

## **Adding truth-constants to logics of continuous t-norms: Axiomatization and completeness results**

This is a pre print version of the following article:

*Original:*

Esteva, F., Gispert, J., Godo, L., Noguera, C. (2007). Adding truth-constants to logics of continuous t-norms: Axiomatization and completeness results. FUZZY SETS AND SYSTEMS, 158(6), 597-618 [10.1016/j.fss.2006.11.010].

*Availability:*

This version is available <http://hdl.handle.net/11365/1200742> since 2022-04-11T15:05:48Z

*Published:*

DOI: <http://doi.org/10.1016/j.fss.2006.11.010>

*Terms of use:*

**Open Access**

The terms and conditions for the reuse of this version of the manuscript are specified in the publishing policy. Works made available under a Creative Commons license can be used according to the terms and conditions of said license.

For all terms of use and more information see the publisher's website.

(Article begins on next page)

# Adding truth-constants to logics of continuous t-norms: axiomatization and completeness results

Francesc Esteva, Lluís Godo, Carles Noguera  
Institut d'Investigació en Intel·ligència Artificial - CSIC  
Catalonia, Spain

Joan Gispert  
Universitat de Barcelona  
Catalonia, Spain

## Abstract

In this paper we study generic expansions of logics of continuous t-norms with truth-constants, taking advantage of previous results for Łukasiewicz logic and more recent results for Gödel and Product logics. Indeed, we consider algebraic semantics for expansions of logics of continuous t-norms with a set of truth-constants  $\{\bar{r} \mid r \in C\}$ , for a suitable countable  $C \subseteq [0, 1]$ , and provide a full description of completeness results when (i) the t-norm is a finite ordinal sum of Łukasiewicz, Gödel and Product components, (ii) the set of truth-constants *covers* all the unit interval in the sense that each component of the t-norm contains at least one value of  $C$  different from the bounds of the component, and (iii) the truth-constants in Łukasiewicz components *behave* as rational numbers.

**Keywords:** Basic Fuzzy logic BL, Gödel, Łukasiewicz and Product Logics, t-norm-based logic, expansions with truth-constants, standard completeness.

## 1 Introduction

T-norm based fuzzy logics are basically logics of *comparative truth*. In fact, the residuum  $\Rightarrow$  of a (left-continuous) t-norm  $*$  satisfies the condition  $x \Rightarrow y = 1$  if, and only if,  $x \leq y$  for all  $x, y \in [0, 1]$ . This means that a formula  $\varphi \rightarrow \psi$  is a logical consequence of a theory if the truth degree of  $\psi$  is at least as high as the truth degree of  $\varphi$  in any interpretation which is a model of the theory. This is fine, but in some situations one might be also interested to explicitly represent and reason with partial degrees of truth. To do so, one convenient and elegant way is introducing truth-constants into the language. This approach actually goes back to Pavelka [29] who built a propositional many-valued logical system which turned out to be equivalent to the expansion of Łukasiewicz Logic by adding into the language a truth-constant  $\bar{r}$  for each real  $r \in [0, 1]$ , together with a number of additional axioms. Although the resulting logic is not strongly complete with respect to the intended semantics defined by the Łukasiewicz t-norm, (like the original Łukasiewicz logic), Pavelka proved that his logic, denoted here PL, is complete in a different sense. Namely, he defined the *truth degree* of a formula  $\varphi$  in a theory  $T$  as  $\|\varphi\|_T = \inf\{e(\varphi) \mid e \text{ is a PL-evaluation model of } T\}$ , and the *provability degree* of  $\varphi$  in  $T$  as  $|\varphi|_T = \sup\{r \mid T \vdash_{\text{PL}} \bar{r} \rightarrow \varphi\}$  and proved that these two degrees coincide. This kind of completeness is usually known as Pavelka-style completeness, and strongly relies on the

continuity of Łukasiewicz truth functions. Novák extended Pavelka's approach to Łukasiewicz first order logic [25, 26].

Later, Hájek [18] showed that Pavelka's logic PL could be significantly simplified while keeping the completeness results. Indeed he showed that it is enough to extend the language only by a countable number of truth-constants, one for each *rational* in  $[0, 1]$ , and by adding to the logic the two following additional axiom schemata, called book-keeping axioms:

$$\begin{aligned}\bar{r} \&\bar{s} &\leftrightarrow &\overline{r * s} \\ \bar{r} \rightarrow \bar{s} &\leftrightarrow &\overline{r \Rightarrow s}\end{aligned}$$

where  $*$  and  $\Rightarrow$  are the Łukasiewicz t-norm and its residuum respectively. He called this new system Rational Pavelka Logic, RPL for short. Moreover, he proved that RPL is strongly complete for finite theories.

Similar *rational* expansions for other continuous t-norm based fuzzy logics can be analogously defined, but Pavelka-style completeness cannot be obtained since Łukasiewicz Logic is the only fuzzy logic whose truth-functions are a continuous t-norm and a continuous residuum.

However, several expansions with truth-constants of fuzzy logics different from Łukasiewicz have been studied, mainly related to the other two outstanding continuous t-norm based logics, namely Gödel and Product logic. We may cite [18] where an expansion of  $G_\Delta$  (the expansion of Gödel Logic  $G$  with Baaz's projection connective  $\Delta$ ) with a finite number of rational truth-constants, [12] where the authors define logical systems obtained by adding (rational) truth-constants to  $G_\sim$  (Gödel Logic with an involutive negation) and to  $\Pi$  (Product Logic) and  $\Pi_\sim$  (Product Logic with an involutive negation). In the case of the rational expansions of  $\Pi$  and  $\Pi_\sim$  an infinitary inference rule (from  $\{\varphi \rightarrow \bar{r} : r \in \mathbb{Q} \cap (0, 1]\}$  infer  $\varphi \rightarrow \bar{0}$ ) is introduced in order to get Pavelka's style completeness. Rational truth-constants have been also considered in some stronger logics like in the logic  $\mathbb{L}\Pi_{\frac{1}{2}}$  [13], a logic that combines the connectives from both Łukasiewicz and Product logics plus the truth-constant  $\frac{1}{2}$ , and in the logic PL [21], a logic which combines Łukasiewicz Logic connectives plus the Product Logic conjunction (but not implication), as well as in some closely related logics.

More recently, the expansion of Gödel (and of some t-norm based logic related to the Nilpotent Minimum t-norm) with rational truth-constants on the one hand and the expansion of Product logic with countable sets of truth-constants have been respectively studied in [14] and in [28]. In these papers, canonical standard completeness (that is, completeness with respect to the corresponding algebra defined over the real unit interval where the truth-constants are interpreted as their own values) for these logics has been proved for theorems as well as for finite theories when restricted to formulae of the kind  $\bar{r} \rightarrow \varphi$ , where  $\bar{r}$  is the truth-constant associated to  $r$  and  $\varphi$  is a formula without additional truth-constants. Actually, this kind of formulas have been extensively considered in other frameworks for reasoning with partial degrees of truth, like in Novák's evaluated syntax formalism based on Łukasiewicz Logic (see e.g. [27]) or in fuzzy logic programming (see e.g. [30]). In particular, these formulas can be seen as a special kind of Novák's *evaluated* formulas, which are expressions  $a/A$  where  $a$  is a truth value (from a given algebra) and  $A$  is a formula that may contain truth-constants again, and whose interpretation is that the truth-value of  $A$  is at least  $a$ . Hence our formulas  $\bar{r} \rightarrow \varphi$  would be expressed as  $r/\varphi$  in Novák's evaluated syntax. On the other hand, formulas  $\bar{r} \rightarrow \varphi$  when  $\varphi$  is a Horn-like rule of the form  $b_1 \& \dots \& b_n \rightarrow h$  also correspond to typical fuzzy logic programming rules ( $b_1 \& \dots \& b_n \rightarrow h, r$ ), where  $r$  specifies a lower bound for the validity of the rule.

In this paper we study expansions with truth-constants of logics of continuous t-norms in a general setting. Actually, we provide a full description of completeness results for the expansions of logics of continuous t-norms with a set of truth-constants  $\{\bar{r} \mid r \in C\}$ , for a suitable countable  $C \subseteq [0, 1]$ , when (i) the t-norm is a finite ordinal sum of Łukasiewicz, Gödel and Product components, (ii) the set of truth-constants *covers* all the unit interval in the sense that each component of the t-norm contains at least one value of  $C$  different from the bounds of the component, and (iii) the truth-constants in Łukasiewicz components *behave* as rational numbers.

The paper is structured as follows. After this introduction, we provide the necessary background in the next section. In Section 3, we establish general results for axiomatic extensions of BL regarding the equivalence between different kinds of standard completeness and properties of the corresponding algebraic varieties, the partial embeddability property playing an important role. In Section 4 we introduce the expanded logics and their algebraic counterpart. In Sections 5 and 6 we study the structure and relevant algebraic properties of the expanded linearly ordered algebras, which are needed to obtain the different completeness results described in Section 7. Finally, in Section 8, we study the further expansions of the logics with the  $\Delta$  projection connective. We finish with some concluding remarks.

## 2 Preliminaries

The weakest logic that we consider in this paper is the system BL. It was defined by Hájek in [18] by means of a Hilbert-style calculus in the language  $\mathcal{L} = \{\&, \rightarrow, \bar{0}\}$  of type  $\langle 2, 2, 0 \rangle$ , built from a denumerable set of variables, where the only inference rule is *Modus Ponens* and the axiom schemata are the following<sup>1</sup> (taking  $\rightarrow$  as the least binding connective):

- (A1)  $(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$
- (A2)  $\varphi \& \psi \rightarrow \varphi$
- (A3)  $\varphi \& \psi \rightarrow \psi \& \varphi$
- (A4)  $\varphi \& (\varphi \rightarrow \psi) \rightarrow \psi \& (\psi \rightarrow \varphi)$
- (A5a)  $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow (\varphi \& \psi \rightarrow \chi)$
- (A5b)  $(\varphi \& \psi \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))$
- (A6)  $((\varphi \rightarrow \psi) \rightarrow \chi) \rightarrow (((\psi \rightarrow \varphi) \rightarrow \chi) \rightarrow \chi)$
- (A7)  $\bar{0} \rightarrow \varphi$

Some other connectives are defined as follows:

$$\begin{aligned}
\varphi \wedge \psi &:= \varphi \& (\varphi \rightarrow \psi); \\
\varphi \vee \psi &:= ((\varphi \rightarrow \psi) \rightarrow \psi) \& ((\psi \rightarrow \varphi) \rightarrow \varphi); \\
\varphi \leftrightarrow \psi &:= (\varphi \rightarrow \psi) \& (\psi \rightarrow \varphi); \\
\neg \varphi &:= \varphi \rightarrow \bar{0}; \\
\bar{1} &:= \neg \bar{0}.
\end{aligned}$$

The set of well-formed formulae in this language is denoted as  $Fm_{\mathcal{L}}$ . The (finitary) notion of proof is defined as usual from the above axioms and inference rule. If  $\Gamma \subseteq Fm_{\mathcal{L}}$  is an arbitrary theory we shall write  $\Gamma \vdash_{\text{BL}} \varphi$  to denote that there exists a proof of  $\varphi$  from  $\Gamma$ .

<sup>1</sup>Cintula has recently proved [8] that axiom (A3) is actually redundant.

The algebraic counterpart of this logic, the class of *BL-algebras*, is also given in [18]. A BL-algebra is an algebra  $\mathcal{A} = \langle A, \&^{\mathcal{A}}, \rightarrow^{\mathcal{A}}, \wedge^{\mathcal{A}}, \vee^{\mathcal{A}}, \bar{0}^{\mathcal{A}}, \bar{1}^{\mathcal{A}} \rangle$  of type  $\langle 2, 2, 2, 2, 0, 0 \rangle$  which is a bounded integral commutative residuated lattice satisfying the prelinearity equation:

$$(x \rightarrow y) \vee (y \rightarrow x) \approx \bar{1}$$

and the divisibility equation:

$$x \wedge y \approx x \& (x \rightarrow y).$$

The negation operation is defined as  $\neg^{\mathcal{A}} a = a \rightarrow^{\mathcal{A}} \bar{0}^{\mathcal{A}}$ . If the lattice order is total we will say that  $\mathcal{A}$  is a BL-chain. The BL-chains defined over the real unit interval  $[0, 1]$  (with the usual order) are those where the interpretation of  $\&$  is a continuous t-norm and they are called *standard BL-chains*. The class of all BL-algebras is a variety and it will be denoted as  $\mathbb{BL}$ .

The semantical consequence relation is defined in the following way: given  $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$  and some class of BL-algebras  $\mathbb{K} \subseteq \mathbb{BL}$ , we will write  $\Gamma \models_{\mathbb{K}} \varphi$  if, and only if, for every  $\mathcal{A} \in \mathbb{K}$  and every evaluation  $e$  of the formulae in  $\mathcal{A}$  such that  $e[\Gamma] \subseteq \{\bar{1}^{\mathcal{A}}\}$ , we have  $e(\varphi) = \bar{1}^{\mathcal{A}}$ .

With this notation we can write now the algebraic completeness theorem for BL:

**Theorem 2.1** ([18]). *For every  $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$ ,  $\Gamma \vdash_{\mathbb{BL}} \varphi$  if, and only if,  $\Gamma \models_{\mathbb{BL}} \varphi$ .*

The finitely subdirectly irreducible members<sup>2</sup> of  $\mathbb{BL}$  are the BL-chains. This implies the following theorem and its corollary:

**Theorem 2.2** ([18]). *All BL-algebras are representable as a subdirect product of BL-chains.*

**Corollary 2.3** ([18]). *For every  $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$ ,  $\Gamma \vdash_{\mathbb{BL}} \varphi$  if, and only if,  $\Gamma \models_{\{\text{BL-chains}\}} \varphi$ .*

Moreover, it is not difficult to prove that BL is an algebraizable logic in the sense of Blok and Pigozzi (see [2]) and  $\mathbb{BL}$  is its equivalent algebraic semantics. This implies much more than the algebraic completeness. In particular, there is an order-reversing isomorphism between axiomatic extensions of BL and subvarieties of  $\mathbb{BL}$ :

- If  $\Sigma \subseteq Fm_{\mathcal{L}}$  and  $\mathbb{L}$  is the extension of BL obtained by adding the formulae of  $\Sigma$  as schemata, then the equivalent algebraic semantics of  $\mathbb{L}$  is the subvariety of  $\mathbb{BL}$  axiomatized by the equations  $\{\varphi \approx 1 : \varphi \in \Sigma\}$ . We denote this variety by  $\mathbb{L}$  and we call its members *L-algebras*.
- Let  $\mathbb{L} \subseteq \mathbb{BL}$  be the subvariety axiomatized by a set of equations  $\Lambda$ . Then the logic associated to  $\mathbb{L}$  is the axiomatic extension  $\mathbb{L}$  of BL given by the axiom schemata  $\{\varphi \leftrightarrow \psi : \varphi \approx \psi \in \Lambda\}$ .

