Solution of some problems in the arithmetical complexity of first-order fuzzy logics

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Abstract

This short paper addresses the open problems left in the paper [5]. Besides giving solutions to these two problems, some clarification concerning the role of the full vocabulary (including functional symbols) in the proofs there given is also discussed.

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The paper [5] obtains general results providing positions in the arithmetical hierarchy for first-order fuzzy logics. It also formulates two problems that are left without solution:

OPEN PROBLEM 1 (see [5, p. 408]): "Show that for every class $\mathbb{K}$ of chains, the set $\mathrm{SAT}_{\mathrm{pos}}(\mathbb{K})$ is Π_1-hard."

OPEN PROBLEM 2 (see [5, p. 421]): "Is it true that $\mathrm{finTAUT}_{\mathrm{pos}}(\text{Ł}\forall) \subseteq \mathrm{stTAUT}_{\mathrm{pos}}(\text{Ł}\forall)$? This would imply: $\mathrm{stTAUT}_{\mathrm{pos}}(\text{Ł}\forall) = \mathrm{canratTAUT}_{\mathrm{pos}}(\text{Ł}\forall) = \mathrm{finTAUT}_{\mathrm{pos}}(\text{Ł}\forall)$ and $\mathrm{stSAT}_1(\text{Ł}\forall) = \mathrm{canratSAT}_1(\text{Ł}\forall) = \mathrm{finSAT}_1(\text{Ł}\forall)$."

The aim of this short paper is to provide answers for these two open problems. We point out that the answers (including the proofs) given here will be included in the forthcoming [4], which, among other stuff, provides an updated version of the kind of results studied in [5].

Notation and Background. The notation used in this paper corresponds to the one introduced in [5], so we advise the reader to get acquaintance with the terminology there used before reading this paper. It is worth noticing two facts concerning these issues. The first one is that although in that paper the authors consider both the first-order language with and without Δ, since here our concern only involves the previous two open problems we can always assume that we are in the first-order language without Δ. It should also be emphasized that [5] only considered the *full vocabulary*, i.e., the first-order vocabulary which includes a countable number of constant symbols, a countable number of predicate symbols

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and a countable number of functional symbols. In the classical setting it is obvious that there is no distinction, from an expressive power point of view, between considering the full vocabulary or the full predicate one (i.e., the language with a countable number of constant symbols and a countable number of predicate symbols, with no functional symbols); but this is not at all obvious in the fuzzy setting. Thus, the design choice of the full vocabulary in [5] is, for the sake of generality, a drawback.

Structure of the paper. The first open problem is answered positively in Section 1. The proof here given follows the same idea than the one given for proving Σ_1-hardness of positive tautologies (see [5, Theorem 3.15]). The only difference is that the role there played by the algebraic term $2x$ (there defined as $\neg(\neg x \& \neg x)$) is replaced here by the much more common term x^2 (as usual defined by $x \& x$); so in some sense this proof can be considered simpler than the one given for positive tautologies in [5]. In the last part of this section we point out that the proof here given, and also the proof given in [5, Theorem 3.15], rely on the crucial fact that the vocabulary considered is the full one (including functional symbols). Thus, although the statement

> "This theorem, in particular, solves a couple of open problems recently proposed by Hájek in [3]; namely given a set $\mathbb{K}$ of standard BL-chains such that its corresponding logic $L_{\mathbb{K}}\forall$ is recursively axiomatizable show that $\text{genTAUT}_1(L_{\mathbb{K}}\forall)$ and $\text{genTAUT}_{\text{pos}}(L_{\mathbb{K}}\forall)$ are Σ_1-hard." [5, p. 409]

is right,[1] Hájek's problem remains open for the full predicate vocabulary (and also for other vocabularies).

The second open problem is considered in Section 2. This problem is narrower than the other in the sense that it only involves the Łukasiewicz case. In this section the authors notice that the second open problem was indeed answered negatively by P. HÁJEK in [2, Lemma 4].

1 Positive Satisfiability is Π_1-hard

In this section we answer the first open problem positively. For the rest of the section, let us fix $\mathbb{K}$ a non-trivial class of MTL-chains, i.e., $\mathbb{K}$ contains some MTL-chain with at least two elements. Let us remind that the set $\text{SAT}_{\text{pos}}(\mathbb{K})$ of positive satisfiable (first-order) sentences is defined as

$$\text{SAT}_{\text{pos}}(\mathbb{K}) \coloneqq \{\varphi \in \text{Sent}_\Gamma \mid \text{there exist } \boldsymbol{A} \in \mathbb{K} \text{ and an } \boldsymbol{A}\text{-structure } \mathbf{M} \text{ such that } \|\varphi\|_{\mathbf{M}}^{\boldsymbol{A}} > \overline{0}^{\boldsymbol{A}}\}.$$

For the purpose of this section we next introduce two auxiliary sets of sentences:

