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# From the superfluid to the Mott regime and back: triggering a non-trivial dynamics in an array of coupled condensates

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**Abstract.** We consider a system formed by an array of Bose-Einstein condensates trapped in a harmonic potential with a superimposed periodic optical potential. Starting from the boson field Hamiltonian, appropriate to describe dilute gas of bosonic atoms, we reformulate the system dynamics within the Bose-Hubbard model picture. Then we analyse the effective dynamics of the system when the optical potential depth is suddenly varied according to a procedure applied in many of the recent experiments on superfluid-Mott transition in Bose-Einstein condensates.

Initially the condensates' array generated in a weak optical potential is assumed to be in the superfluid ground-state which is well described in terms of coherent states. At a given time, the optical potential depth is suddenly increased and, after a waiting time, it is quickly decreased so that the initial depth is restored. We compute the system-state evolution and show that the potential jump brings on an excitation of the system, incorporated in the final condensate wave functions, whose effects are analysed in terms of two-site correlation functions and of on-site population oscillations. Also we show how a too long waiting time can destroy completely the coherence of the final state making it unobservable.

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## 1. Introduction

Bose-Einstein condensation, originally observed in a dilute atomic gas trapped in a harmonic potential [1, 2], is today obtained in a variety of experimental configurations. Experimental efforts have allowed to realize setup in which condensates are achieved into one-, two- or three-dimensional optical lattices [3, 4, 5, 6], that is arrays of microscopic potentials induced by ac Stark effect due to interfering laser beams. Such microscopic potentials are often superimposed to the trapping harmonic one and give rise to a fragmentation of the condensate.

In the very recent experiments on Bose-Einstein condensates (BECs) in optical lattices, dynamically active states have been generated either by accelerating or by tilting the optical lattice or by shifting the harmonic potential trap [3, 4, 5, 6]. Nevertheless, in such experimental realizations essentially classical/superfluid regimes have been explored and the corresponding dynamics results to be quite well described by the discrete Gross-Pitaevskii equation (GPE). Phenomena as Bloch oscillation, nonlinear Landau-Zener tunneling, Josephson junction current, can be explained in terms of the band structure entailed by the GPE in the spirit of solid-state physics [7].

The opposite regime, where a low number of bosons per well or a strong optical potential require a quantum description of the dynamics, has been recently explored in some experiments [8, 9]. A strong interplay between quantum and classical regime takes place in such experiments when, for example, the quantum phase transition from the superfluid to the Mott insulator regime is generated, or when the collapse and revival of the bosonic wave functions is observed.

In the present paper we consider an experimentally realistic system constituted by a dilute gas of  $N$  ultracold bosonic atoms, trapped in a harmonic potential and loaded into an one-dimensional optical lattice of  $M$  wells. At the beginning, we derive an effective dynamics by reformulating the second-quantized many-body Hamiltonian, that well describes the dynamics of this system, within a generalised Bose-Hubbard model (BHM) picture. In such a way we know the effective Hamiltonian dynamical parameters as a function of the microscopic system constants as well as of external variables such as the magnetic-optical potential strength. Based on such effective picture, we study the system dynamics when the optical potential depth is quickly varied.

Initially, we consider the situation where the system is in the ground-state involved by a weak optical potential. Clearly, the discrete GPE should be applied in this regime to recognize the ground-state configuration [10, 11, 12]. At time  $t = 0$  the lattice potential depth is suddenly increased, so that the tunneling amplitude between neighbouring wells quickly drops to zero. Consequently, the system enters in the *Mott regime* in which the time evolution requires a quantum description. After a waiting time  $t'$ , the optical potential depth is suddenly decreased to the initial value. From this time on, the discrete GPE gives a satisfactory description of the system time evolution. The initial conditions for the following mean-field evolution driven by the GPE thus stem from the quantum state emerging from the Mott regime. We compute the quantum

state describing the system as a function of time and of the Hamiltonian parameters, and show that the phase shift between the condensates of neighbouring wells exhibits a strong time dependence while the well populations undergo oscillations. The system is thus taken in an excited state [13, 14, 15]. Further, we show both that the final state coherence dramatically depends on the waiting time  $t'$ , and that the site wave-function phase coherence is destroyed when  $t'$  is increased.

