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Abstract

This paper is devoted to the problem of existence of saturated models for first-order many-valued logics. We consider a general notion of type as pairs of sets of formulas in one free variable which express properties that an element of a model should, respectively, satisfy and falsify. By means of an elementary chains construction, we prove that each model can be elementarily extended to a κ-saturated model, that is, a model where as many types as possible are realized. In order to prove this theorem we obtain, as by-products, some results on tableaux (understood as pairs of sets of formulas) and their consistency and satisfiability, and a generalization of the Tarski–Vaught theorem on unions of elementary chains. Finally, we provide a structural characterization of κ-saturation in terms of the completion of a diagram representing a certain configuration of models and mappings.

Keywords: mathematical fuzzy logic, first-order graded logics, uninorms, residuated lattices, logic UL, types, saturated models, elementary chains

1 Introduction

Graded logics or *fuzzy logics* are particular kinds of many-valued inference systems and form the subject of mathematical fuzzy logic [13, 26]. Models of first-order *many-valued* logics differ from classical structures by allowing predicates to be evaluated over algebras of truth degrees, beyond the classical two-valued Boolean algebra. In particular, models of first-order *fuzzy* logics are usually evaluated over algebras of *linearly ordered* truth-degrees and are an object of interest in computer science, where

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they are studied as *weighted structures* and used in several areas such as preference modeling [9], argumentation theory [36], models of description logics [6], or valued constraint satisfaction problems [29, 30, 35].

The general study of these models is, to some extent, analogous to classical model theory (see e.g. [28, 31, 34]). Indeed, the study is based on strong completeness theorems [12, 15, 27], which ensure the correspondence between theories and their classes of models, and its literature has so far followed closely the classical agenda: e.g. study of mappings and diagrams [19], ultraproduct constructions [20, 21], characterization of elementary equivalence in terms of elementary mappings [22], Löwenheim–Skolem theorems [23], back-and-forth systems for elementary equivalence [24], preservation theorems and classification of structures [1], Fraïssé limits [4], type omission theorems [3, 17, 33], or relation with continuous model theory [7, 8].

On one hand, this description may give the impression that such research stream might be little more than an exercise in generalization; however, this is very far from the truth. In fact, the transit from the bivalued to the many-valued setting carries a substantial increase in conceptual complexity. Central notions that used to have several equivalent definitions in classical model theory now will split into different well-motivated concepts, as their equivalence does not hold anymore. A paradigmatic example is the very notion of elementary equivalence, which in many-valued models splits in three non-equivalent definitions (see [24]). On the other hand, one may suspect the existence of some translation of classical results into many-valued model theory. There is indeed an interesting formal connection between models of the two kinds (see [12, 23, 24]) that describes many-valued structures as classical two-sorted structures with one sort for the first-order domain and another accounting for truth-values in the algebra. This connection certainly allows to import to the many-valued setting some classical results, although very often they will be uninteresting (even unformulated) from the classical point of view, as we discuss in the concluding remarks of the article.

This paper is devoted to another important item in the classical agenda: saturated models, that is, the construction of structures rich in elements satisfying many expressible properties. In the classical equality-free context the problem was addressed in [18]. In continuous model theory the construction of such models is well known (cf. [5, 8]). However, the problem has not yet received a systematic treatment in mathematical fuzzy logic. It was only formulated in [21], where Dellunde suggested that saturated models of fuzzy logics could be built as an application of the ultraproduct construction. This idea followed the classical tradition found in [11]. However, in other classical standard references such as [28, 31, 34] the construction of saturated structures is obtained by other methods. Based on the initial results of [2], the goal of the present article is twofold:

1. to show that, albeit elusive and hard to pin down in particular many-valued examples, saturated models for first-order fuzzy logics can always be guaranteed to exist as elementary extensions of each given model,

2. to characterize saturated models in terms of the completion of a diagram representing a certain configuration of models and mappings.

The paper is organized as follows: after this introduction, Section 2 presents the necessary preliminaries we need by recalling several semantical notions from mathematical fuzzy logic, namely, the algebraic counterpart of extensions of the uninorm logic UL, first-order fuzzy models based on such algebras, and some basic model-theoretic notions. Section 3 introduces the notion of tableaux (necessary for our treatment of types) as pairs of sets of formulas and proves that each consistent tableau has a model. Section 4 defines types as pairs of sets of formulas in one free variable (roughly speaking, expressing the properties that an element should satisfy and falsify) and contains the main results of the paper: a fuzzy version of the Tarski–Vaught theorem for unions of elementary chains, the existence theorem for κ-saturated models, and their characterization by diagrams of mappings. Finally, Section 5 ends the paper with some concluding remarks.

2 Preliminaries

In this section we introduce the object of our study, fuzzy first-order models, and several necessary related notions for the development of the paper. For comprehensive information on the subject, one may consult the Handbook of Mathematical Fuzzy Logic [13] (e.g. Chapters 1 and 2).

We choose, as the underlying propositional basis for the first-order setting, the class of residuated uninorm-based logics [32]. This class contains most of the well-studied particular systems of fuzzy logic that can be found in the literature and has been recently proposed as a suitable framework for reasoning with graded predicates in [16], while it retains important properties, such as associativity and commutativity of the residuated conjunction, that will be used to obtain the results of this paper.

The algebraic semantics of such logics is based on UL-*algebras*, that is, algebraic structures $\boldsymbol{A}$ in the language $\mathcal{L} = \{\wedge, \vee, \&, \to, \overline{0}, \overline{1}, \bot, \top\}$ such that

- $\langle A, \wedge^{\boldsymbol{A}}, \vee^{\boldsymbol{A}}, \bot^{\boldsymbol{A}}, \top^{\boldsymbol{A}} \rangle$ is a bounded lattice,
- $\langle A, \&^{\boldsymbol{A}}, \overline{1}^{\boldsymbol{A}} \rangle$ is a commutative monoid,
- for each $a, b, c \in A$, we have:

$$a \,\&^{\boldsymbol{A}}\, b \leq c \quad \text{iff} \quad b \leq a \to^{\boldsymbol{A}} c, \tag{res}$$

$$((a \to^{\boldsymbol{A}} b) \wedge \overline{1}^{\boldsymbol{A}}) \vee^{\boldsymbol{A}} ((b \to^{\boldsymbol{A}} a) \wedge^{\boldsymbol{A}} \overline{1}^{\boldsymbol{A}}) = \overline{1}^{\boldsymbol{A}} \tag{lin}$$

$\boldsymbol{A}$ is called a UL-*chain* if its underlying lattice is linearly ordered. *Standard* UL-chains are those defined over the real unit interval $[0, 1]$ with its usual order; in this case the operation $\&^{\boldsymbol{A}}$ is a residuated uninorm, that is, a left-continuous binary associative commutative monotonic operation with a neutral element $\overline{1}^{\boldsymbol{A}}$ (which need not coincide with the element 1 of $[0, 1]$).

Let $Fm_{\mathcal{L}}$ denote the set of propositional formulas written in the language of UL-algebras with a denumerable set of variables and let $\boldsymbol{Fm}_{\mathcal{L}}$ be the absolutely free algebra defined on such set. Given a UL-algebra $\boldsymbol{A}$, we say that an $\boldsymbol{A}$-*evaluation* is a homomorphism from $\boldsymbol{Fm}_{\mathcal{L}}$ to $\boldsymbol{A}$. The logic of all UL-algebras is defined by establishing, for each $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$, $\Gamma \models \varphi$ if and only if, for each UL-algebra $\boldsymbol{A}$ and each $\boldsymbol{A}$-evaluation e, we have $e(\varphi) \geq \overline{1}^{\boldsymbol{A}}$, whenever $e(\psi) \geq \overline{1}^{\boldsymbol{A}}$ for each $\psi \in \Gamma$. The logic UL is, hence, defined as preservation of truth over all UL-algebras, where the notion of truth is given by the set of designated elements, or *filter*, $\mathcal{F}^{\boldsymbol{A}} = \{a \in A \mid a \geq \overline{1}^{\boldsymbol{A}}\}$. The standard completeness theorem of UL proves that the logic is also complete with respect to its intended semantics: the class of UL-chains defined over $[0, 1]$ by residuated uninorms (the standard UL-chains); this justifies the name of UL (uninorm logic).

Most well-known propositional fuzzy logics can be obtained by extending the logic UL with additional axioms and rules, possibly written an expanded language; therefore, from now on we will take $\mathcal{L}$ to be any language containing the operations of UL-algebras and we will refer to any expansion of a UL-chain to such language simply as a *chain*. Important examples of fuzzy logics covered in this framework are MTL, BL, Gödel–Dummett logic G, Łukasiewicz logic Ł, their expansions with the projection connective $\triangle$, their expansions with truth-constants, etc. (see e.g. [12–14, 25]).

A *predicate language* $\mathcal{P}$ is a triple $\langle \mathbf{P}, \mathbf{F}, \mathbf{ar} \rangle$, where $\mathbf{P}$ is a non-empty set of predicate symbols, $\mathbf{F}$ is a set of function symbols, and $\mathbf{ar}$ is a function assigning to each symbol a natural number called the *arity* of the symbol. Let us further fix a denumerable set V whose elements are called *object variables*. The sets of $\mathcal{P}$*-terms*, *atomic* $\mathcal{P}$*-formulas*, and $\langle \mathcal{L}, \mathcal{P} \rangle$*-formulas* are defined as in classical logic. A $\mathcal{P}$*-structure* $\mathfrak{M}$ is a pair $\langle \boldsymbol{A}, \mathbf{M} \rangle$ where $\boldsymbol{A}$ is a chain and $\mathbf{M} = \langle M, \langle P_{\mathbf{M}} \rangle_{P \in \mathbf{P}}, \langle F_{\mathbf{M}} \rangle_{F \in \mathbf{F}} \rangle$, where M is a non-empty domain; $P_{\mathbf{M}}$ is a function $M^n \to A$, for each n-ary predicate symbol $P \in \mathbf{P}$; and $F_{\mathbf{M}}$ is a function $M^n \to M$ for each n-ary function symbol $F \in \mathbf{F}$. An $\mathfrak{M}$*-evaluation* of the object variables is a mapping $\mathrm{v} \colon V \to M$; by $\mathrm{v}[x{\to}a]$ we denote the $\mathfrak{M}$-evaluation where $\mathrm{v}[x{\to}a](x) = a$ and $\mathrm{v}[x{\to}a](y) = \mathrm{v}(y)$ for each object variable $y \neq x$. We define the *values* of the terms and the *truth values* of the formulas as (where $\circ$ stands for any n-ary connective in $\mathcal{L}$):

$$
\begin{aligned}
\|x\|^{\mathfrak{M}}_{\mathrm{v}} &= \mathrm{v}(x), \\
\|F(t_1, \dots, t_n)\|^{\mathfrak{M}}_{\mathrm{v}} &= F_{\mathbf{M}}(\|t_1\|^{\mathfrak{M}}_{\mathrm{v}}, \dots, \|t_n\|^{\mathfrak{M}}_{\mathrm{v}}), \\
\|P(t_1, \dots, t_n)\|^{\mathfrak{M}}_{\mathrm{v}} &= P_{\mathbf{M}}(\|t_1\|^{\mathfrak{M}}_{\mathrm{v}}, \dots, \|t_n\|^{\mathfrak{M}}_{\mathrm{v}}), \\
\|\circ(\varphi_1, \dots, \varphi_n)\|^{\mathfrak{M}}_{\mathrm{v}} &= \circ^{\boldsymbol{A}}(\|\varphi_1\|^{\mathfrak{M}}_{\mathrm{v}}, \dots, \|\varphi_n\|^{\mathfrak{M}}_{\mathrm{v}}), \\
\|(\forall x)\varphi\|^{\mathfrak{M}}_{\mathrm{v}} &= \inf_{\leq_{\boldsymbol{A}}} \{\|\varphi\|^{\mathfrak{M}}_{\mathrm{v}[x \to m]} \mid m \in M\}, \\
\|(\exists x)\varphi\|^{\mathfrak{M}}_{\mathrm{v}} &= \sup_{\leq_{\boldsymbol{A}}} \{\|\varphi\|^{\mathfrak{M}}_{\mathrm{v}[x \to m]} \mid m \in M\}.
\end{aligned}
$$