A lot of expansions of BL are also algebraizable. Indeed, let  $\mathbb{L}$  be an axiomatic extension of BL, let  $\mathcal{L}'$  be a language extending  $\mathcal{L}$ , consider a set  $\Sigma \subseteq Fm_{\mathcal{L}'}$  and let  $\mathbb{L}'$  be the expansion of  $\mathbb{L}$  obtained by adding the formulae of  $\Sigma$  as axiom schemata. Assume that for every new  $n$ -ary connective  $\lambda$  in the language  $\mathcal{L}'$ ,

$$\{p_1 \leftrightarrow q_1, \dots, p_n \leftrightarrow q_n\} \vdash_{\mathbb{L}'} \lambda(p_1, \dots, p_n) \leftrightarrow \lambda(q_1, \dots, q_n)$$

---

<sup>2</sup>For the definition of this and any other notion of Universal Algebra used in this paper, the reader is referred to [3].

Logic	Axiom schema
Lukasiewicz logic $\mathbf{L}$	$\neg\neg\varphi \rightarrow \varphi$
Product logic $\Pi$	$\neg\varphi \vee ((\varphi \rightarrow \varphi \& \psi) \rightarrow \psi)$
Gödel logic $\mathbf{G}$	$\varphi \rightarrow \varphi \& \varphi$

Table 1: Main axiomatic extensions of BL.

Then,  $L'$  is algebraizable and its equivalent algebraic semantics is the variety of algebras in the language  $\mathcal{L}'$  axiomatized by the equational base of  $\mathbb{L}$  plus the equations  $\{\varphi \approx \bar{1} : \varphi \in \Sigma\}$ . We call the members of this variety  $L'$ -algebras. In general  $L'$  needs not to be a conservative expansion of  $L$ ; in fact, we can extract from [2] the following criterion.

**Proposition 2.4.** *Under the previous hypothesis,  $L'$  is a conservative expansion of  $L$  if, and only if, every  $L$ -algebra is a subreduct of some  $L'$ -algebra.*

The three main axiomatic extensions of BL are gathered in Table 1 together with their defining axiom schemata.<sup>3</sup>

These logics are complete with respect to the semantics given by the Lukasiewicz, the product and the minimum t-norm respectively and their residua. We will denote the standard algebras defined by them as  $[0, 1]_{\mathbf{L}}$ ,  $[0, 1]_{\Pi}$  and  $[0, 1]_{\mathbf{G}}$ , respectively. It is well-known (see [22] and [24]) that every standard BL-algebra is decomposable as an ordinal sum of isomorphic copies of these three basic components.

In general, we will denote by  $[0, 1]_*$  the standard BL-chain given by a continuous t-norm  $*$  and its residuum  $\Rightarrow$ , i.e.  $[0, 1]_* = \langle [0, 1], *, \Rightarrow, \min, \max, 0, 1 \rangle$ , and  $L_*$  will denote the axiomatic extension of BL whose equivalent algebraic semantics is  $\mathbf{V}([0, 1]_*)$ , the variety generated by  $[0, 1]_*$ . In [13] an algorithm for finding an axiomatization for such a logic is given. Moreover, it also shown that different finite ordinal sums of the basic components yield different logics.

**Theorem 2.5** ([13]). *Let  $\langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle$  and  $\langle \mathcal{B}_1, \dots, \mathcal{B}_m \rangle$  be different sequences of the basic components, not containing two consecutive Gödel components. Let  $[0, 1]_* = \bigoplus_{i=1}^n \mathcal{A}_i$  and  $[0, 1]_{\circ} = \bigoplus_{i=1}^m \mathcal{B}_i$  be their respective ordinal sums. Then,  $\mathbf{V}([0, 1]_*) \neq \mathbf{V}([0, 1]_{\circ})$ , and hence  $L_* \neq L_{\circ}$ .*

Applying Jónsson's Lemma (see for instance [3]), the structure of all finitely subdirectly irreducible elements (that is all chains) of these varieties is described by the following result, which was originally given in terms of ordinal sums of Wajsberg hoops (slightly more general structures than BL-algebras) in [1], but it can be reformulated in our setting as follows.

**Theorem 2.6.** *Let  $\langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle$  be a sequence of the basic components and let  $[0, 1]_* = \bigoplus_{i=1}^n \mathcal{A}_i$  be their ordinal sum. Then, the class of finitely subdirectly irreducible members of  $\mathbf{V}([0, 1]_*)$  is  $\mathbf{HSP}_U([0, 1]_*) = \mathbf{HSP}_U(\mathcal{A}_1) \cup (\mathbf{ISP}_U(\mathcal{A}_1) \oplus \mathcal{HSP}_U(\mathcal{A}_2)) \cup \dots \cup (\mathbf{ISP}_U(\mathcal{A}_1) \oplus \bigoplus_{i=2}^{n-1} \mathcal{ISP}_U(\mathcal{A}_i) \oplus \mathcal{HSP}_U(\mathcal{A}_n))$ , where  $\mathbf{H}, \mathbf{I}, \mathbf{S}, \mathbf{P}_U$  denote the operators homomorphic images, isomorphic images, subalgebras and ultraproducts in the language of BL-algebras while  $\mathcal{H}, \mathcal{I}, \mathcal{S}, \mathcal{P}_U$  denote the same operators in the language of Basic hoops (that is, in the  $\bar{0}$ -free language).*

<sup>3</sup>These axiomatizations are introduced in [18] and in [23].

Moreover, since for the three basic cases we have that  $\mathbf{ISP}_U([0, 1]_{\mathbf{L}})$ ,  $\mathbf{ISP}_U([0, 1]_{\mathbf{H}})$  and  $\mathbf{ISP}_U([0, 1]_{\mathbf{G}})$  are respectively all MV-chains, all product chains and all Gödel chains [10, 15, 11], it follows that  $\mathcal{ISP}_U([0, 1]_{\mathbf{L}})$ ,  $\mathcal{ISP}_U([0, 1]_{\mathbf{H}})$  and  $\mathcal{ISP}_U([0, 1]_{\mathbf{G}})$  are all  $\bar{0}$ -free subreducts of MV-chains, product chains and Gödel chains. Therefore, from the above Theorem 2.6 and the results in [13] on canonical standard BL-chains it follows that we can generalize those results to any standard BL-algebra.

**Corollary 2.7.** *Let  $[0, 1]_*$  be a standard BL-algebra. Then the class of all finitely subdirectly irreducible algebras in  $\mathbf{V}([0, 1]_*)$  is  $\mathbf{ISP}_U([0, 1]_*)$ .*

A filter in a BL-algebra  $\mathcal{A}$  is any subset  $F \subseteq A$  such that:

- $\bar{1}^{\mathcal{A}} \in F$
- If  $a \in F$  and  $a \leq b$ , then  $b \in F$
- If  $a, b \in F$ , then  $a \& b \in F$ .

$F(a)$  will denote the principal filter generated by the element  $a$ . It can be described as follows:  $F(a) = \{b : a^n \leq b \text{ for some } n \geq 1\}$ . There is the usual correspondence between filters and congruences in BL-algebras:

**Proposition 2.8.** *Let  $\mathcal{A}$  be a BL-algebra. For every filter  $F \subseteq A$  we define  $\Theta(F) := \{\langle a, b \rangle \in A^2 : a \leftrightarrow b \in F\}$ , and for every congruence  $\theta$  of  $\mathcal{A}$  we define  $Fi(\theta) := \{a \in A : \langle a, 1 \rangle \in \theta\}$ . Then,  $\Theta$  is an order isomorphism from the set of filters onto the set of congruences and  $Fi$  is its inverse.*

Given a filter  $F$  of a BL-algebra  $\mathcal{A}$  and an element  $a \in A$ ,  $[a]_F$  will denote its equivalence class with respect to the congruence  $\Theta(F)$ .

### 3 Standard completeness properties for BL extensions revisited

We recall the definitions of three different kinds of completeness with respect to the standard algebras. If a logic  $L$  is an axiomatic expansion of BL in a language  $\mathcal{L}'$ , we say that  $L$  has the (finitely) strong standard completeness property, (F)SSC for short<sup>4</sup>, when for every (finite) set of formulae  $T \subseteq Fm_{\mathcal{L}'}$  and every formula  $\varphi$  it holds that  $T \vdash_L \varphi$  iff  $T \models_{\mathcal{A}} \varphi$  for every standard  $L$ -algebra  $\mathcal{A}$ . We say that  $L$  has the standard completeness property, SC for short, when the equivalence is true for  $T = \emptyset$ . Of course, the SSC implies the FSSC, and the FSSC implies the SC.

On the scope of algebraizable logics [2], these properties have their equivalent algebraic property.

**Theorem 3.1.** *Let  $L$  be an axiomatic extension (or algebraizable axiomatic expansion) of BL and let  $\mathbb{L}$  be its equivalent variety semantics. Then:*

1.  $L$  has the SC if, and only if,  $\mathbb{L} = \mathbf{V}(\text{Stand}_L)$
2.  $L$  has the FSSC if, and only if,  $\mathbb{L} = \mathbf{Q}(\text{Stand}_L)$

---

<sup>4</sup>We drop the P (for “property”) from the acronym for the sake of a simpler notation.

3.  $\mathbf{L}$  has the SSC if, and only if, every countable (finite or numerable) chain of  $\mathbf{L}$  belongs to  $\mathbf{ISP}(\mathbf{Stand}_{\mathbf{L}})$

where  $\mathbf{Stand}_{\mathbf{L}}$  is the class of all standard algebras in  $\mathbf{L}$  and  $\mathbf{Q}(\mathbf{Stand}_{\mathbf{L}})$  denotes the quasivariety generated by  $\mathbf{Stand}_{\mathbf{L}}$ .

The three main fuzzy logics as well as Basic Logic enjoy the FSSC; it is proved in [20, Lemma B, p. 84] for  $\mathbf{L}$ , in [19] for Product logic, in [11] for Gödel logic and in [4] for BL, but only  $\mathbf{G}$  enjoys the SSC. In some cases (see for instance [19, 4]), rather than using the equivalences stated above, some of these standard completeness results have been obtained by proving first that every chain of the equivalent variety semantics is partially embeddable into a standard algebra. As we shall see this property is also equivalent to the FSSC, when the language is finite.

We recall that given two algebras  $\mathcal{A}$  and  $\mathcal{B}$  of the same language we say that  $\mathcal{A}$  is *partially embeddable* into  $\mathcal{B}$  when every finite partial subalgebra of  $\mathcal{A}$  is embeddable into  $\mathcal{B}$ . Generalizing this notion to classes of algebras, we say that a class  $\mathbb{K}$  of algebras is *partially embeddable into a class*  $\mathbb{M}$  if every finite partial subalgebra of a member of  $\mathbb{K}$  is embeddable into a member of  $\mathbb{M}$ . If the language is finite, this turns out to be equivalent to say that  $\mathbb{K}$  belongs to the universal class generated by  $\mathbb{M}$  (see for instance [17]). That is, by recalling Los' theorem (see [3]) of characterization of universal classes, we have the following equivalence.

**Proposition 3.2** ([17]). *Let  $\mathbb{K}$  and  $\mathbb{M}$  be classes of algebras of the same finite language. Then the following conditions are equivalent:*

- $\mathbb{K}$  is partially embeddable into  $\mathbb{M}$
- $\mathbb{K} \subseteq \mathbf{ISP}_U(\mathbb{M})$

The following result states the equivalence of the FSSC and partial embeddability properties in the frame of BL-algebras.<sup>5</sup>

**Proposition 3.3.** *Let  $\mathbf{L}$  be an axiomatic extension (or algebraizable axiomatic expansion in a finite language) of BL. Then  $\mathbf{L}$  has the FSSC if, and only if, the class of all  $\mathbf{L}$ -chains is partially embeddable into the class of all standard  $\mathbf{L}$ -algebras.*

*Proof.* If  $\mathbf{L}$  satisfies the FSSC then, by Theorem 3.1, its equivalent variety semantics  $\mathbb{L}$  is such that  $\mathbb{L} = \mathbf{Q}(\mathbf{Stand}_{\mathbf{L}})$ . It follows from [9, Lemma 1.5] that every relatively finitely subdirectly irreducible member of  $\mathbf{Q}(\mathbf{Stand}_{\mathbf{L}})$  belongs to  $\mathbf{ISP}_U(\mathbf{Stand}_{\mathbf{L}})$ . We recall that given a class  $K$  of algebras we say that  $A \in K$  is *relatively finitely subdirectly irreducible* if it satisfies that whenever  $A$  is representable as finite subdirect product of  $\{A_0, \dots, A_{n-1}\} \subseteq K$  then  $A \cong A_i$  for some  $i < n$ . Since  $\mathbf{Q}(\mathbf{Stand}_{\mathbf{L}})$  is a variety, relatively finitely subdirectly irreducible members coincide with finitely subdirectly irreducible algebras in the absolute sense, hence with  $\mathbf{L}$ -chains. Therefore, if  $\mathbf{L}$  satisfies the FSSC then every  $\mathbf{L}$ -chain belongs to  $\mathbf{ISP}_U(\mathbf{Stand}_{\mathbf{L}})$ , which is equivalent to partial embeddability by Proposition 3.2.

If the class of all  $\mathbf{L}$ -chains is partially embeddable into the class of all standard  $\mathbf{L}$ -algebras, then by Proposition 3.2 every  $\mathbf{L}$ -chain belongs to  $\mathbf{ISP}_U(\mathbf{Stand}_{\mathbf{L}})$ . Now, since every  $\mathbf{L}$ -algebra is representable as subdirect product of  $\mathbf{L}$ -chains we have that

$$\mathbb{L} \subseteq \mathbf{IP}_{SD}(\mathbf{ISP}_U(\mathbf{Stand}_{\mathbf{L}})) \subseteq \mathbf{Q}(\mathbf{Stand}_{\mathbf{L}}) \subseteq \mathbb{L},$$

<sup>5</sup>In fact the same result is valid and the proof is similar in the scope of MTL-algebras



where  $\mathbf{P}_{SD}$  stands for the operator of subdirect products. Therefore by Theorem 3.1,  $L$  has the FSSC.  $\square$

Given any continuous t-norm  $*$ , it follows from Corollary 2.7 that all chains in  $\mathbf{V}([0, 1]_*)$  are partially embeddable into  $[0, 1]_*$ . Therefore, after the above result we get the following direct corollary.