## 2. Space-mode approximation.

The Hamiltonian operator for a dilute gas of bosonic atoms in a harmonic trapping potential  $V_H(\mathbf{r}) = \sum_{j=1}^3 m\Omega_j^2 r_j^2/2$  with the additional one-dimensional optical lattice potential  $V_L(\mathbf{r}) = \hbar^2 \omega^2 \sin^2(kr_1)/(4E_r)$ , ( $k$  is the laser mode and  $E_r = \hbar^2 k^2/(2m)$  is the recoil energy) has the following form

$$\hat{H} = \int d^3\mathbf{r} \hat{\psi}^\dagger(\mathbf{r}) \left[ -\frac{\hbar^2}{2m} \nabla^2 + V_H(\mathbf{r}) + V_L(\mathbf{r}) \right] \hat{\psi}(\mathbf{r}) + \frac{4\pi\hbar^2 a_s}{2m} \int d^3\mathbf{r} \hat{\psi}^\dagger(\mathbf{r}) \hat{\psi}^\dagger(\mathbf{r}) \hat{\psi}(\mathbf{r}) \hat{\psi}(\mathbf{r}),$$

where  $\hat{\psi}(\mathbf{r})$  ( $\hat{\psi}^\dagger(\mathbf{r})$ ) is the annihilation (creation) boson-field operator for the atoms in a given internal state,  $a_s$  is the  $s$ -wave scattering length and  $m$  is the atomic mass. The space-mode approximation [16], which allows us to reformulate the system dynamics within the BHM picture, is performed as follows. Let  $V_j$  be the parabolic approximation to  $V = V_L + \sum_{j=2}^3 m\Omega_j^2 r_j^2/2$  in  $\mathbf{r}_j = (j\pi/k, 0, 0)$  the locations of  $V$  local minima. We assume the energies involved in the system dynamics to be small compared to the excitations of the single well ground-state. Thus we can expand the boson field operators in terms of Wannier functions  $\hat{\psi}(\mathbf{r}, t) = \sum_j u_j^*(\mathbf{r}) \hat{a}_j(t)$ . In the last equation  $j$  runs on the optical lattice sites,  $u_j$  is the single-particle ground-state mode of  $V_j$  with energy eigenvalue  $\epsilon(\omega) = \hbar(\omega + \Omega_2 + \Omega_3)/2$ . By substituting the previous expression of  $\hat{\psi}(\mathbf{r}, t)$  in the previous Hamiltonian and keeping the lowest order in the overlap between the single-well modes, we find the Bose-Hubbard Hamiltonian

$$H = \sum_i [U n_i(n_i - 1) + \lambda_i n_i] - \frac{T}{2} \sum_{\langle ij \rangle} (a_i^\dagger a_j + h.c.), \quad (1)$$

where the operators  $n_i = a_i^\dagger a_i$  count the number of bosons at the  $i$ -site of the lattice and the annihilation and creation operators  $a_j$  and  $a_j^\dagger$  satisfy the standard commutation relations  $[a_i, a_j^\dagger] = \delta_{i,j}$ . In Hamiltonian 1 parameters are defined as follows.  $U := a_s \Omega_0 \sqrt{m\hbar\Omega_0}/(2\pi)$  is the strength of the on-site repulsion, in which we have set  $\Omega_0 = \sqrt[3]{\omega\Omega_2\Omega_3}$ . The site external potential is  $\lambda_j := \epsilon(\omega) + j^2 \pi^2 \hbar^2 \Omega_1^2/(4E_r)$ , and  $T := -2 \int d^3\mathbf{r} \bar{u}_j [V - V_{j\pm 1}] u_{j\pm 1}$  is the tunneling amplitude between neighbouring sites. The indices  $i, j \in Z$  label the local minima  $x_j = \pi j/k$  of  $V$  throughout the lattice and  $V_j = m\omega^2(r_1 - x_j)^2/2 + \sum_{\ell=2}^3 m\Omega_\ell^2 r_\ell^2/2$ . The total number of bosons  $N = \sum_j n_j$  is conserved. From now on we shall consider a gas of repulsive atoms, thus  $U > 0$ .