If the infimum or supremum does not exist, the corresponding value is undefined. We say that $\mathfrak{M}$ is a *safe* if $\|\varphi\|^{\mathfrak{M}}_{\mathrm{v}}$ is defined for each $\mathcal{P}$-formula φ and each $\mathfrak{M}$-evaluation v. Formulas without free variables are called *sentences* and a set of sentences is called a *theory*. Observe that if φ is a sentence, then its value does not depend on a particular $\mathfrak{M}$-evaluation; we denote its value as $\|\varphi\|^{\boldsymbol{A}}_{\mathbf{M}}$. If φ has free variables among $\{x_1, \dots, x_n\}$ we will denote it as $\varphi(x_1, \dots, x_n)$; then the value of the formula under a certain evaluation v depends only on the values given to the free variables; if $v(x_i) = d_i \in M$ we denote $\|\varphi\|^{\mathfrak{M}}_{\mathrm{v}}$ as $\|\varphi(d_1, \dots, d_n)\|^{\boldsymbol{A}}_{\mathbf{M}}$. When $\|\varphi(d_1, \dots, d_n)\|^{\boldsymbol{A}}_{\mathbf{M}} \geq \overline{1}^{\boldsymbol{A}}$, we can say that $d_1, \dots, d_n$ *satisfy* the formula φ in $\mathfrak{M}$, in symbols, $\mathfrak{M} \models \varphi[d_1, \dots, d_n]$. We say that $\mathfrak{M}$ is a *model* of a theory T, in symbols $\mathfrak{M} \models T$, if it is safe and for each $\varphi \in T$, $\|\varphi\|^{\boldsymbol{A}}_{\mathbf{M}} \geq \overline{1}^{\boldsymbol{A}}$. Observe that every safe structure is the model of *some* theory, so we can simply talk about *models* when referring to safe structures.

A structure $\langle \boldsymbol{A}, \mathbf{M} \rangle$ is said to be *exhaustive* if every element of $\boldsymbol{A}$ is the value of some formula for some tuple of objects from M. These models are instrumental in the study of elementary diagrams (see below) and they have proven useful in characterizations of elementary equivalence [22] (Theorem 29). Henceforth, we will assume that all models are exhaustive. For that purpose we need to make sure that our constructions always give us back exhaustive models.

It is worth observing a couple of points. First, we allow arbitrary chains and we do not focus in any kind of standard completeness properties. Second, we do not have, in general, any distinguished (crisp or otherwise) equality symbol in our logical language. This is because we will help ourselves to previous literature where the focus has been on equality-free languages; see e.g. [15, 23, 27].

Using the semantics just defined, the notion of semantical consequence is lifted from the propositional to the first-order level in the obvious way. Such first-order logics satisfy three important properties that we will use in the paper (see e.g. [14]), for each theory $T \cup \{\varphi, \psi, \chi\}$ (inductively defining for each formula α: $\alpha^0 = \overline{1}$, and for each natural n, $\alpha^{n+1} = \alpha^n \,\&\, \alpha$):

1. Local deduction theorem: $T, \varphi \models \psi$ if, and only if, there is a natural number n such that $T \models (\varphi \wedge \overline{1})^n \to \psi$.
2. Proof by cases: If $T, \varphi \models \chi$ and $T, \psi \models \chi$, then $T, \varphi \vee \psi \models \chi$.
3. Finitarity: If $T \models \varphi$, then for some finite $T_0 \subseteq T$, $T_0 \models \varphi$.

Observe that alternatively we could have introduced calculi and a corresponding notion of proof for these logics, but we prefer to keep the focus of the paper on the semantics.

3 Tableaux

In [10] semantical tableaux are described by means of pairs of sets of formulas (writing on the left what needs to be verified, and on the right formulas to be falsified), as a useful syntactical device in the intuitionistic setting where Boolean negation is absent. For the same reason, in our framework we define a *tableau* as a pair $\langle T, U\rangle$ such that T and U are sets of formulas in a predicate language. A tableau $\langle T_0, U_0\rangle$ is called a *subtableau* of $\langle T, U\rangle$ if $T_0 \subseteq T$ and $U_0 \subseteq U$. We say that $\langle T, U\rangle$ is *satisfied* by a model $\mathfrak{M} = \langle \boldsymbol{A}, \mathbf{M}\rangle$ if there is an $\mathfrak{M}$-evaluation v such that for each $\varphi \in T$, $\|\varphi\|^{\mathfrak{M}}_{\mathrm{v}} \geq \overline{1}^{\boldsymbol{A}}$, and for all $\psi \in U$, $\|\psi\|^{\mathfrak{M}}_{\mathrm{v}} < \overline{1}^{\boldsymbol{A}}$. Also, we write $\langle T, U\rangle \models \varphi$ meaning that for any model and evaluation that satisfies $\langle T, U\rangle$, the model and the evaluation must make φ true as well. A tableau $\langle T, U\rangle$ is said to be *consistent* if there is no finite subset $U_0 \subseteq U$ such that $T \models \bigvee U_0$. In the extreme case, we define $\bigvee \emptyset$ as $\perp$.

The next theorem shows that each consistent tableau has a model, which will be necessary in the next section. For its proof, we will use a Henkin canonical model construction and the following related notions from [27]. We say that a set of sentences T is an $\exists$-*Henkin theory* if, whenever $T \models (\exists x)\varphi(x)$, there is a constant c such that $T \models \varphi(c)$. T is a *Henkin theory* if $T \not\models (\forall x)\varphi(x)$ implies that there is a constant c such that $T \not\models \varphi(c)$. We say that T is *doubly Henkin* if it is both $\exists$-Henkin and Henkin. T is a *linear theory* if for any pair of sentences φ, ψ either $T \models \varphi \to \psi$ or $T \models \psi \to \varphi$.

We will prove the next result for countable languages, though the generalization to arbitrary cardinals is straightforward and left to the reader. It is easy to check that the model constructed in the compactness theorem below is exhaustive.

Theorem 1. (Tableaux compactness / Model Existence Theorem) *Let $\langle T, U\rangle$ be a tableau. If for every finite $T_0 \subseteq T$ and $U_0 \subseteq U$, $\langle T_0, U_0\rangle$ is satisfiable, then $\langle T, U\rangle$ is satisfied in some model.*

Proof. First, we observe that $\langle T, U\rangle$ is consistent. Suppose otherwise, that is, there is a finite $U_0 \subseteq U$ such that $T \models \bigvee U_0$. But then for some finite $T_0 \subseteq T$, $T_0 \models \bigvee U_0$. Moreover, this implies that $\langle T_0, \{\bigvee U_0\}\rangle$ cannot be satisfiable, but this is a contradiction with the fact that $\langle T_0, U_0\rangle$ has a model.

We start by adding a countable set C of new constants to the language. We enumerate as $\varphi_0, \varphi_1, \varphi_2, \dots$ all the formulas of the expanded language, and we enumerate as $\langle \theta_0, \psi_0\rangle, \langle \theta_1, \psi_1\rangle, \langle \theta_2, \psi_2\rangle, \dots$ all pairs of such formulas. We modify the proofs of Theorem 4 and Lemma 2 from [27] by building two chains of theories $T_0 \subseteq \cdots \subseteq T_n \subseteq \ldots$ and $U_0 \subseteq \cdots \subseteq U_n \subseteq \ldots$ such that $\langle \bigcup_{i<\omega} T_i, \bigcup_{i<\omega} U_i \rangle$ is a consistent tableau (checking that at every stage we obtain a consistent tableau $\langle T_i, U_i \rangle$), plus $\bigcup_{i<\omega} T_i$ is a linear doubly Henkin theory. Then, we will simply construct the canonical model as in Lemma 3 from [27]. We proceed by induction:

STAGE 0 : Define $T_0 = T$ and $U_0 = U$.

STAGE $s+1 = 3i+1$: At this stage, we make sure that our final theory will be Henkin. To this end, we follow the proof of Lemma 2 (1) from [27]. If φ_i is not of the form $(\forall x)\chi(x)$, then we define $T_{s+1} = T_s$ and $U_{s+1} = U_s$. Assume now that $\varphi_i = (\forall x)\chi(x)$. Then, we consider the following two cases:

(i) There is a finite $U'_s \subseteq U_s$ such that $T_s \models (\bigvee U'_s) \vee (\forall x)\chi(x)$. Then, we define $T_{s+1} = T_s \cup \{(\forall x)\chi(x)\}$ and $U_{s+1} = U_s$.

(ii) Otherwise, let $T_{s+1} = T_s$ and $U_{s+1} = U_s \cup \{\chi(c)\}$ (where c is the first unused constant from C up to this stage).

We have to check that $\langle T_{s+1}, U_{s+1} \rangle$ is consistent in both cases. Suppose that (i) holds and that $T_{s+1} = T_s \cup \{(\forall x)\chi(x)\} \models \bigvee U'_s$ for some finite $U'_s \subseteq U_s$. By construction, we must have that $T_s \models (\bigvee U''_s) \vee (\forall x)\chi(x)$ for some finite $U''_s \subseteq U_s$. Take the finite set $\overline{U_s} = U'_s \cup U''_s$; clearly we also have $T_s \models (\bigvee \overline{U_s}) \vee (\forall x)\chi(x)$. Now, by the local deduction theorem, $T_s \models ((\forall x)\chi(x) \wedge \bar{1})^n \to \bigvee \overline{U_s}$ for some n, so $T_s \cup \{((\forall x)\chi(x) \wedge \bar{1})^n\} \models \bigvee \overline{U_s}$. On the other hand, $T_s \cup \{\bigvee \overline{U_s}\} \models \bigvee \overline{U_s}$. Recall that $(\forall x)\chi(x) \models ((\forall x)\chi(x) \wedge \bar{1})^n$ (this follows from the facts that $\varphi \models \varphi \wedge \bar{1}$ and $\varphi, \psi \models \varphi \,\&\, \psi$). So, by proof by cases, we have that $T_s \cup \{(\bigvee \overline{U_s}) \vee (\forall x)\chi(x)\} \models \bigvee \overline{U_s}$, which means that $T_s \models \bigvee \overline{U_s}$, a contradiction since by induction hypothesis $\langle T_s, U_s \rangle$ is consistent. If (ii) holds, suppose that $\langle T_s, U_s \cup \{\chi(c)\} \rangle$ is not consistent; then, $T_s \models (\bigvee U'_s) \vee \chi(c)$ for some finite $U'_s \subseteq U_s$. Since c is a constant new to T_s, we must have that in any model of T_s, any element e can be made to satisfy $((\bigvee U'_s) \vee \chi(x))$, so $T_s \models (\forall x)((\bigvee U'_s) \vee \chi(x))$, so $T_s \models (\bigvee U'_s) \vee ((\forall x)\chi(x))$, a contradiction.