**Corollary 3.4.** *For every continuous t-norm  $*$ , the logic  $L_*$  has the FSSC.*

Among all the continuous t-norm logics, only  $G$  enjoys the SSC. Actually, for every continuous t-norm  $*$   $\neq \min$ , hence containing at least one Łukasiewicz or product component in its decomposition, the logic  $L_*$  does not enjoy the SSC.<sup>6</sup>

To end up this section we show that if an axiomatic extension of BL does not enjoy the SC, the FSSC or the SSC, then any of its conservative expansions neither does.

**Proposition 3.5.** *Suppose that  $L'$  is a conservative expansion of  $L$  (in the hypothesis of Proposition 2.4). Then:*

- *If  $L'$  enjoys the SC, then  $L$  enjoys the SC.*
- *If  $L'$  enjoys the FSSC, then  $L$  enjoys the FSSC.*
- *If  $L'$  enjoys the SSC, then  $L$  enjoys the SSC.*

*Proof.* All the implications are proved in a similar way. Let us prove as an example the first one. Suppose that  $L$  does not enjoy the SC. Then, there is a formula  $\varphi \in Fm_{\mathcal{L}}$  such that  $\not\models_L \varphi$  and  $\models_{\mathcal{C}} \varphi$  for every standard  $L$ -chain  $\mathcal{C}$ . Let  $\mathcal{A}$  be a standard  $L'$ -chain. Then, its  $\mathcal{L}$ -reduct is a model of  $\varphi$ , thus  $\models_{\mathcal{A}} \varphi$  and, since  $L'$  is a conservative expansion of  $L$ , we also have  $\models_{L'} \varphi$ . Therefore,  $L'$  does not enjoy the SC.  $\square$

## 4 Adding truth-constants

Now we define expansions with truth-constants for those extensions of BL that are the logic of a concrete continuous t-norm.

**Definition 4.1.** *Let  $[0, 1]_* = \langle [0, 1], *, \Rightarrow, \min, \max, 0, 1 \rangle$  be a standard BL-chain and let  $L_*$  be its corresponding axiomatic extension of BL. Take now a countable subalgebra  $\mathcal{C} \subseteq [0, 1]_*$  and consider an expanded language  $\mathcal{L}_{\mathcal{C}} = \mathcal{L} \cup \{\bar{r} : r \in \mathcal{C} \setminus \{0, 1\}\}$  with a new constant for every element in  $\mathcal{C} \setminus \{0, 1\}$ . By  $L_*(\mathcal{C})$  we will denote the expansion of  $L_*$  in the language  $\mathcal{L}_{\mathcal{C}}$  obtained by adding the so-called 'book-keeping axioms':*

$$\begin{aligned} \bar{r} \&\bar{s} &\leftrightarrow \bar{r} * \bar{s} \\ (\bar{r} \rightarrow \bar{s}) &\leftrightarrow \bar{r} \Rightarrow \bar{s} \end{aligned}$$

for every  $r, s \in \mathcal{C}$ .

The algebraic counterpart of the  $L_*(\mathcal{C})$  logics is defined in the natural way.

---

<sup>6</sup>As a matter of example, let  $\Gamma = \{q \rightarrow p^n \mid n \in \mathbb{N}\}$  and  $\varphi = (q \rightarrow q^2) \vee (p \rightarrow p^2) \vee (q \rightarrow p \& q)$ , and consider the following semantical deduction:  $\Gamma \models_{[0, 1]_*} \varphi$ . One can check that this deduction holds for every continuous t-norm  $*$ , but  $\Gamma \not\models_{L_*} \varphi$ , since  $\Gamma_0 \not\models_{[0, 1]_*} \varphi$ , for every finite  $\Gamma_0 \subseteq \Gamma$  when  $*$   $\neq \min$ . In fact, this proves that the only axiomatic extension of BL that enjoys the SSC is  $G$ .

**Definition 4.2.** Let  $*$  be a continuous t-norm and let  $\mathcal{C}$  be a countable subalgebra of  $[0, 1]_*$ . A structure  $\mathcal{A} = \langle A, \&^{\mathcal{A}}, \rightarrow^{\mathcal{A}}, \wedge^{\mathcal{A}}, \vee^{\mathcal{A}}, \{\bar{r}^{\mathcal{A}} : r \in C\} \rangle$  is a  $L_*(\mathcal{C})$ -algebra if:

1.  $\langle A, \&^{\mathcal{A}}, \rightarrow^{\mathcal{A}}, \wedge^{\mathcal{A}}, \vee^{\mathcal{A}}, \bar{0}^{\mathcal{A}}, \bar{1}^{\mathcal{A}} \rangle$  is an  $L_*$ -algebra, and
2. for every  $r, s \in C$  the following identities hold:

$$\begin{aligned} \bar{r}^{\mathcal{A}} \&^{\mathcal{A}} \bar{s}^{\mathcal{A}} &= \overline{r * s}^{\mathcal{A}} \\ \bar{r}^{\mathcal{A}} \rightarrow^{\mathcal{A}} \bar{s}^{\mathcal{A}} &= \overline{r \Rightarrow s}^{\mathcal{A}}. \end{aligned}$$

Given  $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}_C}$ , we define  $\Gamma \models_{\mathcal{A}} \varphi$  iff for all evaluations  $e$  in  $\mathcal{A}$  (i.e. such that  $e(\bar{r}) = \bar{r}^{\mathcal{A}}$ ), we have  $e(\varphi) = \bar{1}^{\mathcal{A}}$  whenever  $e(\psi) = \bar{1}^{\mathcal{A}}$  for all  $\psi \in \Gamma$ .

The canonical standard  $L_*(\mathcal{C})$ -chain is the algebra  $[0, 1]_{L_*(\mathcal{C})} = \langle [0, 1], *, \Rightarrow, \min, \max, \{r : r \in C\} \rangle$ , i. e. the  $\mathcal{L}_C$ -expansion of  $[0, 1]_*$  where the truth-constants are interpreted by themselves.

It is easy to prove that  $L_*(\mathcal{C})$  is also an algebraizable logic (in the sense of [2]) and its equivalent algebraic semantics is the variety of  $L_*(\mathcal{C})$ -algebras. Furthermore, reasoning as in the BL case, we can prove that all  $L_*(\mathcal{C})$ -algebras are representable as a subdirect product of  $L_*(\mathcal{C})$ -chains, hence we also have completeness of  $L_*(\mathcal{C})$  with respect to  $L_*(\mathcal{C})$ -chains.

**Theorem 4.3.** For any  $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}_C}$ ,  $\Gamma \vdash_{L_*(\mathcal{C})} \varphi$  if, and only if,  $\Gamma \models_{\mathcal{A}} \varphi$  for every  $L_*(\mathcal{C})$ -chain  $\mathcal{A}$ .

Since these logics are expansions of BL, sharing Modus Ponens as the only inference rule, they have the same local deduction-detachment theorem as BL has. In fact, the proof for BL (in [18]) also applies here.

**Theorem 4.4.** For every  $\Gamma \cup \{\varphi, \psi\} \subseteq Fm_{\mathcal{L}_C}$ ,  $\Gamma, \varphi \vdash_{L_*(\mathcal{C})} \psi$  if, and only if, there is a natural  $k \geq 1$  such that  $\Gamma \vdash_{L_*(\mathcal{C})} \varphi^k \rightarrow \psi$ .

**Proposition 4.5.**  $L_*(\mathcal{C})$  is a conservative expansion of  $L_*$ .

*Proof.* Let  $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$  be arbitrary formulae and suppose that  $\Gamma \vdash_{L_*(\mathcal{C})} \varphi$ . Then, there is a finite  $\Gamma_0 \subseteq \Gamma$  such that  $\Gamma_0 \vdash_{L_*(\mathcal{C})} \varphi$ , and this implies that  $\Gamma_0 \models_{[0,1]_{L_*(\mathcal{C})}} \varphi$ . Since the new truth-constants do not occur in  $\Gamma_0 \cup \{\varphi\}$ , we have  $\Gamma_0 \models_{[0,1]_*} \varphi$ , and by the FSSC of  $L_*$ ,  $\Gamma_0 \vdash_{L_*} \varphi$ , and hence  $\Gamma \vdash_{L_*} \varphi$ .  $\square$

In the rest of the paper we will study the SC, FSSC and SSC properties for these logics with truth-constants but we will also consider another special kind of standard completeness. Namely, we say that  $L_*(\mathcal{C})$  enjoys the *canonical (finite) strong standard completeness* if, and only if, for every (finite) set of formulae  $T \subseteq Fm_{\mathcal{L}_C}$  and every formula  $\varphi$ ,  $T \vdash_{L_*(\mathcal{C})} \varphi$  iff  $T \models_{[0,1]_{L_*(\mathcal{C})}} \varphi$ . We say that  $L_*(\mathcal{C})$  enjoys the *canonical standard completeness* if, and only if, the equivalence is true for  $T = \emptyset$ .

We have seen (Theorem 4.4) that the logics  $L_*(\mathcal{C})$  are complete with respect to the  $L_*(\mathcal{C})$ -chains. Thus to study standard completeness results we need to get a deeper insight into  $L_*(\mathcal{C})$ -chains. This is done in the next two sections. Actually, for technical reasons (that will be justified later on) we will restrict ourselves to logics  $L_*(\mathcal{C})$  satisfying the following three conditions unless stated otherwise:

- (C1) the continuous t-norm  $*$  is a *finite* ordinal sum of the basic components
- (C2) each component of the t-norm contains at least one value of  $C$  different from the bounds of the component
- (C3) every  $r \in C$  belonging to a Łukasiewicz component of  $[0, 1]_*$  generates a finite MV-chain

## 5 About the structure of standard $L_*(\mathcal{C})$ -chains

Suppose that  $*$  is a continuous t-norm whose decomposition as ordinal sum of isomorphic copies of the three basic components is  $\bigoplus_{i \in I} [a_i, b_i]_{*i}$ .

**Definition 5.1.** Let  $\mathcal{A}$  be an  $L_*(\mathcal{C})$ -chain.  $\mathcal{C}^{\mathcal{A}}$  will denote the subalgebra of  $\mathcal{A}$  defined over  $\{\bar{r}^{\mathcal{A}} : r \in C\}$  and  $F_{\mathcal{C}}(\mathcal{A})$  will denote the set of the truth-constants interpreted as 1 in  $\mathcal{A}$ , i. e.  $F_{\mathcal{C}}(\mathcal{A}) = \{r \in C \mid \bar{r}^{\mathcal{A}} = \bar{1}^{\mathcal{A}}\}$ .

**Lemma 5.2.** Let  $\mathcal{A}$  and  $\mathcal{B}$  be non-trivial  $L_*(\mathcal{C})$ -chains with the same  $\mathcal{L}$ -reduct. Then:

- (i)  $F_{\mathcal{C}}(\mathcal{A})$  is a proper filter of  $\mathcal{C}$ .
- (ii)  $\mathcal{C}/F_{\mathcal{C}}(\mathcal{A}) \cong \mathcal{C}^{\mathcal{A}}$ .
- (iii) If  $\mathcal{A} \cong \mathcal{B}$ , then  $F_{\mathcal{C}}(\mathcal{A}) = F_{\mathcal{C}}(\mathcal{B})$ .
- (iv) If  $r, s \in C \setminus F_{\mathcal{C}}(\mathcal{A})$  and  $r < s$ , then  $\bar{r}^{\mathcal{A}} < \bar{s}^{\mathcal{A}}$ .

*Proof.* (i) Clearly  $1 \in F_{\mathcal{C}}(\mathcal{A})$ . If  $r \in F_{\mathcal{C}}(\mathcal{A})$  and  $s \in C$ ,  $s > r$ , then  $s \in F_{\mathcal{C}}(\mathcal{A})$  because by the book-keeping axioms and the definability of min and max we have  $\bar{s}^{\mathcal{A}} = \max\{\bar{r}^{\mathcal{A}}, \bar{s}^{\mathcal{A}}\} = \bar{1}^{\mathcal{A}}$ . Moreover if  $r, s \in F_{\mathcal{C}}(\mathcal{A})$  then  $r * s \in F_{\mathcal{C}}(\mathcal{A})$  since  $\overline{r * s}^{\mathcal{A}} = \bar{r}^{\mathcal{A}} \& \bar{s}^{\mathcal{A}} = \bar{1}^{\mathcal{A}}$ .

(ii) Consider the function  $f : \mathcal{C} \rightarrow \mathcal{C}^{\mathcal{A}}$  defined by  $f(r) = \bar{r}^{\mathcal{A}}$ . It is clear that  $f$  is a surjective homomorphism and  $\text{Ker } f = F_{\mathcal{C}}(\mathcal{A})$ , so  $\mathcal{C}/F_{\mathcal{C}}(\mathcal{A}) \cong \mathcal{C}^{\mathcal{A}}$ .

(iii) If  $\mathcal{A} \cong \mathcal{B}$ , then it is clear that  $\mathcal{C}^{\mathcal{A}} \cong \mathcal{C}^{\mathcal{B}}$ , so  $F_{\mathcal{C}}(\mathcal{A}) = F_{\mathcal{C}}(\mathcal{B})$ .

(iv) If  $r < s \notin F_{\mathcal{C}}(\mathcal{A})$ , then  $\bar{r}^{\mathcal{A}} \leq \bar{s}^{\mathcal{A}}$  since the book-keeping axioms imply that the order must be preserved. On the other hand, if  $\bar{r}^{\mathcal{A}} = \bar{s}^{\mathcal{A}}$ , then  $[r]_{F_{\mathcal{C}}(\mathcal{A})} = [s]_{F_{\mathcal{C}}(\mathcal{A})}$  which implies  $s \rightarrow r \in F_{\mathcal{C}}(\mathcal{A})$ , and this leads to a contradiction. Indeed, consider the following subcases:

- If  $r, s \in (a_i, b_i)$  and  $[a_i, b_i]$  is a Łukasiewicz component, then  $s \rightarrow r \in F_{\mathcal{C}}(\mathcal{A})$  implies that the minimum of the component also belongs to  $F_{\mathcal{C}}(\mathcal{A})$  and therefore  $[a_i, b_i] \subseteq F_{\mathcal{C}}(\mathcal{A})$ , a contradiction.
- If  $r, s \in (a_i, b_i)$  and  $[a_i, b_i]$  is a Product component, then the assumption  $s \rightarrow r \in F_{\mathcal{C}}(\mathcal{A})$  implies that there exists  $n$  such that  $r > (s \rightarrow r)^n$  and thus  $r, s \in F_{\mathcal{C}}(\mathcal{A})$ , a contradiction.
- finally, if  $r * s = \min\{r, s\}$  then  $r = s \rightarrow r \in F_{\mathcal{C}}(\mathcal{A})$ , a contradiction.