BHM (1) [17, 18, 19, 20] was introduced as model for superconducting films, granular superconductors, short-length superconductors, and arrays of Josephson

junctions [21, 22]. More recently, some authors [23, 24, 25] suggested to describe the dynamics of ultracold dilute gas of bosonic atoms trapped in an optical lattices by means of a BHM. Experimental results have shown that the essential physics of arrays of coupled BECs is captured by BHM [8, 9]. At zero temperature, the ground state of the homogeneous version ( $\lambda_j = \text{const}$ ) of system described by Hamiltonian (1) undergoes a quantum phase transition from the superfluid (SF) phase to the Mott insulator (MI) one [17, 18, 19, 20, 21, 22]. For values of  $T/U$  strong enough the ground-state of (1) is a SF and it is well described by a wave function exhibiting a site independent phase [13, 14, 15]. When the lattice potential depth  $\omega$  is increased and, correspondingly,  $T/U$  is decreased the system ground-state can manifest two behaviours. If the total number of atoms  $N$  is commensurate with the site number  $M$  the system ground-state is a MI with vanishing global compressibility, otherwise it is a SF. In the case of inhomogeneous potentials, as that resulting from the confining trap in system (1), there exist Mott insulating regions for  $T/U$  below a threshold even without commensurate filling [10, 26].

### 3. Superfluid initial state.

We consider a set of initial conditions in which the Hamiltonian parameters entail a superfluid ground-state. These are achieved in the limit where  $T/(NU) \gg 1$ . For  $N$  large and  $T/(NU)$  over a suitable threshold, it is widely accepted that the low-temperature dynamics of (1) can be described by a discrete version of GPE [10]. This semiclassical limit can be accomplished recalling that when the tunneling term dominates the on-site repulsion one (namely  $T/(UN) \gg 1$ ), the Glauber coherent states give rise to effective solutions for the quantum problem entailed by Hamiltonian (1) [19, 20, 25]. Thus, a reasonable solution for the BHM ground-state in the regime of interest, can be obtained within a *coherent state variational picture* based on applying a time-dependent variational principle on coherent-state trial state. By means of this procedure (for details see [27, 19, 20]), the quantum dynamics generated by Hamiltonian (1) can be reformulated in terms of a classical dynamics generated by an effective Hamiltonian  $\mathcal{H}$ . Hence, following this procedure, we assume the system dynamics to be described by the trial state  $|\Psi\rangle = \exp(iS/\hbar)|Z\rangle$ , where  $|Z\rangle := \prod_i |z_i\rangle$  (see [19, 20]) is written in terms of Glauber coherent states as

$$|z_i\rangle := e^{-\frac{1}{2}|z_i|^2} \sum_{n=0}^{\infty} \frac{z_i^n}{n!} (a_i^\dagger)^n |0\rangle \quad (2)$$

(recall that  $a_i|0\rangle = 0$ , and that their defining equation is  $a_i|z_i\rangle = z_i|z_i\rangle$  with  $z_i \in \mathbf{C}$ ). The effective equations of motion are achieved by a variational principle from the effective action  $S = \int dt [i\Sigma_j(\dot{z}_j z_j^* - \dot{z}_j^* z_j)/2 - \mathcal{H}]$ , associated to the classical Hamiltonian  $\mathcal{H}(Z, Z^*) := \langle Z|H|Z\rangle$ , through a variation respect to  $z_j$  and  $z_j^*$ . Hence the time-dependent trial-state parameters  $z_j = \langle \Psi|a_j|\Psi\rangle$  represent the classical canonical variables of the effective Hamiltonian dynamics and satisfy to the Poisson brackets