- $\text{TAUT}_0(\mathbb{K}) \coloneqq \{\varphi \in \text{Sent}_\Gamma \mid \text{for every } \boldsymbol{A} \in \mathbb{K} \text{ and every } \boldsymbol{A}\text{-structure } \mathbf{M}, \|\varphi\|_{\mathbf{M}}^{\boldsymbol{A}} = \overline{0}^{\boldsymbol{A}}\}.$

- $\text{TAUT}_{<1}(\mathbb{K}) \coloneqq \{\varphi \in \text{Sent}_\Gamma \mid \text{for every } \boldsymbol{A} \in \mathbb{K} \text{ and every } \boldsymbol{A}\text{-structure } \mathbf{M}, \|\varphi\|_{\mathbf{M}}^{\boldsymbol{A}} < \overline{1}^{\boldsymbol{A}}\}.$

The following step in our proof is the following lemmata and their consequences. We remind again the reader that this proof is very close to the one given in [5, Theorem 3.15].

Lemma 1.1. *The equation* $x^2 \wedge (\neg x)^2 \approx \overline{0}$ *holds in all* MTL-*algebras.*

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[1]It is worth emphasizing that this quotation refers to the full vocabulary

Proof. It is obvious that MTL-chains satisfy that for every element a, it holds that $a^2 \wedge (\neg a)^2 = (a \wedge \neg a)^2 \leq a * \neg a = 0$. $\qquad\qquad\square$

Corollary 1.2. *Let $\mathbb{K}$ be a class of MTL-chains. Then, for every sentence φ, it holds that $\varphi^2 \wedge (\neg\varphi)^2 \in \mathrm{TAUT}_0(\mathbb{K})$.*

Definition 1.3. *For every formula φ (in the classical setting) we define the formula $\varphi^\star$ (in the fuzzy setting) through the following clauses:*

- *if φ is a literal (i.e., either an atomic formula or the negation of an atomic formula), then $\varphi^\star := \varphi^2$ (i.e., $\varphi \& \varphi$).*

- $(\varphi_1 \wedge \varphi_2)^\star := \varphi_1^\star \wedge \varphi_2^\star$,

- $(\varphi_1 \vee \varphi_2)^\star := \varphi_1^\star \vee \varphi_2^\star$,

- $(\forall x \varphi)^\star := \forall x (\varphi^\star)$,

- $(\exists x \varphi)^\star := \exists x (\varphi^\star)$.

It is obvious from the previous definition that φ and $\varphi^\star$ always have same free variables. In particular, if φ is a sentence, then φ^* is also a sentence.

Lemma 1.4. *Let $\mathbb{K}$ be a non-trivial class of MTL-chains, and let φ be a lattice combination of literals. The following are equivalent:*

(1) φ is a classical propositional contradiction.

(2) $\varphi^\star \in \mathrm{TAUT}_0(\mathbb{K})$.

Proof. First of all we show $(1) \Rightarrow (2)$. By distributivity, φ can be equivalently written as $\bigvee_{i=1}^{n} \bigwedge_{j=1}^{n_i} \alpha_{i,j}$, where $\alpha_{i,j}$ are literals. Thus, φ is a classical contradiction iff for every $i \in \{1, \ldots, n\}$, $\bigwedge_{j=1}^{n_i} \alpha_{i,j}$ is a classical contradiction. Therefore, for every $i \in \{1, \ldots, n\}$ there are $j_1, j_2 \in \{1, \ldots, n_i\}$ such that $\alpha_{i,j_1} = \neg\alpha_{i,j_2}$. Hence, $\alpha_{i,j_1}^2 \wedge \alpha_{i,j_2}^2$ belongs to $\mathrm{TAUT}_0(\mathbb{K})$ by Lemma 1.1. Since this formula is implied by $\bigwedge_{j=1}^{n_i} \alpha_{i,j}^2$, we have that $\varphi^\star$ also belongs to $\mathrm{TAUT}_0(\mathbb{K})$.

$(2) \Rightarrow (1)$ can be easily proved by contraposition. If φ is not a classical propositional contradiction, then there is an evaluation e on $\boldsymbol{B}_2$ such that $e(\varphi) = 1$. Since $\varphi^\star$ and φ are equivalent in classical logic, we also have $e(\varphi^\star) = 1$. Now, given any $\boldsymbol{A} \in \mathbb{K}$, it is clear that e can also be seen as an evaluation on $\boldsymbol{A}$ and $e(\varphi^\star) = \overline{1}$. $\qquad\square$

Lemma 1.5 (Dual Herbrand's Theorem). *A purely universal sentence $\forall x_1 \ldots \forall x_n\ \psi(x_1, \ldots, x_n)$ is a classical contradiction if, and only if, there exists m and closed terms $\{t_1^i, \ldots, t_n^i \mid i = 1, \ldots m\}$ such that $\bigwedge_{i=1}^{m} \psi(t_1^i, \ldots, t_n^i)$ is a classical propositional contradiction.*

Proof. We notice that each one of the following statements is equivalent to the others.

1. $\forall x_1 \ldots \forall x_n\ \psi(x_1, \ldots, x_n)$ is a classical contradiction.