□

Notice that the first three properties in the previous lemma would also hold for the more general case of  $*$  being left-continuous t-norm. However, we do make use of the continuity of  $*$  in the proof of the last one. Actually this lemma describes all the possible interpretations of the truth-constants over  $L_*(\mathcal{C})$ -chains. All these interpretations are quotients of the algebra  $\mathcal{C}$  such that a filter of truth-constants is identified to 1 and all the remaining truth-constants are interpreted in pairwise different elements of the chain. For instance, for every filter  $F$  we can define an  $L_*(\mathcal{C})$ -algebra over  $[0, 1]_*$  interpreting  $\bar{r}$  by 1 if  $r \in F$  and by  $r$  otherwise. We will denote this algebra by  $[0, 1]_{L_*(\mathcal{C})}^F$ . An easy computation shows that it is indeed an  $L_*(\mathcal{C})$ -chain. Remark that the canonical standard algebra corresponds to the case  $F = \{1\}$ .

**Proposition 5.3.** *Let  $[0, 1]_*$  be a finite ordinal sum of Łukasiewicz and product components and  $\mathcal{C} \subseteq [0, 1]_*$  a countable subalgebra satisfying condition (C2). Let  $X = \{[\mathcal{A}] : \mathcal{A} \text{ standard } L_*(\mathcal{C})\text{-algebra over } [0, 1]_*\}$  be the set of isomorphism classes of  $L_*(\mathcal{C})$ -algebras over  $[0, 1]_*$  and let  $Fi(\mathcal{C})$  be the set of proper filters of  $\mathcal{C}$ . Then, the function  $\Phi : X \rightarrow Fi(\mathcal{C})$  such that for every  $[\mathcal{A}] \in X$ ,  $\Phi([\mathcal{A}]) = F_{\mathcal{C}}(\mathcal{A})$ , is a bijection.*

*Proof.*  $\Phi$  is well-defined because of Lemma 5.2 (iii), and for an easier notation we will simply write  $\Phi(\mathcal{A})$  instead of  $\Phi([\mathcal{A}])$ . It is clearly onto because  $\Phi([0, 1]_{L_*(\mathcal{C})}^F) = F$ . We must prove that  $\Phi$  is also injective. Suppose that  $\Phi(\mathcal{A}) = \Phi(\mathcal{B})$ , i. e.  $F_{\mathcal{C}}(\mathcal{A}) = F_{\mathcal{C}}(\mathcal{B})$ . Then, we have  $\mathcal{C}^{\mathcal{A}} \cong \mathcal{C}/F_{\mathcal{C}}(\mathcal{A}) = \mathcal{C}/F_{\mathcal{C}}(\mathcal{B}) \cong \mathcal{C}^{\mathcal{B}}$ . In the following, denoting by  $h$  the isomorphism between  $\mathcal{C}^{\mathcal{A}}$  and  $\mathcal{C}^{\mathcal{B}}$ , we show how to extend it as a function  $h : [0, 1] \rightarrow [0, 1]$  making  $\mathcal{A}$  and  $\mathcal{B}$  isomorphic as well.

1. If  $*$  is the Łukasiewicz t-norm, the only proper filter of  $\mathcal{C}$  is  $\{1\}$ , and thus  $\mathcal{C}^{\mathcal{A}} \cong \mathcal{C}^{\mathcal{B}} \cong \mathcal{C}$ . If  $h \neq Id$ , then there is  $a \neq h(a)$ . Let  $b = h(a)$ . Taking the restriction of  $h$ , it is clear that the generated subalgebras are also isomorphic, i. e.  $\langle a \rangle \cong \langle b \rangle$ , so  $a$  and  $b$  are either both rational or either both irrational (otherwise, the rational one would generate a finite subalgebra, and the irrational one would generate an infinite subalgebra). If  $a$  and  $b$  are irrational, then by [16, proofs of Proposition 2 and Theorem 3]  $a = 1 - b$ . Therefore one of them must be positive; suppose that it is  $a$ . Then  $2a = 1$ , so  $2(1 - b) = 1$ . But, due to the isomorphism, we also have  $2b = 1$ , a contradiction. If  $a$  and  $b$  are rational we reason analogously.
2. If  $*$  is the product t-norm there are only two proper filters,  $\{1\}$  and  $C \setminus \{0\}$  and thus we have two types of  $\Pi(\mathcal{C})$ -chains over  $[0, 1]_{\Pi}$  corresponding to the cases that  $F = \{1\}$  (the corresponding type of  $\Pi(\mathcal{C})$ -chains are the ones such that for each pair  $r < s$  in  $\mathcal{C}$ , then  $\bar{r}^{\mathcal{A}} < \bar{s}^{\mathcal{A}}$ ) and the case  $C \setminus \{0\}$  (the corresponding type of  $\Pi(\mathcal{C})$ -chains are such that  $\bar{r}^{\mathcal{A}} = \bar{1}^{\mathcal{A}}$  for all  $r \neq 0$ ).<sup>7</sup> If  $F_{\mathcal{C}}(\mathcal{A}) = F_{\mathcal{C}}(\mathcal{B}) = \{1\}$ , then  $\mathcal{C}^{\mathcal{A}} \cong \mathcal{C}^{\mathcal{B}} \cong \mathcal{C}$ , and by [28, Theorem 2] we obtain  $\mathcal{A} \cong \mathcal{B}$ . If  $F_{\mathcal{C}}(\mathcal{A}) = F_{\mathcal{C}}(\mathcal{B}) = C \setminus \{0\}$ , the result is trivial.
3. If  $*$  has more than one component, then all possible proper filters are either of the form  $[a, 1]$  where  $a$  is an idempotent element, or of the form  $(a, 1]$  where  $a$  is the minimum of a product component. The result is proved by applying the previous cases to each component of its decomposition not included in the filter.

□

However, this result is not valid for continuous t-norms containing a Gödel component as the following example shows: let  $*$  = min,  $C = \mathbb{Q} \cap [0, 1]$  and  $F = \{1\}$ . The standard  $G(\mathcal{C})$ -algebra  $\mathcal{A}$  where:

$$\bar{r}^{\mathcal{A}} = \begin{cases} r, & \text{if } r \geq 1/2 \\ r/2, & \text{otherwise} \end{cases}$$

is such that  $F_{\mathcal{C}}(\mathcal{A}) = F$  but clearly it is not isomorphic to the canonical  $G(\mathcal{C})$ -algebra.

From now on we will refer to  $L_*(\mathcal{C})$ -chains  $\mathcal{A}$  such that  $F_{\mathcal{C}}(\mathcal{A}) = F$  as  $L_*(\mathcal{C})$ -algebras of type  $F$ , either over  $[0, 1]_*$  (called *standard chains of type  $F$* ) or over any other  $L_*$ -chain. The *quasi-canonical* examples of standard chains of type  $F$  are the above introduced chains  $[0, 1]_{L_*(\mathcal{C})}^F$ .

<sup>7</sup>In [28] these two types of chains are denoted as of type I (when  $F = \{1\}$ ) and type II (when  $F = (0, 1]$ ).

## 6 Partial embeddability property

In this section we prove that logics  $L_*(\mathcal{C})$  satisfying the conditions (C1), (C2) and (C3) enjoy the partial embeddability property, which, by Proposition 3.3, implies the FSSC.

**Definition 6.1.** *The logic  $L_*(\mathcal{C})$  has the partial embeddability property if, and only if, for every filter  $F$  of  $\mathcal{C}$  and every subdirectly irreducible  $L_*(\mathcal{C})$ -chain  $\mathcal{A}$  of type  $F$ ,  $\mathcal{A}$  is partially embeddable into  $[0, 1]_{L_*(\mathcal{C})}^F$ .*

**Proposition 6.2.** *Suppose that  $*$  is the minimum  $t$ -norm or the product  $t$ -norm, and  $\mathcal{C} \subseteq [0, 1]_*$  is a countable subalgebra. Then,  $L_*(\mathcal{C})$  has the partial embeddability property.*

*Proof.* The case of the minimum follows from the proof of [14, Theorem 7] and the case of the product follows from the proof of [28, Theorem 5].  $\square$

**Proposition 6.3.** *Suppose that  $\mathcal{C} \subseteq [0, 1]_L$  is a countable subalgebra of the standard MV-algebra such that  $C \subseteq \mathbb{Q} \cap [0, 1]$ . Then, every  $L(\mathcal{C})$ -chain is partially embeddable into  $[0, 1]_{L(\mathcal{C})}$ .*

*Proof.* Let  $\mathcal{A}$  be an  $L(\mathcal{C})$ -chain and take a finite partial subalgebra  $X \subseteq A$ . Let  $C' = \{r_i = \frac{m_i}{n_i} \mid \overline{r_i}^{\mathcal{A}} \in X\}$  and let  $n = \text{lcm}\{n_i \mid r_i = \frac{m_i}{n_i} \in C'\}$ . Then  $C' \subseteq L_{n+1}$ , where  $L_{n+1}$  denotes the subalgebra of  $[0, 1]_L$  whose universe is  $\{0, \frac{1}{n}, \dots, \frac{n-1}{n}, 1\}$ . Therefore,  $K = \{(\frac{i}{n})^{\mathcal{A}} \mid 0 \leq i \leq n\}$  forms a subalgebra of  $\mathcal{A}$  isomorphic to  $L_{n+1}$ . Moreover the set  $X \cup K$  is a finite set of  $A$  and by the partial embedding theorem for MV-algebras (see [5]), there exists a partial embedding  $h : X \cup K \rightarrow [0, 1]_L$  such that the elements of  $K$  are isomorphically mapped into elements of  $L_{n+1}$ . Thus  $h$  is also a partial embedding of  $X \cup K$  into  $[0, 1]_{L(\mathcal{C})}$ , as desired.  $\square$

**Open problem:** In the previous proposition, is the condition  $C \subseteq \mathbb{Q} \cap [0, 1]$  necessary?

**Theorem 6.4.** *Let  $*$  be a continuous  $t$ -norm and let  $\mathcal{C} \subseteq [0, 1]_*$  be a countable subalgebra satisfying conditions (C1), (C2) and (C3). Then every subdirectly irreducible  $L_*(\mathcal{C})$ -chain of type  $F$  is partially embeddable into  $[0, 1]_{L_*(\mathcal{C})}^F$ . Therefore  $L_*(\mathcal{C})$  has the partial embeddability property.*

*Proof.* Suppose that  $[0, 1]_* = \bigoplus_{i=1}^n \mathcal{A}_i$ . By Theorem 2.6 we know that the subdirectly irreducible chains of  $\mathbf{V}([0, 1]_*)$  are members of  $\mathbf{HSP}_U(\mathcal{A}_1) \cup (\mathbf{ISP}_U(\mathcal{A}_1) \oplus \mathbf{HSP}_U(\mathcal{A}_2)) \cup \dots \cup (\bigoplus_{i=1}^{n-1} \mathbf{ISP}_U(\mathcal{A}_i) \oplus \mathbf{HSP}_U(\mathcal{A}_n))$ . Knowing this structure of  $[0, 1]_*$  as ordinal sum of the three basic components, we can use the two previous results concerning expansions of  $L$  and  $\Pi$  to prove the theorem:

- (i) If there is no Łukasiewicz component in  $[0, 1]_*$  we just apply Proposition 6.2.
- (ii) Otherwise, for every Łukasiewicz component  $\mathcal{A}_i$ , condition (C3) (i.e. every  $r \in C \cap \mathcal{A}_i$  generates a finite MV-chain) amounts that, in the isomorphic copy of this component over  $[0, 1]_L$ , every  $r \in C \cap \mathcal{A}_i$  is mapped to a rational number. Therefore, using Propositions 6.2 and 6.3 the theorem is proved.

$\square$

## 7 Completeness results

In this section we study the different kinds of standard completeness properties we have considered for the family of logics  $L_*(\mathcal{C})$  when  $*$  and  $\mathcal{C}$  satisfy the conditions (C1), (C2) and (C3) as defined in Section 4. Observe that condition (C3) has been crucial for our proof of Theorem 6.4 above. In the first subsection we focus on (finite) strong completeness results and in the second we refine the results by determining which logics are canonical standard complete. Finally, in the third, we study the completeness properties when we restrict to evaluated formulas.

### 7.1 About strong standard completeness

The partial embeddability property allows us to prove some standard completeness results.

As for the FSSC, it easily follows from Proposition 3.3 that all the logics  $L_*(\mathcal{C})$  enjoying the partial embeddability property in the sense of Definition 6.1, i.e. with respect to the family  $\{[0, 1]_{L_*(\mathcal{C})}^F \mid F \text{ proper filter of } \mathcal{C}\}$ , have the FSSC (in particular those falling under the conditions of Theorem 6.4).

**Theorem 7.1.** *If  $L_*(\mathcal{C})$  satisfies the partial embeddability property, then  $L_*(\mathcal{C})$  has the FSSC. In such a case, for every finite set of formulae  $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}_\mathcal{C}}$ ,*

$$\Gamma \vdash_{L_*(\mathcal{C})} \varphi \text{ iff } \Gamma \models_{\{[0, 1]_{L_*(\mathcal{C})}^F \mid F \text{ proper filter of } \mathcal{C}\}} \varphi.$$

From this, it follows that  $L(\mathcal{C})$  also enjoys the canonical FSSC, because the algebra  $\mathcal{C}$  is simple and hence there is only one standard algebra: the canonical one.

As for the SSC, the following result will be useful.

**Theorem 7.2.** *For any continuous  $t$ -norm  $*$ ,  $L_*(\mathcal{C})$  has the SSC if, and only if,  $L_*$  has the SSC. Hence  $L_*(\mathcal{C})$  has the SSC iff  $*$  = min.*

*Proof.* From left to right it is a straightforward generalization of [7, Lemma 3.4.4.], while the converse is a consequence of  $L_*(\mathcal{C})$  being a conservative expansion of  $L_*$  and Proposition 3.5.  $\square$

As a consequence, while  $G(\mathcal{C})$  has the SSC,  $\Pi(\mathcal{C})$  and  $L(\mathcal{C})$  do not, but they still enjoy the FSSC. Actually,  $L(\mathcal{C})$  is the only logic  $L_*(\mathcal{C})$  enjoying the canonical FSSC.