$\{z_j^*, z_\ell\} = i\delta_{j\ell}/\hbar$ . After some algebra we obtain the classical Hamiltonian

$$\mathcal{H} = \sum_j \left[ U|z_j|^4 + \lambda_j|z_j|^2 - \frac{T}{2} (z_j^* z_{j+1} + \text{c.c.}) \right], \quad (3)$$

where  $j$  runs on the chain sites:  $j \in I_M$  with  $I_M = \{0, \pm 1, \dots, (M-1)/2\}$  or  $I_M = \{\pm 1/2, \pm 3/2, \dots, M/2\}$  when  $M$  is odd or even, respectively. The related equations of motion are the following

$$i\hbar \dot{z}_j = (2U|z_j|^2 + \lambda_j)z_j - \frac{T}{2}(z_{j-1} + z_{j+1}), \quad (4)$$

with  $j \in I_M$ , and with the complex conjugate equations.

The ground state of Hamiltonian (3) is determined by studying the variation of  $\mathcal{H} - \chi \mathcal{N}$ , where the Lagrange multiplier  $\chi$  has been introduced to explicitly incorporate the conserved quantity  $\mathcal{N} = \sum_j |z_j|^2$ . In the limit where the on-site chemical potential  $\lambda_j$  is slowly varying with the site index  $j$ , namely  $|\lambda_j - \lambda_{j+1}|/U \ll 1$ , we have  $z_{j-1} + z_{j+1} \approx 2z_j$ . An approximate solution for the SF configuration is thus given by

$$M'\chi = 2UN + M'\bar{\lambda} - M'T, \quad z_j = \sqrt{\frac{N}{M'} - \frac{(\lambda_j - \bar{\lambda})}{2U}} e^{i\phi}, \quad (5)$$

where  $\bar{\lambda} = \sum_{|j| \in I_{M'}} \lambda_j / M'$  and  $M' = \min(M, q)$  is determined by finding out the maximum integer  $q$  such that  $2UN + q(\bar{\lambda} - \lambda_q) \geq 0$ . This solution represents the discrete version for the Thomas-Fermi approximation [28]. The corresponding energy is

$$E_{gs} = N/\sigma [1 - M'\tau - \sigma(\bar{\lambda} + \sigma/4(\delta\bar{\lambda})^2)], \quad (6)$$

in which we have set  $M'(\delta\bar{\lambda})^2 := \sum_{j \in I_{M'}} (\lambda_j - \bar{\lambda})^2$ ,  $\sigma := M'/(UN)$  and  $\tau := T/(UN)$ . Concerning the three coupled condensates system within the SF regime, a thorough study has been made in Ref. [29].

#### 4. Superfluid to Mott-insulator transition.

At the time  $t = 0$  the optical-lattice potential is suddenly increased. This is achieved by varying the potential intensity according to, for example, a tilted slope:  $\omega(t) = \omega[1 + (w-1)t/\tau_b]$ , where  $\tau_b$  is the time scale for the jump and  $w$  is the amplification factor. Consequently, the tunneling amplitude goes to zero  $T[\omega(t)] \rightarrow 0$  as an exponential. In fact, standing the definition  $T := -2 \int d^3\mathbf{r} \bar{u}_j [V - V_{j\pm 1}] u_{j\pm 1}$ , where  $u_j$  is the harmonic oscillator ground state involved by the quadratic single particle potential  $V_j$ , by direct analytical calculations we get

$$T[\omega(t)] = \frac{\hbar^2 \omega^2(t)}{4E_r} \left[ \frac{\pi^2}{2} - 1 + \frac{2E_r}{\hbar\omega(t)} - e^{-\frac{2E_r}{\hbar\omega(t)}} \right] e^{-\frac{\pi^2 \hbar\omega(t)}{8E_r}}. \quad (7)$$

Also  $U$  and  $\lambda_j$  are modified when changing the optical potential amplitude, but their dependence on  $\omega(t)$  is much less dramatic. In fact we have  $U[\omega(t)] = U[\omega(t)/\omega]^{1/2}$  and  $\lambda_j[\omega(t)] = \lambda_j + \hbar[\omega(t) - \omega]/2$ . While the potential depth is changed,  $0 < t < \tau_b$ , we suppose the state of the system to be not modified.