2. $\neg\forall x_1 \ldots \forall x_n\ \psi(x_1, \ldots, x_n)$ is a classical tautology.

3. $\exists x_1 \ldots \exists x_n\ \neg\psi(x_1, \ldots, x_n)$ is a classical tautology.

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4. There are closed terms $\{t_1^i, \ldots, t_n^i \mid i = 1, \ldots m\}$ such that $\bigvee_{i=1}^m \neg\psi(t_1^i, \ldots, t_n^i)$ is a classical propositional tautology.

5. There are closed terms $\{t_1^i, \ldots, t_n^i \mid i = 1, \ldots m\}$ such that $\bigwedge_{i=1}^m \psi(t_1^i, \ldots, t_n^i)$ is a classical propositional contradiction.

The only non trivial step is the one between 3 and 4, and this one is obtained by Herbrand's Theorem. $\qquad\square$

Lemma 1.6. *Let $\mathbb{K}$ be a non-trivial class of MTL-chains, and let φ be $\forall x_1 \ldots \forall x_n \ \psi(x_1, \ldots, x_n)$ where ψ is a lattice combination of literals. The following are equivalent:*

(1) $\varphi \in \mathrm{TAUT}_0(\boldsymbol{B}_2)$.

(2) $\varphi^\star \in \mathrm{TAUT}_0(\mathbb{K})$.

(3) $\varphi^\star \in \mathrm{TAUT}_{<1}(\mathbb{K})$.

Proof. The only non trivial implication is (1) $\Rightarrow$ (2). Suppose that φ is a classical contradiction. By the dual Herbrand's Theorem, there are closed terms t_j^i such that $\bigwedge_{i=1}^m \psi(t_1^i, \ldots, t_n^i)$ is a classical propositional contradiction. By Lemma 1.4, recalling that * commutes with $\wedge$, we have that $\bigwedge_{i=1}^m \psi^\star(t_1^i, \ldots, t_n^i) \in \mathrm{TAUT}_0(\mathbb{K})$. Therefore, $\varphi^\star = \forall x_1 \ldots \forall x_n \ \psi^\star(x_1, \ldots, x_n) \in \mathrm{TAUT}_0(\mathbb{K})$. $\qquad\square$

Lemma 1.7. *The set of classical purely universal first-order contradictions is Σ_1-hard.*

Proof. First observe that the set all contradictions is Σ_1-hard. Indeed, the set of all tautologies is Π_1-hard and we have that for any sentence φ, φ is a contradiction iff $\neg\varphi$ is a tautology. Now given any sentence φ we can write the following chain of equivalencies: φ is a contradiction iff $\neg\varphi$ is a tautology iff its Herbrand form (purely existential) $(\neg\varphi)^H$ is a tautology iff $\neg(\neg\varphi)^H$ is a contradiction. The latter is a purely universal form, so we are done. $\qquad\square$

Theorem 1.8. *Let $\mathbb{K}$ be a non-trivial class of MTL-chains. The set $\mathrm{TAUT}_0(\mathbb{K})$ is Σ_1-hard and thus $\mathrm{SAT}_{\mathrm{pos}}(\mathbb{K})$ is Π_1-hard.*

Proof. It follows from the previous two lemmata and the fact that $\mathrm{SAT}_{\mathrm{pos}}(\mathbb{K})$ is the complementary set of $\mathrm{TAUT}_0(\mathbb{K})$. $\qquad\square$

To finish this section we point out that this proof, and the same for the proof in [5, Theorem 3.15], does not work for the full predicate vocabulary. The reason is that in this vocabulary the set of purely universal contradictions is indeed decidable. This is a particular case of the decidability of the satisfiability problem (in the classical setting) for relational (i.e., without functional symbols) $\exists^*\forall^*$-sentences without equality. This decidability result was proved long ago by Bernays and Schönfinkel (the reader interested on this topic can find more details in [1, Section 6.2.2]).

2 The Second Open Problem

Rutledge proved in [6] that the set of 1-tautologies over the standard MV-chain coincide with the intersection of the sets of 1-tautologies over finite MV-chains.

The fact that concerning 1-satisfiability we can distinguish the standard MV-chain from the finite ones, can be obtained using the sentence

$$\Phi := \quad \exists x(P(x) \leftrightarrow \neg P(x)) \,\&\, \forall x \exists y(P(x) \leftrightarrow (P(y)\&P(y))),$$

which was already considered in [2, Lemma 4]. It is quite simple to check that Φ is 1-satisfiable in some structure over the standard MV-chain, while it cannot be 1-satisfiable in structures over a finite MV-chain. In other words, $\Phi \in \mathrm{stSAT}_1(\text{Ł}\forall)$ while $\Phi \notin \mathrm{finSAT}_1(\text{Ł}\forall)$. An immediate corollary of this fact is that $\neg\Phi \in \mathrm{finTAUT}_{\mathrm{pos}}(\text{Ł}\forall)$, while $\neg\Phi \notin \mathrm{stTAUT}_{\mathrm{pos}}(\text{Ł}\forall)$, which settles negatively the second open problem stated above.

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