**Theorem 7.3.** *Let  $*$  any continuous  $t$ -norm  $*$ . Then  $L_*(\mathcal{C})$  has the canonical FSSC if, and only if,  $*$  is the Lukasiewicz  $t$ -norm.*

*Proof.* It is already known that  $L(\mathcal{C})$  has the canonical FSSC. Conversely, assume  $L_*(\mathcal{C})$  is not  $L(\mathcal{C})$ . Then  $\mathcal{C}$  is not simple and thus it has a non-trivial proper filter  $F$  and there exists  $r \in F$ ,  $r \notin \{0, 1\}$ . Then the following semantical deduction<sup>8</sup> is valid over the canonical standard  $L_*(\mathcal{C})$ -chain but not over  $[0, 1]_{L_*(\mathcal{C})}^F$ :

$$(p \rightarrow q) \rightarrow \bar{r} \models q \rightarrow p.$$

To prove it, take into account that for every evaluation  $e$  over the canonical standard chain,  $e((p \rightarrow q) \rightarrow \bar{r}) = 1$  iff  $e(p \rightarrow q) \leq r < 1$ , and this implies  $e(q) < e(p)$ , so the deduction

---

<sup>8</sup>Actually there are simpler examples that could have been used in this proof, like  $\bar{r} \models \bar{0}$ , but the one chosen here will be useful later in Section 7.3.

	$G(\mathcal{C})$	$\Pi(\mathcal{C})$	$L(\mathcal{C})$	$L_*(\mathcal{C})$
SC	Yes	Yes	Yes	Yes
FSSC	Yes	Yes	Yes	Yes
SSC	Yes	No	No	No
Canonical FSSC	No	No	Yes	No
Canonical SSC	No	No	No	No

Table 2: Standard completeness results for logics with truth-constants enjoying the partial embeddability property.

is valid. But over the chain  $\mathcal{A} = [0, 1]_{L_*(\mathcal{C})}^F$  the formula  $(p \rightarrow q) \rightarrow \bar{r}$  is always satisfied (remember that  $\bar{r}^{\mathcal{A}} = 1$ ) and thus the deduction is not valid. Therefore, due to the algebraic completeness of  $L_*(\mathcal{C})$ ,  $q \rightarrow p$  is not provable from  $(p \rightarrow q) \rightarrow \bar{r}$  in  $L_*(\mathcal{C})$ , and thus this logic has not the canonical FSSC.  $\square$

Notice that no logic  $L_*(\mathcal{C})$  (for any continuous t-norm  $*$ ) has the canonical SSC. Indeed, if  $L_*(\mathcal{C})$  would have the canonical SSC, then it would also enjoy the SSC and hence it should be  $*$  = min, but  $G(\mathcal{C})$  does not enjoy the canonical FSSC.

All these results are collected in Table 2 (where  $*$  denotes a continuous t-norm which is the ordinal sum of at least two basic components):

## 7.2 About canonical standard completeness

Except for  $L_*(\mathcal{C})$ , the logics considered in the last section do not have the canonical FSSC, although some of them enjoy the canonical SC, i. e. their theorems are exactly the tautologies of their corresponding canonical standard algebra, as we will see in this subsection. Actually, this is the case for  $G(\mathcal{C})$  and  $\Pi(\mathcal{C})$ , already proved in [14] and in [28].

**Theorem 7.4** ([14]).  *$G(\mathcal{C})$  has the canonical SC.*

**Theorem 7.5** ([28]).  *$\Pi(\mathcal{C})$  has the canonical SC.*

But the canonical SC does not hold in general for other expansions of a logic of a continuous t-norm and its residuum. Namely, we will show that it fails for a large family of logics by providing counterexamples, i. e. by exhibiting a formula  $\varphi$  that is a tautology of the canonical standard algebra but not of the algebra  $[0, 1]_{L_*(\mathcal{C})}^F$  for some proper filter  $F$  of  $\mathcal{C}$ . In the following we assume that the first component of  $[0, 1]_*$  is defined on the interval  $[0, a]$ .

1. If the first component of the t-norm  $*$  is a copy of Łukasiewicz t-norm (and  $a \in C$ ), then an easy computation shows that the formula

$$\bar{a} \rightarrow (\neg\neg p \rightarrow p)$$

is valid in the canonical standard algebra but it is not valid in the standard chain defined by the filter  $F = [a, 1] \cap C$  (where  $\bar{a}$  is interpreted as 1).

2. If the first component of the t-norm  $*$  is a copy of product t-norm, take  $b$  as any element of  $C \cap (0, a)$ . Then an easy computation shows that the formula

$$\bar{b} \rightarrow \neg p \vee ((p \rightarrow p \& p) \rightarrow p)$$

is valid in the canonical standard algebra but it is not valid in the standard chain defined by the filter  $F = (0, 1] \cap F$  (where  $\bar{b}$  is interpreted as 1).

3. If the first component is minimum t-norm, take  $b$  as any element of  $C \cap (0, a)$ . Then the formula

$$\bar{b} \rightarrow (p \rightarrow p \& p)$$

is valid in the canonical standard algebra but it is not valid in the standard chain where  $\bar{b}$  is interpreted as 1.

Observe that for a t-norm whose decomposition begins with two copies of Lukasiewicz t-norm, the idempotent element  $a$  separating them has to belong to the truth-constants sub-algebra  $\mathcal{C}$ . Indeed, take into account that, by assumption,  $C$  must contain a non idempotent element  $c$  of the second component and for this element there exists a natural number  $n$  such that  $c^n = a$  and thus  $a \in C$ . Hence this case is subsumed in the above first item.

The remaining cases (when the first component is Lukasiewicz but its upper bound  $a$  does not belong to  $C$ ) will be divided in two different groups:

- (1) If  $[0, 1]_* = [0, a]_{\mathbf{L}} \oplus [a, 1]_{\mathbf{G}}$  or  $[0, 1]_* = [0, a]_{\mathbf{L}} \oplus [a, 1]_{\mathbf{\Pi}}$ , then the logic  $L_*(\mathcal{C})$  has the canonical SC. Actually, in that case the filters of  $\mathcal{C}$  are the same as the filters of  $C \cap [a, 1]_{\mathbf{G}}$  or  $C \cap [a, 1]_{\mathbf{\Pi}}$  respectively, and thus a modified version (given in the next two theorems) of the proof of the canonical SC for  $\mathbf{G}(\mathcal{C})$  and  $\mathbf{\Pi}(\mathcal{C})$  applies.
- (2) If  $[0, 1]_*$  is an ordinal sum of three or more components, then  $L_*(\mathcal{C})$  has not the canonical SC as the following examples show:

2.1.- If  $[0, 1]_* = [0, a]_{\mathbf{L}} \oplus [a, b]_{\mathbf{G}} \oplus \mathcal{A}$ , take  $d \in (a, b)$  and  $F = (a, 1] \cap C$ . Then the formula,

$$\bar{d} \rightarrow (\neg \neg p \rightarrow p) \vee (p \rightarrow p \& p)$$

is a tautology of the canonical standard algebra but not of  $[0, 1]_{L_*(\mathcal{C})}^F$ .

2.2.- If  $[0, 1]_* = [0, a]_{\mathbf{L}} \oplus [a, b]_{\mathbf{\Pi}} \oplus \mathcal{A}$ , take  $d \in (a, b)$  and  $F = (a, 1] \cap C$ . Then the formula,

$$\bar{d} \rightarrow (\neg \neg p \& \neg \neg q \& ((p \rightarrow p \& q) \rightarrow q) \& (q \rightarrow p) \& (p \rightarrow p \& p) \rightarrow p)$$

is a tautology of the canonical standard algebra and not of  $[0, 1]_{L_*(\mathcal{C})}^F$ .

The rest of the section is devoted to provide proofs for the canonical SC in the cases  $[0, 1]_* = [0, a]_{\mathbf{L}} \oplus [a, 1]_{\mathbf{\Pi}}$  and  $[0, 1]_* = [0, a]_{\mathbf{L}} \oplus [a, 1]_{\mathbf{G}}$ .

**Theorem 7.6.** *If  $[0, 1]_* = [0, a]_{\mathbf{L}} \oplus [a, 1]_{\mathbf{\Pi}}$ , the logic  $L_*(\mathcal{C})$  has the canonical SC if, and only if,  $a \notin C$ .*

*Proof.* The proof is rather analogous (with adequate changes) to the proof of the canonical standard completeness for the expansion of Product logic with truth-constants given in [28]. Nevertheless we give the proof for the reader's convenience.

If  $a \in C$  we have proved (by a counterexample) that  $L_*(\mathcal{C})$  has not the canonical SC. We will prove that if  $a \notin C$ , then  $L_*(\mathcal{C})$  has the canonical SC. It is obvious that there are only two proper filters of  $\mathcal{C}$ , which define two  $L_*(\mathcal{C})$ -chains over  $[0, 1]$ : the canonical one (defined by the trivial filter) where each element of  $C$  is interpreted as itself, and the chain defined by



the filter  $F = (a, 1] \cap C$  where each element of  $C$  is interpreted as itself if it belongs to the first component and as 1 if it belongs to  $F$ .

Take an arbitrary formula  $\varphi \in Fm_{\mathcal{L}_C}$  and suppose that  $\not\models_{L_*(C)} \varphi$ . We want to show that  $\not\models_{[0,1]_{L_*(C)}} \varphi$ . By the FSSC, the fact  $\not\models_{L_*(C)} \varphi$  implies that  $\not\models_{\{[0,1]_{L_*(C)}, [0,1]_{L_*(C)}^F\}} \varphi$ , hence what we need to prove is the following statement:

$$\text{if } \not\models_{[0,1]_{L_*(C)}^F} \varphi, \text{ then } \not\models_{[0,1]_{L_*(C)}} \varphi.$$

We will prove it in four steps.

Let the restriction of the t-norm  $*$  on the interval  $[a, 1]$  be defined by

$$u * v = h^{-1}(h(u) \cdot h(v))$$

for some increasing bijection  $h : [a, 1] \rightarrow [0, 1]$ . Let  $t > 0$  and define  $k_t : [0, 1] \rightarrow [0, 1]$  by

$$k_t(z) = \begin{cases} z & \text{if } z \in [0, a], \\ h^{-1}((h(z))^t) & \text{otherwise.} \end{cases}$$

Furthermore, for any evaluation  $e$  into  $[0, 1]_{L_*(C)}^F$  we consider:

(i)  $e'_t$  as the evaluation over the canonical standard chain  $[0, 1]_{L_*(C)}$  defined for any propositional variable  $x$  by,

$$e'_t(x) = k_t(e(x))$$

(ii)  $e_t^*$  as the mapping defined by  $e_t^*(\varphi) = k_t(e(\varphi))$ .

**Claim 7.7.** *For any formulae  $\varphi, \psi$ ,*

- (i)  $e_t^*(\varphi \& \psi) = e_t^*(\varphi) * e_t^*(\psi)$
- (ii)  $e_t^*(\varphi \rightarrow \psi) = e_t^*(\varphi) \Rightarrow e_t^*(\psi)$

*Proof.*

(i.1) If  $e_t^*(\varphi \& \psi) > a$  then  $e_t^*(\varphi), e_t^*(\psi) > a$ , and hence  $e(\varphi), e(\psi) > a$  as well. In this case,  $e_t^*(\varphi \& \psi) = h^{-1}((h(e(\varphi \& \psi)))^t) = h^{-1}((h(e(\varphi) * e(\psi)))^t) = h^{-1}((h(h^{-1}(h(e(\varphi))) \cdot h(e(\psi))))^t) = h^{-1}((h(e(\varphi)) \cdot h(e(\psi)))^t) = h^{-1}(h(e(\varphi))^t \cdot h(e(\psi))^t) = e_t^*(\varphi) * e_t^*(\psi)$ .

(i.2) If  $e_t^*(\varphi \& \psi) \leq a$ , then  $e_t^*(\varphi \& \psi) = e(\varphi \& \psi) = e(\varphi) * e(\psi)$ , and hence  $e(\varphi) \leq a$  or  $e(\psi) \leq a$ . W.l.o.g., assume  $e(\varphi) = \min(e(\varphi), e(\psi)) \leq a$ , and hence  $e_t^*(\varphi) = e(\varphi)$ . Then, if  $e(\psi) > a$  then  $e_t^*(\psi) > a$  and  $e(\varphi) * e(\psi) = e(\varphi) = e_t^*(\varphi) * e_t^*(\psi)$ . Otherwise, if  $e(\psi) \leq a$ , then  $e_t^*(\psi) = e(\psi)$ .

(ii) It follows from (i). □

**Claim 7.8.** *For any formula  $\psi$ ,*

- (i) if  $e(\psi) \in (a, 1]$ , then  $e'_t(\psi) \in (a, 1]$ ,
- (ii) if  $e(\psi) \in [0, a]$ , then  $e'_t(\psi) = e(\psi)$ .

*Proof.* The proof is by induction:

- If  $\psi$  is a propositional variable, the statement is obviously true by definition of  $e'_t$ .
- If  $\psi$  is a truth-constant  $\bar{r}$ , either  $r > a$  and then  $e(\bar{r}) = 1$  and  $e'_t(\bar{r}) = r > a$ , or  $r < a$  and then  $e(\bar{r}) = r = e'_t(\bar{r})$ .

- If  $\psi = \delta \& \gamma$ , then we have two cases:
  - 1.- If  $e(\psi) \in (a, 1]$  then it is so for  $e(\delta), e(\gamma)$ . and thus for  $e'_t(\delta), e'_t(\gamma)$  and, as a consequence, for  $e'_t(\psi)$ .
  - 2.- If  $e(\psi) \in [0, a]$ , then at least one of  $e(\delta), e(\gamma)$  must belong to  $[0, a]$ . Suppose that  $e(\delta) \in [0, a]$ , hence by hypothesis  $e'_t(\delta) \in [0, a]$  as well, hence  $e'_t(\psi) = e'_t(\delta) * e'_t(\gamma) \leq a$ .
- If  $\psi = \delta \rightarrow \gamma$ , then we have several cases:
  - 1.- If  $e(\psi) = 1$ , then  $e(\delta) \leq e(\gamma)$  and we have two cases:
    - 1.1.- If  $e(\delta), e(\gamma)$  belong to the same subinterval the statement is obvious.
    - 1.2.- If  $e(\delta), e(\gamma)$  belong to different subintervals, the statement also holds true by the induction hypothesis.
  - 2.- If  $e(\psi) < 1$  then  $e(\delta) > e(\gamma)$  and we have also two cases:
    - 2.1.- If  $e(\psi) > a$ , then  $e(\delta) > e(\gamma) > a$  and thus  $e'_t(\psi) \in (a, 1]$ .
    - 2.2.- If  $e(\psi) \leq a$ , then  $e(\delta) > e(\gamma) \in [0, a]$  and we have two possibilities depending on which component  $e(\gamma)$  belongs. But, in any case, the induction hypothesis proves easily that  $e'_t(\psi) = e(\psi)$ .

□

The set  $[0, 1]^{\mathbb{R}^+}$  of all functions from  $\mathbb{R}^+$  into  $[0, 1]$  becomes an  $L_*$ -algebra with the operations  $*$  and  $\Rightarrow_*$  defined pointwise and with the constant function 0 as bottom and the constant function 1 as top.