To apply the sudden approximation, the time scale  $\tau_b$  characterising the potential-depth jump must be fast compared with the tunneling time between neighbouring wells, but slow enough to prevent the condensate excitations in each well, namely  $2\pi/\omega \ll \tau_b \ll \hbar/T$  (where  $T$  is the larger hopping amplitude). In fact, doing so the system persists in the lowest band and the effects due to the hopping term result negligible. Thus, a general initial state  $\sum_{\{\mathbf{n}\}} c_{\mathbf{n}} |\mathbf{n}\rangle$ , will have the straightforward to compute time evolution  $\sum_{\{\mathbf{n}\}} \exp\{-i/\hbar \sum_j [Un_j^2 + \lambda_j n_j]\} c_{\mathbf{n}} |\mathbf{n}\rangle$ . Concerning the sudden approximation, we recall that it is usually used for calculating transition probabilities in the case when the Hamiltonian changes rapidly within a short time interval (presently this is identified by  $t = 0$  and  $t = \tau_b > 0$ ). One simply assumes the reaction of the initial state to the quick Hamiltonian change to be negligible. So one can approximate the transition amplitude by assuming:  $\langle out|U(\tau_b, 0)|in\rangle \approx \langle out|in\rangle$ . Such an argument works only if impulsive forces are absent, which, otherwise, could generate finite states change even if applied for infinitesimally long time. In our system such kind of forces are absent. In fact, in the time interval  $(0, \tau_b)$ , where the Hamiltonian parameters are time dependent and the jump of the optical-potential depth is driven by  $w$ , by means of the coherent state representation of the path integral, we have [30]

$$\langle z|U(\tau_b, 0)|z\rangle = \int \mathcal{D}[z] \exp \left[ \frac{i}{\hbar} \int_0^{\tau_b} dt L(t) \right]. \quad (8)$$

where

$$L(t) = \left\{ \frac{i}{2} \sum_k [z_k^*(t) \dot{z}_k(t) - \dot{z}_k^*(t) z_k(t)] - \mathcal{H}[z(t), z^*(t), t] \right\}$$

is the Lagrangian of the effective path-integral action. Since  $L(t)$  can be shown to have no singular behaviours as  $\tau_b \rightarrow 0$ , namely  $\lim_{\tau_b \rightarrow 0^+} \int_0^{\tau_b} dt L(t) = 0$ , the above formula implies that the dynamical evolution of the initial state is driven by  $\tau_b$ . So the shorter  $\tau_b$  implies the smaller changes of  $U(\tau_b, 0)|z\rangle$ .

In the Mott regime ( $T/U \ll 1$ ), namely after the potential jump ( $t > \tau_b$ , we will assume  $\tau_b = 0$ ), the classical description of dynamics is no longer valid and the Schrödinger equation is necessary to describe the time evolution. The latter is generated by Hamiltonian (1) where  $T = 0$ ,  $U = U(w\omega) =: \tilde{U}$  and  $\lambda_j = \lambda_j(w\omega) =: \tilde{\lambda}_j$  are assumed. As assumed above, the system initial state is described by the coherent state (2) whose the quantum evolution is given by

$$|(t)\rangle = e^{-\frac{i}{\hbar} H t} \prod_{i \in I_M} |z_i\rangle = \prod_{i \in I_M} e^{-\frac{|z_i|^2}{2}} \sum_{n_i=0}^{+\infty} \frac{[z_i \nu_i(t)]^{n_i}}{\sqrt{n_i!}} e^{-in_i^2 u(t)} |n_i\rangle, \quad (9)$$