STAGE $s+1 = 3i+2$: At this stage we make sure that we will eventually obtain an $\exists$-Henkin theory. If φ_i is not of the form $(\exists x)\chi(x)$, then let $T_{s+1} = T_s$ and $U_{s+1} = U_s$. Otherwise, as in Lemma 2 (2) from [27], we have two cases to consider:

(i) There is a finite $U'_s \subseteq U_s$ such that $T_s \cup \{\varphi_i\} \models \bigvee U'_s$, then we define $T_{s+1} = T_s$ and $U_{s+1} = U_s$.

(ii) Otherwise, define $T_{s+1} = T_s \cup \{\chi(c)\}$ (where c is the first unused constant from C) and $U_{s+1} = U_s$.

Again, in both cases $\langle T_{s+1}, U_{s+1} \rangle$ is consistent (check the proof of Lemma 2 (2) from [27]).

STAGE $s+1 = 3i+3$: At this stage we work to ensure that our final theory will be linear. So given the pair $\langle \theta_i, \psi_i \rangle$ proceed as in Lemma 2 (3) from [27]. That is, we start from the assumption that $\langle T_s, U_s \rangle$ is consistent and letting $U_{s+1} = U_s$ we look to add one of $\theta_i \to \psi_i$ or $\psi_i \to \theta_i$ to T_s to obtain T_{s+1} while making the resulting tableau $\langle T_{s+1}, U_{s+1} \rangle$ consistent. Note that if $T_s \cup \{\theta_i \to \psi_i\} \models \bigvee U'_{s+1}$ and $T_s \cup \{\psi_i \to \theta_i\} \models \bigvee U''_{s+1}$, then $T_s \cup \{\theta_i \to \psi_i\} \models (\bigvee U'_{s+1}) \vee (\bigvee U''_{s+1})$ and $T_s \cup \{\psi_i \to \theta_i\} \models (\bigvee U'_{s+1}) \vee (\bigvee U''_{s+1})$. Hence, $T_s \cup \{(\psi_i \to \theta_i) \vee (\theta_i \to \psi_i)\} \models (\bigvee U'_{s+1}) \vee (\bigvee U''_{s+1})$ by proof by cases and, since $\models (\psi_i \to \theta_i) \vee (\theta_i \to \psi_i)$, we obtain that $T_s \models (\bigvee U'_{s+1}) \vee (\bigvee U''_{s+1})$, a contradiction. $\square$

In classical model theory, the standard definition of type is simply given as a set of formulas with n-free variables, in the presence of a theory that provides a context for the described properties of n-tuples of elements that one wants to verify/falsify. Here we follow the same idea but, in order to account for the sentences that need to be falsified, in the absence of a Boolean negation to formalize them, we use a two-sided notion of type given as a tableau.

Definition 1. *A tableau $\langle p, p' \rangle$ in some free variables is a* type *of a tableau $\langle T, U \rangle$ if $\langle T \cup p, U \cup p' \rangle$ is satisfiable. We call $\langle p, p' \rangle$ an n-*type *to signify that $p \cup p'$ has n free variables. Finally, $\langle p, p' \rangle$ is called* complete *if for any formula φ, either $\varphi \in p$ or $\varphi \in p'$.*

4 Saturated models

Let us recall the notions of (elementary) mappings and substructures between fuzzy first-order structures (see e.g. [23]). Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be chains and let $\langle \boldsymbol{A}, \mathbf{M} \rangle$ and $\langle \boldsymbol{B}, \mathbf{N} \rangle$ be $\mathcal{P}$-structures. Let f be a mapping from A to B, and g be a mapping from M to N. The pair $\langle f, g \rangle$ is said to be a *mapping* from $\langle \boldsymbol{A}, \mathbf{M} \rangle$ to $\langle \boldsymbol{B}, \mathbf{N} \rangle$. A mapping $\langle f, g \rangle$ is said to be a *strong homomorphism* if:

1. f is a homomorphism of (expansions of) UL-algebras.
2. $g \colon M \to N$ is a homomorphism between the algebraic reducts of the first-order structures, that is, for every n-ary function symbol $F \in \mathcal{P}$ and $d_1, \ldots, d_n \in M$,
$$g(F_{\mathbf{M}}(d_1, \ldots, d_n)) = F_{\mathbf{N}}(g(d_1), \ldots, g(d_n)).$$
3. For every n-ary predicate symbol $P \in \mathcal{P}$ and $d_1, \ldots, d_n \in M$,
$$f(P_{\mathbf{M}}(d_1, \ldots, d_n)) = P_{\mathbf{N}}(g(d_1), \ldots, g(d_n)).$$

We say that a strong homomorphism $\langle f, g \rangle$ is an *elementary homomorphism* if for every formula $\varphi(x_1, \ldots, x_n)$, and $d_1, \ldots, d_n \in M$,
$$f(\|\varphi(d_1, \ldots, d_n)\|^{\boldsymbol{A}}_{\mathbf{M}}) = \|\varphi(g(d_1), \ldots, g(d_n))\|^{\boldsymbol{A}}_{\mathbf{N}}.$$

A strong homomorphism $\langle f, g \rangle$ is an *embedding* if both mappings f and g are one-to-one.

$\langle \boldsymbol{A}, \mathbf{M} \rangle$ is a *substructure* of $\langle \boldsymbol{B}, \mathbf{N} \rangle$ if the following conditions are satisfied:

1. $M \subseteq N$.
2. For each n-ary function symbol $F \in \mathbf{F}$, and elements $d_1, \ldots, d_n \in M$,
$$F_{\mathbf{M}}(d_1, \ldots, d_n) = F_{\mathbf{N}}(d_1, \ldots, d_n).$$
3. $\boldsymbol{A}$ is a subalgebra of $\boldsymbol{B}$.
4. For every quantifier-free formula $\varphi(x_1, \ldots, x_n)$, and $d_1, \ldots, d_n \in M$,
$$\|\varphi(d_1, \ldots, d_n)\|^{\boldsymbol{A}}_{\mathbf{M}} = \|\varphi(d_1, \ldots, d_n)\|^{\boldsymbol{B}}_{\mathbf{N}}.$$

Moreover, $\langle \boldsymbol{A}, \mathbf{M}\rangle$ is an *elementary substructure* of $\langle \boldsymbol{B}, \mathbf{N}\rangle$ if condition 4 holds for arbitrary formulas. In this case, we also say that $\langle \boldsymbol{B}, \mathbf{N}\rangle$ is an *elementary extension* of $\langle \boldsymbol{A}, \mathbf{M}\rangle$.

A sequence $\{\mathfrak{M}_{\mathrm{i}} \mid i < \gamma\}$ of models where $\mathfrak{M}_{\mathrm{i}} = \langle \boldsymbol{A}_i, \mathbf{M}_i\rangle$ is a called a *chain* when for all $i < j < \gamma$ we have that $\langle \boldsymbol{A}_i, \mathbf{M}_i\rangle$ is a substructure of $\langle \boldsymbol{A}_j, \mathbf{M}_j\rangle$. If, moreover, these substructures are elementary, we speak of an *elementary chain*. The *union* of the chain $\{\mathfrak{M}_{\mathrm{i}} \mid i < \gamma\}$ is the structure $\langle \boldsymbol{A}, \mathbf{M}\rangle$ where $\boldsymbol{A}$ is the classical union of the chain of algebras $\{\boldsymbol{A}_i \mid i < \gamma\}$, while $\mathbf{M}$ is defined by taking as its domain $\bigcup_{i<\gamma} M_i$, interpreting the constants of the language as they were interpreted in each $\mathbf{M}_i$ and similarly with the relational symbols of the language. Let us note that since all the classes of algebras under consideration are classically $\forall_1$-axiomatizable, $\boldsymbol{A}$ will always be an algebra of the appropriate sort. Observe as well that $\mathbf{M}$ is well defined given that $\{\langle \boldsymbol{A}_i, \mathbf{M}_i\rangle \mid i < \gamma\}$ is a chain.

Theorem 2. (Tarski–Vaught theorem on unions of elementary chains) *Let $\boldsymbol{A} = \langle \boldsymbol{A}, \mathbf{M}\rangle$ be the union of an elementary chain $\{\langle \boldsymbol{A}_i, \mathbf{M}_i\rangle \mid i < \gamma\}$. Then, for each sequence $\overline{a}$ of elements of $\mathbf{M}_i$ and each formula $\varphi(\overline{x})$, $\|\varphi(\overline{a})\|^{\boldsymbol{A}}_{\mathbf{M}} = \|\varphi(\overline{a})\|^{\boldsymbol{A}_i}_{\mathbf{M}_i}$. Moreover, the union $\boldsymbol{A} = \langle \boldsymbol{A}, \mathbf{M}\rangle$ is a safe structure.*

Proof. We proceed by induction on the complexity of φ. When φ is atomic, the result follows by definition of $\boldsymbol{A}$. For any n-ary connective $\circ$,

$$\|\circ(\psi_0(\overline{a}), \dots, \psi_n(\overline{a}))\|^{\boldsymbol{A}}_{\mathbf{M}} = \circ^{\boldsymbol{A}}(\|\psi_0(\overline{a})\|^{\boldsymbol{A}}_{\mathbf{M}}, \dots, \|\psi_n(\overline{a})\|^{\boldsymbol{A}}_{\mathbf{M}}) =$$
$$\circ^{\boldsymbol{A}_i}(\|\psi_0(\overline{a})\|^{\boldsymbol{A}_i}_{\mathbf{M}_i}, \dots, \|\psi_n(\overline{a})\|^{\boldsymbol{A}_i}_{\mathbf{M}_i}) = \|\circ(\psi_0(\overline{a}), \dots, \psi_n(\overline{a}))\|^{\boldsymbol{A}_i}_{\mathbf{M}_i},$$

where the second equality follows by the induction hypothesis and the definition of $\boldsymbol{A}$.

Let $\varphi = (\exists x)\psi$ (the case of $\varphi = (\forall x)\psi$ is analogous). Consider $\|\psi(\overline{a}, b)\|^{\boldsymbol{A}}_{\mathbf{M}}$ for $b \in M^n$. Take $j > i$ sufficiently large such that $b \in M^n_j$. By induction hypothesis, $\|\psi(\overline{a}, b)\|^{\boldsymbol{A}_j}_{\mathbf{M}_j} = \|\psi(\overline{a}, b)\|^{\boldsymbol{A}}_{\mathbf{M}}$. By the elementarity of the chain, $\|(\exists x)\psi(\overline{a})\|^{\boldsymbol{A}_i}_{\mathbf{M}_i} = \|(\exists x)\psi(\overline{a})\|^{\boldsymbol{A}_j}_{\mathbf{M}_j}$. Hence, $\|\psi(\overline{a}, b)\|^{\boldsymbol{A}}_{\mathbf{M}} \leq^{\boldsymbol{A}} \|(\exists x)\psi(\overline{a})\|^{\boldsymbol{A}_i}_{\mathbf{M}_i}$. Then $\|(\exists x)\psi(\overline{a})\|^{\boldsymbol{A}_i}_{\mathbf{M}_i}$ is an upper bound for

$$\{\|\psi(\overline{a}, x)\|^{\boldsymbol{A}}_{\mathbf{M}} \mid \overline{b} \in M^n\}$$

in $\boldsymbol{A}$. Moreover, suppose that u is another such upper bound in $\boldsymbol{A}$. This means that we can find $j \geqslant i$ such that $u \in A_j$. Then u is an upper bound in $\boldsymbol{A}_j$ of

$$\{\|\psi(\overline{a}, x)\|^{\boldsymbol{A}_j}_{\mathbf{M}_j} \mid \overline{b} \in M_j{}^n\},$$

which means that

$$\|(\exists x)\psi(\overline{a})\|^{\boldsymbol{A}_i}_{\mathbf{M}_i} = \|(\exists x)\psi(\overline{a})\|^{\boldsymbol{A}_j}_{\mathbf{M}_j} \leq^{\boldsymbol{A}_j} u,$$

so

$$\|(\exists x)\psi(\overline{a})\|^{\boldsymbol{A}_i}_{\mathbf{M}_i} \leq^{\boldsymbol{A}} u.$$

Therefore:

$$\|(\exists x)\psi(\overline{a})\|^{\boldsymbol{A}_i}_{\mathbf{M}_i} = \|(\exists x)\psi(\overline{a})\|^{\boldsymbol{A}}_{\mathbf{M}}.$$

This establishes as well that the union of this chain of models is a safe structure, and hence, a model. □

Corollary 3. *The union of an elementary chain of exhaustive models is itself exhaustive.*

Proof. Suppose that $x \in \boldsymbol{A}$. Then, $x \in \boldsymbol{A}_i$ for some i, so $x = \|\varphi(\overline{a})\|^{\boldsymbol{A}_i}_{\mathbf{M}_i}$ for some sequence $\bar{a}$ of elements of M_i and some formula φ; but then $x = \|\varphi(\overline{a})\|^{\boldsymbol{A}}_{\mathbf{M}}$ by Theorem 2.