Let  $F \subseteq [0, 1]^{\mathbb{R}^+}$  be the set of all functions  $f : \mathbb{R}^+ \rightarrow [0, 1]$  satisfying the following condition:

(E) There exists  $c$  such that  $a < c \leq 1$  and  $t_0 > 0$  such that  $c \leq f(t)$  for all  $t \geq t_0$ .

It is immediate to verify that  $F$  is an implicative filter (as defined in [6, Lemma 1.5]) on the  $L_*$ -algebra  $[0, 1]^{\mathbb{R}^+}$ . The congruence relation defined by  $F$  on  $[0, 1]^{\mathbb{R}^+}$ ,  $f \sim g$  iff  $f \Rightarrow g \in F$  and  $g \Rightarrow f \in F$ , is defined by

$$f \sim g \text{ iff there exist } c, d \in (a, 1] \text{ and } t_0 > 0 \text{ such that} \\ c * g(t) \leq f(t) \leq d \Rightarrow g(t) \text{ for all } t > t_0.$$

Then, one can check that  $\sim$  satisfies the following properties, where  $f_a$  stands for the constant function with value  $a$ .

**Claim 7.9.** *The congruence relation  $\sim$  satisfies:*

- (i)  $f \sim f_a$  if, and only if, there exists  $t_0$  such that  $f(t) = a$  for all  $t \geq t_0$ .
- (ii) Suppose  $f \sim g$ . Then  $\lim_{t \rightarrow \infty} g(t) = a$  if, and only if,  $\lim_{t \rightarrow \infty} f(t) = a$ .

*Proof.* Just recall that, if  $c, d \in (a, 1]$ , then  $c * a = d \Rightarrow a = a$ . □

**Claim 7.10.** *Let  $e$  and  $e'_t$  as above be given. For every formula  $\phi$  such that  $a < e(\phi) < 1$ , let  $g_\phi(t) = e_t^*(\phi)$  and  $f_\phi(t) = e'_t(\phi)$ . Then we have  $f_\phi \sim g_\phi$ . In particular,  $\lim_{t \rightarrow \infty} e'_t(\phi) = a$ .*

*Proof.* Let us proceed by induction on the complexity of  $\phi$ .

1.  $\phi$  is a constant  $\bar{r}$ . Then it must be  $r > a$ , hence  $e(\bar{r}) = 1$ , and then  $g_{\bar{r}}(t) = k_t(e(\bar{r})) = k_t(1) = 1$  and  $f_{\bar{r}}(t) = e'_t(\bar{r}) = r$ , and obviously  $r \sim 1$ .

2.  $\phi$  is a propositional variable. Then it is a direct consequence of the definition ( $f_x(t) = g_x(t)$ ).

3.  $\phi = (\psi_1 \& \psi_2)$ . If  $e(\psi_1 \& \psi_2) > a$  then  $e(\psi_1), e(\psi_2) > a$ , hence  $g_{\psi_1}, g_{\psi_2}, f_{\psi_1}, f_{\psi_2} \in F$ . Then:

$$g_{\psi_1 \& \psi_2}(t) = e_t^*(\psi_1 \& \psi_2) = e_t^*(\psi_1) * e_t^*(\psi_2) = g_{\psi_1}(t) * g_{\psi_2}(t).$$

$$f_{\psi_1 \& \psi_2}(t) = e_t'(\psi_1 \& \psi_2) = e_t'(\psi_1) * e_t'(\psi_2) = f_{\psi_1}(t) * f_{\psi_2}(t).$$

Since  $\sim$  is a congruence, if we suppose that  $f_{\psi_1} \sim g_{\psi_1}$  and  $f_{\psi_2} \sim g_{\psi_2}$ , we can conclude that  $f_{\psi_1 \& \psi_2} \sim g_{\psi_1 \& \psi_2}$ .

4.  $\phi = (\psi_1 \rightarrow \psi_2)$ . If  $a < e(\psi_1 \rightarrow \psi_2) < 1$  then  $e(\psi_1), e(\psi_2) > a$ , hence  $g_{\psi_1}, g_{\psi_2}, f_{\psi_1}, f_{\psi_2} \in F$ . Then:

$$g_{\psi_1 \rightarrow \psi_2}(t) = e_t^*(\psi_1 \rightarrow \psi_2) = e_t^*(\psi_1) \Rightarrow e_t^*(\psi_2) = g_{\psi_1}(t) \Rightarrow g_{\psi_2}(t).$$

$$f_{\psi_1 \rightarrow \psi_2}(t) = e_t'(\psi_1 \rightarrow \psi_2) = e_t'(\psi_1) \Rightarrow e_t'(\psi_2) = f_{\psi_1}(t) \Rightarrow f_{\psi_2}(t).$$

Using again the fact that  $\sim$  is a congruence, from the hypothesis  $f_{\psi_1} \sim g_{\psi_1}$  and  $f_{\psi_2} \sim g_{\psi_2}$ , we obtain  $f_{\psi_1 \rightarrow \psi_2} \sim g_{\psi_1 \rightarrow \psi_2}$ .

The first statement of the claim is proved. The second statement follows from the first statement and (ii) of Claim 7.9.  $\square$

Finally, we can obtain the result we are looking for:

Let  $\varphi$  be not valid in  $[0, 1]_{L_*(C)}^F$ . There exists an evaluation  $e$  such that  $e(\varphi) < 1$ . By Claim 7.10,  $\lim_{t \rightarrow \infty} e_t'(\varphi) = a$  as well, hence for some large enough  $t$ ,  $e_t'(\varphi) < 1$ . Thus  $\varphi$  is not valid in the canonical standard chain.  $\square$

**Theorem 7.11.** *If  $[0, 1]_* = [0, a]_L \oplus [a, 1]_G$ , the logic  $L_*(C)$  has the canonical SC if, and only if,  $a \notin C$ .*

*Proof.* If  $a \in C$  we have proved that the logic  $L_*(C)$  has not the canonical SC. Now we will prove the canonical standard completeness of  $L_*(C)$  in the case that  $a \notin C$ .

The proof is analogous (with adequate changes) to the one given in [14] for proving canonical standard completeness for the expansion of Gödel logic with truth-constants. We will sketch it. We know  $L_*(C)$  enjoys the FSSC and thus we have to prove that the tautologies of the canonical standard chain are contained in the tautologies of any other standard chain. The proof is by contraposition. Suppose that there is a formula  $\varphi$  and an evaluation  $e$  over a standard chain defined by a proper filter  $F$  such that  $e(\varphi) < 1$  and we have to prove that there is an evaluation  $e'$  over the canonical standard chain such that  $e'(\varphi) < 1$ .

Take  $X = \{e(\psi) \mid \psi \text{ subformula of } \varphi\} \cup \{0, 1\}$  and let  $\alpha = \min\{r \in F \mid \bar{r} \text{ appears in } \varphi\}$ . Now, define  $f : X \rightarrow [0, \alpha]$  by stipulating that its restriction over  $X \cap [0, a]$  is the identity function and its restriction over  $X \cap [a, 1]$  is an increasing function with  $f(a) = a$  and  $f(1) = \alpha$ . Then define  $e'$  as the evaluation over the canonical standard algebra such that

$$e'(x) = \begin{cases} f(e(x)), & \text{if } x \text{ propositional variable in } \varphi \\ 1, & \text{if } x \text{ propositional variable not in } \varphi \\ r, & \text{if } x = \bar{r} \end{cases}$$

By induction we can prove that for each subformula  $\psi$  of  $\varphi$  we have:

$[0, 1]_*$	Canonical SC for $L_*(\mathcal{C})$
$[0, 1]_{\mathbf{L}}$	Yes
$[0, 1]_{\mathbf{G}}$	Yes
$[0, 1]_{\mathbf{\Pi}}$	Yes
$[0, a]_{\mathbf{G}} \oplus \mathcal{A}$	No
$[0, a]_{\mathbf{\Pi}} \oplus \mathcal{A}$	No
$[0, a]_{\mathbf{L}} \oplus \mathcal{A}, \quad a \in C$	No
$[0, a]_{\mathbf{L}} \oplus [a, 1]_{\mathbf{G}}, \quad a \notin C$	Yes
$[0, a]_{\mathbf{L}} \oplus [a, 1]_{\mathbf{\Pi}}, \quad a \notin C$	Yes
$[0, a]_{\mathbf{L}} \oplus [a, b]_{\mathbf{G}} \oplus \mathcal{A}, \quad a \notin C$	No
$[0, a]_{\mathbf{L}} \oplus [a, b]_{\mathbf{\Pi}} \oplus \mathcal{A}, \quad a \notin C$	No

Table 3: Canonical standard completeness results for logics  $L_*(\mathcal{C})$  when  $*$  is a finite ordinal sum of the three basic components.

- $e'(\psi) \geq \alpha$ , if  $e(\psi) = 1$
- $a < e'(\psi) < \alpha$ , if  $a < e(\psi) < 1$
- $e'(\psi) = e(\psi)$ , if either  $e(\psi) \in [0, a]$  or  $e(\psi) = e(x)$  for some  $x$  which is a propositional variable or truth-constant appearing in  $\varphi$ .

In particular, from these properties, we see that the evaluation  $e'$  over the canonical standard chain is such that  $e'(\varphi) < 1$ , which ends the proof.  $\square$

Summarizing (see Table 3) the canonical SC holds for the expansion of the logic of a continuous t-norm  $*$  by a set of truth constants satisfying the three conditions (C1), (C2) and (C3) if, and only if,  $[0, 1]_*$  is either one of the three basic algebras  $([0, 1]_{\mathbf{L}}, [0, 1]_{\mathbf{G}}$  or  $[0, 1]_{\mathbf{\Pi}})$  or  $[0, 1]_* = [0, a]_{\mathbf{L}} \oplus [a, 1]_{\mathbf{\Pi}}$  or  $[0, 1]_* = [0, a]_{\mathbf{L}} \oplus [a, 1]_{\mathbf{G}}$  with  $a \notin C$ .

### 7.3 Completeness results for evaluated formulae

This section deals with completeness results when we restrict to what we call *evaluated formulae*, formulae of type  $\bar{r} \rightarrow \varphi$ , where  $\varphi$  is a formula without new truth-constants. From the previous sections we know that FSSC is true for the expansion of  $L_*$  with a subalgebra of truth-constants (not only for evaluated formulae), but the canonical FSSC is only true for expansions of Łukasiewicz logic. Next theorem<sup>9</sup> states the canonical FSSC restricted to evaluated formulae for the expansions of Gödel and Product logics with truth-constants.

**Theorem 7.12.**  *$G(\mathcal{C})$  and  $\Pi(\mathcal{C})$  have the canonical FSSC if we restrict the language to evaluated formulae, i. e. given a natural number  $n \geq 1$ , formulae  $\varphi_1, \dots, \varphi_n, \psi \in Fm_{\mathcal{L}}$ , and  $r_1, \dots, r_n, s \in C$ , we have:*

- $\{\bar{r}_i \rightarrow \varphi_i : 1 \leq i \leq n\} \vdash_{G(\mathcal{C})} \bar{s} \rightarrow \psi$  if, and only if,  
 $\{\bar{r}_i \rightarrow \varphi_i : 1 \leq i \leq n\} \models_{[0,1]_{G(\mathcal{C})}} \bar{s} \rightarrow \psi$ .

<sup>9</sup>The proof can be found in [14] for the case of  $G(\mathcal{C})$  and in [28] for  $\Pi(\mathcal{C})$ .

- $\{\bar{r}_i \rightarrow \varphi_i : 1 \leq i \leq n\} \vdash_{\Pi(\mathcal{C})} \bar{s} \rightarrow \psi$  if, and only if,  
 $\{\bar{r}_i \rightarrow \varphi_i : 1 \leq i \leq n\} \models_{[0,1]_{\Pi(\mathcal{C})}} \bar{s} \rightarrow \psi$ .

One could wonder whether these restricted completeness results hold for formulae of type  $\varphi \rightarrow \bar{r}$  such that  $\varphi$  does not contain a truth-constant different from  $\bar{0}$  and  $\bar{1}$ . Actually the situation is different for  $G(\mathcal{C})$  and  $\Pi(\mathcal{C})$ :

- As for  $G(\mathcal{C})$ , the result does not hold. For instance, it is easy to check that

$$\neg\neg p \rightarrow \bar{0.7} \models_{[0,1]_{G(\mathcal{C})}} p \rightarrow \bar{0.2}$$

since the premise is only true if  $e(p) = 0$  while

$$\neg\neg p \rightarrow \bar{0.7} \not\models_{G(\mathcal{C})} p \rightarrow \bar{0.2}.$$

In fact, by the deduction-detachment theorem and the canonical SC of  $G(\mathcal{C})$  this is equivalent to show that

$$\not\models_{[0,1]_{G(\mathcal{C})}} (\neg\neg p \rightarrow \bar{0.7}) \rightarrow (p \rightarrow \bar{0.2}),$$

and this is true since, if  $e(p) = c$  for  $c > 0.2$ , an easy computation shows that  $e((\neg\neg p \rightarrow \bar{0.7}) \rightarrow (p \rightarrow \bar{0.2})) = 0.2$ .

- As for  $\Pi(\mathcal{C})$ , the result holds true when the formulae  $\varphi \rightarrow \bar{r}$  are such that  $r > 0$  (see [28]), since in such a case these formulae are trivially satisfied in the (unique) non-canonical standard  $\Pi(\mathcal{C})$ -algebra  $[0, 1]_{\Pi(\mathcal{C})}^F$  for  $F = (0, 1]$ .

In any case, the result is not true if we allow formulae of both types together. Indeed, given  $r \neq 1$ , it is obvious that the semantical deduction (already used in the proof of Theorem 7.3)

$$(p \rightarrow q) \rightarrow \bar{r} \models \bar{1} \rightarrow (q \rightarrow p)$$

is valid over the canonical standard chain but not over a standard chain where  $\bar{r}$  is interpreted as 1.

Now we will study the canonical SC and the canonical FSSC restricted to evaluated formulae for other logics. Suppose that  $*$  is a t-norm which is a non-trivial finite ordinal sum of the basic components and suppose that the first component is defined on the interval  $[0, a]$ . For the following cases we can refute the canonical SC (and hence the canonical FSSC as well):

1. The first component of the t-norm  $*$  is a copy of Łukasiewicz t-norm and  $a \in C$ .
2. The first component of the t-norm  $*$  is a copy of product t-norm.
3. The first component of the t-norm  $*$  is a copy of minimum t-norm.
4. There are more than two components and the second component is a copy of minimum t-norm.
5. There are more than two components and the second component is a copy of product t-norm.

Indeed, for all these cases we can use the same counterexample that was given in the previous section to show that the corresponding logics do not enjoy the canonical SC, because the counterexamples were actually evaluated formulae.