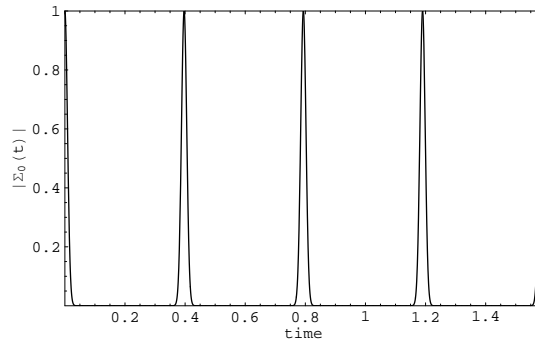
where we have set  $\nu_i(t) := \exp[i/\hbar(\tilde{U} + \tilde{\lambda}_i)t]$  and  $u(t) = \tilde{U}t/\hbar$ . The quantum time evolution does not preserve the coherent state structure of state (2), in fact, the term  $\exp[-in_i^2 u(t)]$  in (9) breaks the coherent state form. By direct calculations, it is easy to show that the quantum evolution in the Mott-regime time interval entails the following expectation values

$$\langle (t) | a_j^\dagger a_j | (t) \rangle = |z_j|^2,$$

$$z_j(t) := \langle(t)|a_j|(t)\rangle = z_j \exp \left\{ \frac{i}{\hbar} \tilde{\lambda}_j t - i|z_j|^2 \sin^2[u(t)] \right\} \exp \left\{ -2|z_j|^2 \sin^2[u(t)] \right\}, \quad (10)$$

$$\begin{aligned} \langle(t)|a_{j+1}^+ a_j|(t)\rangle &= z_{j+1}^* z_j \exp \left\{ -2(|z_j|^2 + |z_{j+1}|^2) \sin^2[u(t)] \right\} \times \\ &\times \exp \left\{ \frac{i}{\hbar} (\tilde{\lambda}_j - \tilde{\lambda}_{j+1}) t - i(|z_j|^2 - |z_{j+1}|^2) \sin^2[u(t)] \right\}. \end{aligned} \quad (11)$$

Such equations display that, during the quantum time evolution, the wells population does not change, whereas the site wave-functions  $z_j(t)$  are dynamically active and driven by more then one characteristic times. The modulus of  $z_j(t)$  is a periodic function of  $t$   $|z_j(t)| = |z_j| \exp\{-2|z_j|^2 \sin^2(\tilde{U}t/\hbar)\}$  with period  $T_m = \pi\hbar/\tilde{U}$ . The phase of the site



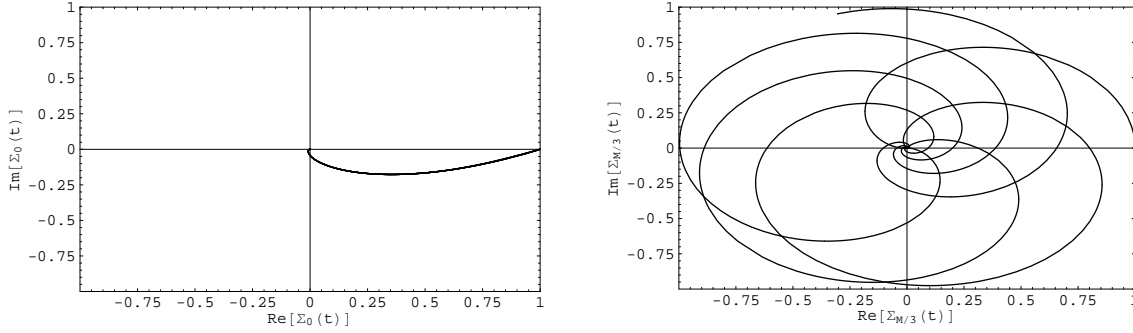
**Figure 1.** This figure shows the time-evolution of the modulus of the central-site wave-function  $\Sigma_0(t) = z_0(t)/z_0$  as given by Eq. (10). It displays a periodic behavior whose period is  $T_m = \pi\hbar/\tilde{U} \approx 0.396$  sec. Notice the “almost impulsive” periodic behavior of  $\Sigma_0(t)$  which is mostly almost vanishing.

wave-functions  $\varphi_j := \arg[z_j(t)/|z_j(t)|] = \tilde{\lambda}_j t/\hbar - |z_j|^2 \sin^2[u(t)]$  is driven by  $T_m$ , the period of  $\sin^2[u(t)]$ , and by the  $T_{\tilde{\lambda}_j} = 2\pi\hbar/\tilde{\lambda}_j$  the site dependent periods involved by the external potential  $\tilde{\lambda}_j$ . Furthermore, in each site  $j$  where  $|z_j|^2$  exceeds the value  $2\pi$  the condition  $|z_j|^2 \sin^2[u(t + T_{z_j})] = |z_j|^2 \sin^2[u(t)] + 2\pi$  implies a further characteristic time  $T_{z_j}$  for  $j \in I_M$ .

