of the divergent coalition is small enough. A general result on the polychromatic case with unbounded connectivity is under investigation by the authors.

Although the result is grounded in the framework of game theory, we have proved it using an entirely new approach to that adopted in the literature for sub-results of the current one. Specifically, it is obtained by splitting the vertices of the graph into three subsets: the coalition itself, the coalition boundary and the nodes without relationship with the coalition.

In order to refine the present findings, it would be interesting to investigate how the conditions vary for graphs with particular properties, such as regular, planar graphs, and so on. Alternatively, one could try to exploit defective colorings [13]: a (k, m) *defective coloring* (or (k, m)-coloring) for a graph G is a coloring of G with k colors, such that each node has at most m neighbors of the same color as itself. We conjecture that any optimal coloring that is a $(k,1)$-coloring (respectively, $(k,2)$-coloring) for a $(k,1)$-colorable graph that is not k-chromatic (respectively, for a $(k,2)$-colorable graph that is not $(k,1)$-colorable) is a strong equilibrium.

Moreover, since in [14] the max k-cut game was extended to hypergraphs, by examining two possible extensions of the payoff function, it could be interesting to investigate hypergraphs using an approach similar to that proposed in this paper.

Lastly, it would be interesting to analyze how strong an optimal coloring is in games which are similar to the max k-cut game, e.g., the generalized graph k-coloring games [11], and the efficient and strategic graph colorings [15,16].

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