□

Given a model $\mathfrak{M} = \langle \boldsymbol{A}, \mathbf{M}\rangle$ and a collection $D \subseteq M$, we denote by $\mathrm{Th}_D(\mathfrak{M})$ the theory of $\mathfrak{M}$ relative to D, that is, the collection of all sentences φ (in the language augmented with constants to denote the elements of D) such that $\|\varphi\|^{\boldsymbol{A}}_{\mathbf{M}} \geq \overline{1}^{\boldsymbol{A}}$. On the other hand, $\overline{\mathrm{Th}}_D(\mathfrak{M})$ will simply denote the set-theoretic complement of $\mathrm{Th}_D(\mathfrak{M})$.

Our next aim is to show an application of tableaux compactness: elementary amalgamation of models. To this end, we need to introduce and recall several notions. We will write $\langle \boldsymbol{B}_1, \mathbf{M}_1, \bar{a}\rangle \Rrightarrow \langle \boldsymbol{B}_2, \mathbf{M}_2, \bar{b}\rangle$ whenever for every formula φ, $\langle \boldsymbol{B}_1, \mathbf{M}_1\rangle \vDash \varphi[\bar{a}]$ only if $\langle \boldsymbol{B}_2, \mathbf{M}_2\rangle \vDash \varphi[\bar{b}]$. Similarly, when $\bar{a}$ and $\bar{b}$ are empty we write $\langle \boldsymbol{B}_1, \mathbf{M}_1\rangle \Rrightarrow \langle \boldsymbol{B}_2, \mathbf{M}_2\rangle$. Given a model $\langle \boldsymbol{A}, \mathbf{M}\rangle$, by the *elementary diagram* of $\langle \boldsymbol{A}, \mathbf{M}\rangle$, in symbols $\mathrm{Eldiag}(\boldsymbol{A}, \mathbf{M})$, we will denote the theory of $\langle \boldsymbol{A}, \mathbf{M}\rangle$ relative to the whole of M. In a nutshell, $\mathrm{Eldiag}(\boldsymbol{A}, \mathbf{M}) = \mathrm{Th}_M(\boldsymbol{A}, \mathbf{M})$. This notion has been studied in detail in [19, 22, 27] and we refer the reader to those papers for further information. On the other hand, $\overline{\mathrm{Eldiag}}(\boldsymbol{A}, \mathbf{M})$ will denote the set-theoretic complement of $\mathrm{Eldiag}(\boldsymbol{A}, \mathbf{M})$. The important fact for our purposes is the following:

Fact 4. *If a* canonical model *(those models obtained by the Model Existence Theorem) realizes $\langle \mathit{Eldiag}(\boldsymbol{A}, \mathbf{M}), \overline{\mathit{Eldiag}}(\boldsymbol{A}, \mathbf{M})\rangle$ and $\langle \boldsymbol{A}, \mathbf{M}\rangle$ is exhaustive, then we can build an embedding from $\langle \boldsymbol{A}, \mathbf{M}\rangle$ into the new canonical model (cf. [22], Corollary 27).*

Theorem 5. (Elementary amalgamation) *Let $\mathfrak{M}_1 = \langle \boldsymbol{B}_1, \mathbf{M}_1\rangle$ and $\mathfrak{M}_2 = \langle \boldsymbol{B}_2, \mathbf{M}_2\rangle$ be two models and let $\mathfrak{M}_0 = \langle \boldsymbol{A}, \mathbf{M}\rangle \subseteq \langle \boldsymbol{B}_2, \mathbf{M}_2\rangle$ be a substructure whose domain is generated by a sequence of elements $\overline{b}$ and such that there is an embedding $\langle i, h\rangle \colon \langle \boldsymbol{A}, \mathbf{M}\rangle \longrightarrow \langle \boldsymbol{B}_1, \mathbf{M}_1\rangle$. Moreover, suppose that $\overline{a}$ is a sequence of elements of M_1 of the same length as $\overline{b}$ such that $\langle \boldsymbol{B}_1, \mathbf{M}_1, \overline{a}\rangle \Rrightarrow \langle \boldsymbol{B}_2, \mathbf{M}_2, \overline{b}\rangle$. Then, there is a model $\mathfrak{M} = \langle \boldsymbol{C}, \mathbf{N}\rangle$ into which $\langle \boldsymbol{B}_1, \mathbf{M}_1\rangle$ is $\mathcal{P}$-elementarily mapped by $\langle f, g\rangle$ while $\langle \boldsymbol{B}_2, \mathbf{M}_2\rangle$ is $\mathcal{P}$-elementarily embedded. Furthermore, we can guarantee that $g(\overline{a}) = \overline{b}$. The situation is described by the following picture:*

$$\begin{tikzcd}
& \mathfrak{M} = \langle \boldsymbol{C}, \mathbf{N}\rangle & \\
\mathfrak{M}_1 = \langle \boldsymbol{B}_1, \mathbf{M}_1, \overline{a}\rangle \arrow[ur, dashed, "\langle f, g\rangle"] & \Rrightarrow & \mathfrak{M}_2 = \langle \boldsymbol{B}_2, \mathbf{M}_2, \overline{b}\rangle \arrow[ul, dashed, "\preccurlyeq"'] \\
& \mathfrak{M}_0 = \langle \boldsymbol{A}, \mathbf{M}\rangle \arrow[ul, "\langle i, h\rangle"] \arrow[ur, "\subseteq"'] &
\end{tikzcd}$$

In particular, the result is true when the sequence of elements $\overline{b}$ is empty.

Proof. We take isomorphic copies, if necessary, to guarantee that $\overline{a} = \overline{b}$ and the structures $\langle \boldsymbol{B}_1, \mathbf{M}_1\rangle$ and $\langle \boldsymbol{B}_2, \mathbf{M}_2\rangle$ have no other elements in common. Furthermore, taking isomorphic copies, we may assume that $\langle \boldsymbol{B}_2, \mathbf{M}_2\rangle$ is just a $\mathcal{P}$-elementary substructure. It is not a difficult to show that

$$\langle \mathrm{Eldiag}(\boldsymbol{B}_1, \mathbf{M}_1) \cup \mathrm{Eldiag}(\boldsymbol{B}_2, \mathbf{M}_2), \overline{\mathrm{Eldiag}}(\boldsymbol{B}_2, \mathbf{M}_2)\rangle$$

(where we let the elements of the domain serve as constants to name themselves) has a model, which suffices for the purposes of the result. Otherwise, by Theorem 1, for some finite

$$\text{Eldiag}_0(\boldsymbol{B}_2, \mathbf{M}_2) \subseteq \text{Eldiag}(\boldsymbol{B}_2, \mathbf{M}_2)$$

and

$$\overline{\text{Eldiag}}_0(\boldsymbol{B}_2, \mathbf{M}_2) \subseteq \overline{\text{Eldiag}}(\boldsymbol{B}_2, \mathbf{M}_2),$$

we have that

$$\text{Eldiag}(\boldsymbol{B}_1, \mathbf{M}_1) \vDash ((\bigwedge \text{Eldiag}_0(\boldsymbol{B}_2, \mathbf{M}_2)) \wedge \overline{1})^k \to \bigvee \overline{\text{Eldiag}}(\boldsymbol{B}_2, \mathbf{M}_2)$$

for some k by the local deduction theorem. Since $\langle \boldsymbol{B}_1, \mathbf{M}_1, \overline{a}\rangle \Rrightarrow \langle \boldsymbol{B}_2, \mathbf{M}_2, \overline{b}\rangle$, given that

$$\langle \boldsymbol{B}_2, \mathbf{M}_2\rangle \vDash ((\bigwedge \text{Eldiag}_0(\boldsymbol{B}_2, \mathbf{M}_2)) \wedge \overline{1})^k \to \bigvee \overline{\text{Eldiag}}(\boldsymbol{B}_2, \mathbf{M}_2)[\overline{b}],$$

we get a contradiction.

Observe that the proof can be similarly carried out, *mutatis mutandis*, when the sequence of elements $\overline{b}$ is empty. $\square$

It is worth noticing that if we have models $\mathfrak{M}, \mathfrak{M}_1$ and $\mathfrak{M}_2$ with mappings between them as described in Theorem 5, then $\mathfrak{M}_1 \Rrightarrow \mathfrak{M}_2$ holds. This, in conjunction with Theorem 5, gives a characterization of the relation $\Rrightarrow$ (in the style of Theorem 29 from [22]). Observe that the relation $\Rrightarrow$ in a Boolean setting with classical negation around would coincide with typical elementary equivalence (the relation $\equiv$, studied in [22] for the fuzzy setting). However, both relations come apart in non-classical frameworks.