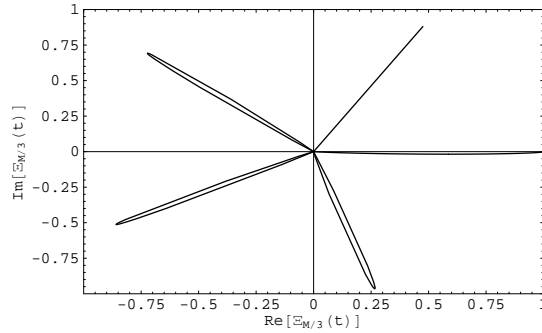
Also, the site-dependent external potentials  $\tilde{\lambda}_j$  induce a dephasing between the  $z_j(t)$  and  $z_{j+1}(t)$  representing the condensate states (namely the site wave function) at sites  $j$  and  $j + 1$ . In fact, it results  $\varphi_j - \varphi_{j+1} = -(\pi\hbar\Omega_1/2)^2(2j+1)t/(\hbar E_r) - (|z_j|^2 - |z_{j+1}|^2) \sin^2[u(t)]$  that shows as the dephasing increases along the lattice. The difference between the phases of the condensates announces that the system is no more in the ground-state [13, 14, 15] and give rise to excited configurations.

Figures 1-3 have been achieved by considering a realistic experimental configuration with  $10^4$   $^{85}\text{Rb}$  atoms, a harmonic trapping potential frequency  $\Omega_j \approx 50$  Hz ( $j=1,2,3$ ), and a laser mode  $k \approx 10^7$  m. The optical potential amplitude, initially of the order  $E_r$ , is suddenly increased to  $30E_r$ .





**Figure 2.** Trajectories of the site wave-functions  $\Sigma_j(t) = z_j(t)/z_j$  in the complex plane (abscissae and ordinates refer to real and imaginary part, respectively) for  $0 < t < 4T_m$ . Left and right figures refer to site  $j = 0$  (the central one) and site  $j = M/3$ , respectively. Notice that, unlike  $\Sigma_0(t)$ ,  $\Sigma_{M/3}(t)$  is not periodic.



**Figure 3.** The evolution of the spatial correlation of adjacent sites is represented by the parametric plot of  $\Xi_j(t) = \langle (t) | a_{j+1}^+ a_j | (t) \rangle / z_{j+1}^* z_j$ , with  $j = M/3$  and  $0 < t < 4T_m$ . A strong dephasing between the sites under concern is evident.

## 5. Mott-insulator to superfluid transition.

Times  $t > t'$  correspond to the third stage of the dynamics. The optical-potential depth is quickly decreased (in a time of order  $\tau_b \approx 0$ ) to the original value  $\omega$  thus restoring the regime with  $T/(NU) \gg 1$ . In this case an approximate description of the system dynamics within the semiclassical variational picture applied in the superfluid regime should be again applicable. The initial conditions for the third stage of the system evolution can be easily shown to be represented by a superposition of Glauber's coherent states. In fact, the initial state is obtained from Eq. (9) by setting  $t = t'$

$$| (t') \rangle = \prod_{j \in I_M} \mathcal{E}_j \sum_{n_j=0}^{+\infty} \frac{[z'_j]^{n_j}}{\sqrt{n_j!}} e^{-in_j^2 u'} |n_j\rangle \quad .$$