The following theorem deals with the remaining case of ordinal sums of two basic components. The case  $[0, 1]_* = [0, a]_L \oplus [a, 1]_L$  is not considered here since in such a situation, under the working hypothesis that there exists  $b \in (a, 1]$  such that  $b \in C$ , necessarily  $a \in C$  as well.

**Theorem 7.13.** *The restriction to evaluated formulae of the logic  $L_*(C)$  when either  $[0, 1]_* = [0, a]_L \oplus [a, 1]_G$  or  $[0, 1]_* = [0, a]_L \oplus [a, 1]_\Pi$  and  $a \notin C$  has the canonical FSSC.*

*Proof.* The proof is an easy modification of the proofs given in [14] for  $G(C)$  and in [28] for  $\Pi(C)$ . Here we only sketch the proof for  $[0, 1]_* = [0, a]_L \oplus [a, 1]_\Pi$ .

What we want to prove is:

$$\begin{aligned} \{\bar{r}_i \rightarrow \varphi_i \mid i = 1, \dots, n\} \vdash_{L_*(C)} \bar{s} \rightarrow \psi \\ \text{if, and only if,} \\ \{\bar{r}_i \rightarrow \varphi_i \mid i = 1, \dots, n\} \models_{[0,1]_{L_*(C)}} \bar{s} \rightarrow \psi \end{aligned}$$

where  $\varphi_i$  and  $\psi$  are  $L_*(C)$ -formulae, i.e., formulae not containing truth-constants different from  $\bar{0}$  and  $\bar{1}$ . Actually, as always, one direction (soundness) is obvious. To prove the converse direction, i.e.

$$\text{If } \{\bar{r}_i \rightarrow \varphi_i \mid i = 1, \dots, n\} \models_{[0,1]_{L_*(C)}} \bar{s} \rightarrow \psi, \text{ then } \{\bar{r}_i \rightarrow \varphi_i \mid i = 1, \dots, n\} \vdash_{L_*(C)} \bar{s} \rightarrow \psi,$$

it is enough to combine the FSSC of  $L_*(C)$  with the following result:

**Claim 7.14.** *If  $\{\bar{r}_i \rightarrow \varphi_i \mid i = 1, \dots, n\} \models_{[0,1]_{L_*(C)}} \bar{s} \rightarrow \psi$  then  $\{\bar{r}_1 \rightarrow \varphi_1, \dots, \bar{r}_n \rightarrow \varphi_n\} \models_{[0,1]_{L_*(C)}^F} \bar{s} \rightarrow \psi$ , where  $F = (a, 1] \cap C$ .*

To prove it we may assume without loss of generality that  $r_i > 0$  for all  $i$  and  $s > 0$ . Suppose  $\{\bar{r}_1 \rightarrow \varphi_1, \dots, \bar{r}_n \rightarrow \varphi_n\} \not\models_{[0,1]_{L_*(C)}^F} \bar{s} \rightarrow \psi$ . Then there exists a  $[0, 1]_{L_*(C)}^F$ -evaluation  $e$  such that  $e(\bar{r}_1 \rightarrow \varphi_1) = \dots = e(\bar{r}_n \rightarrow \varphi_n) = 1$  and  $e(\bar{s} \rightarrow \psi) < 1$ .

(i) If  $s \in (0, a]$ , and hence  $e(\bar{s}) = s$  and  $e(\psi) < s$ , then take the evaluation  $e'$  over the canonical standard chain defined by  $e'(p) = e(p)$  for any propositional variable  $p$ . Notice that, since  $e(\bar{r}) \geq e'(\bar{r})$  and  $e(\varphi) = e'(\varphi)$ , it is easy to compute that  $e'(\bar{r}_1 \rightarrow \varphi_1) = \dots = e'(\bar{r}_n \rightarrow \varphi_n) = 1$  and  $e'(\bar{s} \rightarrow \psi) = e(\bar{s} \rightarrow \psi) < 1$ .

(ii) If  $s \in (a, 1]$ , and hence  $e(\bar{s}) = 1$  and  $e(\psi) < 1$ , we can assume  $e(\psi) \geq s$ , otherwise the above evaluation  $e'$  does the job. Then take the family of evaluations  $e'_t$  over the canonical standard chain defined by  $e'_t(p) = k_t(e(p))$  for any propositional variable  $p$ , where  $k_t : [0, 1] \rightarrow [0, 1]$  is the mapping defined in the proof of Theorem 7.6, i.e.

$$k_t(z) = \begin{cases} z & \text{if } z \in [0, a], \\ h^{-1}((h(z))^t) & \text{otherwise.} \end{cases}$$

By definition of  $k_t$  it is easy to find a large enough  $t$  such that  $a < e'_t(\psi) < s$ , and hence  $e'_t(\bar{s} \rightarrow \psi) < 1$ . Moreover, it is easy to check that we still have  $e'_t(\bar{r}_1 \rightarrow \varphi_1) = \dots = e'_t(\bar{r}_n \rightarrow \varphi_n) = 1$ . Indeed, if  $r_i \in (a, 1]$ , then  $e(r_i) = 1$  and  $e(\varphi) = 1$ , hence  $e'_t(\varphi) = 1$  as well. If  $r_i \in (0, a]$ , then  $e'_t(\bar{r}_i) = e(\bar{r}_i) = r_i$  and  $e(\varphi_i) \geq r_i$ . Now, if  $e(\varphi_i) \leq a$  then  $e'_t(\varphi_i) = e(\varphi_i)$ , otherwise, if  $e(\varphi_i) > a$  then  $e'_t(\varphi_i) > a$  as well. In any case,  $e'_t(\varphi_i) \geq r_i$ , hence  $e'_t(\bar{r}_i \rightarrow \varphi_i) = 1$ .  $\square$

Restricted to evaluated formulae of $L_*(\mathcal{C})$		
$[0, 1]_*$	Canonical SC	Canonical FSSC
$[0, 1]_L$	Yes	Yes
$[0, 1]_G$	Yes	Yes
$[0, 1]_\Pi$	Yes	Yes
$[0, a]_G \oplus \mathcal{A}$	No	No
$[0, a]_\Pi \oplus \mathcal{A}$	No	No
$[0, a]_L \oplus \mathcal{A}, \quad a \in C$	No	No
$[0, a]_L \oplus [a, 1]_G, \quad a \notin C$	Yes	Yes
$[0, a]_L \oplus [a, 1]_\Pi, \quad a \notin C$	Yes	Yes
$[0, a]_L \oplus [a, b]_G \oplus \mathcal{A}, \quad a \notin C$	No	No
$[0, a]_L \oplus [a, b]_\Pi \oplus \mathcal{A}, \quad a \notin C$	No	No

Table 4: Canonical SC and FSSC results restricted to evaluated formulae for logics  $L_*(\mathcal{C})$  when  $*$  is a finite ordinal sum of the three basic components.

All these results are summarized in Table 4 where, interestingly, it turns out that both standard completeness properties (SC and FSSC) restricted to evaluated formulae are equivalent. Furthermore, comparing Table 4 with Table 3 we realise that for a logic  $L_*(\mathcal{C})$  where  $*$  is a finite ordinal sum of basic components, the canonical SC turns out to be equivalent to the canonical SC (and to the canonical FSSC) restricted to evaluated formulae.

**Open problem:** Are these equivalences valid for wider classes of  $L_*(\mathcal{C})$  logics?

Finally, it is easy to see that none of the considered logics enjoy the canonical SSC when restricted to evaluated formulae. Indeed, for every continuous t-norm  $*$ , we have  $\{(\frac{n}{n+1}) \rightarrow p \mid n \in \mathbb{N}\} \models_{[0,1]_{L_*(\mathcal{C})}} p$ . But if  $\{(\frac{n}{n+1}) \rightarrow p \mid n \in \mathbb{N}\} \vdash_{L_*(\mathcal{C})} p$ , then, since  $L_*(\mathcal{C})$  is a finitary logic, there would exist  $n_0 \in \mathbb{N}$  such that  $(\frac{n_0}{n_0+1}) \rightarrow p \vdash_{L_*(\mathcal{C})} p$ , hence, by soundness we would have  $(\frac{n_0}{n_0+1}) \rightarrow p \models_{[0,1]_{L_*(\mathcal{C})}} p$ ; a contradiction.

## 8 Adding truth-constants to expansions with $\Delta$ connective

For every continuous t-norm  $*$ , one can define an expansion of the logic  $L_*$  by adding to the language a unary connective  $\Delta$ , and adding to the Hilbert-style system of  $L_*$  the following axiom schemata:

$$(\Delta 1) \quad \Delta\varphi \vee \neg\Delta\varphi$$

$$(\Delta 2) \quad \Delta(\varphi \vee \psi) \rightarrow (\Delta\varphi \vee \Delta\psi)$$

$$(\Delta 3) \quad \Delta\varphi \rightarrow \varphi$$

$$(\Delta 4) \quad \Delta\varphi \rightarrow \Delta\Delta\varphi$$

$$(\Delta 5) \quad \Delta(\varphi \rightarrow \psi) \rightarrow (\Delta\varphi \rightarrow \Delta\psi)$$

and the rule of necessitation:

$$\frac{\varphi}{\Delta\varphi}$$

This logic is denoted by  $L_{*\Delta}$ . It is algebraizable and its equivalent algebraic semantics is the variety of  $L_{*\Delta}$ -algebras, i.e. expansions with  $\Delta$  of  $L_*$ -algebras satisfying the translation of the axioms  $(\Delta 1), \dots, (\Delta 5)$  and the equation  $\Delta \bar{1} \approx \bar{1}$ . It is easy to prove that all  $L_{*\Delta}$ -algebras are representable as subdirect products of  $L_{*\Delta}$ -chains. The interpretation of the  $\Delta$  connective in these chains is very simple, namely if  $\mathcal{A}$  is an  $L_{*\Delta}$ -chain, then  $\Delta^{\mathcal{A}}(\bar{1}^{\mathcal{A}}) = \bar{1}^{\mathcal{A}}$  and  $\Delta^{\mathcal{A}}(a) = \bar{0}^{\mathcal{A}}$  for every  $a \in A \setminus \{\bar{0}^{\mathcal{A}}\}$ . We will denote by  $[0, 1]_{L_{*\Delta}}$  the expansion of  $[0, 1]_*$  with the Delta operation.

**Proposition 8.1.** *For every continuous t-norm  $*$ ,  $L_{*\Delta}$  is a conservative expansion of  $L_*$ .*

*Proof.* It is obvious that every  $L_*$ -chain is the reduct of an  $L_{*\Delta}$ -chain (just take the same chain and define  $\Delta$  in the only possible way for chains), thus we can apply Proposition 2.4.  $\square$

All the results about  $L_{*\Delta}$  logics so far mentioned can be found in the literature (see for instance [18, 7]).

**Proposition 8.2.** *For every continuous t-norm  $*$ , the logic  $L_{*\Delta}$  enjoys the FSSC.*

*Proof.*<sup>10</sup> Let  $*$  be a continuous t-norm. We know that every  $L_*$ -chain is partially embeddable into  $[0, 1]_*$ . Let  $\mathcal{A}$  be an  $L_{*\Delta}$ -chain and take any finite set  $X \subseteq A$ . Since the  $\mathcal{L}$ -reduct of  $\mathcal{A}$  is an  $L_*$ -chain, then there is a partial embedding  $f$  (in the language  $\mathcal{L}$ ) of  $X$  into  $[0, 1]_*$ . It is straightforward that  $f$  is also a partial embedding (in the language  $\mathcal{L} \cup \{\Delta\}$ ) of  $X$  into  $[0, 1]_{L_{*\Delta}}$ .  $\square$

**Proposition 8.3.** *Let  $*$  be a continuous t-norm.  $L_{*\Delta}$  enjoys the SSC if, and only if,  $*$  = min.*

*Proof.* Suppose that  $*$  = min and take a countable  $L_{*\Delta}$ -chain  $\mathcal{A}$ . Then its  $\mathcal{L}$ -reduct is a G-chain and it is well-known that it can be embedded into the standard G-chain  $[0, 1]_*$ . Clearly, the same embedding works for  $\mathcal{A}$  and  $[0, 1]_{L_{*\Delta}}$ . Conversely, if  $*$   $\neq$  min, then  $L_*$  has not the SSC, and since  $L_{*\Delta}$  is a conservative expansion of  $L_*$ , then by Proposition 3.5 also  $L_{*\Delta}$  has not the SSC.  $\square$

Now we will consider expansions with truth-constants for these logics with  $\Delta$ . Given a continuous t-norm  $*$  and a countable subalgebra  $\mathcal{C} \subseteq [0, 1]_*$ , we define the logic  $L_{*\Delta}(\mathcal{C})$  as the expansion of  $L_{*\Delta}$  in the language  $\mathcal{L}_{\mathcal{C}}$  obtained by adding the following book-keeping axioms:

$$\begin{aligned} \bar{r} \&\bar{s} &\leftrightarrow &\bar{r} * \bar{s} \\ (\bar{r} \rightarrow \bar{s}) &\leftrightarrow &\bar{r} \Rightarrow \bar{s} \\ \Delta \bar{r} &\leftrightarrow &\Delta \bar{r} \end{aligned}$$

for every  $r, s \in \mathcal{C}$ .

Again, using the general facts mentioned in the preliminaries we know that  $L_{*\Delta}(\mathcal{C})$  is an algebraizable logic and we can axiomatize its equivalent algebraic semantics, the variety of  $L_{*\Delta}(\mathcal{C})$ -algebras. Moreover, it can be easily checked that  $L_{*\Delta}(\mathcal{C})$ -algebras are representable as subdirect product of chains.

---

<sup>10</sup>An alternative proof of this can be found in [7].



**Proposition 8.4.** *For every continuous t-norm  $*$  and every countable subalgebra  $\mathcal{C} \subseteq [0, 1]_*$ , the logic  $L_{*\Delta}(\mathcal{C})$  is a conservative expansion of  $L_{*\Delta}$ .*

*Proof.* It is analogous to the proof of Proposition 4.5.  $\square$

**Lemma 8.5.** *Let  $\mathcal{A}$  be a non-trivial  $L_{*\Delta}(\mathcal{C})$ -chain,  $*$  be a continuous t-norm and  $\mathcal{C} \subseteq [0, 1]_*$  be a countable subalgebra. Then, for every  $r, s \in \mathcal{C}$  such that  $r < s$ , we have  $\bar{r}^{\mathcal{A}} < \bar{s}^{\mathcal{A}}$ .*

*Proof.* Suppose  $\bar{r}^{\mathcal{A}} = \bar{s}^{\mathcal{A}}$ . Then,  $\bar{1}^{\mathcal{A}} = \Delta \bar{1}^{\mathcal{A}} = \Delta \bar{s} \rightarrow \bar{r}^{\mathcal{A}} = \overline{\Delta(s \rightarrow t)}^{\mathcal{A}} = \bar{0}^{\mathcal{A}}$ ; a contradiction.  $\square$

Therefore, in the variety of  $L_{*\Delta}(\mathcal{C})$ -algebras all standard chains over  $[0, 1]_*$  are of type  $F = \{1\}$ , among them the canonical chain that we denote by  $[0, 1]_{L_{*\Delta}(\mathcal{C})}$ .