The next result is an example of an application of elementary amalgamation. Suppose that our language has a binary predicate R. We will say that an element a of a structure $\langle \boldsymbol{B}_1, \mathbf{M}_1\rangle$ is *R-algebraic over X* (a finite subset of M_1) if, where $\bar{c}$ lists the elements of X, there is a formula φ such that $\langle \boldsymbol{B}_1, \mathbf{M}_1\rangle \vDash \varphi[a, \bar{c}]$ and

$$\langle \boldsymbol{B}_1, \mathbf{M}_1\rangle \vDash (\exists y_0, \dots, y_n)(\forall x)(\varphi(x, \bar{c}) \to Rxy_0 \vee \cdots \vee Rxy_n).$$

Corollary 6. *Let $\langle \boldsymbol{B}_1, \mathbf{M}_1\rangle$ be a model in a language with a binary relation R, let $\overline{a}$ be a sequence listing the elements of a finite set $X \subseteq M_1$, and $b \in M_1$. If b is not algebraic over X, then*

- *(i) There is a model $\langle \boldsymbol{B}_2, \mathbf{M}_2\rangle$ into which $\langle \boldsymbol{B}_1, \mathbf{M}_1\rangle$ is elementarily mapped by a pair of maps $\langle f, g\rangle$ such that for some $c \notin g(M_1)$ and sequence $\overline{d}$ of M_2, $\langle \boldsymbol{B}_1, \mathbf{M}_1, \overline{a}, b\rangle \Rrightarrow \langle \boldsymbol{B}_2, \mathbf{M}_2, \overline{d}, c\rangle$.*
- *(ii) There is a model $\langle \boldsymbol{B}_3, \mathbf{M}_3\rangle$ into which $\langle \boldsymbol{B}_1, \mathbf{M}_1\rangle$ is elementarily mapped by a pair of maps $\langle f, g\rangle$ such that $\langle \boldsymbol{B}_3, \mathbf{M}_3\rangle$ has an elementary substructure $\langle \boldsymbol{B}_2, \mathbf{M}_2\rangle$ containing $g(X)$ such that $g(b) \notin M_2$.*

Proof. (i): Let $\Delta(x)$ be the collection of all formulas with parameters in X satisfied by b. Add a new constant c to the language and consider the theory:

$$T_0 = \text{Eldiag}(\boldsymbol{B}_1, \mathbf{M}_1) \cup \Delta(x).$$

Consider the tableau $\langle T_0, \{Rxd \mid d \in M_1\}\rangle$. We claim that it is consistent. For otherwise,

$$\text{Eldiag}(\boldsymbol{B}_1, \mathbf{M}_1) \cup \Delta(x) \models Rxd_0 \vee \cdots \vee Rxd_n$$

for some n. Using the local deduction theorem, we conclude that there is m such that for some $\wedge$-conjunction φ of formulas from $\Delta(x)$, we have that

$$\text{Eldiag}(\boldsymbol{B}_1, \mathbf{M}_1) \models (\varphi(x) \wedge \overline{1})^m \to Rxd_0 \vee \cdots \vee Rxd_n,$$

and, then

$$\text{Eldiag}(\boldsymbol{B}_1, \mathbf{M}_1) \models (\forall x)((\varphi(x) \wedge \overline{1})^m \to Rxd_0 \vee \cdots \vee Rxd_n).$$

However, $\langle \boldsymbol{B}_1, \mathbf{M}_1\rangle \models \varphi^m[b]$ and given that

$$\langle \boldsymbol{B}_1, \mathbf{M}_1\rangle \models (\forall x)((\varphi(x) \wedge \overline{1})^m \to Rxd_0 \vee \cdots \vee Rxd_n),$$

b is algebraic over X contrary to our assumption.

(ii): By (i) and the elementary amalgamation theorem. $\square$

We are finally ready to define the intended notion of type with respect to a model $\mathfrak{M}$ (observe that it is the particular case of Definition 1 in which the tableau would be $\langle \text{Th}_D(\mathfrak{M}), \overline{\text{Th}}_D(\mathfrak{M})\rangle$).

Definition 2. *Let $\mathfrak{M} = \langle \boldsymbol{A}, \mathbf{M}\rangle$ be a model. If $\langle p, p'\rangle$ is a pair of sets of formulas in some variable x and parameters over some $D \subseteq M$, we will call $\langle p, p'\rangle$ a* type *of $\langle \boldsymbol{A}, \mathbf{M}\rangle$ over D if the tableau $\langle \text{Th}_D(\mathfrak{M}) \cup p, \overline{\text{Th}}_D(\mathfrak{M}) \cup p'\rangle$ is satisfiable (consistent). We will denote the set of all such types by $S^{\langle \boldsymbol{A}, \mathbf{M}\rangle}(D)$.*

The following definition captures the notion of a model realizing as many types as possible (under a certain cardinal restriction).

Definition 3. *For any cardinal κ, a model $\mathfrak{M} = \langle \boldsymbol{A}, \mathbf{M}\rangle$ is said to be κ-*saturated *if for any $D \subseteq M$ such that $|D| < \kappa$, any type in $S^{\mathfrak{M}}(D)$ is satisfiable in $\mathfrak{M}$. In particular, $\mathfrak{M}$ is said to be* saturated *if it is $|M|$-saturated.*

We can observe that in the above definition it suffices to consider types in one free variable. Indeed, the more general case of finitely many variables, say, $x_0, \ldots, x_n$ can be reduced to the one variable case by a standard argument. Indeed, suppose that the tableau $\langle \text{Th}_D(\mathfrak{M}) \cup p, \overline{\text{Th}}_D(\mathfrak{M}) \cup p'\rangle$ is satisfiable in some model $\langle \mathbf{B}, \mathbf{N}\rangle$ (obtained by the model existence theorem) by a sequence $e_0, \ldots, e_n \in N$. Thus, the type of e_0 with parameters over D is realized in $\mathfrak{M} = \langle \boldsymbol{A}, \mathbf{M}\rangle$ by an element e_0'. But then we can also realize in $\langle \boldsymbol{A}, \mathbf{M}\rangle$ the type $\langle T, U\rangle$ where

$$T = \{\varphi(x, e_0') \mid \langle \mathbf{B}, \mathbf{N}\rangle \models \varphi(e_1, e_0)\}$$

$$U = \{\psi(x, e_0') \mid \langle \mathbf{B}, \mathbf{N}\rangle \not\models \psi(e_1, e_0)\}$$

since it is satisfied in $\langle \mathbf{B}, \mathbf{N}\rangle$ by interpreting e_0' as e_0. We keep going this way until we finally realize the type of an element e_n' with parameters in $D \cup \{e_0', \ldots, e_{n-1}'\}$.

Example 7. *Let $n \in \mathbb{N}$. We will build an example of a saturated model with domain of cardinality 2^n. Consider a signature with unary predicates $\{P_i \mid i < n\} \cup \{U\}$. We consider a model $\mathfrak{M} = \langle \boldsymbol{A}, \mathbf{M}\rangle$ constructed as follows. We take an appropriate finite chain $\boldsymbol{A}$, namely, the Łukasiewicz three-element chain over the set $\{0, \frac{1}{2}, 1\}$ with the operations defined for each $a, b \in \{0, \frac{1}{2}, 1\}$ as:*

1. $a \wedge^{\boldsymbol{A}} b = \min\{a, b\}$

2. $a \vee^{\boldsymbol{A}} b = \max\{a, b\}$

3. $a \,\&^{\boldsymbol{A}} b = \max\{a + b - 1, 0\}$

4. $a \rightarrow^{\boldsymbol{A}} b = \min\{1 - a + b, 1\}$

5. $\overline{0}^{\boldsymbol{A}} = \bot^{\boldsymbol{A}} = 0$

6. $\overline{1}^{\boldsymbol{A}} = \top^{\boldsymbol{A}} = 1$

Next, let $\mathbf{M}$ *have as domain* $\wp(n)$ *(where we understand* n *as its von Neumann counterpart* $\{0, \ldots, n-1\}$*). We define the interpretation of* U *to take value* $\frac{1}{2}$ *for every element of the domain of* $\mathbf{M}$ *(this is just to guarantee exhaustiveness in an effortless way). The remaining unary predicates are defined as follows: for* $d \in \wp(n)$*,* $\|P_i[d]\|^{\mathfrak{M}} = 1$ *iff* $i \in d$*, and* $\|P_i[d]\|^{\mathfrak{M}} = 0$ *iff* $i \notin d$*. But, then, for any formula* $\varphi(x, \overline{z})$ *of the language,*

$$\mathfrak{M} \models (\forall x)(\forall y)((\bigwedge_{i<n}(P_i x \leftrightarrow P_i y)) \rightarrow (\varphi(x, \overline{z}) \leftrightarrow \varphi(y, \overline{z}))).$$

This is because $\left\|\bigwedge_{i<n}(P_i[d] \leftrightarrow P_i[e])\right\|^{\mathfrak{M}} = 1$ *iff* $\|P_i[d]\|^{\mathfrak{M}} = \|P_i[e]\|^{\mathfrak{M}}$ *(*$i < n$*) iff* $d = e$*. Moreover, since we are working with crisp predicates, if* $\|P_i[d]\|^{\mathfrak{M}} \neq \|P_i[e]\|^{\mathfrak{M}}$ *for some* i*,* $\|(P_i[d] \leftrightarrow P_i[e])\|^{\mathfrak{M}} = 0$*, and hence,* $\left\|\bigwedge_{i<n}(P_i[d] \leftrightarrow P_i[e])\right\|^{\mathfrak{M}} = \overline{0}^{\boldsymbol{A}}$*. Hence, in either case*

$$\left\|\bigwedge_{i<n}(P_i[d] \leftrightarrow P_i[e])\right\|^{\mathfrak{M}} \leq \|\varphi(d, \overline{c}) \leftrightarrow \varphi(e, \overline{c})\|^{\mathfrak{M}}.$$

Let $D \subsetneq \wp(n)$*. Take any model* $\mathfrak{M}' = \langle \boldsymbol{A}', \mathbf{M}' \rangle$ *where* $\langle \mathrm{Th}_D(\mathfrak{M}), \overline{\mathrm{Th}}_D(\mathfrak{M}) \rangle$ *is satisfied and* b *is an element that satisfies a type* $\langle p, p' \rangle$ *over* D*. By (iii) in the proof of Theorem 10 below, we can assume, without loss of generality, that* $\mathfrak{M}'$ *is an elementary extension. Observe that all the* P_i*s are also crisp in the new model. Suppose that* $\|P_j[b]\|^{\mathfrak{M}'} = \overline{1}^{\boldsymbol{A}'}$ *for exactly* $j \in X \subseteq \{0, \ldots, n-1\}$*. Then the same holds for* $b' = X$ *in* $\mathfrak{M}$ *by definition of* $\mathfrak{M}$*. Hence,*

$$\mathfrak{M}' \models \bigwedge_{i<n}(P_i[b] \leftrightarrow P_i[b']),$$

which implies that $\langle p, p' \rangle$ *is actually satisfied by* b' *in* $\mathfrak{M}$*. Therefore, we have seen that* $\mathfrak{M}$ *is* 2^n*-saturated. Intuitively, the point of this model is that whichever configuration of* P_i*s an element satisfies will determine its type, and in* $\mathfrak{M}$ *we made sure that every configuration was covered by some element.*

Saturated models can be found relatively easily in classical first-order logic (with equality). Simple examples are finite models (see [28], p. 484) or $\langle \mathbb{Q}, < \rangle$, the ordering of the rationals. Typically, for finite models this follows simply because there is a theory pinning down the isomorphism type of the model, so there is only one elementary extension of a finite model, namely, itself. The case of $\langle \mathbb{Q}, < \rangle$ follows by the Cantor back-and-forth argument because any countable ω-categorical structure is saturated (Exercise 7.2.11 from [28]).

The situation in the present non-classical setting is a bit trickier. Observe that since we do not have a formula characterizing cardinality (as it can be shown by running an argument using the Upwards Löwenheim–Skolem Theorem from [23]), we cannot just write a theory pinning down the isomorphism type of a finite model as we would do in the classical case. However, something else is true. For a type $\langle p, p' \rangle$ of a model $\mathfrak{M} = \langle \boldsymbol{A}, \mathbf{M} \rangle$ with domain of finite cardinality n, if every finite subtableau $\langle p_0, p_0' \rangle$ of $\langle p, p' \rangle$ is realized in $\mathfrak{M} = \langle \boldsymbol{A}, \mathbf{M} \rangle$, then $\langle p, p' \rangle$ is too. Otherwise, for each element $a_i \in M$ $(1 \leq i \leq n)$, there is a formula φ_i such that either (1) $\varphi_i \in p$ and $\mathfrak{M} \not\models \varphi_i[a_i]$ or (2) $\varphi_i \in p'$ and $\mathfrak{M} \models \varphi_i[a_i]$. Take the sets $X = \{\varphi_i \mid \text{(1) holds}\}$ and $Y = \{\varphi_i \mid \text{(2) holds}\}$. Then the finite subtableau $\langle X, Y \rangle$ is not realizable in $\mathfrak{M}$, which is a contradiction. Using this fact, we can see that finite models are always saturated.