Here  $\mathcal{E}_j = \exp\{-|z_j|^2/2\}$ ,  $z'_j = z_j \nu_j(t')$ , and  $u' = u(t')$ . By using the identity

$$\lim_{\epsilon \rightarrow 0^+} \int_{-\infty}^{\infty} dx \exp[-(p + \epsilon)x^2 - inx] = \exp[-n^2/(4p)] \sqrt{\pi/p},$$

with  $p = -i/(4u')$ , this state can be written in a very suggestive form as a superposition of coherent states. Direct calculations give

$$|(t')\rangle = \prod_{j \in I_M} \int_{-\infty}^{\infty} \frac{dx_j}{2\sqrt{\pi u'}} e^{-i\pi/4} e^{ix_j^2/(4u')} |z'_j e^{-ix_j}\rangle = \prod_{j \in I_M} \int_{-\infty}^{\infty} dx_j K(x_j, u') |z'_j e^{-ix_j}\rangle, \quad (12)$$

where the state labeled by  $z''_j = z'_j e^{-ix_j}$  is the normalized coherent state given in Eq. (2) with  $z_j = z''_j$ . If one expresses  $|(t')\rangle$  as  $|(t')\rangle = \prod_{j \in I_M} |(t')_j\rangle$ , the new trial state accounting for the evolution after the second potential-depth change, might be expressed as

$$|(\xi_j)\rangle = \prod_{j \in I_M} D(\xi_j) |(t')_j\rangle$$

with  $D(\xi_j) = \exp(\xi_j a_j^\dagger - a_j \xi_j^*)$ , where the time behaviour of new dynamical parameters  $\xi_j, \xi_j^*$  occurring in the exponential terms (actually these are coherent-state displacement operators) must be reconstructed by implementing once more the time-dependent variational procedure. Due to the properties characterising the displacement-operators action on a coherent state  $|z\rangle$  ( $D(\xi)|z\rangle = \exp[i\text{Im}(z^*\xi)]|z + \xi\rangle$ ) the final form of the trial state  $|(\xi_j)\rangle$  is

$$|\{\xi_j(t)\}\rangle = \prod_{j \in I_M} \int_{-\infty}^{\infty} dx_j K(x_j, u') e^{\phi_j} |\xi_j(t) + z'_j e^{-ix_j}\rangle$$

where the kernel  $K(x_j, u')$  is defined implicitly, and  $\xi_j(t) = 0$  at  $t = 0$  and  $\phi_j = \text{Im}[\xi_j^*(t) z'_j e^{-ix_j}]$ .

## 6. Final discussion.

In the present paper we have considered a performable experimental process exhibiting a strong interplay between classical (SF) and quantum (MI) regimes. We have described the dynamics of an array of BECs when the optical potential depth is quickly varied. The process we have considered forces the system to go through an intermediate quantum regime. As a consequence of this, the system loses its semiclassical character assuming the form of an excited state that cannot be represented as a simple direct product of coherent states. Eqs. (11) show that collapsing/revival phenomena occur whose characteristic time scales have been recognized. Further, the presence of the harmonic external potential appears to be responsible for a strong site dephasing. When the optical potential depth is lowered again, the resulting state has been shown, within the previous section, to be a superposition of coherent states represented by the integrals of eq. (12). In performing the integration on  $x_j$ , at each site, each coherent state  $|z'_j e^{-ix_j}\rangle$  contributes with a phase  $e^{-ix_j}$ . The latter might have a destructive effect for increasing  $u'$  when calculating the expectation value of the physically relevant operators of the model. This can be seen by re-expressing the integrals in (12) in the form

$$\int_{-\infty}^{\infty} \frac{dx}{\sqrt{\pi u'}} e^{ix^2/(4u')} e^{\exp[(-ix)z'a^\dagger]}.$$

Also, since the term  $\exp[ix^2/(4u')]/\sqrt{u'}$  is rapidly oscillating outside the interval  $[-\sqrt{\pi u'}, \sqrt{\pi u'}]$ , the major contributions is expected to come from the integration on this interval. Observing that the corresponding coherent-states phases change in  $-\sqrt{\pi u'} \leq x \leq \sqrt{\pi u'}$ , the simplest way to reduce the decoherence effects may be achieved by imposing  $u' \ll 1$ , that is  $t' \ll \hbar/\tilde{U}$ . In view of their complexity this problem and the evaluation of the time behaviour of the state  $|\{\xi_j(t)\}\rangle$  emerging from the Mott regime will be discussed in a separate paper.

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