**Theorem 8.6.** *Let  $*$  be a continuous t-norm and let  $\mathcal{C} \subseteq [0, 1]_*$  be a countable subalgebra. If  $L_*(\mathcal{C})$  has the partial embeddability property<sup>11</sup>, then  $L_{*\Delta}(\mathcal{C})$  has the canonical FSSC.*

*Proof.* Take an arbitrary  $L_{*\Delta}(\mathcal{C})$ -chain  $\mathcal{A}$ . Then, the  $\mathcal{L}_C$ -reduct of  $\mathcal{A}$  is partially embeddable into  $[0, 1]_{L_*(\mathcal{C})}$ , so it is clear that also  $\mathcal{A}$  is partially embeddable into  $[0, 1]_{L_{*\Delta}(\mathcal{C})}$ .  $\square$

**Proposition 8.7.** *Let  $*$  be a continuous t-norm and  $\mathcal{C}$  a countable subalgebra  $\mathcal{C} \subseteq [0, 1]_*$  such that  $L_*(\mathcal{C})$  satisfies the partial embeddability property. Then,  $L_{*\Delta}(\mathcal{C})$  is a conservative expansion of  $L_*(\mathcal{C})$  if, and only if,  $L_*(\mathcal{C})$  enjoys the canonical FSSC.*

*Proof.* One direction is again analogous to the proof of Proposition 4.5. For the converse, suppose that  $L_*(\mathcal{C})$  does not enjoy the canonical FSSC. Then there is a finite set of formulae  $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}_C}$  such that  $\Gamma \models_{[0,1]_{L_*(\mathcal{C})}} \varphi$  and  $\Gamma \not\models_{L_*(\mathcal{C})} \varphi$ . But then,  $\Gamma \models_{[0,1]_{L_{*\Delta}(\mathcal{C})}} \varphi$  and hence  $\Gamma \vdash_{L_{*\Delta}(\mathcal{C})} \varphi$ , by the canonical FSSC of  $L_{*\Delta}(\mathcal{C})$ . Therefore,  $L_{*\Delta}(\mathcal{C})$  is not a conservative expansion of  $L_*(\mathcal{C})$ .  $\square$

**Theorem 8.8.** *Let  $*$  be a continuous t-norm and let  $\mathcal{C} \subseteq [0, 1]_*$  be a countable subalgebra. Then,  $L_{*\Delta}$  has the SSC if, and only if,  $L_{*\Delta}(\mathcal{C})$  has the canonical SSC.*

*Proof.* From left to right it is again proved by generalizing [7, Lemma 3.4.4.], while the converse is a consequence of Proposition 3.5.  $\square$

**Corollary 8.9.** *Let  $*$  be a continuous t-norm and let  $\mathcal{C} \subseteq [0, 1]_*$  be a countable subalgebra.  $L_{*\Delta}(\mathcal{C})$  enjoys the SSC if, and only if,  $*$  = min.*

## 9 Concluding remarks

In this paper we have provided a complete description of completeness results for the expansions of logics of continuous t-norms with a set of truth-constants  $\{\bar{r} \mid r \in \mathcal{C}\}$ , for a suitable countable  $\mathcal{C} \subseteq [0, 1]$ , when (i) the t-norm is a finite ordinal sum of basic components, (ii) the set of truth-constants *covers* all the unit interval in the sense that each component of the t-norm contains at least one value of  $\mathcal{C}$  different from the bounds of the component, and (iii) the truth-constants in Łukasiewicz components behave as rational numbers. From a practical point of view, it seems these cases are the most interesting for fuzzy logic-based systems, since

<sup>11</sup>In particular for any  $*$  satisfying conditions (C1), (C2) and (C3).

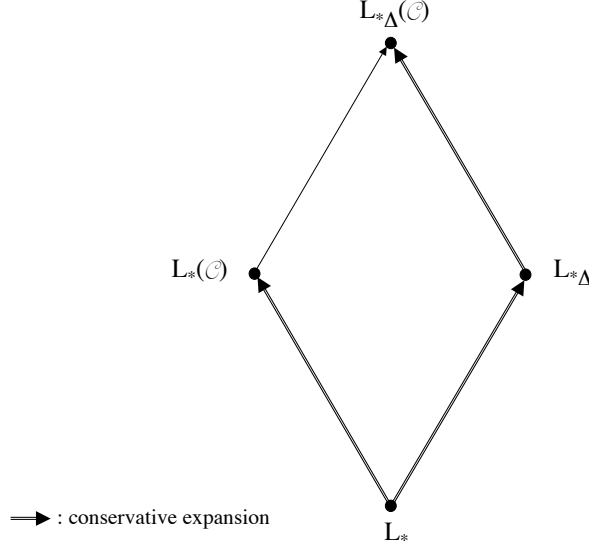


Figure 1: Diagram of expansions showing which are always conservative.

they usually consider a set of truth values spread all over the real unit interval, and hence it is natural to assume there are elements of  $C$  in each component of the t-norm. In all these cases completeness results have been settled, with one important exception. Actually, it remains as open problem the case of a logic of a t-norm with a Lukasiewicz component containing some  $r \in C$  which generates an infinite MV-chain (in other words, when  $r$  corresponds to an irrational value in the isomorphic copy of the component over  $[0, 1]$ ). The case of expanding Lukasiewicz logic when  $C$  contains irrational values is therefore an important particular case.

Of course, all those cases where at least one of the two conditions (i) and (ii) above is not satisfied also remain to be studied. It seems that for these remaining cases (i.e. when either the t-norm has infinitely many components or the set  $C$  does not cover  $[0, 1]$ ), a methodology similar to the one used in this paper could be applied. But in fact there is an explosion of cases to be considered and the need of new definitions and tools seems unavoidable. Let us show a couple of illustrative examples, the first when the set  $C$  does not cover  $[0, 1]$  and the second when the t-norm has infinitely many components.

*Example 1.* Let  $[0, 1]_* = [0, a]_\Pi \oplus [a, 1]_\Pi$  and let  $C = \{0, 1\} \cup \{b^n \mid n \in \mathbb{N}\}$  for some  $b < a$ . Obviously, there are only two proper filters of  $C$ ,  $F_1 = \{1\}$  and  $F_2 = C \setminus \{0\}$  but there are (up to isomorphism) three standard  $L_*(C)$ -chains. One, of type  $F_2$  in the sense used in this paper, is the  $L_*(C)$ -chain over  $[0, 1]_*$  where the constants different from  $\bar{0}$  are interpreted as 1 and  $\bar{0}$  is interpreted as 0. The other two are of type  $F_1$ . They are both  $L_*(C)$ -chains over  $[0, 1]_*$  where all constants are interpreted as different elements, either as powers of an element of the first product component or as powers of an element of the second product component. Of course, these two algebras are not isomorphic. This example shows that in general there is not a bijection between proper filters and standard algebras (hence Proposition 5.3 fails in this case) and, even though it seems possible to have the partial embeddability property,

the notion and treatment of standard chains should be modified in the case that  $C$  does not cover all components.

*Example 2.* Let  $[0, 1]_* = \bigoplus_{n \in \mathbb{N}} [a_n, a_{n+1}]_{\mathbb{L}}$ , where  $a_n = n/(n+1)$ , be an infinite ordinal sum of Łukasiewicz components where the idempotent elements form an increasing sequence with limit 1. For a given  $k > 2$ , let  $C_i$  the carrier of the  $k$ -element MV-subalgebra of  $[a_i, a_{i+1}]_{\mathbb{L}}$  and denote its elements as  $r_{1i} = a_i, r_{2i}, \dots, r_{ki} = a_{i+1}$ . Take  $C = \bigcup_{i \in \mathbb{N}} C_i \cup \{1\}$ . It is clear that  $C$  covers all the components but there are standard algebras where the interpretations of the truth-constants do not cover all the components. Indeed, let  $f$  be any strictly increasing mapping  $f : \mathbb{N} \rightarrow \mathbb{N}$  different from the identity such that  $f(1) = 1$ . One standard  $L_*(C)$ -algebra is the chain over  $[0, 1]_*$  where  $\bar{r}_{ij}$  is interpreted as  $r_{f(i)j}$ . An easy computation shows that this interpretation defines a standard  $L_*(C)$ -chain where the interpretations of truth-constants do not cover the real unit interval. In fact, there exists  $i$  such that  $f(i+1)$  is not the successor of  $f(i)$  (there are some natural numbers in between), and thus the components between the  $f(i)$ -th and the  $f(i+1)$ -th components contain no interpretations of truth-constants.

We conjecture that the study of completeness results for the expansions of the remaining logics of continuous t-norms, like the ones in the above examples, will be more in a case-by-case basis rather than by means of a new general theory.

## Acknowledgements

Esteva, Godo and Noguera acknowledge partial support of the Spanish project MULOG TIN2004-07933-C03-01. Gispert acknowledges partial support to the projects MULOG TIN2004-07933-C03-01/02 and MTM2004-031012001-3329.

## References

- [1] P. AGLIANÓ AND F. MONTAGNA. Varieties of BL-algebras I: general properties, *Journal of Pure and Applied Algebra*, 181 (2003) 105–129.
- [2] W. J. BLOK AND D. PIGOZZI. Algebraizable logics, *Mem. Amer. Math. Soc.* 396, vol 77, 1989.
- [3] S. BURRIS AND H.P. SANKAPPANAVAR, *A Course in Universal Algebra*, Graduate Texts in Mathematics, Springer Verlag, New York, 1981.
- [4] R. CIGNOLI, F. ESTEVA, L. GODO AND A. TORRENS. Basic Fuzzy Logic is the logic of continuous t-norms and their residua, *Soft Computing* 4 (2000) 106–112.
- [5] R. CIGNOLI AND D. MUNDICI. Partial isomorphisms on totally ordered abelian groups and Hájek’s completeness theorem for Basic Logic, *Multiple Valued Logic*, special issue dedicated to the memory of Grigore Moisil, vol. 6 (2001) 89-94
- [6] R. CIGNOLI AND A. TORRENS. An Algebraic Analysis of Product Logic, *Multiple Valued Logic* 5 (2000) 45–65.

- [7] P. CINTULA. *From Fuzzy Logic to Fuzzy Mathematics*. Ph.D. dissertation, Czech Technical University, Prague (Czech Republic), 2005.
- [8] P. CINTULA. Short note: on the redundancy of axiom (A3) in BL and MTL. *Soft Computing - A Fusion of Foundations, Methodologies and Applications*, Volume 9, Issue 12, Dec 2005, Page 942.
- [9] CZELAKOWSKI, J., DZIOBIAK, W. Congruence distributive quasivarieties whose finitely subdirectly irreducible members form a universal class. *Algebra Universalis*, 27 (1990) 128–149.
- [10] A. DI NOLA, MV-Algebras in the Treatment of Uncertainty, In: *Proceedings of the International IFSA Congress*, Bruxelles 1991 (P. Löwen and E. Roubens, eds), Kluwer, Dordrecht, 1993, 123–131.
- [11] M. DUMMETT. A propositional calculus with denumerable matrix, *The Journal of Symbolic Logic* 24 (1959) 97–106.
- [12] F. ESTEVA, L. GODO, P. HÁJEK, M. NAVARA. Residuated fuzzy logic with an involutive negation. *Archive for Mathematical Logic* 39 (2000) 103–124.
- [13] F. ESTEVA, L. GODO AND F. MONTAGNA. Equational characterization of the subvarieties of BL generated by t-norm algebras, *Studia Logica* 76 (2004) 161–200.
- [14] F. ESTEVA, L. GODO AND C. NOGUERA. On rational weak nilpotent minimum logics, *Journal of Multiple-Valued Logic and Soft Computing* 12 (2006), Number 1-2, pp. 9–32.
- [15] I.M.A. FERRERIM. *On varieties and quasivarieties of hoops and their reducts*. Ph.D. dissertation, University of Illinois at Chicago, 1992.
- [16] H. GAITÁN. The number of simple one-generated bounded commutative BCK-chains *Math. Japonica*, Vol. 38, No. 3, pp. 483–486, 1993.
- [17] S. GOTTFWALD. *A treatise on many-valued logics*, Studies in Logic and Computation, 9 Baldock: Research Studies Press, 2000.
- [18] P. HÁJEK. *Metamathematics of fuzzy logic*, Trends in Logic, vol. 4 Kluwer, 1998.
- [19] P. HÁJEK, L. GODO AND F. ESTEVA. A complete many-valued logic with product-conjunction, *Archive for Mathematical Logic* 35 (1996) 191–208.
- [20] L. HAY. Axiomatization of the infinite-valued predicate calculus, *The Journal of Symbolic Logic* 28 (1963) 77–86.
- [21] R. HORČÍK AND P. CINTULA. Product Łukasiewicz Logic. *Archive for Mathematical Logic* 43 (2004) 477–503.
- [22] C. M. LING. Representation of associative functions, *Publ. Math. Debrecen* 12 (1965) 189–212.
- [23] F. MONTAGNA, C. NOGUERA AND R. HORČÍK. On weakly cancellative fuzzy logics. To appear in *Journal of Logic and Computation*.

- [24] P. S. MOSTERT AND A. L. SHIELDS. On the structure of semigroups on a compact manifold with boundary, *Annals of Math.* 65 (1957) 117–143.
- [25] V. NOVÁK. On the syntactico-semantical completeness of first-order fuzzy logic. Part I. Syntax and Semantics. *Kybernetika* 26 (1990), 47-66.
- [26] V. NOVÁK. On the syntactico-semantical completeness of first-order fuzzy logic. Part II. Main Results. *Kybernetika* 26 (1990), 134-154.
- [27] V. NOVÁK, I. PERFILIEVA, J. MOČKOŘ. *Mathematical Principles of Fuzzy Logic*. Kluwer Academic Pub., 1999.
- [28] P. SAVICKÝ, R. CIGNOLI, F. ESTEVA, L. GODO AND C. NOGUERA. On product logic with truth-constants. *Journal of Logic and Computation* 2006 16: 205-225.
- [29] J. PAVELKA. On Fuzzy Logic I, II, III. *Zeitschrift fur Math. Logik und Grundlagen der Math.* 25 (1979) 45-52, 119-134, 447-464.
- [30] P. VOJTÁŠ. Fuzzy logic programming. *Fuzzy Sets and Systems* 124 (2001), 361-370.