Proposition 8. *For a type $\langle p, p' \rangle$ of a model $\mathfrak{M} = \langle \boldsymbol{A}, \mathbf{M} \rangle$ with finite domain, every finite subtableau $\langle p_0, p_0' \rangle$ of $\langle p, p' \rangle$ is realized in $\mathfrak{M} = \langle \boldsymbol{A}, \mathbf{M} \rangle$.*

Proof. [1] Since $\langle p, p' \rangle$ is a type of model $\mathfrak{M}$, we have an elementary extension $\mathfrak{M}^*$ of $\mathfrak{M}$ where $\langle p, p' \rangle$ is realized by an element b. Trivially, it follows then that for any finite subtableau $\langle p_0, p_0' \rangle$ of $\langle p, p' \rangle$,

$$\mathfrak{M}^* \models \bigwedge p_0[b] \text{ and } \mathfrak{M}^* \not\models \bigvee p_0'[b].$$

Assume now for reductio that for some such finite $\langle p_0, p_0' \rangle$, $\mathfrak{M}$ does not realize $\langle p_0, p_0' \rangle$. So for each $a \in M$

$$\mathfrak{M} \not\models \bigwedge p_0[a] \text{ or } \mathfrak{M} \models \bigvee p_0'[a],$$

which implies that either $\bigvee p_0'(a) \in \mathit{Th}_M(\mathfrak{M})$ or $\bigwedge p_0(a) \in \overline{\mathit{Th}}_M(\mathfrak{M})$. But then the tableau

$$\langle \mathit{Th}_M(\mathfrak{M}) \cup \{\bigwedge p_0(a)\}, \overline{\mathit{Th}}_M(\mathfrak{M}) \cup \{\bigvee p_0'(a)\} \rangle$$

is not satisfiable, and, by Theorem 1, we must have that for some finite $\Phi_0^a \subseteq \overline{\mathit{Th}}_M(\mathfrak{M})$,

$$\mathit{Th}_M(\mathfrak{M}) \cup \{\bigwedge p_0(a)\} \vDash (\bigvee \Phi_0^a(a)) \vee (\bigvee p_0'(a)).$$

Then, for some k,

$$\mathit{Th}_M(\mathfrak{M}) \vDash (\bigwedge p_0(a) \wedge 1)^k \to (\bigvee \Phi_0^a(a)) \vee (\bigvee p_0'(a)).$$

Choosing k sufficiently large, which is possible since there are only finitely many $a \in M$, we can get that for any such a,

$$\mathfrak{M} \models (\bigwedge p_0(a) \wedge 1)^k \to \bigvee_{c \in M} ((\bigvee \Phi_0^c(a)) \vee (\bigvee p_0'(a))),$$

which implies that

$$\mathfrak{M} \models (\forall x)(\bigwedge p_0(x) \wedge 1)^k \to \bigvee_{c \in M} ((\bigvee \Phi_0^c(x)) \vee (\bigvee p_0'(x))).$$

But since $\mathfrak{M}^*$ is an elementary extension of $\mathfrak{M}$,

$$\mathfrak{M}^* \models (\forall x)(\bigwedge p_0(x) \wedge 1)^k \to \bigvee_{c \in M} ((\bigvee \Phi_0^c(x)) \vee (\bigvee p_0'(x))),$$

[1] This proof was essentially suggested by an anonymous referee.

and then

$$\mathfrak{M}^* \models (\bigwedge p_0(b) \wedge 1)^k \to \bigvee_{c \in M} ((\bigvee \Phi_0^c(b)) \vee (\bigvee p_0'(b))).$$

Because we have that $\mathfrak{M}^* \models \bigwedge p_0[b]$, it follows that $\mathfrak{M}^* \models (\bigwedge p_0[b] \wedge 1)^k$, and since $\mathfrak{M}^* \not\models \bigvee \Phi_0^c[b]$, it must be that $\mathfrak{M}^* \models \bigvee p_0'[b]$, a contradiction. □

Corollary 9. *Finite models are saturated.*

Given two theories of our language T and S, we write $T \not\vDash S$ if there is $\varphi \in S$ such that $T \not\vDash \varphi$.

Theorem 10. *For each infinite cardinal κ, each model can be elementarily extended to a κ^+-saturated model.*

Proof. Let $\mathfrak{M} = \langle \boldsymbol{A}, \mathbf{M} \rangle$ be a model. Observe that

$$|\{D \subseteq M \mid |D| \leq \kappa\}| \leq |M|^\kappa.$$

This means, together with the fact that $|S^{\mathfrak{M}}(D)| \leq 2^\kappa$, that we can list all types in $S^{\mathfrak{M}}(D)$ for $D \subseteq M, |D| \leq \kappa$ as $\{\langle p_\alpha, p_{\alpha'} \rangle \mid \alpha < |M|^\kappa\}$.

We will find a model $\langle \boldsymbol{A}', \mathbf{M}' \rangle$ that realizes all types in $S^{\mathfrak{M}}(D)$ for any $D \subseteq M, |D| \leq \kappa$. We will use the union of elementary chains construction, defining a sequence of models $\{\langle \boldsymbol{A}_\alpha, \mathbf{M}_\alpha \rangle \mid \alpha < |M|^\kappa\}$ which is an elementary chain, and where $\langle \boldsymbol{A}_\alpha, \mathbf{M}_\alpha \rangle$ realizes $\langle p_\alpha, p_{\alpha'} \rangle$.

The goal is to build the model $\bigcup_{\alpha < |M|^\kappa} \langle \boldsymbol{A}_\alpha, \mathbf{M}_\alpha \rangle$, which will be our $\langle \boldsymbol{A}', \mathbf{M}' \rangle$.

We let

(i) $\mathfrak{M}_0 = \langle \boldsymbol{A}_0, \mathbf{M}_0 \rangle = \langle \boldsymbol{A}, \mathbf{M} \rangle$

(ii) $\mathfrak{M}_\alpha = \langle \boldsymbol{A}_\alpha, \mathbf{M}_\alpha \rangle = \bigcup_{\beta < \alpha} \langle \boldsymbol{A}_\beta, \mathbf{M}_\beta \rangle$ when α is a limit ordinal.

(iii) $\mathfrak{M}_{\alpha+1} = \langle \boldsymbol{A}_{\alpha+1}, \mathbf{M}_{\alpha+1} \rangle$ is an elementary extension of $\langle \boldsymbol{A}_\alpha, \mathbf{M}_\alpha \rangle$ which realizes $\langle p_\alpha, p_{\alpha'} \rangle$. We build $\langle \boldsymbol{A}_{\alpha+1}, \mathbf{M}_{\alpha+1} \rangle$ using Lemma 3.24 [15], the construction of canonical models from that paper and our tableaux compactness.

We start by showing that

$$\text{Eldiag}(\boldsymbol{A}_\alpha, \mathbf{M}_\alpha) \cup p_\alpha \not\vDash X,$$

where X is an arbitrary finite subset of $\overline{\text{Eldiag}}(\boldsymbol{A}_\alpha, \mathbf{M}_\alpha) \cup p_{\alpha'}$. Observe that the set of theories $\{X\}$ is trivially deductively directed in the sense of Definition 3.21 from [15]. Using the canonical model construction and Lemma 3.24 [15] we can then provide a model for the tableau $\langle \text{Eldiag}(\boldsymbol{A}_\alpha, \mathbf{M}_\alpha) \cup p_\alpha, X \rangle$ for each such X. Hence, an application of tableaux compactness provides us with a model of $\langle \text{Eldiag}(\boldsymbol{A}_\alpha, \mathbf{M}_\alpha) \cup p_\alpha, \overline{\text{Eldiag}}(\boldsymbol{A}_\alpha, \mathbf{M}_\alpha) \cup p_{\alpha'} \rangle$.

Suppose, for a contradiction, that for each $\psi \in X$,

$$\text{Eldiag}(\boldsymbol{A}_\alpha, \mathbf{M}_\alpha) \cup p_\alpha \vDash \psi.$$

Then take $\psi \in X$. There are two possibilities: either (1) $\psi \in \overline{\text{Eldiag}} \langle \boldsymbol{A}_\alpha, \mathbf{M}_\alpha \rangle$ or (2) $\psi \in p_{\alpha'}$. First. let us suppose that (1) holds. Since

$$\text{Eldiag}(\boldsymbol{A}_\alpha, \mathbf{M}_\alpha) \cup p_\alpha \vDash \psi,$$

by the local deduction theorem, $p_\alpha \vDash (\varphi \wedge \bar{1})^n \to \psi$ where φ is $\bigwedge S$ for some finite $S \subseteq \mathrm{Eldiag}(\boldsymbol{A}_\alpha, \mathbf{M}_\alpha)$. Quantifying away the new constants (so only constants from the particular $D \subseteq M$ remain), we obtain that $p_\alpha \vDash (\forall \overline{x})((\varphi \wedge \bar{1})^n \to \psi)$. Now, since

$$\langle p_\alpha, \overline{\mathrm{Th}}_D(\mathfrak{M}_\alpha)\rangle$$

has a model, we have that

$$(\forall \overline{x})((\varphi \wedge \bar{1})^n \to \psi) \notin \overline{\mathrm{Th}}_D(\mathfrak{M}_\alpha),$$

so

$$(\forall \overline{x})((\varphi \wedge \bar{1})^n \to \psi) \in \mathrm{Th}_D(\mathfrak{M}_\alpha).$$

But then $\|\varphi\|^{\boldsymbol{A}_\alpha}_{\mathfrak{M}_\alpha} \geq \bar{1}^{\boldsymbol{A}_\alpha}$, so $\|\varphi \wedge \bar{1}\|^{\boldsymbol{A}_\alpha}_{\mathfrak{M}_\alpha} \geq \bar{1}^{\boldsymbol{A}_\alpha}$ and, hence, $\|(\varphi \wedge \bar{1})^n\|^{\boldsymbol{A}_\alpha}_{\mathfrak{M}_\alpha} \geq \bar{1}^{\boldsymbol{A}_\alpha}$, which leads to a contradiction. On the other hand, suppose that (2) holds ($\psi \in p_{\alpha'}$). Similarly, we can obtain that $\mathrm{Eldiag}(\boldsymbol{A}_\alpha, \mathbf{M}_\alpha) \vDash (\forall \overline{x})((\varphi \wedge \bar{1})^n \to \psi)$ where this time φ is a lattice conjunction of elements from p_α. Then, the formula $(\forall \overline{x})((\varphi \wedge \bar{1})^n \to \psi)$ would have to be in $\mathrm{Th}_D(\mathfrak{M}_\alpha)$. This gives a contradiction with the existence of a model of

$$\langle \mathrm{Th}_D(\mathfrak{M}_\alpha) \cup p_\alpha, \overline{\mathrm{Th}}_D(\mathfrak{M}_\alpha) \cup p_{\alpha'}\rangle.$$

Next we build another elementary chain to get the κ^+-saturated structure $\langle \boldsymbol{D}, \mathbf{O}\rangle$. This time we put:

(i) $\langle \boldsymbol{D}_0, \mathbf{O}_0\rangle = \langle \boldsymbol{A}, \mathbf{M}\rangle$

(ii) $\langle \boldsymbol{D}_\alpha, \mathbf{O}_\alpha\rangle = \bigcup_{\beta<\alpha}\langle \boldsymbol{D}_\beta, \mathbf{O}_\beta\rangle$ when α is a limit ordinal.

(iii) $\langle \boldsymbol{D}_{\alpha+1}, \mathbf{O}_{\alpha+1}\rangle$ is a model that elementarily extends $\langle \boldsymbol{D}_\alpha, \boldsymbol{O}_\alpha\rangle$ and realizes all types in $S^{\langle \boldsymbol{D}_\alpha, \mathbf{O}_\alpha\rangle}(X)$ for any $X \subseteq M_\alpha, |X| \leq \kappa$.

Consider the structure $\bigcup_{\alpha<\kappa^+}\langle \boldsymbol{D}_\alpha, \mathbf{O}_\alpha\rangle$, which will be our $\langle \boldsymbol{D}, \mathbf{O}\rangle$. Suppose that $X \subseteq N, |X| \leq \kappa$ and $\langle p, p'\rangle \in S^{\langle \boldsymbol{D}, \mathbf{O}\rangle}(X)$. By the regularity of the cardinal κ^+, we must have that indeed $X \subseteq O_\alpha$ for some $\alpha < \kappa^+$. But, of course, since $\mathrm{Th}_X(\boldsymbol{D}, \mathbf{O}) = \mathrm{Th}_X(\boldsymbol{D}_\alpha, \mathbf{O}_\alpha)$ and $\overline{\mathrm{Th}}_X(\boldsymbol{D}, \mathbf{O}) = \overline{\mathrm{Th}}_X(\boldsymbol{D}_\alpha, \mathbf{O}_\alpha)$, $\langle p, p'\rangle \in S^{\langle \boldsymbol{D}_\alpha, \mathbf{O}_\alpha\rangle}(X)$, so it is in fact realized in $\langle \boldsymbol{D}_{\alpha+1}, \mathbf{O}_{\alpha+1}\rangle$, and hence in $\langle \boldsymbol{D}, \mathbf{O}\rangle$. $\square$

Observe that, in contrast to the classical theorem, we do not really have a nice bound on the size of the resulting model, since the Downward Löwenheim–Skolem Theorem available to us (Theorem 30 [23]) has a more complicated cardinality calculation when determining the size of the resulting structure. For instance, in the proof of Theorem 10, we would want to make sure that $|\mathbf{M}_\alpha| = |\mathbf{M}_{\alpha+1}|$ via a Löwenheim–Skolem argument. What we would want to do is take $\mathbf{M}_{\alpha+1}$ to be the domain of size $|\mathbf{M}_\alpha|$ of an elementary substructure of the model $\langle \boldsymbol{B}, \mathbf{N}\rangle$ of the tableau

$$\langle \mathrm{Eldiag}(\boldsymbol{A}_\alpha, \mathbf{M}_\alpha) \cup p_\alpha, \overline{\mathrm{Eldiag}}(\boldsymbol{A}_\alpha, \mathbf{M}_\alpha) \cup p_{\alpha'}\rangle$$

obtained by compactness. Classically, this is no problem, but in our context, the possibility of building such an elementary substructure depends on $|\mathbf{M}_\alpha|$ being $\geq p(\boldsymbol{B})$ where $p(\boldsymbol{B})$ is a cardinal depending on $\langle \boldsymbol{B}, \mathbf{N}\rangle$ and the size of $\boldsymbol{B}$ with a calculation which is not at all obvious (Definition 28 [23]). Hence, we do not, in general, seem to get models that are saturated in the sense of κ-saturated with respect to the size κ of their own domain, even under set-theoretic assumptions like the Generalized Continuum Hypothesis, as is the case in classical model theory.

We can provide a structural characterization of κ-saturation under certain conditions. There is a cardinality restriction on the algebra of truth values of the model that is not explicitly stated in the classical case (cf. Theorem 16.6 [34]). The reason for the restriction is again the form the Downward Löwenheim–Skolem Theorem has in the general non-classical context (Theorem 30 [23]).

Theorem 11. *Let κ be an uncountable cardinal, $\mathcal{P}$ be countable, and $\mathfrak{M} = \langle \boldsymbol{A}, \mathbf{M}\rangle$ a model with $\boldsymbol{A}$ of cardinality $< \kappa$. Then, the following are equivalent:*

(i) A model $\mathfrak{M} = \langle \boldsymbol{A}, \mathbf{M}\rangle$ is κ-saturated.

(ii) Every diagram

$$\begin{tikzcd}
 & & \mathfrak{M} = \langle \boldsymbol{A}, \mathbf{M}\rangle & & \\
|M_1| < \kappa & \mathfrak{M}_1 = \langle \boldsymbol{B}_1, \mathbf{M}_1\rangle \arrow[ur, "\preccurlyeq"] \arrow[rr, "\preccurlyeq"'] & & \mathfrak{M}_2 = \langle \boldsymbol{B}_2, \mathbf{M}_2\rangle \arrow[ul, dashed, "\langle f, g\rangle"'] & |M_2| \leq \kappa
\end{tikzcd}$$

can be completed as shown in the picture, where $\langle f, g\rangle$ is a mapping where $f\colon \boldsymbol{B}_2 \longrightarrow \boldsymbol{A}$ is a partial embedding between algebras, and $g\colon \mathbf{M}_2 \longrightarrow \mathbf{M}$ is a mapping such that $\mathfrak{M}_2 \models \varphi[\overline{a}]$ iff $\mathfrak{M} \models \varphi[g(\overline{a})]$.

Proof. $(i) \implies (ii)$: Assume that $\langle \boldsymbol{A}, \mathbf{M}\rangle$ is κ-saturated. Let $\boldsymbol{M}_2 \setminus \boldsymbol{M}_1 = \{c_\alpha \mid \alpha < \kappa\}$. We let f be the partial mapping from $\boldsymbol{B}_2$ to $\boldsymbol{A}$ which is just the inclusion mapping on $\boldsymbol{B}_1$ (recall that $\boldsymbol{B}_1 \subseteq \boldsymbol{B}_2$). We define g by cases: if $x \in \mathbf{M}_1$ then g acts on x as the inclusion, while $g(c_\alpha)$ will be defined inductively. Fix α and assume that $\mathfrak{M}_2 \models \varphi[\overline{a}]$ iff $\mathfrak{M} \models \varphi[g(\overline{a})]$ whenever the sequence $\overline{a}$ has only elements from either $\mathbf{M}_1$ or $\{c_\beta \mid \beta < \alpha\}$. The complete type $\langle p, p'\rangle$ of c_α in $\mathfrak{M}_2$ is a type of $\mathfrak{M}$ over $M_1 \cup \{c_\beta \mid \beta < \alpha\}$, and this latter set is of cardinality $< \kappa$. Hence, by κ-saturation, $\langle p, p'\rangle$ is realized in $\mathfrak{M}$, and the witness to this fact will serve as the our $g(c_\alpha)$.

$(ii) \implies (i)$: Assume that the diagram in the picture can be completed as shown. Let $Y \subseteq M$ be of cardinality $< \kappa$. Assume that Y is such that every element of $\boldsymbol{A}$ is the value of some formula in a variable assignment taking elements from Y, otherwise just add all the necessary elements (since $\mathfrak{M}$ is exhaustive it suffices to add less than κ elements, since $\boldsymbol{A}$ has cardinality $< \kappa$). Then, by Theorem 30 [23], we get an elementary substructure $\mathfrak{M}_1 = \langle \boldsymbol{A}, \mathbf{M}_1\rangle$ of $\mathfrak{M} = \langle \boldsymbol{A}, \mathbf{M}\rangle$ such that $Y \subseteq N$ and $|M_1| < \kappa$, which then will be exhaustive. Take $\langle p, p'\rangle \in S^{\mathfrak{M}}(Y) = S^{\mathfrak{M}_1}(Y)$. By (iii) in the proof of Theorem 10, we have an elementary extension $\mathfrak{M}_2 = \langle \boldsymbol{B}, \mathbf{M}_2\rangle$ of $\mathfrak{M}_1$ which can be taken to be of cardinality $\leq \kappa$ (by construction of the canonical model) where $\langle p, p'\rangle$ is realized by an element d. Then $g(d)$ realizes $\langle p, p'\rangle$ in $\mathfrak{M}$. $\square$

A type of the form $\langle p, \emptyset\rangle$ will be called a *left type*. We might also write it simply as p. Left types are characterizable in the following way:

Proposition 12. *p is a left type of $\langle \boldsymbol{B}, \mathbf{M}\rangle$ with parameters in $X \subseteq M$ iff there is an elementary extension of $\langle \boldsymbol{B}, \mathbf{M}\rangle$ where p is realized.*

Proof. Let $\langle \boldsymbol{B}', \mathbf{M}'\rangle$ be an elementary extension of the model $\langle \boldsymbol{B}, \mathbf{M}\rangle$ and assume that it realizes p. We have that $\mathrm{Th}_X(\boldsymbol{B}, \mathbf{M}) = \mathrm{Th}_X(\boldsymbol{B}', \mathbf{M}')$ and $\overline{\mathrm{Th}}_X(\boldsymbol{B}, \mathbf{M}) =$

$\overline{\mathrm{Th}}_X(\boldsymbol{B}', \mathbf{M}')$, so in fact p is a left type of $\langle \boldsymbol{B}, \mathbf{M}\rangle$. On the other hand, if p is a left type of $\langle \boldsymbol{B}, \mathbf{M}\rangle$ with parameters in $X \subseteq M$, take a model of $\langle \mathrm{Th}_X(\boldsymbol{B}, \mathbf{M}) \cup p, \overline{\mathrm{Th}}_X(\boldsymbol{B}, \mathbf{M})\rangle$. Now use the elementary amalgamation theorem to obtain the model we desire. $\square$

Definition 4. *For any cardinal κ, a model $\langle \boldsymbol{B}, \mathbf{M}\rangle$ is said to be left κ-saturated if for any $X \subseteq M$ such that $|X| < \kappa$, any left type in $S^{\langle \boldsymbol{B}, \mathbf{M}\rangle}(X)$ is satisfiable in $\langle \boldsymbol{B}, \mathbf{M}\rangle$.*

Now we provide a result for the construction of left κ-saturated models which is just a corollary of Theorem 10:

Theorem 13. *For each cardinal κ, each model can be elementarily extended to a left κ^+-saturated model.*

We can end with an application. A model $\langle \boldsymbol{B}, \mathbf{M}\rangle$ is said to be *κ-homogeneous* if for any $X \subseteq M$ and $|X| < \kappa$, if there is a pair $\langle f, g\rangle$ with $g \colon X \longrightarrow M$ and f a homomorphism defined on at least $\{\|\varphi(\bar{a})\|^{\boldsymbol{B}}_{\mathbf{M}} \mid \bar{a} \in X^n \text{ for some } n\}$ such that $f(\|\varphi(\bar{a})\|^{\boldsymbol{B}}_{\mathbf{M}}) = \|\varphi(g(\bar{a}))\|^{\boldsymbol{B}}_{\mathbf{M}}$, then for any $a \in M$, we can find a pair $\langle f', g'\rangle$ with $g' \colon X \cup \{a\} \longrightarrow M$ and f' a homomorphism defined on at least $\{\|\varphi(\bar{a})\|^{\boldsymbol{B}}_{\mathbf{M}} \mid \bar{a} \in (X \cup \{a\})^n \text{ for some } n\}$ such that $f'(\|\varphi(\bar{a})\|^{\boldsymbol{B}}_{\mathbf{M}}) = \|\varphi(g'(\bar{a}))\|^{\boldsymbol{B}}_{\mathbf{M}}$.

Proposition 14. *If $\langle \boldsymbol{B}, \mathbf{M}\rangle$ is left κ-saturated, then $\langle \boldsymbol{B}, \mathbf{M}\rangle$ is κ-homogeneous.*

Proof. Let $b \in M \setminus X$ and consider:

$$\Delta = \{\varphi(x, g(\overline{a})) : \overline{a} \in X^n, \langle \boldsymbol{B}, \mathbf{M}\rangle \models \varphi(b, \overline{a})\}.$$

However, for every $\varphi(x, g(\overline{a})) \in \Delta$ we have that indeed $\langle \boldsymbol{B}, \mathbf{M}\rangle \vDash (\exists x)\varphi(x, \overline{a})$ by definition, so given the existence of the map $\langle f, g\rangle$ with $g \colon X \longrightarrow M$, we see that $\langle \boldsymbol{B}, \mathbf{M}\rangle \vDash (\exists x)\varphi(x, g(\overline{a}))$ (because $\langle \boldsymbol{B}, \mathbf{M}\rangle \vDash \varphi(\overline{a})$ only if $\langle \boldsymbol{B}, \mathbf{M}\rangle \vDash \varphi(g(\overline{a}))$). Hence, the type $\langle \Delta, \emptyset\rangle$ is finitely satisfiable, so satisfiable by some element c. Now expand g by adding the pair $\langle b, c\rangle$. Moreover, expand f to f' in the obvious way that would satisfy the condition $f'(\|\varphi(\bar{a})\|^{\boldsymbol{B}}_{\mathbf{M}}) = \|\varphi(g'(\bar{a}))\|^{\boldsymbol{B}}_{\mathbf{M}}$ for $\bar{a} \in (X \cup \{b\})^n$ (which would also make it a homomorphism). $\square$

5 Concluding Remarks

In this paper we have shown the existence of κ-saturated first-order fuzzy models (Theorem 10), that is, models realizing as many types as possible (given some cardinality restrictions). Furthermore, in Theorem 11, we have provided a structural characterization of κ-saturation in terms of the completion of a diagram representing a certain configuration of models and mappings.

It is natural to wonder if the two-sorted translation of the languages of predicate fuzzy logics into predicate classical logic introduced in [23] can be used to obtain in a direct way our results from their classical counterparts. We will briefly explore now to what extent this can be accomplished. We will see that our results could be interpreted as non-classical proofs of certain theorems in classical model theory for a fragment of a language that, as far as we know, had not drawn specific attention in the literature. Furthermore, these results are not, in general, immediate consequences of their classical counterparts since the fact that we are working with exhaustive models makes it necessary to appeal to the classical Omitting Types Theorem most of the time. It is not clear that one can point out to any place where these facts had been proved simply because they might not be of particular interest from the classical point of view, while they are interesting from the non-classical perspective.

Let us begin by recalling some definitions and facts from [23].

Definition 5. *A* many-sorted predicate language $\mathcal{P}_k$ *is a tuple*

$$\langle \mathcal{S}, Pred_{\mathcal{P}_k}, Func_{\mathcal{P}_k}, Ar_{\mathcal{P}_k}, Sort_{\mathcal{P}_k} \rangle,$$

where $\mathcal{S}$ *is a non-empty set of* sorts *of size* k*,* $Pred_{\mathcal{P}_k}$ *is a non-empty set of* sorted predicate symbols, $Func_{\mathcal{P}_k}$ *is a set (disjoint with* $Pred_{\mathcal{P}_k}$*) of* sorted function symbols, $Ar_{\mathcal{P}_k}$ *is the* arity function, *assigning to each predicate or function symbol a natural number called the* arity *of the symbol, and* $Sort_{\mathcal{P}_k}$ *is a function that maps each* n*-ary* $R \in Pred_{\mathcal{P}_k}$ *to a sequence of* n *sorts and each* n*-ary* $F \in Pred_{\mathcal{P}_k}$ *to a sequence of* $n+1$ *sorts.*

Definition 6. *Given a many-sorted predicate language* $\mathcal{P}_k$*, we define a* $\mathcal{P}_k$*-structure as a tuple* $\mathbf{M} = \langle M, \langle R^{\mathbf{M}} \rangle_{R \in Pred_{\mathcal{P}_k}}, \langle F^{\mathbf{M}} \rangle_{f \in Func_{\mathcal{P}_k}} \rangle$*, where* M *is a family of non-empty domains* $\{S(M) \mid S \in \mathcal{S}\}$*; for each* n*-ary* $R \in Pred_{\mathcal{P}_k}$*, if* $Sort_{\mathcal{P}_k}(R) = \langle S_1, \ldots, S_n \rangle$*,* $R^{\mathbf{M}} \subseteq S_1(M) \times \ldots \times S_n(M)$*; for each* n*-ary function symbol* $F \in Func_{\mathcal{P}_k}$*, if* $Sort_{\mathcal{P}_k}(F) = \langle S_1, \ldots, S_n, S \rangle$*,* $F^{\mathbf{M}}$ *is a function from* $S_1(M) \times \ldots \times S_n(M)$ *to* $S(M)$*.*

Let us show now how we can translate our predicate language $\mathcal{P}$ from §2 into a classical 2-sorted language $\mathcal{P}_2$:

- For each sort $i \in \{1, 2\}$, we take quantifiers $\forall_i$ and $\exists_i$.
- Variables of sort 1 are denoted as $x, y, z, x_1, \ldots, x_n, \ldots$, and those of sort 2 as $v, w, v_1, \ldots, v_n, \ldots$
- For each sort $i \in \{1, 2\}$, we take an equality symbol $\approx_i$.
- For each propositional n-ary connective λ, we take the same symbol λ as a functional of type $\langle 1, \overset{(n)}{\ldots}, 1, 1 \rangle$.
- For each n-ary functional symbol $F \in \mathbf{F}$, we take the same symbol F as a functional of type $\langle 2, \overset{(n)}{\ldots}, 2, 2 \rangle$.
- For each n-ary relational symbol $R \in \mathbf{P}$, we take the same symbol R as a functional of type $\langle 2, \overset{(n)}{\ldots}, 2, 1 \rangle$.

Now, given a $\mathcal{P}$-structure $\langle \boldsymbol{B}, \mathbf{M} \rangle$, we build a 2-sorted $\mathcal{P}_2$-structure $\boldsymbol{B}_{\mathbf{M}}$:

- The universe of sort 1 is B and the universe of sort 2 is M.
- The symbols $\approx_i$ are interpreted as crisp equality in the corresponding sorts.
- For each propositional n-ary connective λ, define $\lambda^{\boldsymbol{B}_{\mathbf{M}}}$ as $\lambda^{\boldsymbol{B}}$.
- For each n-ary functional symbol $F \in \mathbf{F}$, define $F^{\boldsymbol{B}_{\mathbf{M}}}$ as $F_{\mathbf{M}}$.
- For each n-ary relational symbol $P \in \mathbf{P}$, define $P^{\boldsymbol{B}_{\mathbf{M}}}$ as $P_{\mathbf{M}}$.

Fact 15 ([23]). *For each $\mathcal{P}$-formula $\varphi(v_1, \ldots, v_n)$, there is a $\mathcal{P}_2$-formula $E_\varphi(v_1, \ldots, v_n, x)$ such that, for every $\mathcal{P}$-structure $\langle \boldsymbol{B}, \mathbf{M}\rangle$, and $d_1, \ldots, d_n \in M$,*

$$\|\varphi(d_1, \ldots, d_n)\|^{\boldsymbol{B}}_{\mathbf{M}} = b \quad \Leftrightarrow \quad \boldsymbol{B}_{\mathbf{M}} \models E_\varphi(d_1, \ldots, d_n, b).$$

Fact 16 ([23]). *A $\mathcal{P}$-structure $\langle \boldsymbol{B}, \mathbf{M}\rangle$ is safe iff for every $\mathcal{P}$-formula $\varphi(v_1, \ldots, v_n)$,*

$$\boldsymbol{B}_{\mathbf{M}} \models (\forall v_1, \ldots, v_n)(\exists! x) E_\varphi(v_1, \ldots, v_n, x).$$

Fact 17. *For every $\mathcal{P}$-formula $\varphi(v_1, \ldots, v_n)$, and $\mathcal{P}$-structure $\langle \boldsymbol{B}, \mathbf{M}\rangle$,*

$$\langle \boldsymbol{B}, \mathbf{M}\rangle \models \varphi(a_1, \ldots, a_n) \textit{ iff } \boldsymbol{B}_{\mathbf{M}} \models (\forall x)(E_\varphi(a_1, \ldots, a_n, x) \to x \geq \overline{1}).$$

Fact 18. *A $\mathcal{P}$-structure $\langle \boldsymbol{B}, \mathbf{M}\rangle$ is exhaustive iff the classical type*

$$p(x) = \{\neg(\exists v_1, \ldots, v_n) E_\varphi(v_1, \ldots, v_n, x) \mid \varphi(v_1, \ldots, v_n) \textit{ is a } \mathcal{P}\textit{-formula}\}$$

is omitted in $\boldsymbol{B}_{\mathbf{M}}$.

From the above observations, in addition with the observation that the class of UL-chains is axiomatizable in $\mathcal{P}_2$ by a $\forall_1$-theory, we see that the class of $\mathcal{P}_2$-structures $\boldsymbol{B}_{\mathbf{M}}$ which are *safe* is axiomatizable by a theory $\mathbb{T}$ which is $\forall\exists$. Furthermore, the class of safe $\boldsymbol{B}_{\mathbf{M}}$ which are *exhaustive* is comprised of all models of $\mathbb{T}$ which omit the type $p(x)$ from Fact 18. Furthermore, it is not difficult to observe that our elementary extensions are plain superstructures from the point of view of $\mathcal{P}_2$ on models of the theory $\mathbb{T}$ omitting the type $p(x)$. The key observation is to notice that an equality $\approx_1$ holding in a two-sorted model is actually the same as a certain $\leftrightarrow$ formula holding in one of our many-valued models.

Our Theorem 1, from the classical point of view, becomes:

Fact 19. *Let $\langle T, U\rangle$ be a tableau. If every finite subset of the theory*

$$\begin{aligned} S = \{&(\forall x)(E_\varphi(a_1, \ldots, a_n, x) \to x \geq \overline{1}) \mid \varphi \in T\} \cup \\ &\{\neg(\forall x)(E_\psi(a_1, \ldots, a_n, x) \to x \geq \overline{1}) \mid \psi \in U\} \end{aligned}$$

has a model omitting the type $p(x)$ from Fact 18 and satisfying the theory $\mathbb{T}$, then S itself has one such model.

Classically, the proof of Fact 19 would require an application of the Omitting Types Theorem. So we would have to show that the type $p(x)$ is non-isolated. Finally, from the classical perspective, our Theorem 10 becomes:

Fact 20. *For each infinite cardinal κ, each model $\boldsymbol{B}_{\mathbf{M}}$ of the theory $\mathbb{T}$ omitting the type $p(x)$ from Fact 18 is a substructure of another such model $\boldsymbol{B}'_{\mathbf{M}}$ where every free partial type over $< \kappa^+$ many parameters of $\boldsymbol{B}_{\mathbf{M}}$ is satisfied.*

Once more, the proof of this fact would be using the Omitting Types Theorem, so it would not be a direct instance of the classical existence of saturated models; one would have to modify the construction in the classical case to obtain our theorem. We have only seen two examples but they should suffice for the reader to figure out for themselves how to restate the results of the paper in the classical setting.

A complementary task to the one tackled in the present paper is that of building models realizing very few types, which in classical model theory is accomplished by means of the Omitting Types Theorem (already mentioned above). In the context of mathematical fuzzy logic, some work along these lines has been done focusing on types with respect to a theory in [8, 17, 33] and, for the more general two-sided types used in this paper, we prove the theorem in [3]. In the latter article, we also considered topological aspects of this problem.

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